Dynamic Information Aggregation in Asset Prices

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy is entirely my own work, and that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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A mamma Claudia e papá Pier Giorgio
a mio fratello Michele.

A mi novia Rocio
y a nuestro hijo Yeims.

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Contents

Notation

Basic Sets and Spaces

$\mathbb{N}^*$	$\mathbb{N} \setminus \{0\}$
$\mathbb{R}^*$	$\mathbb{R} \setminus \{0\}$
$\mathbb{B}$	$(0, 2/r) \setminus \{1/r\}$
$\mathbb{B}^+$	$(0, 2/r)$

Measures and sigma algebras

$\mathcal{B}$	Borel $\sigma-$algebra
$\lambda_{\vert[0+\infty[}$	Lebesgue measure on $[0 + \infty[$

Constants

$\bar{\alpha}$	$\left(\sum_{i=1}^{n} \frac{1}{\alpha_i}\right)^{-1}$

Operators

$x \otimes y$ for $x, y \in \mathbb{R}^n$	$(x \otimes y)_{ij} := x_i y_j$
$\Vert A\Vert$ for $A \in M_n(\mathbb{R})$	$\max_{1\leq i\leq n} \sum_{j=1}^{n} \vert a_{ij}\vert$
$\Vert v\Vert_\infty$ for $v \in \mathbb{R}^n$	$\max_{1\leq i\leq n} \vert v_i\vert$
$\Vert v\Vert_1$ for $v \in \mathbb{R}^n$	$\sum_{i=1}^{n} \vert v_i\vert$

Abstract

Dynamic Information Aggregation in Asset Prices

Luca Bernardinelli

This thesis investigates how the information dispersed among market participants dynamically aggregates in asset prices, the extent to which prices reflect available information, and how such information affects investors' decisions. The main model considers a population of investors with different absolute risk aversions and time-varying, diverse signals on the growth rate of an asset's dividends. Each investor bids the asset based on the information in his private signal and in the asset price itself, which is determined in equilibrium by the market-clearing condition and partially reflects the signals of other market participants. The dividend stream is driven by a latent variable, which investors strive to estimate based on their individual, private information, and on the common knowledge revealed by prices. We find in closed form equilibrium prices and the optimal behaviour of the agents. Price volatility depends on the volatility of dividends and on the volatility of the estimate of the latent variable, which is revealed to all agents through prices. Equilibrium prices do not reveal all the private signals of market participants, but the same estimate of the state of the economy that an agent with all private signals would be able to obtain. Put differently, prices reveal not all information but all relevant information. The first chapter presents a baseline model, where the only noise in the market is on the stochastic dividend process. In the second chapter dividends become mean reverting to a state variable observed by all agents - the state of the economy - which fluctuates over time. The state of the economy is unobservable in the last chapter, but market participants have individual information, which jointly with asset prices, helps them to estimate the latent variable.

Introduction

Any version of the efficient market hypothesis [14], weak, semi-strong or strong, implies that asset prices reveal part of the information available to market participants. This thesis investigates how the investors' heterogeneous information aggregates in asset prices and to which extent such knowledge spreads to the economy. How good is the information revealed from asset prices? Our models are inspired from the work of Hayek [17], who realises that agents are only aware of their surroundings and not of every change in the economy, and that *"knowledge never exists in concentrated or integrated form, but solely as the dispersed bits of incomplete and frequently contradictory knowledge which all the separate individuals possess"*[17]. The presence of many discordant investors contrasts the classic concept of a central planner, the fully informed rational authority analysing data and making decision on behalf of society.

The main model considers a population of investors with different absolute risk aversions and time-varying, diverse signals on the growth rate of an asset's dividends. Each investor bids the asset based on his private signal and on the asset price itself, which is determined in equilibrium by the market-clearing condition. The dividend stream is driven by a latent variable, which investors strive to estimate based on their individual, private information, and on the common knowledge revealed by prices. Asset prices do not only broadcast the present value of future dividends as in the classic theory, but they become a channel streaming not all the knowledge in the economy, but an aggregate flow of what is important. We prove the existence of an equilibrium and its uniqueness in the family of linear equilibria, we derive in closed form the unique optimal behaviour of the agents and the price of the risky asset, assuming that each investor has constant absolute risk aversion. Price volatility depends on the volatility of dividends and on the volatility of the estimate of the latent variable, which is revealed to all agents through prices.

Methods

While the market models of this thesis become more general and complex as the chapters flow, the proofs of existence and uniqueness of the equilibrium follow a common pattern. Assuming a parametric form of the linear price, we calculate the dynamics of a self-financing portfolio and we find heuristically the Hamilton-Jacobi-Bellman equation.

One of its solutions is our guess for the value function and we use it to conjecture optimal behaviours of the agents and stochastic discount factors. Once we get heuristics for the optimal policies, we verify admissibility and optimality of the consumption-investment processes and we use the market clearing condition, stating the presence of only one risky asset at each point in time, to conclude existence and uniqueness of the linear equilibrium.

Literature

The theory of heterogeneous information, formulated by Hayek [17] in 1945, has been extensively studied in one period models in which agents trade at time 0 and consume at time 1. In 1976 Grossman [15] proposes a model with a *"fully revealing equilibrium"* in which the price *"reveals information to each trader which is of "higher quality" than his own information"*. Hellwig [19] shows the existence of an equilibrium price dependent on the agents' preferences claiming that the less risk averse investors are, the more they act on new information. Diamond and Verrecchia [13] develop a non fully revealing equilibrium and Admati [1] generalizes the findings to more than one risky asset assuming a large number of investors. Vayanos and Wang [34] show heterogeneous information to raise expected returns and to affect several measures of liquidity. All these models analysing the cross-section reveal how prices aggregate knowledge on short terms risks but they do not explain how prices combine information on expectation of future cash flows.

As soon as agents are allowed to trade for more than one period, the price of risky assets becomes a signal from which investors wish to extract information. Prices depend on private and public information of market participants, which, in turn, depend on prices. Such endogenous information structure leads to an infinite regression described by Keynes in [23] and called "Forecasting the forecasts of others" by Townsend [33]. *"An understanding of financial markets requires an understanding not just of market participants' beliefs about assets' future payoffs, but also an understanding of market participants' beliefs about other market participants' beliefs, and higher-order beliefs"*[3]. Many authors have been tackling the issue of higher-order beliefs in multi-period discrete models because such beliefs complicate the search for closed-form solutions. Higher-order beliefs are linear functions of first order beliefs in [18] while Allen, Morris and Shin [3] focus their interest on the failure of the law of iterated expectations when dealing with second order beliefs. Bacchetta and Wincoop [6] show that the presence of higher-order beliefs reduces price volatility and the impact of expected changes in future dividends, thus moving the price away from present value of expected cash flow.

The models of our paper are inspired from those of Wang [36], who focuses on the effects of noise trading on price volatility in continuous time. The presence of noise traders introduces asymmetry of information among two families of agents: those who see the process of noise traders and those who do not. Separating the agents in two classes with homogeneous knowledge, avoids the infinite regression of information because the uninformed traders are the only ones wishing to learn from the price. Veronesi [35] studies a continuous time model with heterogeneous agents, where the dividends' drift is driven by a Markov chain with discrete state space. He shows that lower noise in the private signals increases the risk premium and that the equity premium is bounded from above by a constant not depending on the investors' risk aversion. Market participants, with heterogeneous information and thus endogenous filtrations, trade in continuous time in Qiu and Wang [30]. They argue that *"information heterogeneity tends to lower the level of asset prices, increase price volatility and return variability, and reduce trading volume"*.

Outline of the dissertation

The first chapter presents the simplest model of the thesis, where the only noise in the market is on the stochastic dividend process. We give sufficient conditions for the agents' optimal consumption-investment problem to be well-posed and ill-posed. In case of a well-posed problem, we solve it showing the agents' optimal strategies and the unique linear equilibrium in closed form. For the ill-posed optimal consumption problem we construct a maximizing sequence that yields in the limit zero expected utility, which cannot be attained by any strategy as the utility function is strictly negative.

In the second chapter dividends become mean reverting to a state variable, called state of the economy, stochastic but known by all agents. In the light of the findings in Chapter 1, we show a region in which the investors' optimization problem is well-posed and we find in closed form the unique linear equilibrium for a small noise of the state variable.

The full-blown model appears in Chapter 3, where the stochastic state of the economy is not adapted to the filtrations of the agents. Investors filter such a latent variable with public prices and private signals. We show the existence of an equilibrium and its uniqueness in the family of linear equilibria.

All chapters share a common structure: the formulation of the problem appears in the first section, the main result in the second one, while the third section contains the heuristics for the consumption-investment problem of the agents. The verification starts in the last section of each chapter and culminates with the results of existence of the equilibrium and its uniqueness in the class of linear equilibria.

Chapter 1

Baseline model

1.1 Model and main definitions

The economy has one risky asset in unit supply, which pays a dividend stream $(D_t)_{t\geq 0}$ described as

$$dD_t = (\bar{\pi} - kD_t)dt + \sigma_D dW_t^D. \tag{1.1.1}$$

There is a continuously compounded risk-free asset $(P_t^0)_{t\geq 0}$ with rate of return $r > 0$, at which investors can both lend and borrow. There are $n \in \mathbb{N}$ investors competing for the risky asset, with price $(P_t)_{t\geq 0}$. The $i-$th investor has constant absolute risk aversion $\alpha_i \geq 1$ and initial wealth $x_0^i \in \mathbb{R}$. $W^D = (W_t^D)_{t\geq 0}$ is a Brownian motion and D_0 is a normal random variable, with mean μ_D and variance Σ_D^2, independent of the Brownian motion previously defined. The probability space is $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$, where $\mathcal{G}_t$ is the augmented natural filtration of $D_0, (W_u)_{0\leq u\leq t}$ and $\mathcal{G}$ is the augmented sigma algebra generated by $\bigcup_{t\geq 0} \mathcal{G}_t$[1].

In such economy assume $k, \sigma_D > 0$ and $\bar{\pi} \in \mathbb{R}$. All equalities and inequalities between random variables are understood $\mathbb{P}-$almost surely.

Definition 1.1.1 (Admissibile strategies). $(c_t, \theta_t)_{t\geq 0}$ is an admissible (consumption-investment) strategy for the $i-$th investor if:

(i) $(c_t)_{t\geq 0}$ and $(\theta_t)_{t\geq 0}$ are $(\mathcal{G}_t)_{t\geq 0}-$progressively measurable processes;

(ii) for every $s \geq 0$

$$\limsup_{t\to+\infty} E\left[\int_s^t N_u c_u du \middle| \mathcal{G}_s\right] \leq N_s X_s, \tag{1.1.2}$$

where $(X_t)_{t\geq 0}$ is the self-financing wealth process

$$dX_t = -c_t dt + \theta_t D_t dt + r(X_t - \theta_t P_t)dt + \theta_t dP_t, \qquad X_0 = x_0^i, \tag{1.1.3}$$

[1] Note that all filtrations are augmented with the null sets of the sigma algebra $\mathcal{G}$.

$(N_t)_{t\geq 0}$ is the process

$$N_t = \exp\left(-rt + \int_0^t (\Delta_D D_u + \Delta_0)dW_u^D - \frac{1}{2}\int_0^t (\Delta_D D_u + \Delta_0)^2 du\right) \tag{1.1.4}$$

and Δ_D and Δ_0 are given in Definition 1.4.1 below.

The set of admissible strategies for the $i-$th investor is $\mathcal{U}^i$.

Definition 1.1.2 (Optimality). A (consumption-investment) strategy $(c_t^i, \theta_t^i)_{t\geq 0}$ is optimal for the $i-$th investor if it is admissible and if

$$\sup_{(c,\theta)\in\mathcal{U}^i} E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u)du\middle|\mathcal{G}_0\right] = E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u^i)du\middle|\mathcal{G}_0\right], \tag{1.1.5}$$

where

$$U^i(c) := -\frac{e^{-\alpha_i c}}{\alpha_i}, \qquad i \in \{1, \dots, n\}.$$

The time impatience parameter $\beta > 0$ is common to all agents. The consumption-investment problem of each agent is well-posed if an optimal strategy exists, otherwise the problem is ill-posed.

Remark 1.1.1. The process $(D_t)_{t\geq 0}$ starting with a random variable is not a fundamental feature of the model in this chapter but it will be important in the third chapter, for a stationary filter in Lemma 3.4.1. D_0 being a random variable implies that the $\sigma-$algebra $\mathcal{G}_0$ is different from the trivial $\sigma-$algebra, so a conditional expectation appears in (1.1.5).

Definition 1.1.3. $(c_t^i, \theta_t^i)_{t\geq 0}$ is the unique optimal (consumption-investment) strategy for the $i-$th investor if it is optimal for the $i-$th investor and if

$$(c_t^i, \theta_t^i)_{t\geq 0} = (\bar{c}_t, \bar{\theta}_t)_{t\geq 0} \qquad \lambda_{|[0,+\infty[} \otimes \mathbb{P} - \text{a.s.}$$

for every other optimal strategy $(\bar{c}_t, \bar{\theta}_t)_{t\geq 0}$.

Definition 1.1.4. A linear equilibrium is an $(n+2)-$tuple $(\epsilon_D, C, (S^i)^{1\leq i\leq n})$, where $\epsilon_D \in \mathbb{R}\setminus\{2/r\}, C \in \mathbb{R}$ and $S^i = (c_t^i, \theta_t^i)_{t\geq 0}$ is an optimal strategy for the $i-$th investor for every $i \in \{1, \dots, n\}$ such that for every $t \geq 0$

(i) the price of the risky asset is

$$P_t = C + \epsilon_D D_t; \tag{1.1.6}$$

(ii) the market clearing condition

$$\sum_{i=1}^n \theta_t^i = 1 \tag{1.1.7}$$

holds.

1.2 Existence and uniqueness of the equilibrium

Theorem 1.2.1. *There exists a unique linear equilibrium* $(\epsilon_D, C, (S^i)^{1\leq i\leq n})$, *for which the price is*

$$P_t = C^* + \epsilon_D^* D_t, \quad \text{where} \quad C^* = \frac{\bar{\pi}}{r(k+r)} - \frac{\bar{\alpha}\sigma_D^2}{(k+r)^2}, \epsilon_D^* = \frac{1}{k+r} \quad \text{and} \quad \bar{\alpha} = \left(\sum_{i=1}^{n} \frac{1}{\alpha_i}\right)^{-1}.$$

The unique optimal consumption-investment strategy for the $i-$*th agent is*

$$c_t^{i*} = rX_t^{i*} + \frac{\beta - r}{r\alpha_i} + \frac{\bar{\alpha}^2}{2\alpha_i}\frac{r\sigma_D^2}{(k+r)^2}, \qquad \theta_t^{i*} = \frac{\bar{\alpha}}{\alpha_i}. \tag{1.2.1}$$

For every $\epsilon_D \leq 0$ *or* $\epsilon_D > 2/r$ *the consumption-investment problem of the agents is ill-posed and in particular no linear equilibrium exists.*

Preliminaries and outline of the proof

Remark 1.2.1. If $\epsilon_D = 0$, then (1.1.6) implies $P_t = C$ for every $t \geq 0$. If the assets are two deterministic processes with different interest rates (0 for P_t and $r > 0$ for P_t^0), then the model admits arbitrage, therefore the consumption-investment problem of the agents is ill-posed and in particular no linear equilibrium exists.

Remark 1.2.2. Theorem 1.2.1 specifies that the consumption-investment problem of the agents is ill-posed for every $\epsilon_D \leq 0$ and $\epsilon_D > 2/r$. Furthermore the consumption-investment problem of the agents is well-posed for every $\epsilon_D \in \mathbb{B}^+$ and a unique equilibrium exists for $\epsilon_D \in \mathbb{R} \setminus \{2/r\}$. If $\epsilon_D = 2/r$ the solution of the HJB equation (1.4.3) is not exponential affine any more. We conjecture the consumption-investment problem of the agents to be ill-posed and the existence of a portfolio $(X_t^{iT}, c_t^{iT}, \theta_t^{iT})_{t\geq 0, T\in\mathbb{N}^*}$ satisfying (1.3.5).

Definition 1.2.1. A value function for the $i-$th investor is a function

$$V^i : \mathbb{R}^2 \to [-\infty, 0)$$
$$(\bar{x}, \bar{D}) \to V^i(\bar{x}, \bar{D})$$

such that for every $(\bar{x}, \bar{D}) \in \mathbb{R}^2$

$$V^i(\bar{x}, \bar{D}) = \sup_{(c,\theta)\in\mathcal{U}^i} E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u) du \middle| x_0^i = \bar{x}, D_0 = \bar{D}\right]. \tag{1.2.2}$$

It follows from this definition that if there exists a value function $V^i(\cdot)$ and a strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ optimal for the $i-$th investor, then

$$V^i(x_0^i, D_0) = E\left[\int_0^{+\infty} e^{-\beta s} U^i(c_s^{i*}) ds \middle| \mathcal{G}_0\right].$$

Definition 1.2.2. A stochastic discount factor (SDF) is a positive, continuous, $(\mathcal{G}_t)_{t\geq 0}-$adapted process $(M_t)_{t\geq 0}$ such that for every $0 \leq s \leq t$

$$M_s P_s^0 = E[M_t P_t^0 | \mathcal{G}_s] \tag{1.2.3}$$

and

$$M_s P_s + \int_0^s M_u D_u du = E\left[M_t P_t + \int_0^t M_u D_u du \middle| \mathcal{G}_s\right]. \tag{1.2.4}$$

A stochastic discount factor is normalized if $M_0 = 1$.

We find the (unique) equilibrium in the market in two steps: first we solve the optimal consumption problem of the agents for a generic price with the form of (1.1.6) or, when this is not possible, we show that such problem is ill-posed; then we clear the market with condition (1.1.7) and we deduce that the price of the unique linear equilibrium has parameters

$$C^* = \frac{\bar{\pi}}{r(k+r)} - \frac{\bar{\alpha}\sigma_D^2}{(k+r)^2} \qquad \text{and} \qquad \epsilon_D^* = \frac{1}{k+r}.$$

- **Section 1.3** contains the formal derivation of the results, divided in two subsections;
  - for $\epsilon_D \in \mathbb{B}^+ := (0, 2/r)$, **Subsection 1.3.1** formulates the Hamilton Jacobi Bellman (HJB) equation, which leads to a guess of the value function and the optimal strategies.
  - for $\epsilon_D < 0$ and for $\epsilon_D > 2/r$, **Subsection 1.3.2** formulates the HJB equation for the finite horizon, which leads to a sequence of admissible strategies whose total utility tends to 0 as the horizon approaches $+\infty$.
- **Section 1.4** formalizes the heuristics of the previous section.
  - **Subsection 1.4.1** proves the existence and uniqueness of the optimal portfolio for a generic price function for $\epsilon_D \in \mathbb{B}^+$.
  - **Subsection 1.4.2** proves the consumption-investment problem of the agents to be ill-posed for $\epsilon_D < 0$ and for $\epsilon_D > 2/r$, finding a sequence of admissible strategy whose total utility converges to 0 as the horizon approaches $+\infty$.
  - **Subsection 1.4.3** finds the unique linear equilibrium in the market through the market clearing condition.
- **Appendixes A and B** recall some well known results that are used along the chapter.

1.3 Heuristics

1.3.1 Well posed problem

Guess a value function V^i which depends on the dividend rate and on the wealth; because of the infinite time horizon we guess that V^i does not depend on the initial time $t > 0$, i.e.

$$V^i(X_t^i, D_t) = \sup_{(c^i,\theta^i)\in\mathcal{U}^i} E\left[\int_t^{+\infty} e^{-\beta(s-t)}U^i(c_s^i)ds\Big|\mathcal{G}_t\right].$$

Splitting the integral at time $t+h$ and using the tower property, the value function is

$$V^i(X_t^i, D_t) = \sup_{(c^i,\theta^i)} E\left[\int_t^{t+h} e^{-\beta(s-t)}U^i(c_s^i)ds + e^{-\beta h}E\left[\int_{t+h}^{+\infty} e^{-\beta[s-(t+h)]}U^i(c_s^i)ds\Big|\mathcal{G}_{t+h}\right]\Big|\mathcal{G}_t\right].$$

Multiplying both sides by $e^{-\beta t}$ and since $E\left[\int_{t+h}^{+\infty} e^{-\beta[s-(t+h)]}U^i(c_s^i)ds|\mathcal{G}_{t+h}\right] = V^i(X_{t+h}^i, D_{t+h})$ we get

$$E\left[e^{-\beta(t+h)}V^i(X_{t+h}^i, D_{t+h}) - e^{-\beta t}V^i(X_t^i, D_t)\Big|\mathcal{G}_t\right] = -e^{-\beta t}\sup_{(c^i,\theta^i)} E\left[\int_t^{t+h} e^{-\beta(s-t)}U^i(c_s^i)ds\Big|\mathcal{G}_t\right]. \tag{1.3.1}$$

Applying Itô's formula to the function $e^{-\beta t}V^i(X_t^i, D_t)$ yields

$$\begin{aligned} e^{-\beta(t+h)}V^i(X_{t+h}^i, D_{t+h}) = e^{-\beta t}V^i(X_t^i, D_t) + \sup_{(c^i,\theta^i)}\int_t^{t+h} e^{-\beta s}\Big\{ -\beta V^i + V_x^i\Big[-c_s^i + rX_s^i + \\ \theta_s^i(\epsilon_D\bar{\pi} - rC) + \theta_s^i D_s(1 - r\epsilon_D - \epsilon_D k)\Big] + V_D^i(\bar{\pi} - kD_s) + \frac{1}{2}\Big[V_{xx}^i(\theta_s^i)^2\epsilon_D^2\sigma_D^2 + \\ V_{DD}^i\sigma_D^2 + 2V_{xD}^i\theta_s^i\epsilon_D\sigma_D^2\Big]\Big\}ds + \int_t^{t+h} e^{-\beta s}\left(V_x^i\theta_s^i\epsilon_D\sigma_D + V_D^i\sigma_D\right)dW_s^D. \end{aligned}$$

Assuming sufficient regularity for the value function V^i and for the investment strategy θ^i, the Brownian term is a martingale and therefore taking expectation of both sides and using (1.3.1), we get

$$\begin{aligned} -e^{-\beta t}\sup_{(c^i,\theta^i)} E\left[\int_t^{t+h} e^{-\beta(s-t)}U^i(c_s^i)ds\Big|\mathcal{G}_t\right] = \sup_{(c^i,\theta^i)} E\Big[\int_t^{t+h} e^{-\beta s}\Big\{-\beta V^i + V_x^i\Big[-c_s^i + rX_s^i + \\ + \theta_s^i(\epsilon_D\bar{\pi} - rC) + \theta_s^i D_s(1 - r\epsilon_D - \epsilon_D k)\Big] + V_D^i(\bar{\pi} - kD_s) + \\ + \frac{1}{2}\Big[V_{xx}^i(\theta_s^i)^2\epsilon_D^2\sigma_D^2 + V_{DD}^i\sigma_D^2 + 2V_{xD}^i\theta_s^i\epsilon_D\sigma_D^2\Big]\Big\}ds\Big|\mathcal{G}_t\Big]. \end{aligned}$$

Dividing both sides by h yields

$$0 = \sup_{(c^i,\theta^i)} \Big\{ -\frac{e^{-\alpha_i c^i}}{\alpha_i} - \beta V^i + V^i_x\Big[-c^i + rx + \theta^i(\epsilon_D\bar{\pi} - rC) + \theta^i D(1 - r\epsilon_D - \epsilon_D k)\Big] + \\ + V^i_D(\bar{\pi} - kD) + \frac{1}{2}\Big[V^i_{xx}(\theta^i)^2\epsilon_D^2\sigma_D^2 + V^i_{DD}\sigma_D^2 + 2V^i_{xD}\theta^i\epsilon_D\sigma_D^2\Big]\Big\}. \quad (1.3.2)$$

Differentiating with respect to c^i and θ^i, we find the candidate optimal consumption-investment policy

$$c^{i*} = -\frac{\log(V^i_x)}{\alpha_i}, \quad \theta^{i*} = -\frac{V^i_x\Big[(\epsilon_D\bar{\pi} - rC) + D(1 - r\epsilon_D - \epsilon_D k)\Big] + V^i_{xD}\epsilon_D\sigma_D^2}{V^i_{xx}\epsilon_D^2\sigma_D^2}. \quad (1.3.3)$$

The HJB equation for the $i-$th investor follows by substituting the candidate optimal policies into (1.3.2)

$$0 = -\frac{V^i_x}{\alpha_i} - \beta V^i + V^i_x\Big[\frac{\log(V^i_x)}{\alpha_i} + rx + \theta^{i*}(\epsilon_D\bar{\pi} - rC) + \theta^{i*}D(1 - r\epsilon_D - \epsilon_D k)\Big] + \\ + V^i_D(\bar{\pi} - kD) + \frac{1}{2}\Big[V^i_{xx}(\theta^{i*})^2\epsilon_D^2\sigma_D^2 + V^i_{DD}\sigma_D^2 + 2V^i_{xD}\theta^{i*}\epsilon_D\sigma_D^2\Big]. \quad (1.3.4)$$

Using the Ansatz $V^i(x, D) = -\frac{1}{r\alpha_i}\exp(-r\alpha_i x + \delta_{DD}D^2 + \delta_D D + \delta_0)$, where $\delta_{DD}, \delta_D, \delta_0$ are in Theorem 1.4.1, (1.3.3) leads to the optimal consumption investment strategy

$$c^{i*}_t = rX^{i*}_t - \frac{\delta_{DD}}{\alpha_i} - \frac{\delta_D}{\alpha_i} - \frac{\delta_0}{\alpha_i}, \qquad \theta^{i*}_t = \frac{M_D D_t + M_0}{M\alpha_i},$$

where M_D and M_0 are in Definition 1.4.1.

1.3.2 Ill-posed problem

We construct a sequence $(c^{iT}_t, \theta^{iT}_t)_{t\geq 0, T\in\mathbb{N}, i\in\{1,\ldots,n\}}$ of admissible strategies such that

$$\sup_{T\in\mathbb{N}} E\left[\int_0^{+\infty} e^{-\beta(s-t)}U^i(c^{iT}_s)ds\Big|\mathcal{G}_0\right] = 0 \quad (1.3.5)$$

for every $\epsilon_D < 0$ or $\epsilon_D > 2/r$. Equality (1.3.5) shows the consumption-investment problem of the agents to be ill-posed because utility 0 is not attainable since the utility function is strictly negative.

Guess a value function $V^i = V^i(t, T, x, D)$ for the problem with finite horizon $T > 0$, i.e. suppose that

$$V^i(t, T, X^i_t, D_t) = \sup_{(c^i,\theta^i)\in\mathcal{U}^i} E\left[\int_t^T e^{-\beta(s-t)}U^i(c^i_s)ds\Big|\mathcal{G}_t\right].$$

Splitting the integral at time $t+h$ and using the tower property, the value function is

$$V^i(t,T,X^i_t,D_t) = \sup_{(c^i,\theta^i)} E\left[\int_t^{t+h} e^{-\beta(s-t)}U^i(c^i_s)ds + e^{-\beta h}E\left[\int_{t+h}^T e^{-\beta[s-(t+h)]}U^i(c^i_s)ds\Big|\mathcal{G}_{t+h}\right]\Big|\mathcal{G}_t\right].$$

Multiplying both sides by $e^{-\beta t}$ and since $E\left[\int_{t+h}^T e^{-\beta[s-(t+h)]}U^i(c^i_s)ds|\mathcal{G}_{t+h}\right]$
$= V^i(t,T,X^i_{t+h},D_{t+h})$, we get

$$E\left[e^{-\beta(t+h)}V^i(t+h,T,X^i_{t+h},D_{t+h}) - e^{-\beta t}V^i(t,T,X^i_t,D_t)\Big|\mathcal{G}_t\right] = \\ -e^{-\beta t}\sup_{(c^i,\theta^i)} E\left[\int_t^{t+h} e^{-\beta(s-t)}U^i(c^i_s)ds\Big|\mathcal{G}_t\right]. \quad (1.3.6)$$

Itô's formula, applied to the function $e^{-\beta t}V^i(t,T,X^i_t,D_t)$, yields

$$e^{-\beta(t+h)}V^i(t+h,T,X^i_{t+h},D_{t+h}) = e^{-\beta t}V^i(t,T,X^i_t,D_t) + \sup_{(c^i,\theta^i)}\int_t^{t+h} e^{-\beta s}\Big\{V^i_t - \beta V^i + \\ +V^i_x\Big[-c^i_s+rX^i_s+\theta^i_s(\epsilon_D\bar{\pi}-rC)+\theta^i_sD_s(1-r\epsilon_D-\epsilon_Dk)\Big]+V^i_D(\bar{\pi}-kD_s)+\frac{1}{2}\Big[V^i_{xx}(\theta^i_s)^2\epsilon_D^2\sigma_D^2+ \\ +V^i_{DD}\sigma_D^2 + 2V^i_{xD}\theta^i_s\epsilon_D\sigma_D^2\Big]\Big\}ds + \int_t^{t+h} e^{-\beta s}\left(V^i_x\theta^i_s\epsilon_D\sigma_D + V^i_D\sigma_D\right)dW^D_s.$$

Assuming sufficient regularity for the value function V^i and for the investment strategy θ^i, the Brownian term is a martingale and therefore taking expectations of both sides and using (1.3.6), we get

$$-e^{-\beta t}\sup_{(c^i,\theta^i)} E\left[\int_t^{t+h} e^{-\beta(s-t)}U^i(c^i_s)ds\Big|\mathcal{G}_t\right] = \sup_{(c^i,\theta^i)} E\Big[\int_t^{t+h} e^{-\beta s}\Big\{V^i_t-\beta V^i+V^i_x\Big[-c^i_s+rX^i_s+ \\ +\theta^i_s(\epsilon_D\bar{\pi}-rC)+\theta^i_sD_s(1-r\epsilon_D-\epsilon_Dk)\Big]+V^i_D(\bar{\pi}-kD_s)+\frac{1}{2}\Big[V^i_{xx}(\theta^i_s)^2\epsilon_D^2\sigma_D^2+V^i_{DD}\sigma_D^2+2V^i_{xD}\theta^i_s\epsilon_D\sigma_D^2\Big]\Big\}ds\Big|\mathcal{G}_t\Big].$$

Dividing both sides by h yields

$$0 = \sup_{(c^i,\theta^i)}\Big\{-\frac{e^{-\alpha_i c^i}}{\alpha_i} - \beta V^i + V^i_t + V^i_x\Big[-c^i_s + rx + \theta^i(\epsilon_D\bar{\pi}-rC) + \theta^iD(1-r\epsilon_D-\epsilon_Dk)\Big] + \\ + V^i_D(\bar{\pi}-kD) + \frac{1}{2}\Big[V^i_{xx}(\theta^i)^2\epsilon_D^2\sigma_D^2 + V^i_{DD}\sigma_D^2 + 2V^i_{xD}\theta^i\epsilon_D\sigma_D^2\Big]\Big\}. \quad (1.3.7)$$

Differentiating with respect to c^i and θ^i we find the candidate optimal consumption-investment policy

$$c^{i*} = -\frac{\log(V^i_x)}{\alpha_i}, \quad \theta^{i*} = -\frac{V^i_x\Big[(\epsilon_D\bar{\pi}-rC) + D(1-r\epsilon_D-\epsilon_Dk)\Big] + V^i_{xD}\epsilon_D\sigma_D^2}{V^i_{xx}\epsilon_D^2\sigma_D^2}. \quad (1.3.8)$$

The HJB equation for the $i-$th investor follows substituting the candidate optimal policies into (1.3.7)

$$0 = -\frac{V^i_x}{\alpha_i} - \beta V^i + V^i_t + V^i_x\Big[\frac{\log(V^i_x)}{\alpha_i} + rx + \theta^{i*}(\epsilon_D\bar{\pi} - rC) + \theta^{i*}D(1 - r\epsilon_D - \epsilon_D k)\Big] + \\ + V^i_D(\bar{\pi} - kD) + \frac{1}{2}\Big[V^i_{xx}(\theta^{i*})^2\epsilon_D^2\sigma_D^2 + V^i_{DD}\sigma_D^2 + 2V^i_{xD}\theta^{i*}\epsilon_D\sigma_D^2\Big]. \quad (1.3.9)$$

Using the Ansatz

$$V^i(t,T,x,D) = \frac{(1-e^{r(T-t)})}{r\alpha_i}\exp\left(-\frac{e^{r(T-t)}}{e^{r(T-t)}-1}r\alpha_i x + \delta_{\mathbf{DD}}(T-t)D^2 + \delta_{\mathbf{D}}(T-t)D + \delta_{\mathbf{0}}(T-t)\right),$$

(1.3.8) leads to the optimal consumption investment policies

$$\begin{aligned} c^{iT}_t &= \frac{e^{r(T-t)}}{e^{r(T-t)}-1}rX^{iT}_t - \frac{\delta_{\mathbf{DD}}(T-t)}{\alpha_i} - \frac{\delta_{\mathbf{D}}(T-t)}{\alpha_i} - \frac{\delta_{\mathbf{0}}(T-t)}{\alpha_i} - \frac{(T-t)r}{\alpha_i}, \quad (1.3.10) \\ \theta^{iT}_t &= \frac{\mathbf{M_D}(T-t)D_t + \mathbf{M_0}(T-t)}{M\alpha_i}, \end{aligned}$$

where the constant M and the functions $\delta_{\mathbf{DD}}, \delta_{\mathbf{D}}, \delta_{\mathbf{0}}, \mathbf{M_D}, \mathbf{M_0}$ are those of Definition B.0.1. We show (in Theorems 1.4.7 and 1.4.8 below) that an obvious extension of the policies (1.3.10) gives a sequence of admissible strategies $(c^{iT}_t, \theta^{iT}_t)_{t\geq 0, T\in\mathbb{N}, i\in\{1,\ldots,n\}}$ satisfying (1.3.5). As a consequence the optimal consumption problem is ill-posed whenever $\epsilon_D < 0$ or $\epsilon_D > 2/r$.

1.4 Verification

1.4.1 Well-posed problem

Theorem 1.2.1 identifies the unique linear equilibrium in the market. The first step of the proof is to solve the consumption investment problem of the agents for a generic price with form (1.1.6), when $\epsilon_D \in \mathbb{B}^+ := (0, 2/r)$.

Direct calculations show that the self-financing condition (1.1.3) for an investor with consumption-investment strategy $(c_t^i, \theta_t^i)_{t\geq 0}$ is equivalent to

$$dX_t^i = \Big[- c_t^i + rX_t^i + \theta_t^i(\epsilon_D \bar{\pi} - rC) + \theta_t^i D_t \left(1 - \epsilon_D(k + r)\right) \Big] dt + \theta_t^i \epsilon_D \sigma_D dW_t^D. \quad (1.4.1)$$

The following theorem proves the existence of a solution of the HJB equation, and thus a candidate value function.

Theorem 1.4.1. *Fix $\epsilon_D \in \mathbb{B}^+$ and define*

$$\delta_{DD} = \frac{(-1 + \epsilon_D(k + r))^2}{2\sigma_D^2 \epsilon_D(\epsilon_D r - 2)}, \quad \delta_D = \frac{\Big(-1 + (k + r)\epsilon_D \Big)}{\sigma_D^2 \epsilon_D (r\epsilon_D - 2)} \Big[\epsilon_D \bar{\pi}(r\epsilon_D - 2) + Cr(1 + k\epsilon_D) \Big],$$

$$\delta_0 = \frac{r - \beta}{r} - \frac{(\epsilon_D \bar{\pi} - rC)^2}{2r\epsilon_D^2 \sigma_D^2} + \frac{C\Big(-1 + (k + r)\epsilon_D \Big)}{\sigma_D^2 \epsilon_D^2 (r\epsilon_D - 2)} \Big[\epsilon_D \bar{\pi}(r\epsilon_D - 2) + Cr(1 + k\epsilon_D) \Big] + + \frac{\Big(-1 + (k + r)\epsilon_D \Big)^2}{2r\epsilon_D(r\epsilon_D - 2)}.$$

Then for every $i \in \{1, \ldots, n\}$ the function

$$V^i(x, D) = -\frac{1}{r\alpha_i} \exp\Big(-r\alpha_i x + \delta_{DD} D^2 + \delta_D D + \delta_0 \Big), \quad (1.4.2)$$

solves the Hamilton Jacobi Bellman equation

$$0 = -\frac{V_x^i}{\alpha_i} - \beta V^i + V_x^i \Big[\frac{\log(V_x^i)}{\alpha_i} + rx + \theta^{i*}(\epsilon_D \bar{\pi} - rC) + \theta^{i*} D(1 - r\epsilon_D - \epsilon_D k) \Big] + + V_D^i(\bar{\pi} - kD) + \frac{1}{2} \Big[V_{xx}^i (\theta^{i*})^2 \epsilon_D^2 \sigma_D^2 + V_{DD}^i \sigma_D^2 + 2V_{xD}^i \theta^{i*} \epsilon_D \sigma_D^2 \Big], \quad (1.4.3)$$

where

$$\theta^{i*} = -\frac{V_x^i \Big[(\epsilon_D \bar{\pi} - rC) + D(1 - r\epsilon_D - \epsilon_D k) \Big] + V_{xD}^i \epsilon_D \sigma_D^2}{V_{xx}^i \epsilon_D^2 \sigma_D^2}.$$

Proof. Inserting (1.4.2) into (1.4.3) and comparing coefficients reveals that $V^i(x, D)$ indeed solves the HJB equation. □

The following are technical results for the solution of the consumption-investment problem.

Lemma 1.4.1. There exist constants $\bar{\mu}, \bar{\sigma} > 0$ independent by t such that, for every $t \geq 0$,

$$|E[D_t]| \leq \bar{\mu}, \qquad \text{Var}[D_t] \leq \bar{\sigma}^2.$$

Proof. Apply Itô's formula to $e^{kt}D_t$ to get

$$D_t = e^{-k(t-s)}D_s + \frac{\bar{\pi}}{k}\left(1 - e^{-k(t-s)}\right) + \sigma_D e^{-kt}\int_s^t e^{ku}dW_u^D.$$

Since D_0 is normal with mean μ_D and variance Σ_D^2, then $|E[D_t]| \leq |\mu_D| + 2\frac{|\bar{\pi}|}{k}$ and $\text{Var}[D_t] \leq \Sigma_D^2 + \frac{\sigma_D^2}{k}$. □

The value of the constants Δ_D, Δ_0 will be set later in Definition 1.4.1

Lemma 1.4.2. For every $\Delta_D, \Delta_0 \in \mathbb{R}$, the process

$$H_t = \exp\left(\int_0^t (\Delta_D D_u + \Delta_0)dW_u^D - \frac{1}{2}\int_0^t (\Delta_D D_u + \Delta_0)^2 du\right)$$

is a $\mathbb{P}-$martingale.

Proof. Define $Y_t = \Delta_D D_u + \Delta_0$ and recall Novikov's condition [22, Corollary 5.13], which ensures that H_t is a martingale:

(A) $\mathbb{P}\left[\int_0^t Y_u^2 du < +\infty\right] = 1$;

(B) there exists a sequence $(t_m)_{m\in\mathbb{N}} \subset \mathbb{R}$ increasing to $+\infty$, such that, for every $m \in \mathbb{N}$,

$$E\left[\exp\left(\int_{t_{m-1}}^{t_m} \frac{1}{2}Y_u^2 du\right)\right] < +\infty.$$

The process $(Y_t)_{t\geq 0}$ is $\mathbb{P}-$a.s. continuous, hence (A) is true. By Jensen's inequality [28, Theorem 1.8.1], for every $t, \epsilon \geq 0$,

$$\exp\left(\int_t^{t+\epsilon} \frac{1}{2}Y_u^2 du\right) \leq \frac{1}{\epsilon}\int_t^{t+\epsilon} \exp\left(\frac{\epsilon}{2}Y_u^2\right) du$$

In addition, by Fubini's Theorem [4, Theorem 1.1.7] it follows that

$$E\left[\exp\left(\int_t^{t+\epsilon} \frac{1}{2}Y_u^2 du\right)\right] \leq \frac{1}{\epsilon}\int_t^{t+\epsilon} E\left[\exp\left(\epsilon Y_u^2\right)\right] du.$$

If $\Delta_D \neq 0$, define $\mu_u = E[Y_u]$ and $\sigma_u^2 = \text{Var}[Y_u]$. In view of Lemma 1.4.1, there exist constants $\bar{\mu}$ and $\bar{\sigma}^2$ such that

$$|\mu_u| \leq \bar{\mu}, \qquad \sigma_u^2 \leq \bar{\sigma}^2, \tag{1.4.4}$$

for every $u \geq 0$. For every $u \geq 0$, Y_u is a normally distributed random variable, and in particular

$$E\Big[\exp\Big(\epsilon Y_u^2\Big)\Big] = \frac{\exp\Big(\frac{\mu_u^2 \epsilon}{1-2\sigma_u^2\epsilon}\Big)}{\sqrt{1-2\sigma_u^2\epsilon}}, \qquad \text{if} \qquad 2\sigma_u^2\epsilon \leq 1.$$

Since $\sigma_u^2 \leq \bar{\sigma}^2$, then any $\epsilon < \frac{1}{2}\bar{\sigma}^{-2}$ satisfies $2\sigma_u^2\epsilon < 1$ because

$$2\sigma_u^2\epsilon \leq 2\bar{\sigma}^2\epsilon < 1. \tag{1.4.5}$$

Fix $\epsilon < \frac{1}{2}\bar{\sigma}^{-2}$; if we prove that $E\Big[\exp\Big(\epsilon Y_u^2\Big)\Big]$ is a continuous function, uniformly bounded in t on the interval $[t, t+\epsilon]$, for the ϵ chosen above, then its integral is finite and it is enough to define the sequence $t_m = m\epsilon$. Equation (1.4.5) implies $1 - 2\sigma_u^2\epsilon \geq 1 - 2\bar{\sigma}^2\epsilon$, and both terms are between 0 and 1 because of the choice of ϵ. Thus, defining $\kappa_\epsilon = \frac{1}{1-2\bar{\sigma}^2\epsilon}$, it follows that

$$\frac{1}{1-2\sigma_u^2\epsilon} \leq \kappa_\epsilon \qquad \text{and} \qquad \frac{1}{\sqrt{1-2\sigma_u^2\epsilon}} \leq \kappa_\epsilon,$$

for every $u \geq 0$. As a consequence

$$E\Big[\exp\Big(\epsilon Y_u^2\Big)\Big] \leq \kappa_\epsilon \exp\Big(\kappa_\epsilon \epsilon \bar{\mu}^2\Big) < +\infty.$$

$E\Big[\exp\Big(\epsilon Y_u^2\Big)\Big]$ is a continuous and bounded function on the interval $[t, t+\epsilon]$ and so for every $\epsilon > 0$ and every $t \geq 0$

$$E\left[\frac{1}{\epsilon}\int_t^{t+\epsilon} \exp\Big(\epsilon Y_u^2\Big) du\right] = \frac{1}{\epsilon}\int_t^{t+\epsilon} E\Big[\exp\Big(\epsilon Y_u^2\Big)\Big] du < +\infty.$$

□

Definition 1.4.1. We introduce the following constants

$$M := r\epsilon_D^2\sigma_D^2,$$

$$M_D := 1 - \epsilon_D(k+r) + 2\delta_{DD}\epsilon_D\sigma_D^2 = 1 - \epsilon_D(k+r) + \frac{(1-(k+r)\epsilon_D)^2}{r\epsilon_D - 2},$$

$$M_0 := \epsilon_D\bar{\pi} - rC + \delta_D\epsilon_D\sigma_D^2 = \epsilon_D\bar{\pi} - rC + \frac{(-1+(k+r)\epsilon_D)(rC(1+k\epsilon_D) + \epsilon_D\bar{\pi}(r\epsilon_D - 2))}{r\epsilon_D - 2},$$

and

$$\Delta_D := -\frac{1-\epsilon_D(k+r)}{\epsilon_D\sigma_D}, \qquad \Delta_0 := \frac{1}{\epsilon_D\sigma_D}(rC - \bar{\pi}\epsilon_D). \tag{1.4.6}$$

Corollary 1.4.1. *The process* $(\mathcal{E}_t)_{t\geq 0} = (e^{rt}N_t)_{t\geq 0}$, *in* (1.1.4), *is a* $\mathbb{P}-$*martingale.*

Since $(\mathcal{E}_t)_{t\geq 0}$ is a $\mathbb{P}-$martingale, Girsanov's Theorem [22, Theorem 5.1] holds. In particular, $(\mathcal{E}_t)_{t\geq 0}$ defines a probability measure $\bar{\mathbb{P}} := \bar{\mathbb{P}}^{\Delta}$, such that $\mathcal{E} = d\bar{\mathbb{P}}/d\mathbb{P}$. Any equality or inequality between random variables is understood $\mathbb{P}$ and $\bar{\mathbb{P}}-$almost surely. We denote by $\bar{E}[\cdot]$ and $\bar{\mathrm{V}}\mathrm{ar}[\cdot]$ the conditional expectation and variance under the measure $\bar{\mathbb{P}}$. The process

$$\bar{W}_t^D = W_t^D - \int_0^t (\Delta_D D_u + \Delta_0) du \tag{1.4.7}$$

is a $\bar{\mathbb{P}}-$Brownian motion and furthermore Bayes' formula [22, Lemma 5.3] applies: for every $\mathcal{G}_t-$measurable random variable X satisfying $\bar{E}[|X|] < +\infty$ and for every $0 \leq s \leq t$

$$\bar{E}[X|\mathcal{G}_s] = \frac{1}{\mathcal{E}_s} E[X\mathcal{E}_t|\mathcal{G}_s].$$

The next lemma describes the process $(D_t)_{t\geq 0}$ under the new measure $\bar{\mathbb{P}}$.

Lemma 1.4.3. For every $\epsilon_D \neq 0$, the process $(D_t)_{t\geq 0}$ satisfies the stochastic differential equation

$$dD_t = (AD_t + b)dt + \sigma_D d\bar{W}_t^D; \tag{1.4.8}$$

where

$$A = \left(r - \frac{1}{\epsilon_D}\right) = \frac{r\epsilon_D - 1}{\epsilon_D} \qquad \text{and} \qquad b = \frac{rC}{\epsilon_D}.$$

The unique solution of (1.4.8) is

$$\begin{aligned} D_t &= e^{A(t-s)} D_s + \frac{b}{A}\left(e^{A(t-s)} - 1\right) + \sigma_D e^{At} \int_s^t e^{-Au} d\bar{W}_u^D && (\epsilon_D \neq 1/r) \\ D_t &= D_s + b(t-s) + \sigma_D(\bar{W}_t^D - \bar{W}_s^D) && (\epsilon_D = 1/r). \end{aligned} \tag{1.4.9}$$

For every $0 \leq s \leq t$ there exists a $\mathcal{G}_s-$measurable random variable $\eta_s \geq 0$ and a positive constant η such that

$$\begin{array}{ccc} \epsilon_D \neq 1/r & \epsilon_D = 1/r & (1.4.10) \\ \left|\bar{E}\left[D_t|\mathcal{G}_s\right]\right| \leq \eta_s e^{|A|t} + \eta, & \left|\bar{E}\left[D_t|\mathcal{G}_s\right]\right| \leq \eta_s + bt, & (i) \\ \bar{\mathrm{V}}\mathrm{ar}\left[D_t|\mathcal{G}_s\right] \leq \eta e^{2|A|t} + \eta, & \bar{\mathrm{V}}\mathrm{ar}\left[D_t|\mathcal{G}_s\right] \leq \eta t, & (ii) \\ \left(\bar{E}[D_t|\mathcal{G}_s]\right)^2 \leq \eta_s e^{2|A|t} + \eta, & \left(\bar{E}[D_t|\mathcal{G}_s]\right)^2 \leq \eta_s + \eta_s t^2, & (iii) \\ \bar{E}\left[D_t^2|\mathcal{G}_s\right] \leq \eta_s e^{2|A|t} + \eta, & \bar{E}\left[D_t^2|\mathcal{G}_s\right] \leq \eta_s + \eta_s t^2. & (iv) \end{array}$$

$$\begin{array}{lll} \epsilon_D \neq 1/r & \epsilon_D = 1/r & (1.4.11) \\ \left|\bar{E}\left[D_t\right]\right| \leq \eta e^{|A|t} + \eta, & \left|\bar{E}\left[D_t\right]\right| \leq \eta + \eta t, & (i) \\ \bar{\mathrm{V}}\mathrm{ar}\left[D_t\right] \leq \eta e^{2|A|t} + \eta, & \bar{\mathrm{V}}\mathrm{ar}\left[D_t\right] \leq \eta + \eta t, & (ii) \\ \left(\bar{E}[D_t]\right)^2 \leq \eta e^{2|A|t} + \eta, & \left(\bar{E}[D_t]\right)^2 \leq \eta + \eta t^2, & (iii) \\ \bar{E}\left[D_t^2\right] \leq \eta e^{2|A|t} + \eta, & \bar{E}\left[D_t^2\right] \leq \eta + \eta t^2. & (iv) \end{array}$$

For every $s \geq 0$, for every $\eta_0, \eta_1, \eta_2 \in \mathbb{R}$ and for every $\mathcal{G}_s-$measurable random variables $\eta_{s,0}, \eta_{s,1}, \eta_{s,2}$

(a)

$$E\left[\int_s^t e^{-ru}\mathcal{E}_u D_u du \Big| \mathcal{G}_s\right] = \mathcal{E}_s \bar{E}\left[\int_s^t e^{-ru} D_u du \Big| \mathcal{G}_s\right];$$

(b)

$$-\infty < \bar{E}\left[\int_s^t \eta_2 D_u^2 + \eta_1 D_u + \eta_0 du\right] < +\infty;$$

(c)

$$\int_s^t (\eta_1 D_u + \eta_0) d\bar{W}_u^D \qquad \text{is } \bar{\mathbb{P}}-\text{martingale};$$

(d) for every $\epsilon_D \in \mathbb{B}^+$

$$\lim_{t\to+\infty} e^{-rt}\bar{E}\left[(\eta_{s,2}D_t^2 + \eta_{s,1}D_t + \eta_{s,0})|\mathcal{G}_s\right] = \lim_{t\to+\infty} e^{-rt}\bar{E}\left[\int_s^t (\eta_{s,2}D_u^2 + \eta_{s,1}D_u + \eta_{s,0})du \Big| \mathcal{G}_s\right] = 0;$$

(e) for every $\epsilon_D \in \mathbb{B}^+$ and for every $s \geq 0$

$$\frac{1}{2}\lim_{t\to+\infty} \bar{E}\left[\int_s^t e^{-r(u-s)}(\Delta_D D_u + \Delta_0)^2 du \Big| \mathcal{G}_s\right] = -\delta_{DD}D_s^2 - \delta_D D_s - \delta_0 - \frac{\beta - r}{r}.$$

Proof. The following proofs are for $\epsilon_D \neq 1/r$; the steps verifying the claims for $\epsilon_D = 1/r$ are analogous. (1.4.8) is a direct consequence of (1.4.7). Apply the product rule to $f(D_t) = e^{-At}D_t$ to get (1.4.9).

Proof of inequalities (1.4.10) and (1.4.11).

Equation (1.4.9) and the triangle inequality imply that $\left|\bar{E}\left[D_t|\mathcal{G}_s\right]\right| \leq \eta_s e^{|A|t} + \eta$, $\left|\bar{\mathrm{V}}\mathrm{ar}\left[D_t|\mathcal{G}_s\right]\right| \leq \eta e^{2|A|t} + \eta$ and that $\left(\bar{E}[D_t|\mathcal{G}_s]\right)^2 \leq \eta_s e^{2|A|t} + \eta$. The definition of the conditional variance yields $\bar{E}\left[D_t^2|\mathcal{G}_s\right] \leq \eta_s e^{2|A|t} + \eta$. The unconditional inequalities follow similarly.

Proof of (a): $E\left[\int_s^t e^{-ru}\mathcal{E}_u D_u du \Big| \mathcal{G}_s\right] = \mathcal{E}_s \bar{E}\left[\int_s^t e^{-ru} D_u du \Big| \mathcal{G}_s\right]$

Since $\bar{E}[|D_u|] \leq \bar{E}[D_u^2] + 1$ and because of (1.4.11) (iv),

$$\int_s^t e^{-ru}\bar{E}[|D_u|]du < +\infty. \tag{1.4.12}$$

(a) is true thanks to Bayes' formula and to Fubini's Theorem .

Proof of (b) and (c):

Since $|x| \leq x^2 + 1$ for every $x \in \mathbb{R}$, then for every $u \in [s,t]$, $|\eta_1 D_u| \leq \eta_1^2 D_u^2 + 1$. Thus

$$\int_s^t \bar{E}\left[|\eta_2 D_u^2 + \eta_1 D_u + \eta_0|\right] du \leq (|\eta_{s,2}| + \eta_1^2) \int_s^t \bar{E}[D_u^2] du + (|\eta_0| + 1)(t-s) < +\infty$$

thanks to (1.4.11) (iv). Fubini's Theorem concludes the proof of (b) from which it follows that $\int_s^t (\eta_1 D_u + \eta_0) d\bar{W}_u^D$ is $\bar{\mathbb{P}}-$martingale.

Proof of (d)

The proof of (d) is made of several steps.

$$\text{Claim:} \quad \lim_{t\to+\infty} e^{-rt}\bar{E}\left[\int_s^t D_u du \Big| \mathcal{G}_s\right] = 0 \Longleftrightarrow \lim_{t\to+\infty} e^{-rt}\bar{E}\left[D_t|\mathcal{G}_s\right] = 0$$

Write the explicit dynamics for (1.4.8) and multiply by e^{-rt} to get

$$e^{-rt}\bar{E}\left[D_t|\mathcal{G}_s\right] = e^{-rt}D_s + Ae^{-rt}\bar{E}\left[\int_s^t D_u du \Big| \mathcal{G}_s\right] + e^{-rt}b(t-s).$$

Taking $\lim_{t\to+\infty}$ of both sides, proves the claim.

$$\text{Claim:} \quad \lim_{t\to+\infty} e^{-rt}\bar{E}\left[D_t|\mathcal{G}_s\right] = 0$$

Applying the conditional expectation to (1.4.9) and multiplying by e^{-rt} it follows that

$$e^{-rt}\bar{E}\left[D_t|\mathcal{G}_s\right] = e^{-rt}e^{A(t-s)}D_s + \frac{b}{A}\left(e^{-rt+A(t-s)} - e^{-rt}\right).$$

The result of taking the $\lim_{t\to+\infty}$ of both sides is 0 because $-r + A = -1/\epsilon_D < 0$.

$$\text{Claim:} \quad \lim_{t\to+\infty} e^{-rt}\bar{E}\left[\int_s^t D_u^2 du \Big| \mathcal{G}_s\right] = 0 \quad \Longleftrightarrow \quad \lim_{t\to+\infty} e^{-rt}\bar{E}\left[D_t^2|\mathcal{G}_s\right] du = 0$$

Apply Itô's formula to the function $f(D_t) = D_t^2$, take the conditional expectation of both sides and multiply by e^{-rt} to get

$$e^{-rt}\bar{E}[D_t^2|\mathcal{G}_s] = e^{-rt}D_s^2 + 2Ae^{-rt}\bar{E}\left[\int_s^t D_u^2 du \Big| \mathcal{G}_s\right] + \sigma_D e^{-rt}(t-s) + e^{-rt}\bar{E}\left[\int_s^t D_u d\bar{W}_u^D \Big| \mathcal{G}_s\right]. \tag{1.4.13}$$

Because of (c) it follows that $\bar{E}\left[\int_s^t D_u d\bar{W}_u^D|\mathcal{G}_s\right] = 0$, so taking $\lim_{t\to+\infty}$ of both sides of (1.4.13) proves the claim.

$$\text{Claim:} \quad \lim_{t\to+\infty} e^{-rt}\bar{E}\left[D_t^2|\mathcal{G}_s\right] du = 0$$

Thanks to the definition of the conditional variance and to (1.4.9) it follows that

$$e^{-rt}\bar{E}\left[D_u^2|\mathcal{G}_s\right] = e^{-rt}\bar{\mathrm{V}}\mathrm{ar}\left[D_u|\mathcal{G}_s\right] + e^{-rt}\left(\bar{E}[D_u|\mathcal{G}_s]\right)^2$$
$$= \frac{\sigma_D^2}{2A}\left(e^{2A(t-s)-rt} - e^{-rt}\right) + e^{-rt}\left(e^{2A(t-s)}\left(D_s + \frac{b}{A}\right)^2 + \frac{b^2}{A^2} - \frac{2b}{A}e^{A(t-s)}\left(D_s + \frac{b}{A}\right)\right).$$

$\lim_{t\to+\infty} e^{-rt}\bar{E}\left[D_u^2|\mathcal{G}_s\right] = 0$ because $2A - r = r - 2/\epsilon_D < 0$ and $A - r = -1/\epsilon_D < 0$.
Claim:

$$\lim_{t\to+\infty} e^{-rt}\bar{E}\left[(\eta_2 D_t^2 + \eta_1 D_t + \eta_0)|\mathcal{G}_s\right] = \lim_{t\to+\infty} e^{-rt}\bar{E}\left[\int_s^t (\eta_2 D_u^2 + \eta_1 D_u + \eta_0)du\Big|\mathcal{G}_s\right] = 0.$$

This is a consequence of

$$\lim_{t\to+\infty} e^{-rt}\bar{E}\left[D_t^2|\mathcal{G}_s\right] = \lim_{t\to+\infty} e^{-rt}\bar{E}\left[D_t|\mathcal{G}_s\right]$$
$$= \lim_{t\to+\infty} e^{-rt}\bar{E}\left[\int_s^t D_u^2 du\Big|\mathcal{G}_s\right] = \lim_{t\to+\infty} e^{-rt}\bar{E}\left[\int_s^t D_u du\Big|\mathcal{G}_s\right] = 0.$$

Proof of (e):

$$\frac{1}{2}\lim_{t\to+\infty}\bar{E}\Big[\int_s^t e^{-r(u-s)}(\Delta_D D_u + \Delta_0)^2 du\Big|\mathcal{G}_s\Big] = -\delta_{DD}D_s^2 - \delta_D D_s - \delta_0 - \frac{\beta - r}{r}.$$

Let $0 \le s \le t$; the function $W : [0,t]\times\mathbb{R}\to\mathbb{R}$

$$W(s,D) = -\delta_{DD}D^2 - \delta_D D - \delta_0 - \frac{\beta - r}{r},$$

is the solution of the Cauchy problem in $[0,t]$

$$0 = W_s + W_D(AD + b) + \frac{1}{2}W_{DD}\sigma_D^2 - rW + \frac{1}{2}(\Delta_D D + \Delta_0)^2;$$
$$W(t,D) = -\delta_{DD}D^2 - \delta_D D - \delta_0 - \frac{\beta - r}{r}.$$

In view of [22, Theorem 7.6],

$$W(s,D_s) = \bar{E}\Big[\int_s^t e^{-r(u-s)}\frac{1}{2}(\Delta_D D_u + \Delta_0)^2 du +$$
$$+ e^{-r(t-s)}\Big(-\delta_{DD}D_t^2 - \delta_D D_t - \delta_0 - \frac{\beta - r}{r}\Big)\Big|\mathcal{G}_s\Big].$$

Since W does not depend by t, it follows that for every $t > 0$

$$\bar{E}\left[\int_s^t e^{-r(u-s)}\frac{1}{2}(\Delta_D D_u + \Delta_0)^2 du|\mathcal{G}_0\right] +$$
$$+ e^{-r(t-s)}\bar{E}\left[\Big(-\delta_{DD}D_t^2 - \delta_D D_t - \delta_0 - \frac{\beta - r}{r}\Big)\Big|\mathcal{G}_s\right] = -\delta_{DD}D_s^2 - \delta_D D_s - \delta_0 - \frac{\beta - r}{r}.$$

Take $\lim_{t\to+\infty}$ of both sides and apply (d) to conclude.

□

With the properties of $(D_t)_{t\geq 0}$ shown in Lemma 2.4.3, we prove that $(N_t)_{t\geq 0}$ of (1.1.4) is a stochastic discount factor.

Theorem 1.4.2. *The process $(N_t)_{t\geq 0}$ of (1.1.4) is a normalized stochastic discount factor. The dynamics of the process $(\log \mathcal{E}_t)_{t\geq 0}$ can be written as*

$$\begin{aligned}\log \mathcal{E}_t &= \log \mathcal{E}_s - \frac{1}{2}\int_s^t (\Delta_D D_u + \Delta_0)^2 du + \int_s^t (\Delta_D D_u + \Delta_0) dW_u^D, \\ &= \log \mathcal{E}_s + \frac{1}{2}\int_s^t (\Delta_D D_u + \Delta_0)^2 du + \int_s^t (\Delta_D D_u + \Delta_0) d\bar{W}_u^D.\end{aligned} \tag{1.4.14}$$

For every $t \geq 0$

$$\begin{aligned}&\bar{E}[|\log \mathcal{E}_t|] \leq \eta\left(e^{2|A|t} + t + 1\right) && (\epsilon_D \neq 1/r) \\ &\bar{E}[|\log \mathcal{E}_t|] \leq \eta(t + t^3) && (\epsilon_D = 1/r).\end{aligned} \tag{1.4.15}$$

Proof. The process $(N_t)_{t\geq 0}$ needs to satisfy conditions (1.2.3) and (1.2.4) of Definition 1.2.2 to be a stochastic discount factor. Property (1.2.3) is a direct calculation. The definition of $\mathcal{E}_t = e^{rt}N_t$ implies that

$$E\left[N_t P_t + \int_0^t N_u D_u du \middle| \mathcal{G}_s\right] = \int_0^s N_u D_u du + E[e^{-rt}\mathcal{E}_t(C + \epsilon_D D_t)|\mathcal{G}_s] + E\left[\int_s^t e^{-ru}\mathcal{E}_u D_u du \middle| \mathcal{G}_s\right].$$

Because of Lemma 1.4.3 (a)

$$E\left[N_t P_t + \int_0^t N_u D_u du \middle| \mathcal{G}_s\right] = \int_0^s N_u D_u du + N_s \bar{E}\left[e^{-r(t-s)}(C + \epsilon_D D_t) + \int_s^t e^{-r(u-s)} D_u du \middle| \mathcal{G}_s\right]. \tag{1.4.16}$$

The function $W(s, D) = C + \epsilon_D D$ solves the Cauchy problem on $[0, t]$

$$\begin{aligned}0 &= W_s + W_D \cdot (AD + b) + \frac{1}{2}\sigma_D^2 W_{DD} - rW + D; \\ W(t, D) &= C + \epsilon_D D;\end{aligned}$$

where A and b are in Lemma 1.4.3. By [22, Theorem 7.6], for every $0 \leq s \leq t$

$$W(s, D_s) = \bar{E}\left[\int_s^t e^{-r(u-s)} D_u du + e^{-r(t-s)}(C + \epsilon_D D_t) \middle| \mathcal{G}_s\right] = C + \epsilon_D D_s.$$

Plugging W into (1.4.16) proves (1.2.4), hence $(N_t)_{t\geq 0}$ is a stochastic discount factor.

The stochastic process $(N_t)_{t\geq 0}$ of (1.1.4) solves the initial value problem

$$\frac{dN_t}{N_t} = -rdt + (\Delta_D D_t + \Delta_0)dW_t^D, \qquad N_0 = 1,$$

thus the process $(\mathcal{E}_t)_{t\geq 0}$ solves the initial value problem

$$\frac{d\mathcal{E}_t}{\mathcal{E}_t} = (\Delta_D D_t + \Delta_0)dW_t^D, \qquad \mathcal{E}_0 = 1.$$

Applying Itô's formula to $f(\mathcal{E}_t) = \log \mathcal{E}_t$ we get the first equality of (1.4.14) and because of (1.4.7) we get the second one. Thanks to (1.4.14) and to the triangle inequality it follows that

$$\bar{E}[|\log \mathcal{E}_u|] \leq \frac{1}{2}\bar{E}\left[\int_0^u (\Delta_D D_h + \Delta_0)^2 dh\right] + \bar{E}\left[\left|\int_0^u (\Delta_D D_h + \Delta_0)d\bar{W}_h^D\right|\right].$$

$\int_0^u (\Delta_D D_h + \Delta_0)d\bar{W}_h^D$ is a $\bar{\mathbb{P}}-$normal random variable with mean $\mu_u = 0$ and variance

$$\begin{aligned} \sigma_u^2 = \int_0^u \bar{E}[(\Delta_D D_h + \Delta_0)^2]dh &\leq \eta\left(e^{2|A|t} + t + 1\right) \qquad (\epsilon_D \neq 1/r) \qquad (1.4.17)\\ &\leq \eta(t + t^3) \qquad (\epsilon_D = 1/r) \end{aligned}$$

where η is a positive constant. In view of Lemma A.0.1 (IX) we get

$$\begin{aligned} \bar{E}\left[\left|\int_0^u (\Delta_D D_h + \Delta_0)d\bar{W}_h^D\right|\right] \leq \sigma_u\sqrt{\frac{2}{\pi}} &\leq \eta\left(e^{2|A|t} + t + 1\right) \qquad (\epsilon_D \neq 1/r)\\ &\leq \eta(t + t^3) \qquad (\epsilon_D = 1/r). \end{aligned}$$

Because of (1.4.17), the right side of the inequalities above bound also $\int_0^u \bar{E}[(\Delta_D D_h + \Delta_0)^2]dh$, and (1.4.15) follows. □

The next theorem proves the admissibility of the candidate optimal policies.

Theorem 1.4.3 (Admissibility and utility). *Define* $y^{i*} = e^{-r\alpha_i X_0^{i*} + \delta_{DD}D_0^2 + \delta_D D_0 + \delta_0}$, *the processes* $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ *as*

$$c_t^{i*} = rX_t^{i*} - \frac{\delta_{DD}}{\alpha_i}D_t^2 - \frac{\delta_D}{\alpha_i}D_t - \frac{\delta_0}{\alpha_i}, \qquad \theta_t^{i*} = \frac{M_D D_t + M_0}{M\alpha_i},$$

and the process X_t^{i*} *as*

$$\begin{aligned} X_t^{i*} = x_0^i + \frac{1}{M\alpha_i}\Big\{&\Big[M\delta_{DD} + M_D\big(1 - \epsilon_D(k + r)\big)\Big]\int_0^t D_u^2 du +\\ &+ \Big[M\delta_D + M_D(\epsilon_D\bar{\pi} - rC) + M_0\big(1 - \epsilon_D(k + r)\big)\Big]\int_0^t D_u du +\\ &\Big[M\delta_0 + M_0(\epsilon_D\bar{\pi} - rC)\Big]t + \epsilon_D\sigma_D M_D\int_0^t D_u dW_u^D + \epsilon_D\sigma_D M_0(W_t^D)\Big\}. \end{aligned} \qquad (1.4.18)$$

For every $\epsilon_D \in \mathbb{B}^+$ the following hold.

(A) (First order condition)

$$-\alpha_i c_t^{i*} = \log(y^{i*}) + (\beta - r)t + \log(\mathcal{E}_t); \tag{1.4.19}$$

(B) (Budget equation) $N_t X_t^{i*} + \int_0^t N_u c_u^{i*} du$ *is a* $\mathbb{P}-$*martingale;*

(C) (Saturation) for every $s \geq 0$, $\lim_{t\to+\infty} E[N_t X_t^{i*}|\mathcal{G}_s] = 0$;

(D) (Admissibility) for every $i \in \{1, \dots, n\}$, $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ *is an admissible strategy with wealth process* $(X_t^{i*})_{t\geq 0}$*. The utility of the strategy is*

$$E\left[\int_0^{+\infty} e^{-\beta u} U(c_u^{i*}) du \Big| \mathcal{G}_0\right] = -\frac{1}{r\alpha_i} e^{-r\alpha_i X_0^{i*} + \delta_{DD} D_0^2 + \delta_D D_0 + \delta_0}.$$

Proof. We proceed in several steps.

Proof of (A): First order condition

The equality $-\alpha_i c_0^{i*} = -r\alpha_i x_0^i + \delta_{DD} D_0^2 + \delta_D D_0 + \delta_0$ holds. Apply Itô's formula to both sides of (1.4.19) and check that they are equal.

Proof of the equality $\mathcal{E}_s \bar{E}\left[\int_s^t e^{-ru} c_u^{i*} du \Big| \mathcal{G}_s\right] = E\left[\int_s^t e^{-ru} \mathcal{E}_u c_u^{i*} du \Big| \mathcal{G}_s\right]$

Due to (1.4.19) and to the triangle inequality, there exists $\eta > 0$ such that

$$|c_u^{i*}| \leq \eta |-r\alpha_i x_0^i + \delta_{DD} D_0^2 + \delta_D D_0 + \delta_0| + \eta u + \eta |\log \mathcal{E}_u|. \tag{1.4.20}$$

Applying the conditional expectation to both sides of (1.4.20), the properties of normal random variables and (1.4.15) imply that

$$\begin{aligned} \bar{E}[|c_u^{i*}|] &\leq \eta(e^{2|A|t} + t + 1) & \epsilon_D \neq 1/r, \\ &\leq \eta(t^3 + t + 1) & \epsilon_D = 1/r. \end{aligned}$$

Fubini's Theorem [4, Theorem 1.1.7] yields to

$$\int_s^t e^{-ru} \bar{E}\left[c_u^{i*}\right] du = \bar{E}\left[\int_s^t e^{-ru} |c_u^{i*}| du\right] < +\infty$$

and by the conditional version of Fubini's Theorem we get

$$\mathcal{E}_s \bar{E}\left[\int_s^t e^{-ru} c_u^{i*} du \Big| \mathcal{G}_s\right] = E\left[\int_s^t e^{-ru} \mathcal{E}_u c_u^{i*} du \Big| \mathcal{G}_s\right]. \tag{1.4.21}$$

Proof of (B): $N_t X_t^{i*} + \int_0^t N_u c_u^{i*}$ is a martingale

Direct calculations show that $(X_t^{i*})_{t\geq 0}$ is the wealth process of the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$

and they satisfy equality (1.4.1), equivalent to the self-financing condition. As a consequence applying Itô's formula to the function $f(t, X_t^{i*}) = e^{-rt}X_t^{i*}$ we get

$$e^{-rt}X_t^{i*} = e^{-rs}X_s^{i*} + \int_s^t -e^{-ru}c_u^{i*}du + (\epsilon_D\bar{\pi} - rC)\int_s^t e^{-ru}\theta_u^{i*}du + \\ + (1 - \epsilon_D(k+r))\int_s^t e^{-ru}\theta_u^{i*}D_u du + \epsilon_D\sigma_D\int_s^t e^{-ru}\theta_u^{i*}dW_u^D.$$

Since the equalities

$$\epsilon_D\sigma_D\Delta_D = -1 + \epsilon_D(k+r), \qquad \epsilon_D\sigma_D\Delta_0 = rC - \epsilon_D\bar{\pi},$$

and (1.4.7) hold, it follows that

$$e^{-rt}X_t^{i*} = e^{-rs}X_s^{i*} + \int_s^t -e^{-ru}c_u^{i*}du + \epsilon_D\sigma_D\int_s^t e^{-ru}\theta_u^{i*}d\bar{W}_u^D.$$

Multiply both sides by $\mathcal{E}_t$, add $\int_0^t N_u c_u^{i*}du$, take the conditional expectation and use Bayes' formula to get

$$E\left[N_tX_t^{i*} + \int_0^t N_u c_u^{i*}du\Big|\mathcal{G}_s\right] = N_sX_s^{i*} + \int_0^s N_u c_u^{i*}du + \mathcal{E}_s\epsilon_D\sigma_D\bar{E}\left[\int_s^t e^{-ru}\theta_u^{i*}d\bar{W}_u^D\Big|\mathcal{G}_s\right] + \\ \mathcal{E}_s\bar{E}\left[\int_s^t -e^{-ru}c_u^{i*}du\Big|\mathcal{G}_s\right] + E\left[\int_s^t N_u c_u^{i*}du\Big|\mathcal{G}_s\right].$$

The Brownian term is a martingale by virtue of Lemma 1.4.3 (c) and since (1.4.21) holds, then

$$E\left[N_tX_t^{i*} + \int_0^t N_u c_u^{i*}du\Big|\mathcal{G}_s\right] = \int_0^s N_u c_u^{i*}du + N_sX_s^{i*}. \tag{1.4.22}$$

Proof of (C): $\lim_{t\to+\infty} E[N_tX_t^{i*}|\mathcal{G}_s] = 0$

For the process X_t^{i*} of (1.4.18), there exist $\eta_1, \ldots, \eta_5 \in \mathbb{R}$ such that

$$N_tX_t^{i*} = e^{-rt}\mathcal{E}_tX_s^{i*} + \eta_1 e^{-rt}\mathcal{E}_t\int_s^t D_u^2du + \eta_2 e^{-rt}\mathcal{E}_t\int_s^t D_u du + \eta_3 e^{-rt}\mathcal{E}_t(t-s) + \\ + \eta_4 e^{-rt}\mathcal{E}_t\int_s^t (D_u + \eta_5)dW_u^D.$$

Taking the conditional expectation and using Bayes' formula yields

$$E[N_tX_t^{i*}|\mathcal{G}_s] = e^{-rt}\mathcal{E}_sX_s^{i*} + \eta_1\mathcal{E}_s e^{-rt}\bar{E}\left[\int_s^t D_u^2du\Big|\mathcal{G}_s\right] + \eta_2\mathcal{E}_s e^{-rt}\bar{E}\left[\int_s^t D_u du\Big|\mathcal{G}_s\right] + \\ + \eta_3 e^{-rt}\mathcal{E}_s(t-s) + \eta_4 e^{-rt}\mathcal{E}_s\bar{E}\left[\int_s^t (D_u + \eta_5)d\bar{W}_u^D\Big|\mathcal{G}_s\right].$$

$\int_s^t (D_u + \eta_5)d\bar{W}_u^D$ is a $\bar{\mathbb{P}}-$ martingale because of Lemma 1.4.3 (c); thanks to Lemma 1.4.3 (d), $\lim_{t\to+\infty} E[N_tX_t^{i*}|\mathcal{G}_s] = 0$.

Proof of (D): Admissibility and utility

Property (i) of Definition 1.1.1 is clear and proving that $(X_t^{i*})_{t\geq 0}$ is the wealth process of the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ is a direct calculation. Take $\lim_{t\to+\infty}$ to both sides of (1.4.22) and use (C) to prove property (1.1.2) and thus the admissibility of the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$. (1.4.19) implies

$$E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u^{i*}) du \Big| \mathcal{G}_0\right] = -\frac{1}{\alpha_i} E\left[\int_0^{+\infty} e^{-\beta u} e^{\log y^{i*} - ru + \log \mathcal{E}_u} du \Big| \mathcal{G}_0\right] = -\frac{y^{i*}}{r\alpha_i}.$$

□

Theorem 1.4.4 (Duality Theorem)**.** *Let $(c_t, \theta_t)_{t\geq 0}$ be an admissible strategy for the $i-$th investor and let $(N_t)_{t\geq 0}$ be the process of (1.1.4); then*

$$\begin{aligned} \lim_{t\to+\infty} E\left[\int_0^t e^{-\beta u} U^i(c_u) du \Big| \mathcal{G}_0\right] &= E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u) du \Big| \mathcal{G}_0\right], \\ \lim_{t\to+\infty} E\left[\int_0^t e^{-\beta u} \tilde{U}^i(y e^{\beta u} N_u) du \Big| \mathcal{G}_0\right] &= E\left[\int_0^{+\infty} e^{-\beta u} \tilde{U}^i(y e^{\beta u} N_u) du \Big| \mathcal{G}_0\right]. \end{aligned} \tag{1.4.23}$$

Furthermore

$$E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u) du \Big| \mathcal{G}_0\right] \leq \inf_{y>0} \left\{ E\left[\int_0^{+\infty} e^{-\beta u} \tilde{U}^i(y e^{\beta u} N_u) du \Big| \mathcal{G}_0\right] + x_0^i y \right\}, \tag{1.4.24}$$

where

$$\tilde{U}^i(y) = \begin{cases} \frac{y}{\alpha_i}(\log y - 1) & y > 0 \\ 0 & y = 0. \end{cases} \tag{1.4.25}$$

If there exist $y^{i*} > 0$ and an admissible strategy $(c_t^*, \theta_t^*)_{t\geq 0}$ for which

$$E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u^*) du \Big| \mathcal{G}_0\right] = E\left[\int_0^{+\infty} e^{-\beta u} \tilde{U}^i(y^{i*} e^{\beta u} N_u) du \Big| \mathcal{G}_0\right] + x_0^i y^{i*}, \tag{1.4.26}$$

then $(c_t^*, \theta_t^*)_{t\geq 0}$ is optimal.

Proof. Define the random variables

$$\lambda^m = \int_0^m e^{-\beta u - \alpha_i c_u} du, \qquad \lambda = \int_0^{+\infty} e^{-\beta u - \alpha_i c_u} du,$$

on the probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$. Then $\lambda^m \geq 0$ for every $m \in \mathbb{N}$ and $(\lambda^m)_{m\in\mathbb{N}}$ is an increasing sequence of random variables such that $\lim_{m\to+\infty} \lambda^m = \lambda$. The Conditional Monotone Convergence Theorem yields to

$$\lim_{m\to+\infty} E[\lambda^m | \mathcal{G}_0] = E[\lambda | \mathcal{G}_0],$$

which implies the first equality in (1.4.23). The function $\tilde{U}^i$ defined in (1.4.25) has a global minimum at $y = 1$; apply the Conditional Monotone Convergence Theorem to the random variables

$$\lambda^m = \int_0^m e^{-\beta u} \left(\tilde{U}^i(y e^{\beta u} N_u) + \frac{1}{\alpha_i} \right) du, \qquad \lambda = \int_0^{+\infty} e^{-\beta u} \left(\tilde{U}^i(y e^{\beta u} N_u) + \frac{1}{\alpha_i} \right) du,$$

to conclude the second equality in (1.4.23). For the proof of (1.4.24) apply (A.0.1) to the random variables c_u and $Y_u = y e^{\beta u} N_u$; then for every $y > 0$ we get

$$U^i(c_u) \leq \tilde{U}^i(y e^{\beta u} N_u) + c_u y e^{\beta u} N_u.$$

Multiply both sides by $e^{-\beta u}$, integrate in $[0, t]$ and take conditional expectations; it follows that for every $y > 0$

$$E\left[\int_0^t e^{-\beta u} U^i(c_u) du \Big| \mathcal{G}_0 \right] \leq E\left[\int_0^t e^{-\beta u} \tilde{U}^i(y e^{\beta u} N_u) du \Big| \mathcal{G}_0 \right] + yE\left[\int_0^t c_u N_u du \Big| \mathcal{G}_0 \right].$$

Take $\limsup_{t\to+\infty}$ of both sides and use (1.4.23) and (1.1.2); for every $y > 0$

$$E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u) du \Big| \mathcal{G}_0 \right] \leq E\left[\int_0^{+\infty} e^{-\beta u} \tilde{U}^i(y e^{\beta u} N_u) du \Big| \mathcal{G}_0 \right] + x_0^i y.$$

Take $\inf_{y>0}$ to obtain (1.4.24). If there exist $y^{i*} > 0$ and an admissible strategy $(c_t^*, \theta_t^*)_{t\geq 0}$ for which (1.4.26) holds, then

$$\begin{aligned} E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u^*) du \Big| \mathcal{G}_0 \right] &\leq \inf_{y>0} \left\{ E\left[\int_0^{+\infty} e^{-\beta u} \tilde{U}^i(y e^{\beta u} N_u) du \Big| \mathcal{G}_0 \right] + x_0^i y \right\} \\ &\leq E\left[\int_0^{+\infty} e^{-\beta u} \tilde{U}^i(y^{i*} e^{\beta u} N_u) du \Big| \mathcal{G}_0 \right] + x_0^i y^{i*} = E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u^*) du \Big| \mathcal{G}_0 \right]. \end{aligned}$$

□

Theorem 1.4.5 (Existence)**.** *For every $\epsilon_D \in \mathbb{B}^+$, the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ in Theorem 1.4.3 is optimal for the $i-$th investor for every $i \in \{1, \ldots, n\}$. The function V^i of Theorem 1.4.1 is the value function of the $i-$th investor.*

Proof. Fixing $0 \leq s \leq t, y > 0$ and using the definition of $\tilde{U}(\cdot)$ in (1.4.25) we get

$$\begin{aligned} E\left[\int_s^t e^{-\beta u} \tilde{U}(y e^{\beta u} N_u) du \Big| \mathcal{G}_s \right] &= \frac{y}{\alpha_i} \Big\{ (\log y - 1)\, E\left[\int_s^t N_u du \Big| \mathcal{G}_s \right] + \\ &\quad + \beta E\left[\int_s^t u N_u du \Big| \mathcal{G}_s \right] + E\left[\int_s^t N_u \log N_u du \Big| \mathcal{G}_s \right] \Big\}. \end{aligned}$$

The following integrability conditions hold:

$$\begin{aligned} \int_s^t E\left[|N_u| \right] du &= \int_s^t E\left[N_u \right] du = \int_s^t e^{-ru} du = \frac{(e^{-rs} - e^{-rt})}{r} < +\infty, \\ \int_s^t E\left[|u N_u| \right] du &= \int_s^t u E\left[N_u \right] du = \int_s^t u e^{-ru} du = \frac{e^{-rs}(1 + rs) - e^{-rt}(1 + rt)}{r^2} < +\infty. \end{aligned}$$

The conditional version of Fubini's Theorem [4, Theorem 1.1.8] applies and yields

$$E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\Big|\mathcal{G}_s\right] = \frac{y}{\alpha_i}\Big\{ (\log y - 1)\,\mathcal{E}_s\int_s^t e^{-ru}du + \\ + (\beta - r)\mathcal{E}_s\int_s^t ue^{-ru}du + E\left[\int_s^t e^{-ru}\mathcal{E}_u \log\mathcal{E}_u du\Big|\mathcal{G}_s\right]\Big\}.$$

(1.4.15) implies that

$$\int_s^t e^{-ru}\bar{E}\left[|\log\mathcal{E}_u|\right]du \le \eta(t-s)\left(e^{2|A|t}+t+1\right) < +\infty \qquad (\epsilon_D \neq 1/r)$$
$$\le \eta(t^2+t^4) < +\infty. \qquad (\epsilon_D = 1/r)$$

Fubini's Theorem and Bayes' formula yield to

$$E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\Big|\mathcal{G}_s\right] = \frac{y}{\alpha_i}\Big\{ (\log y - 1)\,\mathcal{E}_s\int_s^t e^{-ru}du + \\ + (\beta - r)\mathcal{E}_s\int_s^t ue^{-ru}du + \mathcal{E}_s\int_s^t e^{-ru}\bar{E}\left[\log\mathcal{E}_u|\mathcal{G}_s\right]du\Big\}$$

and computing the integrals we get

$$E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\Big|\mathcal{G}_s\right] = \frac{y}{r\alpha_i}\mathcal{E}_s\Big\{ (\log y - 1)\,(e^{-rs}-e^{-rt}) + \\ + (\beta - r)\frac{e^{-rs}(1+rs)-e^{-rt}(1+rt)}{r} + r\int_s^t e^{-ru}\bar{E}\left[\log\mathcal{E}_u|\mathcal{G}_s\right]du\Big\}.$$

By virtue of (1.4.14) and Lemma 1.4.3 (c), $\bar{E}\left[\log\mathcal{E}_u|\mathcal{G}_s\right] = \log\mathcal{E}_s + \frac{1}{2}\bar{E}\Big[\int_s^u(\Delta_D D_u + \Delta_0)^2 du\Big|\mathcal{G}_s\Big]$. Defining $Y_t = \int_s^t(\Delta_D D_u + \Delta_0)^2 du$ it follows that

$$r\int_s^t e^{-ru}\bar{E}\left[\log\mathcal{E}_u|\mathcal{G}_s\right]du = r\log\mathcal{E}_s\int_s^t e^{-ru}du + \frac{r}{2}\int_s^t e^{-ru}\bar{E}[Y_u|\mathcal{G}_s]du$$

and thanks to Lemma 1.4.3 (b) and to Fubini's Theorem we get

$$r\int_s^t e^{-ru}\bar{E}\left[\log\mathcal{E}_u|\mathcal{G}_s\right]du = r\log\mathcal{E}_s\int_s^t e^{-ru}du + \frac{r}{2}\bar{E}\left[\int_s^t e^{-ru}Y_u du\Big|\mathcal{G}_s\right].$$

As a consequence,

$$E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\Big|\mathcal{G}_s\right] = \frac{y}{r\alpha_i}\mathcal{E}_s\Big\{ (\log y - 1)\,(e^{-rs}-e^{-rt}) + \\ + (\beta - r)\frac{e^{-rs}(1+rs)-e^{-rt}(1+rt)}{r} + r\log\mathcal{E}_s\int_s^t e^{-ru}du + \frac{r}{2}\bar{E}\left[\int_s^t e^{-ru}Y_u du\Big|\mathcal{G}_s\right]\Big\}.$$

Applying the Itô formula to the function $e^{-rt}Y_t$ and taking the conditional expectation yields

$$r\bar{E}\left[\int_s^t e^{-ru}Y_u du\Big|\mathcal{G}_s\right] = e^{-rs}Y_s - e^{-rt}\bar{E}\Big[\int_s^t (\Delta_D D_u + \Delta_0)^2 du\Big|\mathcal{G}_s\Big] + \\ + \bar{E}\Big[\int_s^t e^{-ru}(\Delta_D D_u + \Delta_0)^2 du\Big|\mathcal{G}_s\Big].$$

Fix $s = 0$ and take $\lim_{t\to+\infty}$ of both sides to get

$$\lim_{t\to+\infty} E\left[\int_0^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\Big|\mathcal{G}_0\right] + x_0^i y = \frac{y}{r\alpha_i}\Big\{ (\log y - 1) + \frac{\beta - r}{r} + \\ -\frac{1}{2}\lim_{t\to+\infty} e^{-rt}\bar{E}\left[\int_0^t (\Delta_D D_u + \Delta_0)^2 du\Big|\mathcal{G}_0\right] + \frac{1}{2}\lim_{t\to+\infty} \bar{E}\left[\int_0^t e^{-ru}(\Delta_D D_u + \Delta_0)^2 du\Big|\mathcal{G}_0\right]\Big\} + x_0^i y.$$

Choosing $y = y^{i*} = \exp(-r\alpha_i x_0^{i*} + \delta_{DD}D_0^2 + \delta_D D_0 + \delta_0)$ and using Lemma 1.4.3 (d) it follows that

$$\lim_{t\to+\infty} E\left[\int_0^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\Big|\mathcal{G}_0\right] + x_0^i y^{i*} = \frac{y^{i*}}{r\alpha_i}\Big\{(-r\alpha_i x_0^{i*} + \delta_{DD}D_0^2 + \delta_D D_0 + \delta_0 + \\ - 1) + \frac{\beta - r}{r} + \frac{1}{2}\lim_{t\to+\infty} \bar{E}\Big[\int_0^t e^{-ru}(\Delta_D D_u + \Delta_0)^2 du\Big|\mathcal{G}_0\Big]\Big\} + x_0^i y^{i*}.$$

Lemma 1.4.3 (e) and (1.4.23) imply that

$$E\left[\int_0^{+\infty} e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\Big|\mathcal{G}_0\right] = -\frac{y^{i*}}{r\alpha_i}.$$

The conclusion follows from Theorems 1.4.3 and 1.4.4. □

Uniqueness of the optimal strategy

Lemma 1.4.4. Let $i \in \{1, \dots, n\}$ and let $(c_t, \theta_t)_{t\geq 0}$ be an optimal strategy for the $i-$th agent, then for every $s \geq 0$

$$\limsup_{t\to+\infty} E\left[\int_s^t N_u c_u du\Big|\mathcal{G}_s\right] = N_s X_s. \tag{1.4.27}$$

Proof. Suppose, for a contradiction, that there exist $i \in \{1, \dots, n\}$, $s \geq 0$, $S \in \mathcal{G}_s$ with $\mathbb{P}(S) > 0$ and an optimal strategy such that

$$\limsup_{t\to+\infty} E\left[\int_s^t N_u^i c_u du\Big|\mathcal{G}_s\right] < N_s X_s \quad \text{on } S.$$

Let η_s be a $\mathcal{G}_s-$adapted random variable and define the new strategy $(\bar{c}_t, \theta_t)_{t\geq 0}$ as

$(\bar{c}_t)_{t\geq 0} = (c_t)_{t\geq 0} + \eta_s \mathbf{1}_{t\geq s}$ and its wealth process

$$\bar{X}_t = X_t\mathbf{1}_{t<s} + \mathbf{1}_{t\geq s}\Big\{X_s + \int_s^t \Big[-\bar{c}_t + r\bar{X}_u + \theta_u^i(\epsilon_D \bar{\pi} - rC) + \\ + \theta_u^i D_u\Big(1-\epsilon_D(k+r)\Big)\Big] du + \epsilon_D\sigma_D \int_s^t \theta_u^i dW_u^D.$$

If $\limsup_{t\to+\infty} E\left[\int_s^t N_u c_u du | \mathcal{G}_s\right] = -\infty$ the claim follows because $\eta_s = 1$ makes $(\bar{c}_t)_{t\geq 0}$ a better strategy, still admissible. Otherwise, if $\limsup_{t\to+\infty} E\left[\int_s^t N_u c_u du | \mathcal{G}_s\right] > -\infty$, define $\epsilon = X_s N_s - \limsup_{t\to+\infty} E\left[\int_s^t N_u c_u du \Big| \mathcal{G}_s\right] > 0$. Choose $\eta_s = \epsilon r(\mathcal{E}_s)^{-1} e^{rs}$ to obtain a better strategy, which is still admissible because

$$X_s N_s - \limsup_{t\to+\infty} E\left[\int_s^t N_u(c_u+\eta_s) du \Big| \mathcal{G}_s\right] = \epsilon - \eta_s \frac{\mathcal{E}_s}{r} e^{-rs} = 0.$$

□

Theorem 1.4.6 (Uniqueness)**.** *For every $\epsilon_D \in \mathbb{B}^+$ and for all $i \in \{1,\dots,n\}$, $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ in Theorem 1.4.3 is the unique optimal strategy for the $i-$th investor.*

Proof. Claim: The consumption process is unique.
Suppose there exist optimal strategies for the $i-$th investor $(c_t^A, \theta_t^A)_{t\geq 0}$ and $(c_t^B, \theta_t^B)_{t\geq 0}$ and suppose, for a contradiction, that there exists $S \in \mathcal{B}\otimes\mathcal{F}^i$ such that $(\lambda_{|[0,+\infty[} \otimes \mathbb{P})(S) > 0$ and $c_t^A \mathbf{1}_S \neq c_t^B \mathbf{1}_S$. The wealth process of the strategy $\frac{1}{2}(c_t^A + c_t^B, \theta_t^A + \theta_t^B)_{t\geq 0}$ is the process $\frac{1}{2}(X_t^A + X_t^B)_{t\geq 0}$, with dynamics

$$\frac{1}{2} d(X_t^A + X_t^B) = \frac{1}{2}\Big[-c_t^A - c_t^B + r(X_t^A + X_t^B) + (\theta_t^A+\theta_t^B)(\epsilon_D\bar{\pi} - rC) + \\ + (\theta_t^A+\theta_t^B) D_t\Big(1-\epsilon_D(k+r)\Big)\Big] dt + \frac{1}{2}(\theta_t^A+\theta_t^B)\epsilon_D\sigma_D dW_t^D.$$

The new strategy has initial wealth x_0^i and is admissible because $(c_t^A, \theta_t^A)_{t\geq 0}$ and $(c_t^B, \theta_t^B)_{t\geq 0}$ are. Since the utility function is strictly concave, $c_t^A \neq c_t^B$ on S implies

$$U\Big(\frac{c_t^A + c_t^B}{2}\Big) > \frac{1}{2}U(c_t^A) + \frac{1}{2}U(c_t^B) \qquad \text{on } S.$$

Define $H := \Big\{w \in \Omega : \lambda_{|[0,+\infty[}(\{t\geq 0 : (t,w)\in S\})\Big\} \in \mathcal{G}$, then $\mathbb{P}(H) > 0$,

$$\int_0^{+\infty} e^{-\beta t} U\Big(\frac{c_t^A + c_t^B}{2}\Big) dt \geq \frac{1}{2}\int_0^{+\infty} e^{-\beta t}[U(c_t^A) + U(c_t^B)] dt \qquad \text{a.s.}$$

and

$$\int_0^{+\infty} e^{-\beta t} U\Big(\frac{c_t^A + c_t^B}{2}\Big) dt > \frac{1}{2}\int_0^{+\infty} e^{-\beta t}[U(c_t^A) + U(c_t^B)] dt \qquad \text{on } H.$$

By Lemma A.0.2 (II) it follows that

$$E\left[\int_0^{+\infty} e^{-\beta t}\Big(\frac{c_t^A+c_t^B}{2}\Big)dt\Big|\mathcal{G}_0\right] > E\left[\int_0^{+\infty} e^{-\beta t}U(c_t^{i*})dt\Big|\mathcal{G}_0\right]$$

on a positive probability set, thus contradicting the optimality of the consumption processes $(c_t^A)_{t\geq 0}$ and $(c_t^B)_{t\geq 0}$.

Claim: Investment and wealth processes are unique.

Thanks to (1.4.27), it follows that

$$X_s^{i*} = (N_s)^{-1}\limsup_{t\to+\infty} E\left[\int_s^t N_u c_u^{i*}du\Big|\mathcal{G}_s\right],$$

which proves the uniqueness of the optimal wealth process. From (1.4.1) it follows that

$$dX_t^{i*} + c_t^{i*}dt - rX_t^{i*}dt = \theta_t^i\left[(\epsilon_D\bar{\pi} - rC) + D_t\Big(1-\epsilon_D(k+r)\Big)\right]dt + \epsilon_D\sigma_D\theta_t^i dW_t^D.$$

If there exist two strategies with wealth process $(X_t^{i*})_{t\geq 0}$ and consumption $(c_t^{i*})_{t\geq 0}$, then drifts and volatilities must be the same. This implies the uniqueness of the optimal investment strategy. □

1.4.2 Ill-posed problem

Suppose $\epsilon_D < 0$ or $\epsilon_D > 2/r$; the next two theorems show that for such a choice of the parameter ϵ_D the agents' consumption-investment problem is ill-posed and it is not possible to find an optimal strategy. If there does not exist a solution of the optimal consumption-investment problem for the agents, no equilibrium is possible when $\epsilon_D < 0$ or $\epsilon_D > 2/r$. The next theorem defines a sequence of strategies and proves their admissibility.

Theorem 1.4.7 (Admissibility). *For every $T \in \mathbb{N}^*$, define the process $(c_t^{iT}, \theta_t^{iT})_{0\leq t<+\infty}$ as*

$$c_t^{iT} = \mathbf{1}_{\{t\leq T\}}\left(\frac{e^{r(T-t)}}{e^{r(T-t)}-1}rX_t^{iT} - \frac{\delta_{\mathbf{DD}}(T-t)}{\alpha_i}D_t^2 - \frac{\delta_{\mathbf{D}}(T-t)}{\alpha_i}D_t - \frac{\delta_{\mathbf{0}}(T-t)}{\alpha_i} - \frac{r(T-t)}{\alpha_i}\right) \tag{1.4.28}$$

$$\theta_t^{iT} = \mathbf{1}_{\{t\leq T\}}\left(\frac{\mathbf{M_D}(T-t)D_t + \mathbf{M_0}(T-t)}{M\alpha_i}\right), \tag{1.4.29}$$

where the functions $\delta_{\mathbf{DD}}, \delta_{\mathbf{D}}, \delta_{\mathbf{0}}, \mathbf{M_D}$ and $\mathbf{M_0}$ are given in Definition B.0.1 and the *process* X_t^{iT} *is defined as* $X_0^{iT} = x_0^i$ *and*

$$X_t^{iT} = \frac{e^{rT} - e^{rt}}{e^{rT} - e^{rs}} X_s^{iT} + \frac{1}{\alpha_i} \int_s^t \left(\frac{e^{rT} - e^{rt}}{e^{rT} - e^{ru}} \right) \Big[\delta_{\mathbf{DD}}(T-u) D_u^2 + \delta_{\mathbf{D}}(T-u) D_u + \\ + \delta_{\mathbf{0}}(T-u) + r(T-u) \Big] du + \epsilon_D \sigma_D \int_s^t \left(\frac{e^{rT} - e^{rt}}{e^{rT} - e^{ru}} \right) \theta_u^{iT} d\bar{W}_u^D. \quad (1.4.30)$$

For every $i \in \{1, \ldots, n\}$, $(c_t^{iT}, \theta_t^{iT})_{t \geq 0}$ *is an admissible strategy with wealth process* $(X_t^{iT})_{t \geq 0}$.

Proof. Property (i) of Definition 1.1.1 is clear; plugging $(c_t^{iT}, \theta_t^{iT})$ into the self-financing condition (1.1.3), we realize the process (1.4.30) to be the wealth dynamic for the strategy $(c_t^{iT}, \theta_t^{iT})$. By dint of the definition of $(c_t^{iT})_{t \geq 0}$ and $(X_t^{iT})_{t \geq 0}$, $X_t^{iT} = 0$ and $c_t^{iT} = 0$ for every $t \geq T$. As a consequence also (1.1.2) is true and immediate for $s \geq T$. If $s \leq T$, then

$$\limsup_{t \to +\infty} E \left[\int_s^t N_u c_u^{iT} du \Big| \mathcal{G}_s \right] = E \left[\int_s^T N_u c_u^{iT} du \Big| \mathcal{G}_s \right].$$

We prove $E \left[\int_s^T N_u c_u^{iT} du \Big| \mathcal{G}_s \right]$ to be equal to $N_s X_s^{iT}$ for every $0 \leq s \leq T$ in several steps.
Claim: for every $s \leq t \leq T$, $\bar{E}[e^{-rt} |c_t^{iT}|] \leq \eta(T)$
Plug the wealth process (1.4.30) into (1.4.28) and multiply by e^{-rt} to get

$$c_t^{iT} = 1_{\{t \leq T\}} \Big\{ -\frac{1}{\alpha_i} \Big[\delta_{\mathbf{DD}}(T-t) D_t^2 + \delta_{\mathbf{D}}(T-t) D_t + \delta_{\mathbf{0}}(T-t) + r(T-u) \Big] + \\ + \frac{r X_0^{iT}}{(1 - e^{-r(T-s)})} + r \int_0^t \left(\frac{1}{\alpha_i (1 - e^{-r(T-u)})} \right) \Big(\delta_{\mathbf{DD}}(T-u) D_u^2 + \delta_{\mathbf{D}}(T-u) D_u + \\ + \delta_{\mathbf{0}}(T-u) + r(T-u) \Big) du + \epsilon_D \sigma_D r \int_s^t \frac{1}{(1 - e^{-r(T-u)})} \theta_u^{iT} d\bar{W}_u^D \Big\}.$$

Apply the absolute value to both sides, the triangle inequality and Lemma B.0.1 (a) and (b). There exists a constant $\eta(T) \geq 0$ such that

$$|c_t^{iT}| \leq 1_{\{t \leq T\}} \Big\{ \eta(T) D_t^2 + \eta(T) + \eta(T) |x_0^i| + \\ + \int_0^t \eta(T) D_u^2 + \eta(T) du + |\epsilon_D \sigma_D r| \left| \int_s^t \frac{1}{(1 - e^{-r(T-u)})} \theta_u^{iT} d\bar{W}_u^D \right| \Big\}.$$

Applying the conditional expectation to both sides, (1.4.11) (iv) and Lemma B.0.1 (e) we get

$$\bar{E} \left[e^{-rt} |c_t^{iT}| \right] \leq \bar{E} \left[|c_t^{iT}| \right] \leq \eta(T) \text{ for every } T > 0 \text{ and for every } t \leq T, \quad (1.4.31)$$

from which follows that

$$\int_s^t \bar{E}\left[e^{-ru}|c_u^{iT}|\right] dt \le \int_s^t \bar{E}\left[|c_u^{iT}|\right] dt \le \eta(T) \text{ for every } T > 0 \text{ and for every } t \le T. \tag{1.4.32}$$

Claim: $\int_0^t e^{-ru} c_u^{iT} du + e^{-rt} X_t^{iT}$ is a martingale for every $0 \le t \le T$.
Plugging (1.4.7) into (1.1.3) and using the equalities

$$\epsilon_D \sigma_D \Delta_D = -1 + \epsilon_D(k + r), \qquad \epsilon_D \sigma_D \Delta_0 = rC - \epsilon_D \bar{\pi},$$

show that the wealth dynamics is

$$dX_t^{iT} = (-c_t^{iT} + rX_t^{iT})dt + \epsilon_D \sigma_D \theta_t^{iT} d\bar{W}_t^D.$$

Applying Itô's rule to $Y_t := \int_0^t e^{-ru} c_u^{iT} du + e^{-rt} X_t^{iT}$ we get

$$Y_t = Y_0 + \epsilon_D \sigma_D \int_0^t e^{-ru} \theta_u^{iT} d\bar{W}_u^D.$$

Y_t is a martingale because

$$\begin{aligned}\int_0^t e^{-2ru} \bar{E}[(\theta_u^{iT})^2] du &\le \frac{1}{M^2 \alpha_i^2} \int_0^t \bar{E}[(\mathbf{M_D}(T-u)D_u + \mathbf{M_0}(T-u))^2] du \\ &\le \frac{1}{M^2 \alpha_i^2} \int_0^t \eta(T) E[D_u^2 + 1] du \le \eta(T),\end{aligned} \tag{1.4.33}$$

where $\eta(T)$ is a constant dependent only by T. (1.4.29) implies the first inequality in (1.4.33), Lemma A.0.1 (X) and Lemma B.0.1 (c) imply the second one while (1.4.11) (iii) implies the third one. Because $(Y_t)_{0 \le t \le T}$ is a martingale, in particular

$$X_s^{iT} = \bar{E}\Big[\int_s^T e^{-r(u-s)} c_u^{iT} du + e^{-r(T-s)} X_T^{iT} \Big| \mathcal{G}_s\Big].$$

(1.4.30) implies that $X_T^{iT} = 0$ and thus for every $0 \le s \le T$

$$X_s^{iT} = \bar{E}\Big[\int_s^T e^{-r(u-s)} c_u^{iT} du \Big| \mathcal{G}_s\Big]. \tag{1.4.34}$$

$(c_t^{iT}, \theta_t^{iT})_{t \ge 0}$ is admissible since

$$\begin{aligned} N_s X_s^{iT} = \mathcal{E}_s \bar{E}\Big[\int_s^T e^{-ru} c_u^{iT} du \Big| \mathcal{G}_s\Big] &= \int_s^T \mathcal{E}_s e^{-ru} \bar{E}[c_u^{iT} | \mathcal{G}_s] du \\ &= \int_s^T e^{-ru} \bar{E}[\mathcal{E}_u c_u^{iT} | \mathcal{G}_s] du = E\Big[\int_s^T N_u c_u^{iT} du \Big| \mathcal{G}_s\Big]. \end{aligned}$$

Multiplying both sides of (1.4.34) by N_s gives the first equality, while the second one follows from Fubini's Theorem and (1.4.32). Fubini's Theorem again and (1.4.31) imply the third equality, while the last one is true due to Bayes' formula and (1.4.31). □

After proving $(c_t^{iT}, \theta_t^{iT})_{t\geq 0, T\in\mathbb{N}}$ to be a sequence of admissible strategy, we show the total utility converging to 0 as T approaches $+\infty$.

Theorem 1.4.8. *For every $\epsilon_D < 0$ or $\epsilon_D > 2/r$, the optimal consumption problem is ill-posed. In particular for the strategy defined in Theorem 1.4.7,*

$$\sup_{T\in\mathbb{N}^*} E\left[\int_0^{+\infty} e^{-\beta u} U\left(c_u^{iT}\right) du \middle| \mathcal{G}_0\right] = 0. \tag{1.4.35}$$

Proof. For every $0 \leq t < T$, define the function

$$V^i(t,T,x,D) = \frac{(1-e^{r(T-t)})}{r\alpha_i} \exp\left(-\frac{e^{r(T-t)}}{e^{r(T-t)}-1} r\alpha_i x + \delta_{\mathbf{DD}}(T-t)D^2 + \delta_{\mathbf{D}}(T-t)D + \delta_{\mathbf{0}}(T-t)\right),$$

where the functions $\delta_{\mathbf{DD}}, \delta_{\mathbf{D}}, \delta_{\mathbf{0}}$ are given in Definition B.0.1 below. This function solves the finite time HJB equation

$$\begin{aligned} 0 = -\frac{V_x^i}{\alpha_i} - \beta V^i + V_t^i + V_x^i \Big[\frac{\log(V_x^i)}{\alpha_i} + rx + \theta^{iT}(\epsilon_D\bar{\pi} - rC) + \theta^{iT} D(1 - r\epsilon_D - \epsilon_D k)\Big] + \\ + V_D^i(\bar{\pi} - kD) + \frac{1}{2}\Big[V_{xx}^i(\theta^{iT})^2\epsilon_D^2\sigma_D^2 + V_{DD}^i\sigma_D^2 + 2V_{xD}^i\theta^{iT}\epsilon_D\sigma_D^2\Big] \end{aligned} \tag{1.4.36}$$

for $0 \leq t < T$, where

$$\theta_t^{iT} = \frac{\mathbf{M_D}(T-t)D_t + \mathbf{M_0}(T-t)}{M}.$$

The definition of the consumption-investment strategy of Theorem 1.4.7 implies that V^i solves also the equivalent equation

$$\begin{aligned} 0 = U(c_t^{iT}) - \beta V^i + V_t^i + V_x \Big[- c_t^{iT} + rx + \theta_t^{iT}(\epsilon_D\bar{\pi} - rC) + \theta_t^{iT} D(1 - r\epsilon_D - \epsilon_D k)\Big] + \\ + V_D^i(\bar{\pi} - kD) + \frac{1}{2}\Big[V_{xx}^i(\theta_t^{iT})^2\epsilon_D^2\sigma_D^2 + V_{DD}^i\sigma_D^2 + 2V_{xD}^i\theta_t^{iT}\epsilon_D\sigma_D^2\Big] \end{aligned} \tag{1.4.37}$$

for $0 \leq t < T$. For every $T \in \mathbb{N}^*$ define the process $(H_t^T)_{0\leq t<T}$ as

$$H_t^T = e^{-\beta t} V^i(t,T,X_t^{iT},D_t) + \int_0^t e^{-\beta u} U^i(c_u^{iT}) du.$$

Since V^i solves (1.4.37), the drift of the process H_t^T is null and this implies H_t^T to be a $\mathbb{P}-$local martingale. Since V and U are both non-positive, then $-H_t^T$ is a non-negative local martingale, thus a supermartingale [22, Problem 1.5.19 (ii)]. Because of [22, Problem 1.3.16 (ii)] there exists a random variable $-H_T$ such that $(-H_t^T)_{0\leq t\leq T}$ is a continuous supermartingale and $E[-H_t^T|\mathcal{G}_0] \leq -H_0^T$ for every $0 \leq t \leq T$. It follows that

$$E\left[-\int_0^T e^{-\beta u}U^i(c_u^{iT})du\Big|\mathcal{G}_0\right] \le E\left[-\lim_{t\to T} e^{\beta t}V(t,T,X_t^{iT},D_t) - \int_0^T e^{-\beta u}U^i(c_u^{iT})du\Big|\mathcal{G}_0\right]$$
$$\le -V(0,T,X_0^{iT},D_0). \quad (1.4.38)$$

The first inequality in (1.4.38) is true because $V^i(t,T,X_t^{iT},D_t)<0$ for every $0\le t<T$ and the second inequality is the supermartingale property. Thanks to (1.4.38) and since $c_t^{iT}=0$ for every $t\ge T$ we get

$$E\left[\int_0^{+\infty} e^{-\beta u}U^i(c_u^{iT})du\Big|\mathcal{G}_0\right] \ge V^i(0,T,x_0^i,D_0) - \frac{1}{\alpha_i\beta}e^{-\beta T}.$$

Apply $\sup_{T\in\mathbb{N}}$ to both sides

$$\sup_{T\in\mathbb{N}} E\left[\int_0^{+\infty} e^{-\beta u}U^i(c_u^{iT})du\Big|\mathcal{G}_0\right] \ge \sup_{T\in\mathbb{N}} V^i(0,T,x_0^i,D_0) - \frac{1}{\alpha_i\beta}e^{-\beta T}$$
$$\ge \lim_{T\to+\infty} V^i(0,T,x_0^i,D_0) - \frac{1}{\alpha_i\beta}e^{-\beta T} = 0.$$

In the last equality $\lim_{T\to+\infty} V(0,T,x_0^i,D_0)=0$ because for every $D_0, x_0^i\in\mathbb{R}$

$$-\frac{e^{r\tau}}{e^{r\tau}-1}r\alpha_i x_0^i + \delta_{\mathbf{DD}}(\tau)D_0^2 + \delta_{\mathbf{D}}(\tau)D_0 + \delta_{\mathbf{0}}(\tau) \sim_{\tau\to+\infty}$$
$$-\frac{r(-1+\epsilon_D(k+r))^2}{2(r\epsilon_D-2)(-1+r\epsilon_D)\sigma_D^2}\left[\left(D_0+\frac{rC}{r\epsilon_D-1}\right)^2 + \frac{\epsilon_D\sigma_D^2}{2(r\epsilon_D-1)}\right]e^{\left(r-\frac{2}{\epsilon_D}\right)\tau}.$$

For every $\epsilon_D<0$ and $\epsilon_D>\frac{2}{r}$

$$-\frac{r(-1+\epsilon_D(k+r))^2}{2(r\epsilon_D-2)(-1+r\epsilon_D)\sigma_D^2}\left[\left(D_0+\frac{rC}{r\epsilon_D-1}\right)^2 + \frac{\epsilon_D\sigma_D^2}{2(r\epsilon_D-1)}\right] < 0 \text{ and } \left(r-\frac{2}{\epsilon_D}\right) > 0.$$

It follows that (1.4.35) holds and it is not possible to reach total utility 0 since the utility function is strictly negative. □

1.4.3 Market clearing and proof of Theorem 1.2.1

The economy has one risky asset, i.e. for every $t\ge 0$

$$\sum_{i=1}^n \theta_t^{i*} = \sum_{i=1}^n \frac{M_D D_t + M_0}{M\alpha_i} = 1, \quad (1.4.39)$$

where M_D, M_0 are given in Definition 1.4.1.

Proof of Theorem 1.2.1. The market clearing condition (1.4.39) implies $M_D=0$ and

$M_0 = M\alpha_i$. The unique solution of such system, with $\epsilon_D \in \mathbb{B}$, is

$$\epsilon_D = \frac{1}{k+r}, \qquad C = \frac{\bar{\pi}}{r(k+r)} - \frac{\bar{\alpha}\sigma_D^2}{(k+r)^2}.$$

□

Chapter 2

Perfect Information

2.1 Model and main definitions

The economy has one risky asset in unit supply, which pays a dividend stream $(D_t)_{t\geq 0}$ described as

$$dD_t = (\pi_t - kD_t)dt + \sigma_D dW_t^D, \tag{2.1.1}$$

where the state of the economy $(\pi_t)_{t\geq 0}$ is an Ornstein–Uhlenbeck process

$$d\pi_t = a(\bar{\pi} - \pi_t)dt + \sigma_\pi dW_t^\pi. \tag{2.1.2}$$

There is a continuously compounded risk-free asset $(P_t^0)_{t\geq 0}$ with rate of return $r > 0$, at which investors can both lend and borrow. There are $n \in \mathbb{N}$ investors competing for the risky asset, with price $(P_t)_{t\geq 0}$. The $i-$th investor has constant absolute risk aversion $\alpha_i \geq 1$ and initial wealth $x_0^i \in \mathbb{R}$. $W = (W_t^\pi, W_t^D)_{t\geq 0}$ is a Brownian motion and (π_0, D_0) is a normal vector, with mean $(\mu_\pi, \mu_D)^T$ and variance $\begin{pmatrix} \Sigma_\pi^2 & 0 \\ 0 & \Sigma_D^2 \end{pmatrix}$, independent of the Brownian motion previously defined. The probability space is $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$, where $\mathcal{G}_t$ is the augmented natural filtration of $D_0, \pi_0, (W_u)_{0\leq u\leq t}$ and $\mathcal{G}$ is the augmented sigma algebra generated by $\bigcup_{t\geq 0} \mathcal{G}_t$[1]. In the light of Chapter 1, we focus for the whole chapter on the following

Assumption 2.1.1. The parameters of the economy are $a, k, \sigma_D > 0, \sigma_\pi \geq 0$ and $\bar{\pi} \in \mathbb{R}$. Furthermore assume

$$\epsilon_D \in \mathbb{B} := (0, 2/r) \setminus \{1/r\}, \qquad \epsilon_\pi \neq 0, \qquad a \neq k.$$

All equalities and inequalities between random variables are understood $\mathbb{P}-$almost surely.

[1] Note that all filtrations are augmented with the null sets of the sigma algebra $\mathcal{G}$.

Definition 2.1.1 (Admissibile strategies). $(c_t, \theta_t)_{t\geq 0}$ is an admissible (consumption-investment) strategy for the $i-$th investor if:

(i) $(c_t)_{t\geq 0}$ and $(\theta_t)_{t\geq 0}$ are $(\mathcal{G}_t)_{t\geq 0}-$progressively measurable processes;

(ii) for every $s \geq 0$

$$\limsup_{t\to+\infty} E\left[\int_s^t N_u c_u du \Big| \mathcal{G}_s\right] \leq N_s X_s, \tag{2.1.3}$$

where $(X_t)_{t\geq 0}$ is the self-financing wealth process

$$dX_t = -c_t dt + \theta_t D_t dt + r(X_t - \theta_t P_t)dt + \theta_t dP_t, \qquad X_0 = x_0^i, \tag{2.1.4}$$

$(N_t)_{t\geq 0}$ is the process

$$N_t = \exp\left(-rt + \int_0^t (\Delta\cdot v_u)dW_u^D + \int_0^t (\iota\cdot v_u)dW_u^\pi - \frac{1}{2}\int_0^t \left[(\Delta\cdot v_u)^2 + (\iota\cdot v_u)^2\right]du\right), \tag{2.1.5}$$

$v_t = (D_t, \pi_t, 1)^T$ and Δ, ι are given in Definition 2.4.1 below.

The set of admissible strategies for the $i-$th investor is $\mathcal{U}^i$.

Definition 2.1.2 (Optimality). A (consumption-investment) strategy $(c_t^i, \theta_t^i)_{t\geq 0}$ is optimal for the $i-$th investor if it is admissible and if

$$\sup_{(c,\theta)\in\mathcal{U}^i} E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u)du \Big| \mathcal{G}_0\right] = E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u^i)du \Big| \mathcal{G}_0\right], \tag{2.1.6}$$

where

$$U^i(c) := -\frac{e^{-\alpha_i c}}{\alpha_i}, \qquad i \in \{1,\dots,n\}.$$

The time impatience parameter $\beta > 0$ is common to all agents. The consumption-investment problem of each agent is well-posed if an optimal strategy exists, otherwise the problem is ill-posed.

Remark 2.1.1. The process $(D_t, \pi_t)_{t\geq 0}$ starting with a random variable is not a fundamental feature of the model in this chapter but it will be important in the third chapter, for a stationary filter in Lemma 3.4.1. (D_0, π_0) being a random variable implies that the $\sigma-$algebra $\mathcal{G}_0$ is different from the trivial $\sigma-$algebra, so a conditional expectation appears in (2.1.6).

Definition 2.1.3. $(c_t^i, \theta_t^i)_{t\geq 0}$ is the unique optimal (consumption-investment) strategy for the $i-$th investor if it is optimal for the $i-$th investor and if

$$(c_t^i, \theta_t^i)_{t\geq 0} = (\bar{c}_t, \bar{\theta}_t)_{t\geq 0} \qquad \lambda_{|[0,+\infty[} \otimes \mathbb{P} - \text{a.s.}$$

for every other optimal strategy $(\bar{c}_t, \bar{\theta}_t)_{t\geq 0}$.

Definition 2.1.4. A linear equilibrium is an $(n+4)-$tuple $(\sigma_\pi, \epsilon_D^{\sigma_\pi}, \epsilon_\pi^{\sigma_\pi}, C^{\sigma_\pi}, (S^i_{\sigma_\pi})^{1\leq i\leq n})$, where $\sigma_\pi \geq 0, \epsilon_D^{\sigma_\pi} \in \mathbb{B}, \epsilon_\pi^{\sigma_\pi} \in \mathbb{R}^*, C^{\sigma_\pi} \in \mathbb{R}$ and $S^i_{\sigma_\pi} = (c^i_t, \theta^i_t)_{t\geq 0}$ is an optimal strategy for the $i-$th investor for every $i \in \{1, \dots, n\}$ such that for every $t \geq 0$

(i) the price of the risky asset is

$$P_t = C^{\sigma_\pi} + \epsilon_D^{\sigma_\pi} D_t + \epsilon_\pi^{\sigma_\pi} \pi_t; \tag{2.1.7}$$

(ii) the market clearing condition

$$\sum_{i=1}^n \theta^i_t = 1 \tag{2.1.8}$$

holds.

2.2 Existence and uniqueness of the equilibrium

Theorem 2.2.1. *Under Assumption 2.1.1 there exists $\bar{\sigma}_\pi > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma}_\pi$ there exists a unique linear equilibrium $(\sigma_\pi, \epsilon_D^{\sigma_\pi}, \epsilon_\pi^{\sigma_\pi}, C^{\sigma_\pi}, (S^i_{\sigma_\pi})^{1\leq i\leq n})$, for which the price is*

$$P_t = C^* + \epsilon^*_D D_t + \epsilon^*_\pi \pi_t, \qquad \textit{where} \qquad \epsilon^*_D = \frac{1}{k+r}, \qquad \epsilon^*_\pi = \frac{1}{(a+r)(k+r)},$$

$$C^* = \frac{a\bar{\pi}}{r(a+r)(k+r)} - \bar{\alpha}\left(\frac{\sigma_D^2}{(k+r)^2} + \frac{\sigma_\pi^2}{(a+r)^2(k+r)^2}\right) \qquad \textit{and} \qquad \bar{\alpha} = \left(\sum_{i=1}^n \frac{1}{\alpha_i}\right)^{-1}.$$

The unique optimal consumption-investment strategy for the $i-$th agent is

$$c^{i*}_t = rX^{i*}_t + \frac{\beta - r}{r\alpha_i} + \frac{r\bar{\alpha}^2}{2\alpha_i}\left(\frac{\sigma_D^2}{(k+r)^2} + \frac{\sigma_\pi^2}{(a+r)^2(k+r)^2}\right), \qquad \theta^{i*}_t = \frac{\bar{\alpha}}{\alpha_i}. \tag{2.2.1}$$

Preliminaries and outline of the proof

Remark 2.2.1. If $(\epsilon_D, \epsilon_\pi) = (0, 0)$ then (2.1.7) implies $P_t = C$ for every $t \geq 0$. If the assets are two deterministic processes with different interest rates (0 for P_t and $r > 0$ for P^0_t), then the model admits arbitrage, therefore the consumption-investment problem of the agents is ill-posed and in particular no linear equilibrium exists.

Definition 2.2.1. A value function for the $i-$th investor is a function

$$V^i : \mathbb{R}^3 \to [-\infty, 0[$$
$$(\bar{x}, \bar{D}, \bar{\Pi}) \to V^i(\bar{x}, \bar{D}, \bar{\Pi})$$

such that for every $(\bar{x}, \bar{D}, \bar{\Pi}) \in \mathbb{R}^3$

$$V^i(\bar{x}, \bar{D}, \bar{\Pi}) = \sup_{(c,\theta)\in\mathcal{U}^i} E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u)du \Big| x_0^i = \bar{x}, D_0 = \bar{D}, \pi_0 = \bar{\Pi}\right]. \tag{2.2.2}$$

It follows from this definition that if there exists a value function $V^i(\cdot)$ and a strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ which is optimal for the $i-$th investor, then

$$V^i(x_0^{i*}, D_0, \pi_0) = E\left[\int_0^{+\infty} e^{-\beta s} U^i(c_s^{i*})ds \Big| \mathcal{G}_0\right].$$

Definition 2.2.2. A stochastic discount factor (SDF) is a positive, continuous, $(\mathcal{G}_t)_{t\geq 0}-$adapted process $(M_t)_{t\geq 0}$ such that for every $0 \leq s \leq t$

$$M_s P_s^0 = E[M_t P_t^0 | \mathcal{G}_s] \tag{2.2.3}$$

and

$$M_s P_s + \int_0^s M_u D_u du = E\left[M_t P_t + \int_0^t M_u D_u du \Big| \mathcal{G}_s\right]. \tag{2.2.4}$$

A stochastic discount factor is normalized if $M_0 = 1$.

We find the (unique) equilibrium in the market in two steps: first we solve the optimal consumption problem of the agents for a generic price with the form of (2.1.7); then we clear the market with condition (2.1.8) and we deduce that the price of the unique linear equilibrium has parameters

$$\epsilon_D^* = \frac{1}{k+r}, \qquad \epsilon_\pi^* = \frac{1}{(a+r)(k+r)} \qquad \text{and}$$
$$C^* = \frac{a\bar{\pi}}{r(a+r)(k+r)} - \bar{\alpha}\left(\frac{\sigma_D^2}{(k+r)^2} + \frac{\sigma_\pi^2}{(a+r)^2(k+r)^2}\right).$$

- **Section 2.3** formulates the Hamilton Jacobi Bellman (HJB) equation, which leads to a guess of the value function and the optimal strategies.
- **Section 2.4** formalizes the heuristics of the previous section proving existence and uniqueness of the optimal portfolio for a generic price function for $\epsilon_D \in \mathbb{B}$.
  - **Subsection 2.4.1** finds the unique linear equilibrium in the market through the market clearing condition.
- **Appendix C** recalls some well known results that are used in this chapter.

2.3 Heuristics

Guess a value function V^i which depends on the dividend rate, on the state of the economy and on the wealth; because of the infinite time horizon we guess that V^i does not depend on the initial time $t > 0$, i.e.

$$V^i(X_t^i, D_t, \pi_t) = \sup_{(c^i,\theta^i)} E\left[\int_t^{+\infty} e^{-\beta(s-t)} U^i(c_s^i) ds | \mathcal{G}_t\right].$$

A similar procedure as in Subsection 1.3.1 leads to

$$\begin{aligned} 0 = \sup_{(c^i,\theta^i)} \Big\{ &- \frac{e^{-\alpha_i c^i}}{\alpha_i} - \beta V^i + V_x^i \Big[- c^i + rx + \theta^i(\epsilon_\pi a\bar{\pi} - rC) + \theta^i D(1 - \epsilon_D(k+r)) + \\ &+ \theta^i \pi(\epsilon_D - \epsilon_\pi(a+r))\Big] + V_D^i(\pi - kD) + aV_\pi^i(\bar{\pi} - \pi) + \frac{1}{2}\Big[V_{xx}^i(\theta^i)^2(\epsilon_D^2\sigma_D^2 + \epsilon_\pi^2\sigma_\pi^2) + \\ &+ V_{DD}^i\sigma_D^2 + V_{\pi\pi}^i\sigma_\pi^2 + 2V_{xD}^i\theta^i\epsilon_D\sigma_D^2 + 2V_{x\pi}^i\theta^i\epsilon_\pi\sigma_\pi^2\Big]\Big\}. \end{aligned} \tag{2.3.1}$$

Differentiating with respect to c^i and θ^i, we find the candidate optimal consumption-investment policy

$$\begin{aligned} c^{i*} &= \frac{\log(V_x^i)}{-\alpha_i}, \\ \theta^{i*} &= -\frac{V_x^i\Big[(\epsilon_\pi a\bar{\pi} - rC) + D\Big(1 - \epsilon_D(k+r)\Big) + \pi\Big(\epsilon_D - \epsilon_\pi(a+r)\Big)\Big] + V_{xD}^i\epsilon_D\sigma_D^2 + V_{x\pi}^i\epsilon_\pi\sigma_\pi^2}{V_{xx}^i(\epsilon_D^2\sigma_D^2 + \epsilon_\pi^2\sigma_\pi^2)}. \end{aligned} \tag{2.3.2}$$

The HJB equation for the $i-$th investor follows by substituting the candidate optimal policies into (2.3.1)

$$\begin{aligned} 0 = &-\frac{V_x}{\alpha_i} - \beta V^i + V_x^i\Big[\Big(\frac{\log V_x}{\alpha_i} + rx\Big) + \theta^{i*}(\epsilon_\pi a\bar{\pi} - rC) + \theta^{i*}D\Big(1 - \epsilon_D(k+r)\Big) + \\ &+\theta^{i*}\pi\Big(\epsilon_D - \epsilon_\pi(a+r)\Big)\Big] + V_D^i(\pi - kD) + aV_\pi^i(\bar{\pi} - \pi) + \frac{1}{2}\Big[V_{xx}^i(\theta^{i*})^2(\epsilon_D^2\sigma_D^2 + \epsilon_\pi^2\sigma_\pi^2) + V_{DD}^i\sigma_D^2 + \\ &+ V_{\pi\pi}^i\sigma_\pi^2 + 2V_{xD}^i\theta^{i*}\epsilon_D\sigma_D^2 + 2V_{x\pi}^i\theta^{i*}\epsilon_\pi\sigma_\pi^2\Big]. \end{aligned} \tag{2.3.3}$$

Using the Ansatz

$$V^i(x, D, \pi) = -\frac{1}{r\alpha_i}\exp\left(-r\alpha_i x + \delta_{DD}D^2 + \delta_{D\pi}D\pi + \delta_{\pi\pi}\pi^2 + \delta_D D + \delta_\pi \pi + \delta_0\right),$$

where $\delta_{DD}, \delta_{\pi\pi}, \delta_{D\pi}, \delta_D, \delta_\pi, \delta_0$ are in Theorem 2.4.1, (2.3.2) leads to the optimal consumption investment strategy

$$\begin{aligned} c_t^{i*} &= rX_t^i - \frac{\delta_{DD}}{\alpha_i} D_t^2 - \frac{\delta_{D\pi}}{\alpha_i} D_t \pi_t - \frac{\delta_{\pi\pi}}{\alpha_i} \pi_t^2 - \frac{\delta_D}{\alpha_i} D_t - \frac{\delta_\pi}{\alpha_i} \pi_t - \frac{\delta_0}{\alpha_i}; \\ \theta_t^{i*} &= \frac{M_D D_t + M_\pi \pi_t M_0}{M \alpha_i}. \end{aligned}$$

and M_D, M_π, M_0 are in Definition 2.4.1.

2.4 Verification

Theorem 2.2.1 identifies the unique linear equilibrium in the market. The first step of the proof is to solve the consumption-investment problem of the agents for a generic price with form (2.1.7),

Direct calculations show that the self-financing condition (2.1.4) for an investor with consumption-investment strategy $(c_t^i, \theta_t^i)_{t\geq 0}$ is equivalent to

$$dX_t^i = \Big[-c_t^i + rX_t^i + \theta_t^i(\epsilon_\pi a\bar{\pi} - rC) + \theta_t^i D_t\left(1 - \epsilon_D(k+r)\right) + \theta_t^i \pi_t\left(\epsilon_D - \epsilon_\pi(a+r)\right)\Big]dt + \\ + \theta_t^i \epsilon_D \sigma_D dW_t^D + \theta_t^i \epsilon_\pi \sigma_\pi dW_t^\pi. \quad (2.4.1)$$

The following theorem proves the existence of a solution of the HJB equation, and thus a candidate value function.

Theorem 2.4.1. *Fix $\epsilon_D^0 \in \mathbb{B}, \epsilon_\pi^0 \neq 0, C^0 \in \mathbb{R}$, define $\delta = (\delta_{DD}, \delta_{\pi\pi}, \delta_{D\pi}, \delta_D, \delta_\pi, \delta_0)$ and let $\bar{\delta}$ be the function in Definition C.0.1; there exist*

(i) $U(\epsilon_D^0, \epsilon_\pi^0, C^0) \subseteq \mathbb{R}^4 \times \mathbb{R}^6$ open neighbourhood of $\left(0, \epsilon_D^0, \epsilon_\pi^0, C^0, \bar{\delta}(\epsilon_D^0, \epsilon_\pi^0, C^0)\right)$;

(ii) $W(\epsilon_D^0, \epsilon_\pi^0, C^0) \subseteq \mathbb{R}^4$ open neighbourhood of $(0, \epsilon_D^0, \epsilon_\pi^0, C^0)$;

such that

(I) for every $(\sigma_\pi, \epsilon_D, \epsilon_\pi, C) \in W(\epsilon_D^0, \epsilon_\pi^0, C^0)$, there exists a unique δ such that $(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta) \in U(\epsilon_D^0, \epsilon_\pi^0, C^0)$ and the function

$$V^i(x, D, \pi) = -\frac{1}{r\alpha_i}\exp\left(-r\alpha_i x + \delta_{DD}D^2 + \delta_{D\pi}D\pi + \delta_{\pi\pi}\pi^2 + \delta_D D + \delta_\pi \pi + \delta_0\right) \quad (2.4.2)$$

solves the Hamilton Jacobi Bellman equation of the $i-$th investor

$$0 = -\frac{V_x^i}{\alpha_i} - \beta V^i + V_x^i\Big[\Big(\frac{\log V_x^i}{\alpha_i} + rx\Big) + \theta^{i*}(\epsilon_\pi a\bar{\pi} - rC) + \theta^{i*}D\Big(1 - \epsilon_D(k+r)\Big) + \\ + \theta^{i*}\pi\Big(\epsilon_D - \epsilon_\pi(a+r)\Big)\Big] + V_D^i(\pi - kD) + aV_\pi^i(\bar{\pi} - \pi) + \frac{1}{2}\Big[V_{xx}^i(\theta^{i*})^2(\epsilon_D^2\sigma_D^2 + \epsilon_\pi^2\sigma_\pi^2) + V_{DD}^i\sigma_D^2 + \\ + V_{\pi\pi}^i\sigma_\pi^2 + 2V_{xD}^i\theta^{i*}\epsilon_D\sigma_D^2 + 2V_{x\pi}^i\theta^{i*}\epsilon_\pi\sigma_\pi^2\Big], \quad (2.4.3)$$

where

$$\theta^{i*} = -\frac{V_x^i\Big[(\epsilon_\pi a\bar{\pi} - rC) + D\Big(1 - \epsilon_D(k+r)\Big) + \pi\Big(\epsilon_D - \epsilon_\pi(a+r)\Big)\Big] + V_{xD}^i\epsilon_D\sigma_D^2 + V_{x\pi}^i\epsilon_\pi\sigma_\pi^2}{V_{xx}^i(\epsilon_D^2\sigma_D^2 + \epsilon_\pi^2\sigma_\pi^2)}.$$

(II) If this δ is defined to be $g(\sigma_\pi, \epsilon_D, \epsilon_\pi, C)$, then $g \in \mathscr{C}^1(W, U)$ and $g(0, \epsilon_D^0, \epsilon_\pi^0, C^0) = \bar{\delta}(\epsilon_D^0, \epsilon_\pi^0, C^0)$.

Proof. Inserting (2.4.2) into (2.4.3) makes the HJB equation an algebraic equation of second order. For every $i \in \{1, \dots, n\}$, the function $V^i(\cdot)$ solves the HJB equation if and only if

$$f(\sigma_\pi, \delta_{DD}, \delta_{D\pi}, \delta_{\pi\pi}, \delta_D, \delta_\pi, \delta_0) = 0,$$

where f is defined in (C.0.2). Lemma C.0.1 concludes the proof. □

The following are technical results for the solution of the consumption-investment problem.

Lemma 2.4.1. For every $\eta_0, \eta_1, \eta_2 \in \mathbb{R}$, there exist constants $\bar{\mu}, \bar{\sigma} > 0$ independent by t such that, for every $t \geq 0$,

$$|E[\eta_2 D_t + \eta_1 \pi_t + \eta_0]| \leq \bar{\mu}, \qquad \operatorname{Var}[\eta_2 D_t + \eta_1 \pi_t + \eta_0] \leq \bar{\sigma}^2.$$

Proof. Apply Itô's formula to $e^{at}\pi_t$ to get

$$\pi_t = e^{-a(t-s)}\pi_s + \bar{\pi}\left(1 - e^{-a(t-s)}\right) + \sigma_\pi e^{-at}\int_s^t e^{au} dW_u^\pi. \tag{2.4.4}$$

Apply Itô's formula to $e^{kt}D_t$ and to $e^{kt}\pi_t$ to get

$$D_t = e^{-k(t-s)}D_s + \frac{\pi_s}{k-a}\left(e^{-a(t-s)} - e^{-k(t-s)}\right) + \frac{\bar{\pi}}{k-a}\left[\left(1 - e^{-a(t-s)}\right) - \frac{a}{k}\left(1 - e^{-k(t-s)}\right)\right] + \\ + \sigma_D \int_s^t e^{-k(t-u)} dW_u^D + \frac{\sigma_\pi}{k-a}\int_s^t \left(e^{-a(t-u)} - e^{-k(t-u)}\right) dW_u^\pi. \tag{2.4.5}$$

Because of (2.4.4), $|E[\pi_t]| \leq |\mu_\pi| + 2|\bar{\pi}|$ and because of (2.4.5), $|E[D_t]| \leq |\mu_D| + 2|\frac{\mu_\pi}{k-a}| + |\frac{\bar{\pi}}{k-a}|\left(2 + 2|\frac{a}{k}|\right)$; the last two inequalities imply $|E[\eta_2 D_u + \eta_1 \pi_u + \eta_0]| \leq \bar{\mu}$. Thanks to (2.4.4) and to (2.4.5) we get

$$\operatorname{Var}[\pi_t] \leq \Sigma_\pi^2 + \frac{\sigma_\pi^2}{a}, \qquad |\operatorname{Cov}(\pi_t, D_t)| \leq \left|\frac{2\Sigma_\pi^2}{a-k}\right| + \frac{\sigma_\pi^2}{(k-a)^2}\left(\frac{1}{a} + \frac{2}{a+k}\right),$$
$$\operatorname{Var}[D_t] \leq \Sigma_D^2 + \frac{4}{(k-a)^2}\Sigma_\pi^2 + \frac{\sigma_D^2}{k} + \frac{\sigma_\pi^2}{(k-a)^2}\left(\frac{1}{a} + \frac{1}{k} + \frac{2}{a+k}\right),$$

which imply $\operatorname{Var}[\eta_2 D_u + \eta_1 \pi_u + \eta_0] \leq \bar{\sigma}^2$. □

The value of the constants Δ, ι will be set later in Definition 2.4.1

Lemma 2.4.2. Define $(v_t)_{t\geq 0} = (D_t, \pi_t, 1)^T_{t\geq 0}$ and fix

$$\Delta = (\Delta_D, \Delta_\pi, \Delta_0)^T \in \mathbb{R}^3, \qquad \iota = (\iota_D, \iota_\pi, \iota_0)^T \in \mathbb{R}^3.$$

The process

$$H_t := \exp\left(\int_0^t \Delta \cdot v_u dW_u^D + \iota \cdot v_u dW_u^\pi - \frac{1}{2}\int_0^t \left[(\Delta \cdot v_u)^2 + (\iota \cdot v_u)^2\right] du\right)$$

is a $\mathbb{P}-$martingale.

Proof. Define $Y_t = \Delta \cdot v_t, Z_t = \iota \cdot v_t$ and recall Novikov's condition [22, Corollary 5.13 and 5.14], which ensures that H_t is a martingale:

(A) $\mathbb{P}\left[\int_0^t Y_u^2 du < +\infty\right] = \mathbb{P}\left[\int_0^t Z_u^2 du < +\infty\right] = 1$;

(B) there exists a sequence $(t_m)_{m\in\mathbb{N}} \subset \mathbb{R}$ increasing to $+\infty$, such that, for every $m \in \mathbb{N}$,

$$E\left[\exp\left(\int_{t_{m-1}}^{t_m} \frac{1}{2}(Y_u^2 + Z_u^2)du\right)\right] < +\infty.$$

The processes $(Y_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are $\mathbb{P}-$a.s. continuous, hence (A) is true. By Jensen's inequality [28, Theorem 1.8.1], for every $t, \epsilon \geq 0$,

$$\exp\left(\int_t^{t+\epsilon} \frac{1}{2}(Y_u^2 + Z_u^2)du\right) \leq \frac{1}{\epsilon}\int_t^{t+\epsilon} \exp\left(\frac{\epsilon}{2}(Y_u^2 + Z_u^2)\right) du.$$

In addition, by Fubini's Theorem

$$E\left[\frac{1}{\epsilon}\int_t^{t+\epsilon} \exp\left(\frac{\epsilon}{2}(Y_u^2 + Z_u^2)\right)du\right] = \frac{1}{\epsilon}\int_t^{t+\epsilon} E\Big[\exp\left(\frac{\epsilon}{2}(Y_u^2 + Z_u^2)\right)\Big]du.$$

Young's inequality [24, Lemma 7.17] yields to

$$E\left[\exp\left(\int_t^{t+\epsilon} \frac{1}{2}(Y_u^2 + Z_u^2)du\right)\right] \leq \frac{1}{\epsilon}\int_t^{t+\epsilon} E\left[\exp\left(\epsilon Y_u^2\right)\right] du + \frac{1}{\epsilon}\int_t^{t+\epsilon} E\left[\exp\left(\epsilon Z_u^2\right)\right] du. \tag{2.4.6}$$

Suppose $(\Delta_D, \Delta_\pi) \neq (0,0)$ and define $\mu_u = E[Y_u]$ and $\sigma_u^2 = \mathrm{Var}[Y_u]$. In view of Lemma 2.4.1, there exist constants $\bar{\mu}$ and $\bar{\sigma}^2$ such that for every $0 \leq u \leq t$

$$|\mu_u| \leq \bar{\mu}, \qquad\qquad \sigma_u^2 \leq \bar{\sigma}^2 \tag{2.4.7}$$

for every $u \geq 0$. For every $u \geq 0$, Y_u is a normally distributed random variable, and in particular

$$E\Big[\exp\left(\epsilon Y_u^2\right)\Big] = \frac{\exp\left(\frac{\mu_u^2\epsilon}{1-2\sigma_u^2\epsilon}\right)}{\sqrt{1-2\sigma_u^2\epsilon}}, \qquad \text{if} \qquad 2\sigma_u^2\epsilon \leq 1.$$

Since $\sigma_u^2 \leq \bar{\sigma}^2$, then any $\epsilon < \frac{1}{2}\bar{\sigma}^{-2}$ satisfies $2\sigma_u^2\epsilon < 1$ because

$$2\sigma_u^2\epsilon \leq 2\bar{\sigma}^2\epsilon < 1. \tag{2.4.8}$$

Fix $\epsilon < \frac{1}{2}\bar{\sigma}^{-2}$; if we prove that $E\Big[\exp\left(\epsilon Y_u^2\right)\Big]$ is a continuous function, uniformly bounded in t on the interval $[t, t+\epsilon]$, for the ϵ chosen above, then its integral is finite and it is enough to define the sequence $t_m := m\epsilon$. Equation (2.4.8) implies $1 - 2\sigma_u^2\epsilon \geq 1 - 2\bar{\sigma}^2\epsilon$, and both terms are between 0 and 1 because of the choice of ϵ. Thus, defining $\kappa_\epsilon = \frac{1}{1-2\bar{\sigma}^2\epsilon}$, it follows that

$$\frac{1}{1-2\sigma_u^2\epsilon} \leq \kappa_\epsilon \qquad \text{and} \qquad \frac{1}{\sqrt{1-2\sigma_u^2\epsilon}} \leq \kappa_\epsilon.$$

As a consequence

$$E\Big[\exp\left(\epsilon Y_u^2\right)\Big] \leq \kappa_\epsilon \exp\left(\kappa_\epsilon \epsilon \bar{\mu}^2\right) < +\infty.$$

$E\Big[\exp\left(\epsilon Y_u^2\right)\Big]$ is a continuous and bounded function on the interval $[t, t+\epsilon]$ and so for every $\epsilon > 0$ and every $t \geq 0$

$$E\left[\frac{1}{\epsilon}\int_t^{t+\epsilon} \exp\left(\epsilon Y_u^2\right) du\right] = \frac{1}{\epsilon}\int_t^{t+\epsilon} E\Big[\exp\left(\epsilon Y_u^2\right)\Big] du < +\infty.$$

The same reasoning shows also that

$$E\left[\frac{1}{\epsilon}\int_t^{t+\epsilon} \exp\left(\epsilon Z_u^2\right) du\right] < +\infty$$

and (B) follows from (2.4.6). □

Definition 2.4.1. We introduce the following constants,

$$\Delta := (\Delta_D, \Delta_\pi, \Delta_0)^T, \qquad \text{and} \qquad \iota := (\iota_D, \iota_\pi, \iota_0)^T,$$

where

$$\Delta_D := \sigma_D\left(2\delta_{DD} - r\epsilon_D \frac{M_D}{M}\right), \quad \Delta_\pi := \sigma_D\left(\delta_{D\pi} - r\epsilon_D \frac{M_\pi}{M}\right), \quad \Delta_0 := \sigma_D\left(\delta_D - r\epsilon_D \frac{M_0}{M}\right),$$
$$\iota_D := \sigma_\pi\left(\delta_{D\pi} - r\epsilon_\pi \frac{M_D}{M}\right), \quad \iota_\pi := \sigma_\pi\left(2\delta_{\pi\pi} - r\epsilon_\pi \frac{M_\pi}{M}\right), \quad \iota_0 := \sigma_\pi\left(\delta_\pi - r\epsilon_\pi \frac{M_0}{M}\right),$$

and

$$\begin{aligned} M &:= r(\epsilon_D^2\sigma_D^2 + \epsilon_\pi^2\sigma_\pi^2), \\ M_D &:= 1 - \epsilon_D(k + r) + 2\delta_{DD}\epsilon_D\sigma_D^2 + \delta_{D\pi}\epsilon_\pi\sigma_\pi^2, \\ M_\pi &:= \epsilon_D - \epsilon_\pi(a + r) + 2\delta_{\pi\pi}\epsilon_\pi\sigma_\pi^2 + \delta_{D\pi}\epsilon_D\sigma_D^2, \\ M_0 &:= (a\bar{\pi}\epsilon_\pi - rC) + \delta_D\epsilon_D\sigma_D^2 + \delta_\pi\epsilon_\pi\sigma_\pi^2. \end{aligned}$$

The constants $\delta_{DD}, \delta_{\pi\pi}, \delta_{D\pi}, \delta_D, \delta_\pi, \delta_0$ are those of Theorem 2.4.1

Corollary 2.4.1. *The process $(\mathcal{E}_t)_{t\geq 0} = (e^{rt}N_t)_{t\geq 0}$, in (2.1.5), is a $\mathbb{P}-$martingale.*

Since $(\mathcal{E}_t)_{t\geq 0}$ is a $\mathbb{P}-$martingale, Girsanov's Theorem [22, Theorem 5.1] holds. In particular, $(\mathcal{E}_t)_{t\geq 0}$ defines a probability measure $\bar{\mathbb{P}} := \bar{\mathbb{P}}^{(\Delta,\iota)}$, such that $\mathcal{E} = d\bar{\mathbb{P}}/d\mathbb{P}$. We denote by $\bar{E}[\cdot]$ and $\bar{\text{V}}\text{ar}[\cdot]$ the conditional expectation and variance under the measure $\bar{\mathbb{P}}$. Any equality or inequality between random variables is understood $\mathbb{P}$ and $\bar{\mathbb{P}}-$almost surely. The process

$$(\bar{W}_t^D, \bar{W}_t^\pi)_{t\geq 0} = \left(W_t^D - \int_0^t (\Delta \cdot v_u)du, W_t^\pi - \int_0^t (\iota \cdot v_u)du\right)_{t\geq 0} \tag{2.4.9}$$

is a $\bar{\mathbb{P}}-$Brownian motion and furthermore Bayes' formula [22, Lemma 5.3] applies: for every $\mathcal{G}_t-$measurable random variable X satisfying $\bar{E}[|X|] < +\infty$ and for every $0 \leq s \leq t$

$$\bar{E}[X|\mathcal{G}_s] = \frac{1}{\mathcal{E}_s} E[X\mathcal{E}_t|\mathcal{G}_s].$$

The next lemma describes the processes $(D_t)_{t\geq 0}$ and $(\pi_t)_{t\geq 0}$ under the new measure $\bar{\mathbb{P}}$.

Lemma 2.4.3 (Joint dynamics). Suppose that Assumption 2.1.1 holds. Then there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$, all the following hold. The process $\chi_t := (\pi_t, D_t)_{t\geq 0}$ satisfies the stochastic differential equation

$$d\chi_t = (b + A\chi_t)dt + \Sigma d\bar{W}_t; \tag{2.4.10}$$

where

$$A = \begin{pmatrix} \sigma_\pi \iota_\pi - a & \sigma_\pi \iota_D \\ \sigma_D \Delta_\pi + 1 & \sigma_D \Delta_D - k \end{pmatrix}, \quad b = \begin{pmatrix} \sigma_\pi \iota_0 + a\bar{\pi} \\ \sigma_D \Delta_0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_\pi & 0 \\ 0 & \sigma_D \end{pmatrix}, \quad \bar{W}_t = \begin{pmatrix} \bar{W}_t^\pi \\ \bar{W}_t^D \end{pmatrix}.$$

The matrix A is invertible, and the unique solution of (2.4.10) is

$$\chi_t = e^{A(t-s)}\chi_s + A^{-1}(e^{A(t-s)} - I_2)b + e^{At}\int_s^t e^{-Au}\Sigma d\bar{W}_u. \tag{2.4.11}$$

For every $0 \leq s \leq t$ there exists a $\mathcal{G}_s-$measurable random variable $\eta_s \geq 0$ and a positive constant η such that

$$\begin{aligned}
\|\bar{E}\left[\chi_t|\mathcal{G}_s\right]\| &\leq \eta_s e^{\|A\|t} + \eta, && (i) \\
\|\bar{\text{V}}\text{ar}\left[\chi_t|\mathcal{G}_s\right]\| &\leq \eta e^{(\|A\|+\|A^T\|)t}, && (ii) \\
\|\bar{E}[\chi_t|\mathcal{G}_s] \otimes \bar{E}_t[\chi_t|\mathcal{G}_s]\| &\leq \eta_s e^{(\|A\|+\|A^T\|)t} + \eta, && (iii) \\
\|\bar{E}\left[\chi_t \otimes \chi_t|\mathcal{G}_s\right]\| &\leq \eta_s e^{(\|A\|+\|A^T\|)t} + \eta. && (iv)
\end{aligned} \tag{2.4.12}$$

$$\begin{aligned} \|\bar{E}\left[\chi_t\right]\| &\leq \eta e^{\|A\|t} + \eta, && (i) \\ \|\bar{\mathrm{V}}\mathrm{ar}\left[\chi_t\right]\| &\leq \eta e^{(\|A\|+\|A^T\|)t} + \eta, && (ii) \\ \|\bar{E}[\chi_t] \otimes \bar{E}[\chi_t]\| &\leq \eta e^{(\|A\|+\|A^T\|)t} + \eta, && (iii) \\ \|\bar{E}\left[\chi_t \otimes \chi_t\right]\| &\leq \eta e^{(\|A\|+\|A^T\|)t} + \eta, && (iv) \\ \bar{E}\left[\|\chi_t\|\right] &< +\infty. && (v) \end{aligned} \tag{2.4.13}$$

For every $s \geq 0$ and for every $\eta_0, \ldots, \eta_5 \in \mathbb{R}$

(a)

$$E\left[N_t(C + \epsilon_D D_t + \epsilon_\pi \pi_t) + \int_s^t N_u D_u du \Big| \mathcal{G}_s\right] = \\ N_s \bar{E}\left[e^{-r(t-s)}(C + \epsilon_D D_t + \epsilon_\pi \pi_t) + \int_s^t e^{-r(u-s)} D_u du \Big| \mathcal{G}_s\right]; \tag{2.4.14}$$

(b)

$$-\infty < \bar{E}\left[\int_s^t \eta_5 D_u^2 + \eta_4 \pi_u^2 + \eta_3 D_u \pi_u + \eta_2 D_u + \eta_1 \pi_u + \eta_0 du\right] < +\infty;$$

(c)

$$\int_s^t (\eta_5 D_u + \eta_4 \pi_u + \eta_3) d\bar{W}_u^D + (\eta_2 D_u + \eta_1 \pi_u + \eta_0) d\bar{W}_u^\pi \qquad \text{is } \bar{\mathbb{P}}\text{−martingale;}$$

(d) for every $s \geq 0$

$$\lim_{t \to +\infty} e^{-rt} \bar{E}\left[\chi_t | \mathcal{G}_s\right] = \lim_{t \to +\infty} e^{-rt} \bar{E}\left[\chi_t \otimes \chi_t | \mathcal{G}_s\right] = \\ = \lim_{t \to +\infty} e^{-rt} \bar{E}\left[\int_s^t \chi_u du \Big| \mathcal{G}_s\right] = \lim_{t \to +\infty} e^{-rt} \bar{E}\left[\int_s^t \chi_u \otimes \chi_u du \Big| \mathcal{G}_s\right] = 0;$$

(e) for every $s \geq 0$

$$\frac{1}{2} \lim_{t \to +\infty} \bar{E}\Big[\int_s^t e^{-r(u-s)} \left((\Delta \cdot v_u)^2 + (\iota \cdot v_u)^2\right) du \Big| \mathcal{G}_s\Big] \\ = -\delta_{DD} D_s^2 - \delta_{D\pi} D_s \pi_s - \delta_{\pi\pi} \pi_s^2 - \delta_D D_s - \delta_\pi \pi_s - \delta_0 - \frac{\beta - r}{r}.$$

Remark 2.4.1. All the above are local results for σ_π in a right neighbourhood of 0.

Proof. (2.4.10) is a direct consequence of (2.1.1), (2.1.2) and (2.4.9). For $\sigma_\pi = 0$, the matrix A becomes

$$\begin{pmatrix} -a & 0 \\ \frac{\epsilon_\pi (a+r)}{\epsilon_D} & r - \frac{1}{\epsilon_D} \end{pmatrix},$$

whose determinant is $a(-r + 1/\epsilon_D) \neq 0$. Due to the continuity of the determinant, there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for all $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$, A is invertible. Because A is invertible, the unique solution of (2.4.10) is (2.4.11).

Proof of inequalities (2.4.12) and (2.4.13).

Equation (2.4.11), Lemma A.0.1 and the triangle inequality imply that $\|\bar{E}\,[\chi_u|\mathcal{G}_s]\| \leq \eta_s e^{\|A\|t} + \eta$, $\|\bar{\mathrm{V}}\mathrm{ar}\,[\chi_u|\mathcal{G}_s]\| \leq \eta e^{(\|A\|+\|A^T\|)t}$ and that $\|\bar{E}[\chi_u|\mathcal{G}_s] \otimes \bar{E}[\chi_u|\mathcal{G}_s]\| \leq \eta_s e^{(\|A\|+\|A^T\|)t} + \eta$. The definition of the conditional variance yields $\|\bar{E}\,[\chi_u \otimes \chi_u|\mathcal{G}_s]\| \leq \eta_s e^{(\|A\|+\|A^T\|)t} + \eta$. The unconditional inequalities follow similarly.

(a) is true thanks to (2.4.13) (v) and to Fubini's Theorem. Likewise, Fubini's Theorem and (2.4.13), yield to equation (b) and hence (c).

Proof of (d)

We proceed in several steps.

Claim: There exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$ and for every $s \geq 0$,

$$\lim_{t \to +\infty} e^{-rt} e^{A(t-s)} = 0$$

If $\epsilon_D \neq \frac{1}{a+r}$ and $\sigma_\pi = 0$, then

$$A = \begin{pmatrix} -a & 0 \\ \frac{\epsilon_\pi(a+r)}{\epsilon_D} & r - \frac{1}{\epsilon_D} \end{pmatrix}$$

is diagonalizable with two different eigenvalues $-a$ and $r - \frac{1}{\epsilon_D}$. By the continuity of the eigenvalues [2, Remark 3.4 and 3.6] there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$, A is diagonalizable with two different eigenvalues and

$$e^{-rt} e^{A(t-s)} = e^{-rt} H \begin{pmatrix} e^{\lambda_1(t-s)} & 0 \\ 0 & e^{\lambda_2(t-s)} \end{pmatrix} H^{-1}, \tag{2.4.15}$$

with

$$\max(Re\{\lambda_1, \lambda_2, 2\lambda_1, 2\lambda_2, \lambda_1 + \lambda_2\}) < r. \tag{2.4.16}$$

The real parts of all exponentials on every entry are negative, therefore

$$\lim_{t \to +\infty} e^{-rt} e^{A(t-s)} = 0. \tag{2.4.17}$$

If $\epsilon_D = \frac{1}{a+r}$ and $\sigma_\pi = 0$, then A is similar to a Jordan block with eigenvalue $-a$. The continuity of the eigenvalues [2, Remark 3.4] and Lemma A.0.4 imply the existence of $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$ (2.4.16) and (2.4.17) hold.

$$\text{Claim:} \quad \lim_{t \to +\infty} e^{-rt} \bar{E}\left[\chi_t | \mathcal{G}_s\right] = 0$$

Apply the conditional expectation to (2.4.11) and multiply by e^{-rt} to get

$$e^{-rt}\bar{E}\left[\chi_t|\mathcal{G}_s\right] = e^{-rt}e^{A(t-s)}\chi_s + A^{-1}(e^{-rt}e^{A(t-s)} - I_2 e^{-rt})b.$$

From (2.4.17) it follows that $\lim_{t\to+\infty} e^{-rt}\bar{E}\left[\chi_t|\mathcal{G}_s\right] = 0$.

$$\text{Claim:} \quad \lim_{t\to+\infty} e^{-rt}\bar{E}\left[\int_s^t \chi_u \otimes \chi_u \Big| \mathcal{G}_s\right] du = 0 \text{ and } \lim_{t\to+\infty} e^{-rt}\bar{E}\left[\chi_t \otimes \chi_t \Big| \mathcal{G}_s\right] du = 0$$

Integrating the definition of the conditional variance [5, Definition 11.23], and multiplying both sides by e^{-rt},

$$e^{-rt}\int_s^t \bar{E}\left[\chi_u \otimes \chi_u|\mathcal{G}_s\right] du = e^{-rt}\int_s^t \bar{\mathrm{V}}\mathrm{ar}_t[\chi_u|\mathcal{G}_s]du + e^{-rt}\int_s^t \bar{E}[\chi_u|\mathcal{G}_s] \otimes \bar{E}[\chi_u|\mathcal{G}_s]du. \tag{2.4.18}$$

By dint of (2.4.11) we get

$$\bar{\mathrm{V}}\mathrm{ar}[\chi_t|\mathcal{G}_s] = \int_s^t e^{A(t-u)}\Sigma\Sigma^T \left(e^{A(t-u)}\right)^T du. \tag{2.4.19}$$

A is triangularizable in $\mathbb{C}$, therefore there exist an invertible matrix H and a nilpotent matrix N (cfr. [20, 3.2.7 and 3.2.8 page 181], [16, Proposition A.6]) such that

$$e^{A(t-u)}\Sigma\Sigma^T\left(e^{A(t-u)}\right)^T = H\begin{pmatrix} e^{\lambda_1(t-u)} & 0 \\ 0 & e^{\lambda_2(t-u)}\end{pmatrix} H^{-1}\left(\sum_{h=0}^{2}\frac{N^h}{h!}(t-u)^h\right)\cdot\Sigma\Sigma^T$$
$$\cdot\left[H\begin{pmatrix} e^{\lambda_1(t-u)} & 0 \\ 0 & e^{\lambda_2(t-u)}\end{pmatrix} H^{-1}\left(\sum_{h=0}^{2}\frac{N^h}{h!}(t-u)^h\right)\right]^T.$$

Due to (2.4.16), each entry of $e^{A(t-u)}\Sigma\Sigma^T\left(e^{A(t-u)}\right)^T$ is a linear combination of powers of t smaller than 4, multiplied by exponentials with real part of the coefficients in t smaller than r. As a consequence, the same holds for $\int_s^t \bar{\mathrm{V}}\mathrm{ar}_t[\chi_u|\mathcal{G}_s]du$, and

$$\lim_{t\to+\infty} e^{-rt}\int_s^t \bar{\mathrm{V}}\mathrm{ar}[\chi_u|\mathcal{G}_s]du = 0.$$

In the same way $\lim_{t\to+\infty} e^{-rt}\int_s^t \bar{E}[\chi_u|\mathcal{G}_s] \otimes \bar{E}[\chi_u|\mathcal{G}_s]du = 0$.
The proof of $\lim_{t\to+\infty} e^{-rt}\bar{E}\left[\chi_t \otimes \chi_t\Big|\mathcal{G}_s\right] du = 0$ is the same as that of $\lim_{t\to+\infty} e^{-rt}\bar{E}\left[\int_s^t \chi_u \otimes \chi_u\Big|\mathcal{G}_s\right] du = 0$, skipping the step of integration in (2.4.18). It follows that (d) holds.

Proof of (e):

$$\frac{1}{2}\lim_{t\to+\infty}\bar{E}\Big[\int_s^t e^{-r(u-s)}\left((\Delta\cdot v_u)^2+(\iota\cdot v_u)^2\right)du\Big|\mathcal{G}_s\Big]$$
$$=-\delta_{DD}D_s^2-\delta_{D\pi}D_s\pi_s-\delta_{\pi\pi}\pi_s^2-\delta_D D_s-\delta_\pi\pi_s-\delta_0-\frac{\beta-r}{r}.$$

Let $0\le s\le t$ and define $v=(D,\pi)^T$. The function $W:[0,t]\times\mathbb{R}^2\to\mathbb{R}$

$$W(s,\pi,D)=-\delta_{DD}D^2-\delta_{D\pi}D\pi-\delta_{\pi\pi}\pi^2-\delta_D D-\delta_\pi\pi-\delta_0-\frac{\beta-r}{r},$$

is the solution of the Cauchy problem in $[0,t]$

$$0=W_s+\left(\nabla_{(\pi,D)}W\right)\cdot\left(A(\pi,D)^T+b\right)+\frac{1}{2}tr\left(\left(\mathrm{He}_{(\pi,D)}W\right)\Sigma\Sigma^T\right)-rW+\frac{1}{2}\left((\Delta\cdot v)^2+(\iota\cdot v)^2\right),$$
$$W(t,\pi,D)=-\delta_{DD}D_s^2-\delta_{D\pi}D_s\pi_s-\delta_{\pi\pi}\pi_s^2-\delta_D D_s-\delta_\pi\pi_s-\delta_0-\frac{\beta-r}{r}.$$

In view of [22, Theorem 7.6],

$$W(s,\pi_s,D_s)=\bar{E}\Big[\int_s^t e^{-r(u-s)}\frac{1}{2}\left((\Delta\cdot v_u)^2+(\iota\cdot v_u)^2\right)du+$$
$$+e^{-r(t-s)}\Big(-\delta_{DD}D_t^2-\delta_{D\pi}D_t\pi_t-\delta_{\pi\pi}\pi_t^2-\delta_D D_t-\delta_\pi\pi_t-\delta_0-\frac{\beta-r}{r}\Big)\Big|\mathcal{G}_s\Big].$$

Since W does not depend by t, for every $t>0$

$$\bar{E}\Big[\int_s^t e^{-r(u-s)}\frac{1}{2}\left((\Delta\cdot v_u)^2+(\iota\cdot v_u)^2\right)du+e^{-r(t-s)}\Big(-\delta_{DD}D_t^2-\delta_{D\pi}D_t\pi_t-\delta_{\pi\pi}\pi_t^2-\delta_D D_t+$$
$$-\delta_\pi\pi_t-\delta_0-\frac{\beta-r}{r}\Big)\Big|\mathcal{G}_s\Big]=-\delta_{DD}D_s^2-\delta_{D\pi}D_s\pi_s-\delta_{\pi\pi}\pi_s^2-\delta_D D_s-\delta_\pi\pi_s-\delta_0-\frac{\beta-r}{r}.$$

Take $\lim_{t\to+\infty}$ of both and apply (d) to conclude. □

Remark 2.4.2. If $\epsilon_D=1/r$ the matrix A is not invertible for $\sigma_\pi=0$ and (2.4.11) no longer holds. In this case we conjecture the existence of a solution for (2.4.10) but we would need a different way of proving the result since the direct calculations become more difficult.

With the properties of $(\chi_t)_{t\ge0}$ shown in Lemma 2.4.3, we prove that $(N_t)_{t\ge0}$ of (2.1.5) is a stochastic discount factor.

Theorem 2.4.2. *Under Assumption 2.1.1 the process $(N_t)_{t\ge0}$ of (2.1.5) is a normalized* *stochastic discount factor. The dynamics of the process* $(\log \mathcal{E}_t)_{t\geq 0}$ *can be written as*

$$\begin{aligned}\log \mathcal{E}_t &= \log \mathcal{E}_s - \frac{1}{2}\int_s^t \left[(\Delta\cdot v_u)^2 + (\iota\cdot v_u)^2\right] du + \int_s^t (\Delta\cdot v_u) dW_u^D + \int_s^t (\iota\cdot v_u) dW_u^\pi, \\ &= \log \mathcal{E}_s + \frac{1}{2}\int_s^t \left[(\Delta\cdot v_u)^2 + (\iota\cdot v_u)^2\right] du + \int_s^t (\Delta\cdot v_u) d\bar{W}_u^D + \int_s^t (\iota\cdot v_u) d\bar{W}_u^\pi.\end{aligned} \tag{2.4.20}$$

For every $t \geq 0$

$$\bar{E}[|\log \mathcal{E}_t|] \leq \eta \left(e^{(\|A\|+\|A^T\|)t} + t + 1\right). \tag{2.4.21}$$

Proof. The process $(N_t)_{t\geq 0}$ needs to satisfy conditions (2.2.3) and (2.2.4) of Definition 2.2.2 to be a stochastic discount factor. Property (2.2.3) is a direct calculation. The definition of $\mathcal{E}_t = e^{rt}N_t$ and Lemma 2.4.3 (a) imply that

$$\begin{aligned} E\left[N_t P_t + \int_0^t N_u D_u du \middle| \mathcal{G}_s\right] &= \int_0^s N_u D_u du + E\left[N_t(C + \epsilon_D D_t + \pi_t) + \int_s^t N_u D_u du \middle| \mathcal{G}_s\right] \\ &= \int_0^s N_u D_u du + N_s \bar{E}\left[e^{-r(t-s)}(C + \epsilon_D D_t + \epsilon_\pi \pi_t) + \int_s^t e^{-r(u-s)} D_u du \middle| \mathcal{G}_s\right]. \end{aligned} \tag{2.4.22}$$

The function $W(s, D, \pi) = C + \epsilon_D D + \epsilon_\pi \pi$ solves the Cauchy problem on $[0, t]$

$$\begin{aligned} 0 &= W_s + \left(\nabla_{(\pi,D)} W\right)\cdot\left(A(\pi, D)^T + b\right) + \frac{1}{2} tr\left(\left(\mathrm{He}_{(\pi,D)}\, W\right)\Sigma\Sigma^T\right) - rW + D \\ W(t, D, \pi) &= C + \epsilon_D D + \epsilon_\pi \pi, \end{aligned}$$

where A, b and Σ are in Lemma 2.4.3. By [22, Theorem 7.6], for every $0 \leq s \leq t$

$$W(s, D_s, \pi_s) = \bar{E}\left[\int_s^t e^{-r(u-s)} D_u du + e^{-r(t-s)}(C + \epsilon_D D_t + \epsilon_\pi \pi_t) \middle| \mathcal{G}_s\right] = C + \epsilon_D D_s + \epsilon_\pi \pi_s.$$

Plugging W into (2.4.22) proves (2.2.4), hence $(N_t)_{t\geq 0}$ is a stochastic discount factor. The stochastic process $(N_t)_{t\geq 0}$ of (2.1.5) solves the initial value problem

$$\frac{dN_t}{N_t} = -rdt + (\Delta\cdot v_t)dW_t^D + (\iota\cdot v_t)dW_t^\pi, \qquad N_0 = 1,$$

thus the process $(\mathcal{E}_t)_{t\geq 0}$ solves the initial value problem

$$\frac{d\mathcal{E}_t}{\mathcal{E}_t} = (\Delta\cdot v_t)dW_t^D + (\iota\cdot v_t)dW_t^\pi, \qquad \mathcal{E}_0 = 1,$$

by virtue of its definition $\mathcal{E}_t = e^{rt}N_t$. Applying Itô's formula to $f(\mathcal{E}_t) = \log \mathcal{E}_t$ we get the first equality of (2.4.20) and because of (2.4.9) we get the second one. Thanks to (2.4.20) and to the triangle inequality

$$\bar{E}[|\log \mathcal{E}_u|] \leq \frac{1}{2}\bar{E}\left[\int_0^u \left((\Delta \cdot v_h)^2 + (\iota \cdot v_h)^2\right) dh\right] + \bar{E}\left[\left|\int_0^u (\Delta \cdot v_h) d\bar{W}_h^D\right|\right] + \bar{E}\left[\left|\int_0^u (\iota \cdot v_h) d\bar{W}_h^\pi\right|\right].$$

$\int_0^u (\Delta \cdot v_h) d\bar{W}_h^D$ is a $\bar{\mathbb{P}}-$normal random variable with mean $\mu_u = 0$ and variance

$$\sigma_u^2 = \int_0^u \bar{E}[(\Delta \cdot v_h)^2] dh \leq \eta(e^{(\|A\|+\|A^T\|)u} + u + 1). \tag{2.4.23}$$

(2.4.13) (iv) implies the last inequality, where η is a positive constant. In view of Lemma A.0.1 (IX) and (X),

$$\bar{E}\left[\left|\int_0^u (\Delta \cdot v_h) d\bar{W}_h^D\right|\right] \leq \sigma_u \sqrt{\frac{2}{\pi}} \leq \sigma_u^2 + 1 \leq \eta(e^{(\|A\|+\|A^T\|)u} + u + 1). \tag{2.4.24}$$

Because of (2.4.23), the right side of (2.4.24) is a bound also for $\bar{E}\left[\left|\int_0^u (\iota \cdot v_h) d\bar{W}_h^\pi\right|\right]$ and for $\bar{E}\left[\int_0^u \left((\Delta \cdot v_h)^2 + (\iota \cdot v_h)^2\right) dh\right]$, thus (2.4.21) follows. ☐

We are ready to prove the admissibility of the candidate optimal policies.

Theorem 2.4.3 (Admissibility and utility)**.** *Define*

$$y^{i*} = e^{-r\alpha_i x_0^i + \delta_{DD} D_0^2 + \delta_{D\pi} D_0 \pi_0 + \delta_{\pi\pi} \pi_0^2 + \delta_D D_0 + \delta_\pi \pi_0 + \delta_0},$$

the processes $(c_t^{i*}, \theta_t^{i*})_{t \geq 0}$ *as*

$$\begin{aligned} c_t^{i*} &= r X_t^{i*} - \frac{\delta_{DD}}{\alpha_i} D_t^2 - \frac{\delta_{D\pi}}{\alpha_i} D_t \pi_t - \frac{\delta_{\pi\pi}}{\alpha_i} \pi_t^2 - \frac{\delta_D}{\alpha_i} D_t - \frac{\delta_\pi}{\alpha_i} \pi_t - \frac{\delta_0}{\alpha_i}, \\ \theta_t^{i*} &= \frac{M_D D_t + M_\pi \pi_t + M_0}{\alpha_i M}, \end{aligned}$$

and the process $(X_t^{i*})_{t \geq 0}$ *as*

$$\begin{aligned} X_t^{i*} = x_0^i + \frac{1}{M\alpha_i} \Big\{ &\Big[\delta_{DD} M + \Big(1 - \epsilon_D(k+r)\Big) M_D\Big] \int_0^t D_u^2 du + \\ &+ \Big[\delta_{D\pi} M + \Big(1 - \epsilon_D(k+r)\Big) M_\pi + \Big(\epsilon_D - \epsilon_\pi(a+r)\Big) M_D\Big] \int_0^t D_u \pi_u du + \\ &+ \Big[\delta_{\pi\pi} M + \Big(\epsilon_D - \epsilon_\pi(a+r)\Big) M_\pi\Big] \int_0^t \pi_u^2 du + \\ &+ \Big[\delta_D M + (\epsilon_\pi a\bar{\pi} - rC) M_D + \Big(1 - \epsilon_D(k+r)\Big) M_0\Big] \int_0^t D_u du + \\ &+ \Big[\delta_\pi M + (\epsilon_\pi a\bar{\pi} - rC) M_\pi + \Big(\epsilon_D - \epsilon_\pi(a+r)\Big) M_0\Big] \int_0^t \pi_u du + \\ &+ \Big[\delta_0 M + (\epsilon_\pi a\bar{\pi} - rC) M_0\Big] t + \\ &+ \epsilon_D \sigma_D M_D \int_0^t D_u dW_u^D + \epsilon_D \sigma_D M_\pi \int_0^t \pi_u dW_u^D + \epsilon_D \sigma_D M_0 (W_t^D) + \\ &+ \epsilon_\pi \sigma_\pi M_D \int_0^t D_u dW_u^\pi + \epsilon_\pi \sigma_\pi M_\pi \int_0^t \pi_u dW_u^\pi + \epsilon_\pi \sigma_\pi M_0 (W_t^\pi) \Big\}. \end{aligned} \tag{2.4.25}$$

Under Assumption 2.1.1 there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ *such that for every* $0 \leq \sigma_\pi \leq$

$\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$, *the following hold.*

(A) (First order condition)

$$-\alpha_i c_t^{i*} = \log(y^{i*}) + (\beta - r)t + \log(\mathcal{E}_t); \tag{2.4.26}$$

(B) (Budget equation) $N_t X_t^{i*} + \int_0^t N_u c_u^{i*} du$ *is a* $\mathbb{P}-$*martingale;*

(C) (Saturation) for every $s \geq 0$, $\lim_{t\to+\infty} E[N_t X_t^{i*}|\mathcal{G}_s] = 0$;

(D) (Admissibility) for every $i \in \{1, \dots, n\}$, $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ *is an admissible strategy with wealth process* $(X_t^{i*})_{t\geq 0}$*. The utility of the strategy is*

$$E\left[\int_0^{+\infty} e^{-\beta u} U(c_u^{i*}) du \middle| \mathcal{G}_0\right] = -\frac{1}{r\alpha_i} e^{-r\alpha_i x_0^i + \delta_{DD} D_0^2 + \delta_{D\pi} D_0 \pi_0 + \delta_{\pi\pi}\pi_0^2 + \delta_D D_0 + \delta_\pi \pi_0 + \delta_0}.$$

Proof. Let $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$ be the minimum between the constants (with the same name) in Theorem 2.4.1 and Lemma 2.4.3. We proceed in several steps.

Proof of (A): First order condition

The equality $-\alpha_i c_0^{i*} = -r\alpha_i x_0^i + \delta_{DD} D_0^2 + \delta_{D\pi} D_0 \pi_0 + \delta_{\pi\pi}\pi_0^2 + \delta_D D_0 + \delta_\pi \pi_0 + \delta_0$ holds. Apply Itô's formula to both sides of (2.4.26) and check that they are equal.

Proof of the equality $\mathcal{E}_s \bar{E}\left[\int_s^t e^{-ru} c_u^{i*} du \middle| \mathcal{G}_s\right] = E\left[\int_s^t e^{-ru} \mathcal{E}_u c_u^{i*} du \middle| \mathcal{G}_s\right]$

Due to (2.4.26) and to the triangle inequality, there exists $\eta > 0$ such that

$$|c_u^{i*}| \leq \eta |-r\alpha_i x_0^i + \delta_{DD} D_0^2 + \delta_{D\pi} D_0 \pi_0 + \delta_{\pi\pi}\pi_0^2 + \delta_D D_0 + \delta_\pi \pi_0 + \delta_0| + \eta u + \eta |\log \mathcal{E}_u|. \tag{2.4.27}$$

Applying the conditional expectation to both sides of (2.4.27), the properties of normal random variables and (2.4.21) imply that

$$\bar{E}[|c_u^{i*}|] \leq \eta \left(e^{(\|A\|+\|A^T\|)t} + t + 1\right).$$

Fubini's Theorem [4, Theorem 1.1.7] yields to

$$\int_s^t e^{-ru} \bar{E}\left[c_u^{i*}\right] du = \bar{E}\left[\int_s^t e^{-ru} |c_u^{i*}| du\right] < +\infty$$

and by the conditional version of Fubini's Theorem we get

$$\mathcal{E}_s \bar{E}\left[\int_s^t e^{-ru} c_u^{i*} du \middle| \mathcal{G}_s\right] = E\left[\int_s^t e^{-ru} \mathcal{E}_u c_u^{i*} du \middle| \mathcal{G}_s\right]. \tag{2.4.28}$$

Proof of (B): $N_t X_t^{i*} + \int_0^t N_u c_u^{i*}$ is a martingale

Direct calculations show that $(X_t^{i*})_{t\geq 0}$ is the wealth process of the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$

and they satisfy equality (2.4.1), equivalent to the self-financing condition. The equalities

$$\begin{aligned}\epsilon_D\sigma_D\Delta_D + \epsilon_\pi\sigma_\pi\iota_D &= -1+\epsilon_D(k+r),\\ \epsilon_D\sigma_D\Delta_\pi + \epsilon_\pi\sigma_\pi\iota_\pi &= -\epsilon_D+\epsilon_\pi(a+r),\\ \epsilon_D\sigma_D\Delta_0 + \epsilon_\pi\sigma_\pi\iota_0 &= rC - a\bar{\pi}\epsilon_\pi,\end{aligned}$$

and (2.4.9) imply that

$$dX_t^{i*} = (-c_t^{i*} + rX_t^{i*})dt + \epsilon_D\sigma_D\theta_t^{i*}d\bar{W}_t^D + \epsilon_\pi\sigma_\pi\theta_t^{i*}d\bar{W}_t^\pi.$$

Applying Itô's formula to the function $f(t, X_t^{i*}) = e^{-rt}X_t^{i*}$ we get

$$e^{-rt}X_t^{i*} = e^{-rs}X_s^{i*} + \int_s^t -e^{-ru}c_u^{i*}du + \epsilon_D\sigma_D\int_s^t e^{-ru}\theta_u^{i*}d\bar{W}_u^D + \epsilon_\pi\sigma_\pi\int_s^t e^{-ru}\theta_u^{i*}d\bar{W}_u^\pi.$$

Multiply both sides by $\mathcal{E}_t$, add $\int_0^t N_u c_u^{i*}du$, take the conditional expectation and use Bayes' formula to get

$$\begin{aligned}E\left[N_tX_t^{i*} + \int_0^t N_u c_u^{i*}du\middle|\mathcal{G}_s\right] &= N_sX_s^{i*} + \int_0^s N_u c_u^{i*}du + \mathcal{E}_s\epsilon_D\sigma_D\bar{E}\left[\int_s^t e^{-ru}\theta_u^{i*}d\bar{W}_u^D\middle|\mathcal{G}_s\right] +\\ &+ \mathcal{E}_s\epsilon_\pi\sigma_\pi\bar{E}\left[\int_s^t e^{-ru}\theta_u^{i*}d\bar{W}_u^\pi\middle|\mathcal{G}_s\right] + \mathcal{E}_s\bar{E}\left[\int_s^t -e^{-ru}c_u^{i*}du\middle|\mathcal{G}_s\right] + E\left[\int_s^t N_u c_u^{i*}du\middle|\mathcal{G}_s\right].\end{aligned}$$

The Brownian terms are martingales because of Lemma 2.4.3 (c) and since (2.4.28) holds, then

$$E\left[N_tX_t^{i*} + \int_0^t N_u c_u^{i*}du\middle|\mathcal{G}_s\right] = \int_0^s N_u c_u^{i*}du + N_sX_s^{i*}. \tag{2.4.29}$$

Proof of (C): $\lim_{t\to+\infty} E[N_tX_t^{i*}|\mathcal{G}_s] = 0$

Because of (2.4.9) for the process X_t^{i*} of (2.4.25), there exist $\eta_1,\dots,\eta_{12}\in\mathbb{R}$ such that

$$\begin{aligned}N_tX_t^{i*} &= e^{-rt}\mathcal{E}_tX_s^{i*} + \eta_1e^{-rt}\mathcal{E}_t\int_s^t D_u^2du + \eta_2e^{-rt}\mathcal{E}_t\int_s^t D_u\pi_udu + \eta_3e^{-rt}\mathcal{E}_t\int_s^t \pi_u^2du+\\ &+ \eta_4e^{-rt}\mathcal{E}_t\int_s^t D_udu + \eta_5e^{-rt}\mathcal{E}_t\int_s^t \pi_udu + \eta_6e^{-rt}\mathcal{E}_t(t-s)+\\ &+ \eta_7e^{-rt}\mathcal{E}_t\int_s^t D_ud\bar{W}_u^D + \eta_8e^{-rt}\mathcal{E}_t\int_s^t \pi_ud\bar{W}_u^D + \eta_9e^{-rt}\mathcal{E}_t\int_s^t d\bar{W}_u^D+\\ &+ \eta_{10}e^{-rt}\mathcal{E}_t\int_s^t D_ud\bar{W}_u^\pi + \eta_{11}e^{-rt}\mathcal{E}_t\int_s^t \pi_ud\bar{W}_u^\pi + \eta_{12}e^{-rt}\mathcal{E}_t\int_s^t d\bar{W}_u^\pi.\end{aligned}$$

Taking the conditional expectation and using Bayes' formula yields

$$E[N_tX_t^{i*}|\mathcal{G}_s] = e^{-rt}\mathcal{E}_sX_s^{i*} + \eta_1 e^{-rt}\mathcal{E}_s\bar{E}\left[\int_s^t D_u^2 du\Big|\mathcal{G}_s\right] + \eta_2 e^{-rt}\mathcal{E}_s\bar{E}\left[\int_s^t D_u\pi_u du\Big|\mathcal{G}_s\right] +$$
$$+\eta_3 e^{-rt}\mathcal{E}_s\bar{E}\left[\int_s^t \pi_u^2 du\Big|\mathcal{G}_s\right]+\eta_4 e^{-rt}\mathcal{E}_s\bar{E}\left[\int_s^t D_u du\Big|\mathcal{G}_s\right]+\eta_5 e^{-rt}\mathcal{E}_s\bar{E}\left[\int_s^t \pi_u du\Big|\mathcal{G}_s\right]+\eta_6 e^{-rt}\mathcal{E}_s(t-s)+$$
$$+\eta_7 e^{-rt}\mathcal{E}_s\bar{E}\left[\int_s^t D_u d\bar{W}_u^D\Big|\mathcal{G}_s\right] + \eta_8 e^{-rt}\mathcal{E}_s\bar{E}\left[\int_s^t \pi_u d\bar{W}_u^D\Big|\mathcal{G}_s\right] + \eta_9 e^{-rt}\mathcal{E}_s\bar{E}\left[\int_s^t d\bar{W}_u^D\Big|\mathcal{G}_s\right] +$$
$$+\eta_{10} e^{-rt}\mathcal{E}_s\bar{E}\left[\int_s^t D_u d\bar{W}_u^\pi\Big|\mathcal{G}_s\right]+\eta_{11} e^{-rt}\mathcal{E}_s\bar{E}\left[\int_s^t \pi_u d\bar{W}_u^\pi\Big|\mathcal{G}_s\right]+\eta_{12} e^{-rt}\mathcal{E}_s\bar{E}\left[\int_s^t d\bar{W}_u^\pi\Big|\mathcal{G}_s\right].$$

All the Brownian terms are $\bar{\mathbb{P}}-$martingales by virtue of Lemma 2.4.3 (c). Thanks to Lemma 2.4.3 (d), $\lim_{t\to+\infty} E[N_tX_t^{i*}|\mathcal{G}_s] = 0$.

Proof of (D): Admissibility and utility

Property (i) of Definition 3.1.1 is clear and proving that $(X_t^{i*})_{t\geq 0}$ is the wealth process of the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ is a direct calculation. Take $\lim_{t\to+\infty}$ to both sides of (2.4.29) and use (C) to prove (2.1.3) and thus the admissibility of the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$. (2.4.26) implies

$$E\left[\int_0^{+\infty} e^{-\beta u}U^i(c_u^{i*})du\Big|\mathcal{G}_0\right] = -\frac{1}{\alpha_i}E\left[\int_0^{+\infty} e^{\log y^{i*} - ru + \log \mathcal{E}_u}du\Big|\mathcal{G}_0\right] = -\frac{y^{i*}}{r\alpha_i}.$$

□

Theorem 2.4.4 (Duality Theorem)**.** *Let $(c_t, \theta_t)_{t\geq 0}$ be an admissible strategy and let $(N_t)_{t\geq 0}$ be the process of (2.1.5); then*

$$\lim_{t\to+\infty} E\left[\int_0^t e^{-\beta u}U^i(c_u)du\Big|\mathcal{G}_0\right] = E\left[\int_0^{+\infty} e^{-\beta u}U^i(c_u)du\Big|\mathcal{G}_0\right],$$
$$\lim_{t\to+\infty} E\left[\int_0^t e^{-\beta u}\tilde{U}^i(ye^{\beta u}N_u)du\Big|\mathcal{G}_0\right] = E\left[\int_0^{+\infty} e^{-\beta u}\tilde{U}^i(ye^{\beta u}N_u)du\Big|\mathcal{G}_0\right]. \tag{2.4.30}$$

Furthermore

$$E\left[\int_0^{+\infty} e^{-\beta u}U^i(c_u)du\Big|\mathcal{G}_0\right] \leq \inf_{y>0}\left\{E\left[\int_0^{+\infty} e^{-\beta u}\tilde{U}^i(ye^{\beta u}N_u)du\Big|\mathcal{G}_0\right] + x_0^i y\right\}, \tag{2.4.31}$$

where

$$\tilde{U}^i(y) = \begin{cases} \frac{y}{\alpha_i}(\log y - 1) & y > 0 \\ 0 & y = 0. \end{cases} \tag{2.4.32}$$

If there exist $y^{i*} > 0$ and an admissible strategy $(c_t^*, \theta_t^*)_{t\geq 0}$ for which

$$E\left[\int_0^{+\infty} e^{-\beta u}U^i(c_u^*)du\Big|\mathcal{G}_0\right] = E\left[\int_0^{+\infty} e^{-\beta u}\tilde{U}^i(y^{i*}e^{\beta u}N_u)du\Big|\mathcal{G}_0\right] + x_0^i y^{i*}, \tag{2.4.33}$$

then $(c_t^*, \theta_t^*)_{t\geq 0}$ is optimal.

Proof. Define the random variables

$$\lambda^m = \int_0^m e^{-\beta u - \alpha_i c_u} du, \qquad \lambda = \int_0^{+\infty} e^{-\beta u - \alpha_i c_u} du,$$

on the probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$. Then $\lambda^m \geq 0$ for every $m \in \mathbb{N}$ and $(\lambda^m)_{m\in\mathbb{N}}$ is an increasing sequence of random variables such that $\lim_{m\to+\infty} \lambda^m = \lambda$. The Conditional Monotone Convergence Theorem yields to

$$\lim_{m\to+\infty} E[\lambda^m|\mathcal{G}_0] = E[\lambda|\mathcal{G}_0],$$

which implies the first equality in (2.4.30). The function $\tilde{U}^i$ defined in (2.4.32) has a global minimum at $y = 1$; apply the Conditional Monotone Convergence Theorem to the random variables

$$\lambda^m = \int_0^m e^{-\beta u} \left(\tilde{U}^i(ye^{\beta u}N_u) + \frac{1}{\alpha_i} \right) du, \qquad \lambda = \int_0^{+\infty} e^{-\beta u} \left(\tilde{U}^i(ye^{\beta u}N_u) + \frac{1}{\alpha_i} \right) du,$$

to conclude the second equality in (2.4.30). For the proof of (2.4.31) apply (A.0.1) to the random variables c_u and $Y_u = ye^{\beta u}N_u$; for every $y > 0$

$$U^i(c_u) \leq \tilde{U}^i(ye^{\beta u}N_u) + c_u y e^{\beta u} N_u.$$

Multiply both sides by $e^{-\beta u}$, integrate in $[0, t]$ and take conditional expectations; for every $y > 0$

$$E\left[\int_0^t e^{-\beta u} U^i(c_u) du \middle| \mathcal{G}_0 \right] \leq E\left[\int_0^t e^{-\beta u} \tilde{U}^i(ye^{\beta u}N_u) du \middle| \mathcal{G}_0 \right] + yE\left[\int_0^t c_u N_u du \middle| \mathcal{G}_0 \right].$$

Take $\limsup_{t\to+\infty}$ of both sides and use (2.4.30) and (2.1.3); for every $y > 0$

$$E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u) du \middle| \mathcal{G}_0 \right] \leq E\left[\int_0^{+\infty} e^{-\beta u} \tilde{U}^i(ye^{\beta u}N_u) du \middle| \mathcal{G}_0 \right] + x_0^i y.$$

Take $\inf_{y>0}$ to obtain (2.4.31). If there exist $y^{i*} > 0$ and an admissible strategy $(c_t^*, \theta_t^*)_{t\geq 0}$ for which (2.4.33) holds, then

$$\begin{aligned} E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u^*) du \middle| \mathcal{G}_0 \right] &\leq \inf_{y>0} \left\{ E\left[\int_0^{+\infty} e^{-\beta u} \tilde{U}^i(ye^{\beta u}N_u) du \middle| \mathcal{G}_0 \right] + x_0^i y \right\} \\ &\leq E\left[\int_0^{+\infty} e^{-\beta u} \tilde{U}^i(y^{i*}e^{\beta u}N_u) du \middle| \mathcal{G}_0 \right] + x_0^i y^{i*} = E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u^*) du \middle| \mathcal{G}_0 \right]. \end{aligned}$$

All the above are equalities therefore $(c_t^*, \theta_t^*)_{t\geq 0}$ is optimal. □

Theorem 2.4.5 (Existence). *Under Assumption 2.1.1 there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$, the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ in Theorem 2.4.3 is optimal for the $i-th$ investor for every $i \in \{1, \ldots, n\}$. The function V^i of Theorem*

2.4.1 is the value function of the $i-$th investor.

Proof. Fix $0 \leq s \leq t$ and $y > 0$; thanks to the definition of $\tilde{U}(\cdot)$ in (2.4.32)

$$E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\Big|\mathcal{G}_s\right] = \frac{y}{\alpha_i}\Big\{(\log y - 1)\,E\left[\int_s^t N_u du\Big|\mathcal{G}_s\right] + \\ + \beta E\left[\int_s^t uN_u du\Big|\mathcal{G}_s\right] + E\left[\int_s^t N_u \log N_u du\Big|\mathcal{G}_s\right]\Big\}.$$

The following integrability conditions hold:

$$\int_s^t E\left[|N_u|\right]du = \int_s^t E\left[N_u\right]du = \int_s^t e^{-ru}du = \frac{(e^{-rs} - e^{-rt})}{r} < +\infty,$$
$$\int_s^t E\left[|uN_u|\right]du = \int_s^t uE\left[N_u\right]du = \int_s^t ue^{-ru}du = \frac{e^{-rs}(1+rs) - e^{-rt}(1+rt)}{r^2} < +\infty.$$

The conditional version of Fubini's Theorem [4, Theorem 1.1.8] applies and yields

$$E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\Big|\mathcal{G}_s\right] = \frac{y}{\alpha_i}\Big\{(\log y - 1)\,\mathcal{E}_s\int_s^t e^{-ru}du + \\ + (\beta - r)\mathcal{E}_s\int_s^t ue^{-ru}du + E\left[\int_s^t e^{-ru}\mathcal{E}_u\log\mathcal{E}_u du\Big|\mathcal{G}_s\right]\Big\}.$$

(2.4.21) implies that

$$\int_s^t e^{-ru}\bar{E}\left[|\log\mathcal{E}_u|\right]du \leq \eta(t-s)\left(e^{(\|A\|+\|A\|_1)t} + t + 1\right) < +\infty.$$

Fubini's Theorem and Bayes' formula yield to

$$E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\Big|\mathcal{G}_s\right] = \frac{y}{\alpha_i}\Big\{(\log y - 1)\,\mathcal{E}_s\int_s^t e^{-ru}du + \\ + (\beta - r)\mathcal{E}_s\int_s^t ue^{-ru}du + \mathcal{E}_s\int_s^t e^{-ru}\bar{E}\left[\log\mathcal{E}_u|\mathcal{G}_s\right]du\Big\}$$

and computing the integrals we get

$$E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\Big|\mathcal{G}_s\right] = \frac{y}{r\alpha_i}\mathcal{E}_s\Big\{(\log y - 1)\left(e^{-rs} - e^{-rt}\right) + \\ + (\beta - r)\frac{e^{-rs}(1+rs) - e^{-rt}(1+rt)}{r} + r\int_s^t e^{-ru}\bar{E}\left[\log\mathcal{E}_u|\mathcal{G}_s\right]du\Big\}.$$

By virtue of (2.4.20) and Lemma 2.4.3 (c),

$$\bar{E}\left[\log\mathcal{E}_u|\mathcal{G}_s\right] = \log\mathcal{E}_s + \frac{1}{2}\bar{E}\Big[\int_s^u\left((\Delta\cdot v_u)^2 + (\iota\cdot v_u)^2\right)du\Big|\mathcal{G}_s\Big].$$

Defining $Y_t = \int_s^t [(\Delta \cdot v_u)^2 + (\iota \cdot v_u)^2]\, du$ it follows that

$$r\int_s^t e^{-ru}\bar{E}\left[\log \mathcal{E}_u | \mathcal{G}_s\right] du = r\log \mathcal{E}_s \int_s^t e^{-ru} du + \frac{r}{2}\int_s^t e^{-ru}\bar{E}[Y_u|\mathcal{G}_s] du$$

and thanks to Lemma 2.4.3 (b) and to Fubini's Theorem we get

$$r\int_s^t e^{-ru}\bar{E}\left[\log \mathcal{E}_u | \mathcal{G}_s\right] du = r\log \mathcal{E}_s \int_s^t e^{-ru} du + \frac{r}{2}\bar{E}\left[\int_s^t e^{-ru} Y_u du \middle| \mathcal{G}_s\right].$$

As a consequence,

$$E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\middle|\mathcal{G}_s\right] = \frac{y}{r\alpha_i}\mathcal{E}_s\Big\{(\log y - 1)\left(e^{-rs} - e^{-rt}\right) + \\ + (\beta - r)\frac{e^{-rs}(1+rs) - e^{-rt}(1+rt)}{r} + r\log\mathcal{E}_s\int_s^t e^{-ru}du + \frac{r}{2}\bar{E}\left[\int_s^t e^{-ru}Y_u du\middle|\mathcal{G}_s\right]\Big\}. \tag{2.4.34}$$

Applying Itô's formula to the function $e^{-rt}Y_t$ and taking the conditional expectation yields

$$r\bar{E}\left[\int_s^t e^{-ru}Y_u du\middle|\mathcal{G}_s\right] = e^{-rs}Y_s - e^{-rt}\bar{E}\Big[\int_s^t \left((\Delta\cdot v_u)^2 + (\iota\cdot v_u)^2\right)du\Big|\mathcal{G}_s\Big] + \\ + \bar{E}\Big[\int_s^t e^{-ru}\left((\Delta\cdot v_u)^2 + (\iota\cdot v_u)^2\right)du\Big|\mathcal{G}_s\Big]. \tag{2.4.35}$$

Plug (2.4.35) into (2.4.34), fix $s = 0$, take $\lim_{t\to+\infty}$ of both sides and add $x_0^i y$ to get

$$\lim_{t\to+\infty} E\left[\int_0^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\middle|\mathcal{G}_0\right] + x_0^i y = \frac{y}{r\alpha_i}\Big\{(\log y - 1) + \frac{\beta - r}{r} + \\ - \frac{1}{2}\lim_{t\to+\infty} e^{-rt}\bar{E}\left[\int_0^t \left((\Delta\cdot v_u)^2 + (\iota\cdot v_u)^2\right)du\middle|\mathcal{G}_0\right] + \\ + \frac{1}{2}\lim_{t\to+\infty}\bar{E}\left[\int_0^t e^{-ru}\left((\Delta\cdot v_u)^2 + (\iota\cdot v_u)^2\right)du\middle|\mathcal{G}_0\right]\Big\} + x_0^i y.$$

Choosing $y = y^{i*} = \exp(-r\alpha_i x_0^i + \delta_{DD}D_0^2 + \delta_{D\pi}D_0\pi_0 + \delta_{\pi\pi}\pi_0^2 + \delta_D D_0 + \delta_\pi\pi_0 + \delta_0)$ and using Lemma 2.4.3 (d) it follows that

$$\lim_{t\to+\infty} E\left[\int_0^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\middle|\mathcal{G}_0\right] + x_0^i y^{i*} = \frac{y^{i*}}{r\alpha_i}\Big\{(-r\alpha_i x_0^i + \delta_{DD}D_0^2 + \delta_{D\pi}D_0\pi_0 + \delta_{\pi\pi}\pi_0^2 + \\ + \delta_D D_0 + \delta_\pi\pi_0 + \delta_0 - 1) + \frac{\beta - r}{r} + \\ + \frac{1}{2}\lim_{t\to+\infty}\bar{E}\Big[\int_0^t e^{-ru}\left((\Delta\cdot v_u)^2 + (\iota\cdot v_u)^2\right)du\Big|\mathcal{G}_0\Big]\Big\} + x_0^i y^{i*}.$$

Lemma 2.4.3 (e) and (2.4.30) imply

$$E\left[\int_0^{+\infty} e^{-\beta u}\tilde{U}(ye^{\beta u}N_u)du\middle|\mathcal{G}_0\right] = -\frac{y^{i*}}{r\alpha_i}.$$

The conclusion follows from Theorems 2.4.3 and 2.4.4. □

Uniqueness of the optimal strategy

Lemma 2.4.4. Let $i \in \{1, \dots, n\}$ and let $(c_t, \theta_t)_{t\geq 0}$ be an optimal strategy for the $i-$th agent, then for every $s \geq 0$

$$\limsup_{t\to+\infty} E\left[\int_s^t N_u c_u du \middle| \mathcal{G}_s\right] = N_s X_s. \tag{2.4.36}$$

Proof. Suppose, for a contradiction, that there exist $i \in \{1, \dots, n\},\ s \geq 0,\ S \in \mathcal{G}_s$ with $\mathbb{P}(S) > 0$ and an optimal strategy such that

$$\limsup_{t\to+\infty} E\left[\int_s^t N_u^i c_u du \middle| \mathcal{G}_s\right] < N_s X_s \quad \text{ on } S.$$

Let η_s be a $\mathcal{G}_s-$adapted random variable and define the new strategy $(\bar{c}_t, \theta_t)_{t\geq 0}$ as $(\bar{c}_t)_{t\geq 0} = (c_t)_{t\geq 0} + \eta_s \mathbf{1}_{t\geq s}$ and its wealth process

$$\begin{aligned}\bar{X}_t = X_t \mathbf{1}_{t<s} + \mathbf{1}_{t\geq s}\Big\{X_s + \int_s^t \Big[&- \bar{c}_u + r\bar{X}_u + \theta_u^i(\epsilon_\pi a \bar{\pi} - rC) + \theta_u^i D_u\Big(1 - \epsilon_D(k+r)\Big) + \\ &+ \theta_u^i \pi_u\Big(\epsilon_D - \epsilon_\pi(a+r)\Big)\Big]du + \epsilon_D\sigma_D \int_s^t \theta_u^i dW_u^D + \epsilon_\pi\sigma_\pi \int_s^t \theta_u^i dW_u^\pi\Big\}.\end{aligned}$$

If $\limsup_{t\to+\infty} E\left[\int_s^t N_u c_u du | \mathcal{G}_s\right] = -\infty$ the claim follows because $\eta_s = 1$ makes $(\bar{c}_t)_{t\geq 0}$ a better strategy, still admissible. Otherwise, if $\limsup_{t\to+\infty} E\left[\int_s^t N_u c_u du | \mathcal{G}_s\right] > -\infty$, define $\epsilon = X_s N_s - \limsup_{t\to+\infty} E\left[\int_s^t N_u c_u du \middle| \mathcal{G}_s\right] > 0$. Choose $\eta_s = \epsilon r (\mathcal{E}_s)^{-1} e^{rs}$ to obtain a better strategy, which is still admissible because

$$X_s N_s - \limsup_{t\to+\infty} E\left[\int_s^t N_u(c_u + \eta_s) du \middle| \mathcal{G}_s\right] = \epsilon - \eta_s \frac{\mathcal{E}_s}{r} e^{-rs} = 0.$$

□

Theorem 2.4.6 (Uniqueness). *Under Assumption 2.1.1 there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$, the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ in Theorem 2.4.3 is the unique optimal strategy for the $i-$th investor for all $i \in \{1, \dots, n\}$.*

Proof. Claim: The consumption process is unique.

Suppose there exist optimal strategies for the $i-$th investor $(c_t^A, \theta_t^A)_{t\geq 0}$ and $(c_t^B, \theta_t^B)_{t\geq 0}$ and suppose, for a contradiction, that there exists $S \in \mathcal{B} \otimes \mathcal{F}^i$ such that $(\lambda_{|[0,+\infty[} \otimes \mathbb{P})(S) > 0$ and $c_t^A \mathbf{1}_S \neq c_t^B \mathbf{1}_S$. The wealth process of the strategy $\frac{1}{2}(c_t^A + c_t^B, \theta_t^A + \theta_t^B)_{t\geq 0}$

is the process $\frac{1}{2}(X_t^A + X_t^B)_{t\geq 0}$, with dynamics

$$\frac{1}{2}d(X_t^A + X_t^B) = \frac{1}{2}\Big[- c_t^A - c_t^B + r(X_t^A + X_t^B) + (\theta_t^A + \theta_t^B)(\epsilon_\pi a\bar{\pi} - rC) + \\ + (\theta_t^A + \theta_t^B)D_t\Big(1 - \epsilon_D(k+r)\Big) + (\theta_t^A + \theta_t^B)\pi_t\Big(\epsilon_D - \epsilon_\pi(a+r)\Big)\Big]dt + \\ + \frac{1}{2}(\theta_t^A + \theta_t^B)\epsilon_D\sigma_D dW_t^D + \frac{1}{2}(\theta_t^A + \theta_t^B)\epsilon_\pi\sigma_\pi dW_t^\pi.$$

The new strategy has initial wealth x_0^i and is admissible because $(c_t^A, \theta_t^A)_{t\geq 0}$ and $(c_t^B, \theta_t^B)_{t\geq 0}$ are. Since the utility function is strictly concave, $c_t^A \neq c_t^B$ on S implies

$$U\Big(\frac{c_t^A + c_t^B}{2}\Big) > \frac{1}{2}U(c_t^A) + \frac{1}{2}U(c_t^B) \qquad \text{on } S.$$

Define $H := \Big\{w \in \Omega : \lambda_{|[0,+\infty[}\big(\{t \geq 0 : (t,w) \in S\}\big)\Big\} \in \mathcal{G}$, then $\mathbb{P}(H) > 0$,

$$\int_0^{+\infty} e^{-\beta t} U\Big(\frac{c_t^A + c_t^B}{2}\Big) dt \geq \frac{1}{2}\int_0^{+\infty} e^{-\beta t}[U(c_t^A) + U(c_t^B)]dt \qquad \text{a.s.}$$

and

$$\int_0^{+\infty} e^{-\beta t} U\Big(\frac{c_t^A + c_t^B}{2}\Big) dt > \frac{1}{2}\int_0^{+\infty} e^{-\beta t}[U(c_t^A) + U(c_t^B)]dt \qquad \text{on } H.$$

By Lemma A.0.2 (II) it follows that

$$E\left[\int_0^{+\infty} e^{-\beta t}\Big(\frac{c_t^A + c_t^B}{2}\Big)dt\Big|\mathcal{G}_0\right] > E\left[\int_0^{+\infty} e^{-\beta t} U(c_t^{i*})dt\Big|\mathcal{G}_0\right]$$

on a positive probability set, thus contradicting the optimality of the consumption processes $(c_t^A)_{t\geq 0}$ and $(c_t^B)_{t\geq 0}$.

Claim: Investment and wealth processes are unique.

Thanks to (2.4.36), it follows that

$$X_s^{i*} = (N_s)^{-1} \limsup_{t\to+\infty} E\left[\int_s^t N_u c_u^{i*} du\Big|\mathcal{G}_s\right],$$

which proves the uniqueness of the optimal wealth process. From (2.4.1) it follows that

$$dX_t^{i*} + c_t^{i*}dt - rX_t^{i*}dt = \theta_t^i\left[(\epsilon_\pi a\bar{\pi} - rC) + D_t\Big(1 - \epsilon_D(k+r)\Big) + \pi_t\Big(\epsilon_D - \epsilon_\pi(a+r)\Big)\right]dt + \\ + \epsilon_D\sigma_D\theta_t^i dW_t^D + \epsilon_\pi\sigma_\pi\theta_t^i dW_t^\pi.$$

If there exist two strategies with wealth process $(X_t^{i*})_{t\geq 0}$ and consumption $(c_t^{i*})_{t\geq 0}$, then drifts and volatilities must be the same. This implies the uniqueness of the optimal investment strategy. □

2.4.1 Market clearing and proof of Theorem 2.2.1

The economy has one risky asset, i.e. for every $t \geq 0$

$$\sum_{i=1}^{n} \theta_t^{i*} = \sum_{i=1}^{n} \frac{M_D D_t + M_\pi \pi_t + M_0}{M \alpha_i} = 1, \tag{2.4.37}$$

where M_D, M_π, M_0 are given in Definition 2.4.1.

Proof of Theorem 2.2.1. The market clearing condition (2.4.37) implies $M_D = 0, M_\pi = 0$ and $M_0 = M\bar{\alpha}$, where $\bar{\alpha} = \left(\sum_{i=1}^{n} \frac{1}{\alpha_i}\right)^{-1}$. From $M_D = 0$ and $M_\pi = 0$ it follows that

$$\begin{aligned} \epsilon_D &= -\frac{a + r - 2\delta_{\pi\pi}\sigma_\pi^2}{\delta_{D\pi}\sigma_\pi^2(1 + \sigma_D^2\delta_{D\pi}) - (k + r - 2\delta_{DD}\sigma_D^2)(a + r - 2\delta_{\pi\pi}\sigma_\pi^2)}, \\ \epsilon_\pi &= -\frac{1 + \delta_{D\pi}\sigma_D^2}{-(a + r)(k + r - 2\delta_{DD}\sigma_D^2) + \sigma_\pi^2(\delta_{D\pi} + 2(k + r)\delta_{\pi\pi} + \sigma_D^2(\delta_{D\pi}^2 - 4\delta_{DD}\delta_{\pi\pi}))}. \end{aligned} \tag{2.4.38}$$

Because of Theorem 2.4.1, $\delta_{DD}, \delta_{\pi\pi}, \delta_{D\pi}, \delta_D, \delta_\pi, \delta_0$ are the solution of the system

$$f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = 0 \tag{2.4.39}$$

where $f_1, \dots, f_6$ are given in Lemma C.0.1. Plugging ϵ_D and ϵ_π of (2.4.38) into the first 5 equation of (2.4.39) we get 4 solutions which may satisfy the market clearing condition:

$$\begin{pmatrix} \epsilon_D \\ \epsilon_\pi \\ \delta_{DD} \\ \delta_{D\pi} \\ \delta_{\pi\pi} \\ \delta_D \\ \delta_\pi \end{pmatrix} \in \left\{ \begin{pmatrix} \frac{1}{k+r} \\ \frac{1}{(a+r)(k+r)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{k+r} \\ -\frac{1}{a(k+r)} \\ * \\ * \\ \frac{2a+r}{2\sigma_\pi^2} \\ * \\ * \\ * \end{pmatrix}, \begin{pmatrix} -\frac{(a-k)^2(a+r)\sigma_D^2+(a-2k)\sigma_\pi^2}{k(a+r)((a-k)^2\sigma_D^2+\sigma_\pi^2)} \\ * \\ * \\ * \\ * \\ * \\ * \end{pmatrix}, \begin{pmatrix} -\frac{a(a+k+r)^2\sigma_D^2+(a+2k+r)\sigma_\pi^2}{ak((a+k+r)^2\sigma_D^2+\sigma_\pi^2)} \\ * \\ * \\ * \\ * \\ * \\ * \end{pmatrix} \right\}.$$

Neither of the last two solution is an equilibrium because $\lim_{\sigma_\pi \to 0} \epsilon_D(\sigma_\pi) = -\frac{1}{k} < 0$ and so $\epsilon_D \notin \mathbb{B}$ in a neighbourhood of $\sigma_\pi = 0$, which contradicts Definition 2.1.4. Exclude also the second solution because $\lim_{\sigma_\pi \to 0} \delta_{\pi\pi}(\sigma_\pi) = +\infty \neq \bar{\delta}_{\pi\pi}$ contradicting Theorem 2.4.1. The only solution left is the first one which leads to the unique equilibrium. The constant $C^* = \frac{a\bar{\pi}}{r(a+r)(k+r)} - \bar{\alpha}\left(\frac{\sigma_D^2}{(k+r)^2} + \frac{\sigma_\pi^2}{(a+r)^2(k+r)^2}\right)$ is the solution of the equation $M_0 = M\bar{\alpha}$, while δ_0 is the solution of $f_6 = 0$. ∎

Chapter 3

Heterogeneous Information

3.1 Model and main definitions

The economy has one risky asset in unit supply, which pays a dividend stream $(D_t)_{t\geq 0}$ described as

$$dD_t = (\pi_t - kD_t)dt + \sigma_D dW_t^D, \tag{3.1.1}$$

where the state of the economy $(\pi_t)_{t\geq 0}$ is an Ornstein–Uhlenbeck process

$$d\pi_t = a(\bar{\pi} - \pi_t)dt + \sigma_\pi dW_t^\pi. \tag{3.1.2}$$

There is a continuously compounded risk-free asset $(P_t^0)_{t\geq 0}$ with rate of return $r > 0$, at which investors can both lend and borrow. There are $n \in \mathbb{N}$ investors competing for the risky asset: the $i-$th investor has constant absolute risk aversion $\alpha_i \geq 1$ and initial wealth $x_0^i \in \mathbb{R}$. The price $(P_t)_{t\geq 0}$ of the risky asset is public information and each investor observes the private signals $(\xi_t^i)_{t\geq 0}$, which offers a noisy estimate of the state of the economy, i.e.,

$$d\xi_t^i = \pi_t dt + \sigma_i dW_t^i, \qquad \xi_0^i = 0, \tag{3.1.3}$$

where $W = (W_t^D, W_t^\pi, W_t^1, \dots, W_t^n)_{t\geq 0}$ is a $(n+2)-$dimensional Brownian motion and (D_0, π_0) is an independent normally distributed random vector with mean and covariance

$$E\left[\begin{pmatrix} D_0 \\ \pi_0 \end{pmatrix}\right] = \begin{pmatrix} \bar{\pi}/k \\ \bar{\pi} \end{pmatrix}, \quad \mathrm{Var}\left[\begin{pmatrix} D_0 \\ \pi_0 \end{pmatrix}\right] = \begin{pmatrix} \frac{\sigma_D^2}{2k} + \frac{\sigma_\pi^2}{(k-a)^2}\left(\frac{1}{2a} + \frac{1}{2k} - \frac{2}{a+k}\right) & \frac{\sigma_\pi^2}{2a(a+k)} \\ \frac{\sigma_\pi^2}{2a(a+k)} & \frac{\sigma_\pi^2}{2a} \end{pmatrix}.$$

The probability space is $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$, where $\mathcal{G}_t$ is the augmented natural filtration of $D_0, \pi_0, (W_u)_{0\leq u\leq t}$ and $\mathcal{G}$ is the augmented sigma algebra generated by $\bigcup_{t\geq 0} \mathcal{G}_t$. Likewise $\mathcal{F}_t^i$, which represent the information of the $i-$th investor at time $t \geq 0$, is the augmented natural filtration of $(D_u, P_u, \xi_u)_{0\leq u\leq t}$[1]. The $i-$th investor's estimate of the state of the economy is $(\hat{\pi}^i_t)_{t\geq 0} = (E[\pi_t|\mathcal{F}^i_t])_{t\geq 0}$. All equalities and inequalities between random variables are understood $\mathbb{P}-$almost surely.

Definition 3.1.1 (Admissibile strategies). $(c_t, \theta_t)_{t\geq 0}$ is an admissible (consumption-investment) strategy for the $i-$th investor if:

(i) $(c_t)_{t\geq 0}$ and $(\theta_t)_{t\geq 0}$ are $(\mathcal{F}^i_t)_{t\geq 0}-$progressively measurable processes;

(ii) for every $s \geq 0$

$$\limsup_{t\to+\infty} E\left[\int_s^t N^i_u c_u du \middle| \mathcal{F}^i_s\right] \leq N^i_s X_s, \tag{3.1.4}$$

where $(X_t)_{t\geq 0}$ is the self-financing wealth process

$$dX_t = -c_t dt + \theta_t D_t dt + r(X_t - \theta_t P_t)dt + \theta_t dP_t, \qquad X_0 = x^i_0, \tag{3.1.5}$$

$(N^i_t)_{t\geq 0}$ is the process

$$\begin{aligned} N^i_t = \exp\Big(- rt + \int_0^t (\Delta^i \cdot v^i_u) dB^{iD}_u + \int_0^t (\iota^i \cdot v^i_u) dB^i_u + \int_0^t (\rho^i \cdot v^i_u) dB^{i\perp}_u + \\ - \frac{1}{2}\int_0^t \left[(\Delta^i \cdot v^i_u)^2 + (\iota^i \cdot v^i_u)^2 + (\rho^i \cdot v^i_u)^2\right] du\Big), \end{aligned} \tag{3.1.6}$$

$(v^i_t)_{t\geq 0} = (D_t, \hat{\pi}^M_t, \hat{\pi}^i_t, 1)^T_{t\geq 0}$ and $\Delta^i, \iota^i, \rho^i$ are given in Definition 3.4.1 below.

The set of admissible strategies for the $i-$th investor is $\mathcal{U}^i$.

Definition 3.1.2 (Optimality). A (consumption-investment) strategy $(c^i_t, \theta^i_t)_{t\geq 0}$ is optimal for the $i-$th investor if it is admissible and if

$$\sup_{(c,\theta)\in\mathcal{U}^i} E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u) du \middle| \mathcal{F}^i_0\right] = E\left[\int_0^{+\infty} e^{-\beta u} U^i(c^i_u) du \middle| \mathcal{F}^i_0\right], \tag{3.1.7}$$

where

$$U^i(c) := -\frac{e^{-\alpha_i c}}{\alpha_i}, \qquad i \in \{1, \ldots, n\}.$$

The time impatience parameter $\beta > 0$ is common to all agents. The consumption-investment problem of each agent is well-posed if an optimal strategy exists, otherwise the problem is ill-posed.

Remark 3.1.1. (D_0, π_0) being a random variable with the distribution defined above is fundamental for $(D_t, \pi_t)_{t\geq 0}$ to be a stationary process and thus for the stationary filter of Lemma 3.4.1 to make sense. As a consequence the $\sigma-$algebra $\mathcal{F}^i_0$ is different from the trivial $\sigma-$algebra and a conditional expectation appears in (3.1.7).

[1] Note that all filtrations are augmented with the null sets of the sigma algebra $\mathcal{G}$.

Definition 3.1.3. $(c_t^i, \theta_t^i)_{t\geq 0}$ is the unique optimal (consumption-investment) strategy for the $i-$th investor if it is optimal for the $i-$th investor and if

$$(c_t^i, \theta_t^i)_{t\geq 0} = (\bar{c}_t, \bar{\theta}_t)_{t\geq 0} \qquad \lambda_{|[0,+\infty[} \otimes \mathbb{P} - \text{a.s.}$$

for every other optimal strategy $(\bar{c}_t, \bar{\theta}_t)_{t\geq 0}$.

Definition 3.1.4. Fix $\epsilon_1, \ldots, \epsilon_n > 0$. The (consensus) market estimate for the state of the economy is

$$\hat{\pi}_t^M := E\left[\pi_t \middle| \left(D_u, \sum_{i=1}^n \epsilon_i \xi_u^i\right)_{0\leq u\leq t}\right].$$

Remark 3.1.2. As the definition of $(\hat{\pi}_t^M)_{t\geq 0}$ is invariant with respect to a common scaling factor of all $\epsilon_1, \ldots, \epsilon_n$, without loss of generality we suppose that $\sum_{i=1}^n \epsilon_i^2 \sigma_i^2 = \sum_{i=1}^n \epsilon_i$.

Lemma 3.1.1. The (consensus) market estimate for the state of the economy $(\hat{\pi}_t^M)_{t\geq 0}$ has dynamics

$$\begin{aligned} d\hat{\pi}_t^M &= \left[a(\bar{\pi} - \hat{\pi}_t^M) + o_M k \sigma_D^{-2} D_t - o_M \left(\sigma_D^{-2} + \sum_{i=1}^n \epsilon_i\right) \hat{\pi}_t^M\right] dt + o_M \left(\sigma_D^{-2} dD_t + \sum_{i=1}^n \epsilon_i d\xi_t^i\right), \\ \hat{\pi}_0^M &= \bar{\pi}, \end{aligned} \tag{3.1.8}$$

where $o_M = \frac{-a + \sqrt{a^2 + \sigma_\pi^2(\sigma_D^{-2} + \sum_{i=1}^n \epsilon_i)}}{\sigma_D^{-2} + \sum_{i=1}^n \epsilon_i}$.

Proof. Apply Theorem D.0.1 (cf. [25, Theorem 10.3]), with the processes $(D_t, \sum_{i=1}^n \epsilon_i \xi_t^i)_{t\geq 0}$ as signals. □

Definition 3.1.5. A linear equilibrium is an $(n+4)-$tuple $(\sigma_\pi, \epsilon_D^{\sigma_\pi}, \epsilon_\pi^{\sigma_\pi}, C^{\sigma_\pi}, (S_{\sigma_\pi}^i)^{1\leq i\leq n})$, where $\sigma_\pi \geq 0, \epsilon_D^{\sigma_\pi} \in \mathbb{B}, \epsilon_\pi^{\sigma_\pi} \in \mathbb{R}^*, C^{\sigma_\pi} \in \mathbb{R}$ and $S_{\sigma_\pi}^i = (c_t^i, \theta_t^i)_{t\geq 0}$ is an optimal strategy for the $i-$th investor for every $i \in \{1, \ldots, n\}$ such that for every $t \geq 0$

(i) the price of the risky asset is

$$P_t = C^{\sigma_\pi} + \epsilon_D^{\sigma_\pi} D_t + \epsilon_\pi^{\sigma_\pi} \hat{\pi}_t^M; \tag{3.1.9}$$

(ii) the market clearing condition

$$\sum_{i=1}^n \theta_t^i = 1 \tag{3.1.10}$$

holds.

In the light of Chapter 1, we focus for the whole chapter on the following

Assumption 3.1.1. The parameters of the economy are $a, k, \sigma_1, \dots \sigma_n, \sigma_D > 0, \sigma_\pi \geq 0$ and $\bar{\pi}, C \in \mathbb{R}$. Furthermore assume

$$\epsilon_D \in \mathbb{B} := (0, 2/r) \setminus \{1/r\}, \qquad \epsilon_\pi \neq 0, \qquad a \neq k.$$

Definition 3.1.6. $(E_{\sigma_\pi})_{0 \leq \sigma_\pi \leq \bar{\sigma_\pi}} = (\sigma_\pi, \epsilon_D^{\sigma_\pi}, \epsilon_\pi^{\sigma_\pi}, C^{\sigma_\pi}, (S^i_{\sigma_\pi})^{1 \leq i \leq n})_{0 \leq \sigma_\pi \leq \bar{\sigma_\pi}}$ is a continuous equilibrium if there exists $\bar{\sigma_\pi} > 0$ such that $(E_{\sigma_\pi})_{0 \leq \sigma_\pi \leq \bar{\sigma_\pi}}$ is a linear equilibrium for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}$, and if

$$\lim_{\sigma_\pi \to 0^+} (\epsilon_D^{\sigma_\pi}, \epsilon_\pi^{\sigma_\pi}, C^{\sigma_\pi}) = (\epsilon_D^0, \epsilon_\pi^0, C^0) \in \mathbb{B} \times \mathbb{R}^* \times \mathbb{R}. \tag{3.1.11}$$

Definition 3.1.7. The continuous equilibrium $(E^A_{\sigma_\pi})_{0 \leq \sigma_\pi \leq \bar{\sigma_\pi}^A}$ is unique if $E^A_{\sigma_\pi} = E^B_{\sigma_\pi}$ for every $0 \leq \sigma_\pi \leq \min\{\bar{\sigma_\pi}^A, \bar{\sigma_\pi}^B\}$, for every $(E^B_{\sigma_\pi})_{0 \leq \sigma_\pi \leq \bar{\sigma_\pi}^B}$ continuous equilibrium.

Definition 3.1.8. We introduce the following constants,

$$\epsilon_{i\perp} = \sqrt{\sum_{j \neq i} \epsilon_j^2 \sigma_j^2}, \qquad \bar{\alpha} = \left(\sum_{i=1}^n \frac{1}{\alpha_i} \right)^{-1} \qquad \sigma_{i\perp} = \frac{\sqrt{\sum_{j \neq i} \epsilon_j^2 \sigma_j^2}}{\sum_{j \neq i} \epsilon_j},$$

$$\nu = \sigma_D^{-2} + \sum_{i=1}^n \epsilon_i, \qquad \nu_i = \sigma_D^{-2} + \sigma_i^{-2} + \sigma_{i\perp}^{-2},$$

$$o_M = \frac{-a + \sqrt{a^2 + \sigma_\pi^2 \nu}}{\nu}, \qquad o_i = \frac{-a + \sqrt{a^2 + \sigma_\pi^2 \nu_i}}{\nu_i}.$$

3.2 Existence and uniqueness of the equilibrium

Theorem 3.2.1. *Under Assumption 3.1.1 there exists a unique continuous equilibrium* $(\sigma_\pi, C^*, \epsilon_D^*, \epsilon_\pi^*, (S^{i*})^{1 \leq i \leq n})$, *for which the price is*

$$P_t = C^* + \epsilon_D^* D_t + \epsilon_\pi^* \hat{\pi}_t^M, \quad \textit{where} \quad \hat{\pi}_t^M = E\left[\pi_t \middle| \left(D_u, \sum_{i=1}^n \sigma_i^{-2} \xi_u^i \right)_{0 \leq u \leq t} \right],$$

$$\epsilon_D^* = \frac{1}{k+r}, \qquad \epsilon_\pi^* = \frac{1}{(a+r)(k+r)} \quad \textit{and}$$

$$C^* = \frac{a\bar{\pi}}{r(a+r)(k+r)} - \bar{\alpha} \left(\frac{\sigma_D^2}{(k+r)^2} + \frac{\sigma_\pi^2}{(a+r)^2(k+r)^2} \left(1 + \frac{2r}{a + \sqrt{a^2 + \sigma_\pi^2 (\sigma_D^{-2} + \sum_{i=1}^n \sigma_i^{-2})}} \right) \right).$$

The unique optimal consumption-investment strategy for the $i-$th agent is

$$c_t^{i*} = rX_t^{i*} + \frac{\beta - r}{r\alpha_i} + \frac{r\bar{\alpha}^2}{2\alpha_i}\left(\frac{\sigma_D^2}{(k+r)^2} + \frac{\sigma_\pi^2}{(a+r)^2(k+r)^2}\left(1 + \frac{2r}{a + \sqrt{a^2 + \sigma_\pi^2(\sigma_D^{-2} + \sum_{i=1}^n \sigma_i^{-2})}}\right)\right), \tag{3.2.1}$$

$$\theta_t^{i*} = \frac{\bar{\alpha}}{\alpha_i}. \tag{3.2.2}$$

Preliminaries and outline of the proof

Remark 3.2.1. If $(\epsilon_D, \epsilon_\pi) = (0,0)$ then (3.1.9) implies $P_t = C$ for every $t \geq 0$. If the assets are two deterministic processes with different interest rates (0 for P_t and $r > 0$ for P_t^0), then the model admits arbitrage, therefore the consumption-investment problem of the agents is ill-posed and in particular no linear equilibrium exists.

Definition 3.2.1. A value function for the $i-$th investor is a function

$$V^i : \mathbb{R}^4 \to [-\infty, 0[$$
$$(\bar{x}, \bar{D}, \bar{\pi}^M, \bar{\pi}^i) \to V^i(\bar{x}, \bar{D}, \bar{\pi}^M, \bar{\pi}^i)$$

such that for every $(\bar{x}, \bar{D}, \bar{\pi}^M, \bar{\pi}^i) \in \mathbb{R}^4$

$$V^i(\bar{x}, \bar{D}, \bar{\pi}^M, \bar{\pi}^i) = \sup_{(c,\theta)\in\mathcal{U}^i} E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u) du \middle| x_0^i = \bar{x}, D_0 = \bar{D}, \hat{\pi}_0^M = \bar{\pi}^M, \hat{\pi}_0^i = \bar{\pi}^i\right]. \tag{3.2.3}$$

It follows from this definition that if there exists a value function $V^i(\cdot)$ and a strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ optimal for the $i-$th investor, then

$$V^i(x_0^i, D_0, \hat{\pi}_0^M, \hat{\pi}_0^i) = E\left[\int_0^{+\infty} e^{-\beta s} U^i(c_s^{i*}) ds \middle| \mathcal{F}_0^i\right].$$

Definition 3.2.2. A stochastic discount factor (SDF) for the $i-$th agent is a positive, continuous, $(\mathcal{F}_t^i)_{t\geq 0}-$adapted process $(M_t)_{t\geq 0}$ such that for every $0 \leq s \leq t$

$$M_s P_s^0 = E[M_t P_t^0 | \mathcal{F}_s^i] \tag{3.2.4}$$

and

$$M_s P_s + \int_0^s M_u D_u du = E\left[M_t P_t + \int_0^t M_u D_u du \middle| \mathcal{F}_s^i\right]. \tag{3.2.5}$$

A stochastic discount factor is normalized if $M_0 = 1$.

We find the (unique) equilibrium in the market in two steps: first we solve the optimal consumption problem of the agents for a generic price with the form of (3.1.9);

then we clear the market with condition (3.1.10) and we deduce that the price of the unique continuous equilibrium has parameters

$$\epsilon_D^* = \frac{1}{k+r}, \qquad \epsilon_\pi^* = \frac{1}{(a+r)(k+r)} \qquad \text{and}$$
$$C^* = \frac{a\bar{\pi}}{r(a+r)(k+r)} - \bar{\alpha}\left(\frac{\sigma_D^2}{(k+r)^2} + \frac{\sigma_\pi^2}{(a+r)^2(k+r)^2}\left(1 + \frac{2r}{a + \sqrt{a^2 + \sigma_\pi^2(\sigma_D^{-2} + \sum_{i=1}^n \sigma_i^{-2})}}\right)\right).$$

- **Section 3.3** formulates the Hamilton Jacobi Bellman (HJB) equation, which leads to a guess of the value function and the optimal strategies.
- **Section 3.4** formalizes the heuristics of the previous section proving existence and uniqueness of the optimal portfolio for a generic price function for $\epsilon_D \in \mathbb{B}$.
  - **Subsection 3.4.1** finds the unique linear equilibrium in the market through the market clearing condition.
- **Appendix D** recalls some well known results that are used in this chapter.

3.3 Heuristics

If the price of the risky asset is linear in D and $\hat{\pi}^M$, standard results in filtering theory [25, Theorem 10.3] imply that the dynamics of the state of the economy for the $i-$th investor is

$$d\hat{\pi}_t^i = a(\bar{\pi} - \hat{\pi}_t^i)dt + o_i\left(\sigma_D^{-1}dB_t^{iD} + \sigma_i^{-1}dB_t^i + \sigma_{i\perp}^{-1}dB_t^{i\perp}\right),$$

where $(B_t)_{t\geq 0} = (B_t^{iD}, B_t^i, B_t^{i\perp})_{t\geq 0}$ is the $\mathbb{P}-$Brownian motion defined in (3.4.1) below. Guess a value function V^i which depends on the dividend rate, on the opinions about the state of the economy and on the wealth; because of the infinite time horizon we guess that V^i does not depend on the initial time $t > 0$, i.e.

$$V^i(X_t^i, D_t, \hat{\pi}_t^M, \hat{\pi}_t^i) = \sup_{(c^i,\theta^i)} E\left[\int_t^{+\infty} e^{-\beta(s-t)}U^i(c_s^i)ds|\mathcal{F}_t^i\right].$$

A similar procedure as in Subsection 1.3.1 (replacing $\mathcal{G}_t$ with $\mathcal{F}_t^i$) leads to

$$\begin{aligned}0 = \sup_{(c^i,\theta^i)}\Bigg\{ &-\frac{e^{-\alpha_i c^i}}{\alpha_i} - \beta V^i + V_x^i\Big[-c^i + rx + \theta^i(\epsilon_\pi a\bar{\pi} - rC) + \theta^i D(1-\epsilon_D(k+r)) + \\ &+\theta^i\hat{\pi}^M(-\epsilon_\pi(a+r) - \epsilon_\pi o_M\nu) + \theta^i\hat{\pi}^i(\epsilon_D + \epsilon_\pi o_M\nu)\Big] + V_D^i(\hat{\pi}^i - kD) + V_{\hat{\pi}^M}^i(a\bar{\pi} - a\hat{\pi}^M + o_M\nu\hat{\pi}^i - o_M\nu\hat{\pi}^M) + \\ &+V_{\hat{\pi}^i}^i(a\bar{\pi} - a\hat{\pi}^i) + \frac{1}{2}(\theta^i)^2 V_{xx}^i(\nu\epsilon_\pi^2 o_M^2 + \epsilon_D^2\sigma_D^2 + 2\epsilon_D\epsilon_\pi o_M) + \frac{1}{2}V_{DD}^i\sigma_D^2 + \frac{1}{2}V_{\hat{\pi}^M\hat{\pi}^M}^i o_M^2\nu + \frac{1}{2}V_{\hat{\pi}^i\hat{\pi}^i}^i o_i^2\nu_i + \\ &+V_{xD}^i\theta^i(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + V_{x\hat{\pi}^M}^i\theta^i o_M(\epsilon_D + \epsilon_\pi o_M\nu) + V_{x\hat{\pi}^i}^i\theta^i o_i(\epsilon_D + \epsilon_\pi o_M\nu) + V_{D\hat{\pi}^M}^i o_M + \\ &V_{D\hat{\pi}^i}^i o_i + V_{\hat{\pi}^M\hat{\pi}^i}^i o_M o_i\nu\Bigg\}. \qquad (3.3.1)\end{aligned}$$

Differentiating with respect to c^i and θ^i, we find the candidate optimal consumption-investment policy

$$\begin{aligned}c^{i*} &= \frac{\log(V_x^i)}{-\alpha_i}, \\ \theta^{i*} &= -\frac{1}{V_{xx}^i(\nu\epsilon_\pi^2 o_M^2 + \epsilon_D^2\sigma_D^2 + 2\epsilon_D\epsilon_\pi o_M)}\Bigg\{V_x^i\Big[(\epsilon_\pi a\bar{\pi} - rC) + D(1-\epsilon_D(k+r)) + \hat{\pi}^M(-\epsilon_\pi(a+r) - \epsilon_\pi o_M\nu) + \\ &+\hat{\pi}^i(\epsilon_D + \epsilon_\pi o_M\nu)\Big] + V_{xD}^i(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + V_{x\hat{\pi}^M}^i o_M(\epsilon_D + \epsilon_\pi o_M\nu) + V_{x\hat{\pi}^i}^i o_i(\epsilon_D + \epsilon_\pi o_M\nu)\Bigg\}. \qquad (3.3.2)\end{aligned}$$

The HJB equation for the $i-$th investor follows by substituting the candidate optimal policies into (3.3.1)

$$0 = -\frac{V_x^i}{\alpha_i} - \beta V^i + V_x^i\Big[\Big(\frac{\log V_x^i}{\alpha_i} + rx\Big) + \theta^{i*}(\epsilon_\pi a\bar{\pi} - rC) + \theta^{i*}D(1-\epsilon_D(k+r)) + \\ + \theta^{i*}\hat{\pi}^M(-\epsilon_\pi(a+r) - \epsilon_\pi o_M\nu) + \theta^{i*}\hat{\pi}^i(\epsilon_D + \epsilon_\pi o_M\nu)\Big] + V_D^i(\hat{\pi}^i - kD) + V_{\hat{\pi}^M}^i(a\bar{\pi} - a\hat{\pi}^M + \\ + o_M\nu\hat{\pi}^i - o_M\nu\hat{\pi}^M) + V_{\hat{\pi}^i}^i(a\bar{\pi} - a\hat{\pi}^i) + \frac{1}{2}(\theta^{i*})^2V_{xx}^i(\nu\epsilon_\pi^2 o_M^2 + \epsilon_D^2\sigma_D^2 + 2\epsilon_D\epsilon_\pi o_M) + \frac{1}{2}V_{DD}^i\sigma_D^2 + \\ + V_{\hat{\pi}^M\hat{\pi}^M}^i o_M^2\nu + \frac{1}{2}V_{\hat{\pi}^i\hat{\pi}^i}^i o_i^2\nu_i + V_{xD}^i\theta^{i*}(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + V_{x\hat{\pi}^M}^i\theta^{i*}o_M(\epsilon_D + \epsilon_\pi o_M\nu) + \\ + V_{x\hat{\pi}^i}^i\theta^{i*}o_i(\epsilon_D + \epsilon_\pi o_M\nu) + V_{D\hat{\pi}^M}^i o_M + V_{D\hat{\pi}^i}^i o_i + V_{\hat{\pi}^M\hat{\pi}^i}^i o_M o_i\nu, \quad (3.3.3)$$

Using the Ansatz

$$V^i(x, D, \hat{\pi}^M, \hat{\pi}^i) = -\frac{1}{r\alpha_i}\exp\Big(-r\alpha_i x + \delta_{DD}^i D^2 + \delta_{MM}^i(\hat{\pi}^M)^2 + \delta_{ii}(\hat{\pi}^i)^2 + \\ + \delta_{DM}^i D\hat{\pi}^M + \delta_{Di}D\hat{\pi}^i + \delta_{Mi}\hat{\pi}^M\hat{\pi}^i + \delta_D^i D + \delta_M^i\hat{\pi}^M + \delta_i\hat{\pi}^i + \delta_0^i\Big),$$

where $\delta_{DD}^i, \delta_{MM}^i, \delta_{ii}, \delta_{DM}^i, \delta_{Di}, \delta_{Mi}, \delta_D^i, \delta_M^i, \delta_i, \delta_0^i$ are in Theorem 3.4.1, (3.3.2) leads to the optimal consumption investment strategy

$$c_t^{i*} = rX_t^i - \frac{\delta_{DD}^i}{\alpha_i}D_t^2 - \frac{\delta_{MM}^i}{\alpha_i}(\hat{\pi}_t^M)^2 - \frac{\delta_{ii}}{\alpha_i}(\hat{\pi}_t^i)^2 - \frac{\delta_{DM}^i}{\alpha_i}D_t\hat{\pi}_t^M - \frac{\delta_{Di}}{\alpha_i}D_t\hat{\pi}_t^i - \frac{\delta_{Mi}}{\alpha_i}\hat{\pi}_t^M\hat{\pi}_t^i + \\ - \frac{\delta_D^i}{\alpha_i}D_t - \frac{\delta_M^i}{\alpha_i}\hat{\pi}_t^M - \frac{\delta_i}{\alpha_i}\hat{\pi}_t^i - \frac{\delta_0^i}{\alpha_i};$$
$$\theta_t^{i*} = \frac{M_D^i D_t + M_M^i\hat{\pi}_t^M + M_i\hat{\pi}_t^i + M_0^i}{M\alpha_i},$$

where M_D^i, M_M^i, M_i, M_0^i are in Definition 3.4.1.

3.4 Verification

Theorem 3.2.1 identifies a unique continuous equilibrium in the market. The first step of the proof is to solve the consumption-investment problem of the agents for a generic price with form (3.1.9). We start by finding the agents' views about the state of the economy.

Lemma 3.4.1 (Filtering). Define

$$\xi_t^{i\perp} = \frac{1}{\sum_{j\neq i}\epsilon_j}\sum_{j\neq i}\epsilon_j\xi_t^j, \qquad \hat{\pi}_t^i = E[\pi_t|\mathcal{F}_t^i],$$

$$(B_t)_{t\geq 0} = (B_t^{iD}, B_t^i, B_t^{i\perp})_{t\geq 0} = \\ \left(W_t^D + \int_0^t \frac{\pi_u - \hat{\pi}_u^i}{\sigma_D}du, W_t^i + \int_0^t \frac{\pi_u - \hat{\pi}_u^i}{\sigma_i}du, \frac{1}{\epsilon_{i\perp}}\sum_{j\neq i}\epsilon_j\sigma_j W_t^j + \int_0^t \frac{\pi_u - \hat{\pi}_u^i}{\sigma_{i\perp}}du\right)_{t\geq 0}. \tag{3.4.1}$$

The following hold

(A) For every $t \geq 0$

$$\mathcal{F}_t^i = \sigma\{D_u, \hat{\pi}_u^M, \xi_u^i\}_{0\leq u\leq t} = \sigma\left\{D_u, \sum_{j\neq i}\epsilon_j\xi_u^j, \xi_u^i\right\}_{0\leq u\leq t} = \sigma\left\{D_u, \xi_u^{i\perp}, \xi_u^i\right\}_{0\leq u\leq t}.$$

(B) The $i-$th investor's (stationary) filter for the state of the economy is

$$d\hat{\pi}_t^i = a(\bar{\pi} - \hat{\pi}_t^i)dt + o_i\left(\sigma_D^{-1}dB_t^{iD} + \sigma_i^{-1}dB_t^i + \sigma_{i\perp}^{-1}dB_t^{i\perp}\right), \qquad \hat{\pi}_0^i = \bar{\pi}, \tag{3.4.2}$$

where $B = (B_t^{iD}, B_t^i, B_t^{i\perp})_{t\geq 0}$ is a $\mathbb{P}-$Brownian motion.

(C) For every $i \in \{1, \ldots, n\}$ the processes $(\hat{\pi}_t^M)_{t\geq 0}$ and $(D_t)_{t\geq 0}$ follow the dynamics

$$d\hat{\pi}_t^M = [a(\bar{\pi} - \hat{\pi}_t^M) + o_M\nu(\hat{\pi}_t^i - \hat{\pi}_t^M)]dt + o_M\left(\sigma_D^{-1}dB_t^{iD} + \epsilon_i\sigma_i dB_t^i + \epsilon_{i\perp}dB_t^{i\perp}\right), \quad \hat{\pi}_0^M = \bar{\pi}, \tag{3.4.3}$$

$$dD_t = (\hat{\pi}_t^i - kD_t)dt + \sigma_D dB_t^{iD}.$$

Proof. Proof of (A)
Define $\mathcal{H}_t^i = \sigma\{D_u, \hat{\pi}_u^M, \xi_u^i : 0 \leq u \leq t\}$ and $\mathcal{L}_t^i = \sigma\left\{D_u, \sum_{j\neq i}\epsilon_j\xi_u^j, \xi_u^i : 0 \leq u \leq t\right\}$. Equation (3.1.9) implies $(\mathcal{F}_t^i)_{t\geq 0} = (\mathcal{H}_t^i)_{t\geq 0}$. Defining $\lambda = (a + o_M\nu)$ and applying Itô's

rule to $e^{\lambda t}\hat{\pi}^M_t, e^{\lambda t}D_t$, and $e^{\lambda t}\xi^i_t$, it follows that

$$\hat{\pi}^M_t = e^{-\lambda(t-s)}\hat{\pi}^M_s + \frac{a\bar{\pi}}{\lambda}\left(1-e^{-\lambda(t-s)}\right) + o_M k\sigma_D^{-2}e^{-\lambda t}\int_s^t e^{\lambda u}D_u du + o_M\sigma_D^{-2}e^{-\lambda t}\Big[e^{\lambda t}D_t + \\ -e^{\lambda s}D_s - \lambda\int_s^t e^{\lambda u}D_u du\Big] + o_M\epsilon_i e^{-\lambda t}\left[e^{\lambda t}\xi^i_t - e^{\lambda s}\xi^i_s - \lambda\int_s^t e^{\lambda u}\xi^i_u du\right] + o_M e^{-\lambda t}\Big[e^{\lambda t}\sum_{j\neq i}\epsilon_j\xi^j_t + \\ - e^{\lambda s}\sum_{j\neq i}\epsilon_j\xi^j_s - \lambda\int_s^t e^{\lambda u}\sum_{j\neq i}\epsilon_j\xi^j_u du\Big],$$

which implies $\mathcal{H}^i_t \subseteq \mathcal{L}^i_t$. In view of (3.1.8),

$$\hat{\pi}^M_t = \hat{\pi}^M_0 + a\bar{\pi}t - (a+o_M\nu)\int_0^t \hat{\pi}^M_u du + o_M k\sigma_D^{-2}\int_0^t D_u du + o_M\sigma_D^{-2}D_t - o_M\sigma_D^{-2}D_0 + \\ + o_M\epsilon_i\xi^i_t - o_M\epsilon_i\xi^i_0 + o_M\left(\sum_{j\neq i}\epsilon_j\xi^j_t\right) - o_M\left(\sum_{j\neq i}\epsilon_j\xi^j_0\right),$$

which implies $\mathcal{L}^i_t \subseteq \mathcal{H}^i_t$ and thus $\mathcal{L}^i_t = \mathcal{H}^i_t$. The last equality of (A) is true because $\xi^{i\perp}_t$ is a multiple of the process $\sum_{j\neq i}\epsilon_j\xi^j_t$.

Applying Theorem D.0.1 (cf. [25, Theorem 10.3]), with the process $(D_t, \xi^i_t, \xi^{i\perp}_t)_{t\geq 0}$ as signal, (B) follows, while (C) is a direct consequence of the definition of $(B^{iD}_t, B^i_t, B^{i\perp}_t)_{t\geq 0}$. □

The market estimate of the state of the economy expresses a weighted average of the private information available. The following theorem shows that each agent considers $(\hat{\pi}^M_t)_{t\geq 0}$ their best approximation for $(\pi_t)_{t\geq 0}$ if and only if the weight of their private signal in the process $(\hat{\pi}^M_t)_{t\geq 0}$ is the inverse of the square of their signal's noise, i.e. $\epsilon_i = \sigma_i^{-2}$.

Lemma 3.4.2 (Properties of the filters). Let $\sigma_\pi > 0$ and $i \in \{1,\dots,n\}$; the following are equivalent

1. The $(\mathcal{G}_t)_{t\geq 0}$−measurable processes $(\hat{\pi}^M_t)_{t\geq 0}$ and $(\hat{\pi}^i_t)_{t\geq 0}$ are indistinguishable;
2. $\epsilon_i = \sigma_i^{-2}$;
3. $\nu_i = \nu$;
4. $o_i = o_M$;
5. $\nu_i o_i = o_M\nu$;

If $\sigma_\pi = 0$, then $(\hat{\pi}^M_t)_{t\geq 0} = (\hat{\pi}^i_t)_{t\geq 0} = (\pi_t)_{t\geq 0}$ for every $i \in \{1,\dots,n\}$.

Proof. (3) $\iff$ (4)
We defined

$$o_M = \frac{-a+\sqrt{a^2+\sigma_\pi^2\nu}}{\nu}, \qquad o_i = \frac{-a+\sqrt{a^2+\sigma_\pi^2\nu_i}}{\nu_i},$$

in Definition 3.1.8 so (3) implies (4). For the reverse implication observe that the derivative of the function $f(x) = \frac{-a+\sqrt{a^2+\sigma_\pi^2 x}}{x}$ is strictly negative for $a, \sigma_\pi, x > 0$, then such function is decreasing and thus injective.

(2) $\iff$ (3)

Since $\sum_{i=1}^n \epsilon_i = \sum_{i=1}^n \epsilon_i^2\sigma_i^2$, $\nu = \nu_i$ if and only if $\sum_{i=1}^n \epsilon_i = \sigma_i^{-2} + \frac{\sum_{j\neq i}\epsilon_j}{\epsilon_{i\perp}^2}$. Multiplying both sides by $\epsilon_{i\perp}^2 > 0$ and using the equality $\epsilon_{i\perp}^2 = \sum_{i=1}^n \epsilon_i - \epsilon_i^2\sigma_i^2$ we get an algebraic equation of second order whose unique solution is $\epsilon_i = \sigma_i^{-2}$.

(1) $\implies$ (2)

Thanks to (3.1.8) it follows that

$$d\hat{\pi}_t^M = [a\bar{\pi} + o_M\nu\pi_t - (a + o_M\nu)\hat{\pi}_t^M]dt + o_M\left(\sigma_D^{-1}dW_t^D + \sum_{i=1}^n \epsilon_i\sigma_i dW_t^i\right) \tag{3.4.4}$$

and (3.4.2) yields

$$d\hat{\pi}_t^i = \left\{a\bar{\pi} + o_i\nu_i\pi_t - [a + o_i\nu_i]\,\hat{\pi}_t^i\right\}dt + o_i\left[\sigma_D^{-1}dW_t^D + \sigma_i^{-1}dW_t^i + \left(\frac{\sum_{j\neq i}\epsilon_j}{\epsilon_{i\perp}^2}\right)\sum_{j\neq i}\epsilon_j\sigma_j dW_t^j\right]. \tag{3.4.5}$$

For the terms in dW_t^D of (3.4.4) and (3.4.5) to match we need $o_i = o_M$ which we proved to be equivalent to $\epsilon_i = \sigma_i^{-2}$.

(2) $\implies$ (1)

Comparing (3.4.4) and (3.4.5), it is suffices to show that $\epsilon_h\sigma_h = \left(\frac{\sum_{j\neq i}\epsilon_j}{\epsilon_{i\perp}^2}\right)\epsilon_h\sigma_h$ for every $i \in \{1, \dots, n\}$ and $h \neq i$. This is equivalent to $1 = \frac{\sum_{i=1}^n \epsilon_i - \sigma_i^{-2}}{\sum_{i=1}^n \epsilon_i - \sigma_i^{-2}}$ which is always true. Observe that the denominator is always different from 0 under Assumption 3.1.1 because $\sum_{i=1}^n \epsilon_i - \sigma_i^{-2} = \sum_{j\neq i}\epsilon_j > 0$.

$(1-4) \implies (5)$

Multiply the equalities (3) and (4).

$(5) \implies (1-4)$

Since the derivative of the function $f(\nu) = -a + \sqrt{a^2 + \sigma_\pi^2\nu}$ is strictly positive for $a, \sigma_\pi, \nu > 0$, then $f(\nu)$ is increasing and thus injective. It follows that (5) implies (4). □

The dynamics (3.4.3) and direct calculations imply that the self-financing condition (3.1.5) for an investor with consumption-investment strategy $(c_t^i, \theta_t^i)_{t\geq 0}$ is equivalent to

$$dX_t^i = \Big[-c_t^i + rX_t^i + \theta_t^i(\epsilon_\pi a\bar{\pi} - rC) + \theta_t^i D_t\,(1 - \epsilon_D(k + r)) + \theta_t^i\hat{\pi}_t^M\,(-\epsilon_\pi(a + r) - \epsilon_\pi o_M\nu) + \\ + \theta_t^i\hat{\pi}_t^i\,(\epsilon_D + \epsilon_\pi o_M\nu)\Big]dt + \theta_t^i(\epsilon_D\sigma_D + \epsilon_\pi o_M\sigma_D^{-1})dB_t^{iD} + \theta_t^i o_M\epsilon_\pi\epsilon_i\sigma_i dB_t^i + \theta_t^i o_M\epsilon_\pi\epsilon_{i\perp}dB_t^{i\perp}. \tag{3.4.6}$$

The following theorem proves the existence of a solution of the HJB equation and thus a candidate value function.

Theorem 3.4.1. *Fix $\epsilon_D^0 \in \mathbb{B}, \epsilon_\pi^0 \neq 0, C^0 \in \mathbb{R},$ define*

$$\delta^i = (\delta^i_{DD}, \delta^i_{MM}, \delta_{ii}, \delta^i_{DM}, \delta_{Di}, \delta_{Mi}, \delta^i_D, \delta^i_M, \delta_i, \delta^i_0)$$

and let $\bar{\delta}$ be the function in Definition D.0.1. For every $i \in \{1, \dots, n\}$, there exist

(i) $U(\epsilon_D^0, \epsilon_\pi^0, C^0) \subseteq \mathbb{R}^4 \times \mathbb{R}^{10}$ open neighbourhood of $\left(0, \epsilon_D^0, \epsilon_\pi^0, C^0, \bar{\delta}(\epsilon_D^0, \epsilon_\pi^0, C^0)\right)$;

(ii) $W(\epsilon_D^0, \epsilon_\pi^0, C^0) \subseteq \mathbb{R}^4$ open neighbourhood of $(0, \epsilon_D^0, \epsilon_\pi^0, C^0)$;

such that

(I) for every $(\sigma_\pi, \epsilon_D, \epsilon_\pi, C) \in W(\epsilon_D^0, \epsilon_\pi^0, C^0)$, there exists a unique δ^i such that $(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) \in U(\epsilon_D^0, \epsilon_\pi^0, C^0)$ and the function

$$V^i(x, D, \hat{\pi}^M, \hat{\pi}^i) = -\frac{1}{r\alpha_i} \exp\Big(- r\alpha_i x + \delta^i_{DD} D^2 + \delta^i_{MM} (\hat{\pi}^M)^2 + \delta_{ii} (\hat{\pi}^i)^2 + \\ + \delta^i_{DM} D\hat{\pi}^M + \delta_{Di} D\hat{\pi}^i + \delta_{Mi} \hat{\pi}^M \hat{\pi}^i + \delta^i_D D + \delta^i_M \hat{\pi}^M + \delta_i \hat{\pi}^i + \delta^i_0 \Big) \quad (3.4.7)$$

solves the Hamilton Jacobi Bellman equation of the $i-$th investor

$$0 = -\frac{V^i_x}{\alpha_i} - \beta V^i + V^i_x \Big[\Big(\frac{\log V^i_x}{\alpha_i} + rx \Big) + \theta^{i*} (\epsilon_\pi a\bar{\pi} - rC) + \theta^{i*} D(1 - \epsilon_D(k + r)) + \\ + \theta^{i*} \hat{\pi}^M (-\epsilon_\pi (a+r) - \epsilon_\pi o_M \nu) + \theta^{i*} \hat{\pi}^i (\epsilon_D + \epsilon_\pi o_M \nu) \Big] + V^i_D (\hat{\pi}^i - kD) + V^i_{\hat{\pi}^M} (a\bar{\pi} - a\hat{\pi}^M + \\ + o_M \nu \hat{\pi}^i - o_M \nu \hat{\pi}^M) + V^i_{\hat{\pi}^i} (a\bar{\pi} - a\hat{\pi}^i) + \frac{1}{2} (\theta^{i*})^2 V^i_{xx} (\nu \epsilon_\pi^2 o_M^2 + \epsilon_D^2 \sigma_D^2 + 2\epsilon_D \epsilon_\pi o_M) + \frac{1}{2} V^i_{DD} \sigma_D^2 + \\ + V^i_{\hat{\pi}^M \hat{\pi}^M} o_M^2 \nu + \frac{1}{2} V^i_{\hat{\pi}^i \hat{\pi}^i} o_i^2 \nu_i + V^i_{xD} \theta^{i*} (\epsilon_D \sigma_D^2 + \epsilon_\pi o_M) + V^i_{x\hat{\pi}^M} \theta^{i*} o_M (\epsilon_D + \epsilon_\pi o_M \nu) + \\ + V^i_{x\hat{\pi}^i} \theta^{i*} o_i (\epsilon_D + \epsilon_\pi o_M \nu) + V^i_{D\hat{\pi}^M} o_M + V^i_{D\hat{\pi}^i} o_i + V^i_{\hat{\pi}^M \hat{\pi}^i} o_M o_i \nu, \quad (3.4.8)$$

where

$$\theta^{i*} = -\frac{1}{V^i_{xx} (\nu \epsilon_\pi^2 o_M^2 + \epsilon_D^2 \sigma_D^2 + 2\epsilon_D \epsilon_\pi o_M)} \Big\{ V^i_x \Big[(\epsilon_\pi a\bar{\pi} - rC) + D(1 - \epsilon_D(k + r)) + \\ + \hat{\pi}^M (-\epsilon_\pi (a+r) - \epsilon_\pi o_M \nu) + \hat{\pi}^i (\epsilon_D + \epsilon_\pi o_M \nu) \Big] + V^i_{xD} (\epsilon_D \sigma_D^2 + \epsilon_\pi o_M) + V^i_{x\hat{\pi}^M} o_M (\epsilon_D + \epsilon_\pi o_M \nu) + \\ + V^i_{x\hat{\pi}^i} o_i (\epsilon_D + \epsilon_\pi o_M \nu) \Big\}.$$

(II) If this δ^i is defined to be $g^i(\sigma_\pi, \epsilon_D, \epsilon_\pi, C)$, then $g^i \in \mathscr{C}^1(W, U)$ and $g^i(0, \epsilon_D^0, \epsilon_\pi^0, C^0) = \bar{\delta}(\epsilon_D^0, \epsilon_\pi^0, C^0)$.

Proof. Inserting (3.4.7) into (3.4.8) makes the HJB equation an algebraic equation of second order. For every $i \in \{1, \dots, n\}$, the function $V^i(\cdot)$ solves the HJB equation if and only if

$$f^i(\sigma_\pi, \delta^i_{DD}, \delta^i_{MM}, \delta_{ii}, \delta^i_{DM}, \delta_{Di}, \delta_{Mi}, \delta^i_D, \delta^i_M, \delta_i, \delta^i_0) = 0,$$

where f^i is defined in (D.0.2). Lemma D.0.1 concludes the proof. □

The following are technical results for the solution of the consumption-investment problem.

Lemma 3.4.3. For every $\eta_0, \eta_1, \eta_2, \eta_3 \in \mathbb{R}$, there exist constants $\bar{\mu}, \bar{\sigma} > 0$ independent by t such that, for every $t \geq 0$,

$$|E[\eta_3 D_t + \eta_2 \hat{\pi}^M_t + \eta_1 \hat{\pi}^i_t + \eta_0]| \leq \bar{\mu}, \qquad \mathrm{Var}[\eta_3 D_t + \eta_2 \hat{\pi}^M_t + \eta_1 \hat{\pi}^i_t + \eta_0] \leq \bar{\sigma}^2.$$

Proof. In view of the Cauchy-Swartz inequality, it is enough to prove that the mean and the variance of the processes $(D_t)_{t\geq 0}, (\hat{\pi}^M_t)_{t\geq 0}, (\hat{\pi}^i_t)_{t\geq 0}$ are bounded from above and from below. Apply Itô's formula to $e^{at}\hat{\pi}^i_t, e^{kt}D_t$ and to $e^{kt}\hat{\pi}^i_t$ to get

$$\hat{\pi}^i_t = e^{-a(t-s)}\hat{\pi}^i_s + \bar{\pi}(1 - e^{-a(t-s)}) + o_i\sigma_D^{-1}e^{-at}\int_s^t e^{au}dB^{iD}_u + \\ + o_i\sigma_i^{-1}e^{-at}\int_s^t e^{au}dB^i_u + o_i\sigma_{i\perp}^{-1}e^{-at}\int_s^t e^{au}dB^{i\perp}_u$$

and

$$D_t = D_s e^{-k(t-s)} + \frac{e^{-a(t-s)} - e^{-k(t-s)}}{k-a}\hat{\pi}^i_s + \frac{\bar{\pi}}{k-a}\left[\left(1 - e^{-a(t-s)}\right) - \frac{a}{k}\left(1 - e^{-k(t-s)}\right)\right] + \\ + \frac{1}{k-a}\int_s^t \left[o_i\sigma_D^{-1}e^{-a(t-u)} - o_i\sigma_D^{-1}e^{-k(t-u)} + (k-a)\sigma_D e^{-k(t-u)}\right] dB^{iD}_u + \\ + \frac{o_i\sigma_i^{-1}}{k-a}\int_s^t \left[e^{-a(t-u)} - e^{-k(t-u)}\right] dB^i_u + \frac{o_i\sigma_{i\perp}^{-1}}{k-a}\int_s^t \left[e^{-a(t-u)} - e^{-k(t-u)}\right] dB^{i\perp}_u.$$

Defining $\lambda = a + o_M\nu$ and using the product rule on $e^{\lambda t}\hat{\pi}^M_t$ it follows that

$$\hat{\pi}^M_t = e^{-\lambda(t-s)}\hat{\pi}^M_s + \left(e^{-a(t-s)} - e^{-\lambda(t-s)}\right)\hat{\pi}^i_s + \bar{\pi}\left(1 - e^{-a(t-s)}\right) + \int_s^t \Big[o_M\sigma_D^{-1}e^{-\lambda(t-u)} + \\ + o_i\sigma_D^{-1}\left(e^{-a(t-u)} - e^{-\lambda(t-u)}\right)\Big]dB^{iD}_u + \int_s^t \left[o_M\epsilon_i\sigma_i e^{-\lambda(t-u)} + o_i\sigma_i^{-1}\left(e^{-a(t-u)} - e^{-\lambda(t-u)}\right)\right] dB^i_u + \\ + \int_s^t \left[o_M\epsilon_{i\perp}e^{-\lambda(t-u)} + o_i\sigma_{i\perp}^{-1}\left(e^{-a(t-u)} - e^{-\lambda(t-u)}\right)\right] dB^{i\perp}_u.$$

Mean and variance are a direct calculations and are uniformly bounded because all the exponential functions in the processes above have negative exponent. □

The value of the constants $\Delta^i, \iota^i, \rho^i$ will be set later in Definition 3.4.1

Lemma 3.4.4. Define $(v^i_t)_{t\geq 0} = (D_t, \hat{\pi}^M_t, \hat{\pi}^i_t, 1)^T_{t\geq 0}$ and fix

$$\Delta^i = (\Delta^i_D, \Delta^i_M, \Delta_i, \Delta^i_0)^T \in \mathbb{R}^4, \quad \iota^i = (\iota_i, \iota^i_M, \iota_i, \iota^i_0)^T \in \mathbb{R}^4, \quad \rho^i = (\rho^i_D, \rho^i_M, \rho_i, \rho^i_0)^T \in \mathbb{R}^4.$$

The process

$$H_t := \exp\Bigg(\int_0^t \Delta^i \cdot v_u^i dB_u^{iD} + \iota^i \cdot v_u^i dB_u^i + \rho^i \cdot v_u^i dB_u^{i\perp} + \\ -\frac{1}{2}\int_0^t \Big[(\Delta^i \cdot v_u^i)^2 + (\iota^i \cdot v_u^i)^2 + (\rho^i \cdot v_u^i)^2\Big] du\Bigg)$$

is a $\mathbb{P}-$martingale.

Proof. Define $Y_t = \Delta^i \cdot v_t^i, Z_t = \iota^i \cdot v_t^i, K_t = \rho^i \cdot v_t^i$ and recall Novikov's condition [22, Corollary 5.13 and 5.14], which ensures that H_t is a martingale:

(A) $\mathbb{P}\left[\int_0^t Y_u^2 du < +\infty\right] = \mathbb{P}\left[\int_0^t Z_u^2 du < +\infty\right] = \mathbb{P}\left[\int_0^t K_u^2 du < +\infty\right] = 1$;

(B) there exists a sequence $(t_m)_{m\in\mathbb{N}} \subset \mathbb{R}$ increasing to $+\infty$, such that, for every $m \in \mathbb{N}$,

$$E\left[\exp\left(\int_{t_{m-1}}^{t_m} \frac{1}{2}(Y_u^2 + Z_u^2 + K_u^2) du\right)\right] < +\infty.$$

The processes $(Y_t)_{t\geq 0}, (Z_t)_{t\geq 0}$ and $(K_t)_{t\geq 0}$ are $\mathbb{P}-$a.s. continuous, hence (A) is true. By Jensen's inequality [28, Theorem 1.8.1], for every $t, \epsilon \geq 0$,

$$\exp\left(\int_t^{t+\epsilon} \frac{1}{2}(Y_u^2 + Z_u^2 + K_u^2) du\right) \leq \frac{1}{\epsilon}\int_t^{t+\epsilon} \exp\left(\frac{\epsilon}{2}(Y_u^2 + Z_u^2 + K_u^2)\right) du.$$

In addition, by Fubini's Theorem

$$E\left[\frac{1}{\epsilon}\int_t^{t+\epsilon} \exp\left(\frac{\epsilon}{2}(Y_u^2 + Z_u^2 + K_u^2)\right) du\right] = \frac{1}{\epsilon}\int_t^{t+\epsilon} E\left[\exp\left(\frac{\epsilon}{2}(Y_u^2 + Z_u^2 + K_u^2)\right)\right] du.$$

Young's inequality [24, Lemma 7.15] yields

$$E\left[\exp\left(\int_t^{t+\epsilon} \frac{1}{2}(Y_u^2 + Z_u^2 + K_u^2) du\right)\right] \leq \frac{1}{\epsilon}\int_t^{t+\epsilon} E\left[\exp\left(2\epsilon Y_u^2\right)\right] du + \\ + \frac{1}{\epsilon}\int_t^{t+\epsilon} E\left[\exp\left(2\epsilon Z_u^2\right)\right] du + \frac{1}{\epsilon}\int_t^{t+\epsilon} E\left[\exp\left(2\epsilon K_u^2\right)\right] du. \quad (3.4.9)$$

Suppose $(\Delta_D^i, \Delta_M^i, \Delta_i) \neq (0, 0, 0)$ and define $\mu_u = E[Y_u]$ and $\sigma_u^2 = \text{Var}[Y_u]$. In view of Lemma 3.4.3, there exist constants $\bar{\mu}$ and $\bar{\sigma}^2$ such that for every $0 \leq u \leq t$

$$|\mu_u| \leq \bar{\mu}, \qquad \sigma_u^2 \leq \bar{\sigma}^2, \quad (3.4.10)$$

for every $u \geq 0$. For every $u \geq 0$, Y_u is a normally distributed random variable, and in particular

$$E\left[\exp\left(2\epsilon Y_u^2\right)\right] = \frac{\exp\left(\frac{2\mu_u^2\epsilon}{1-4\sigma_u^2\epsilon}\right)}{\sqrt{1-4\sigma_u^2\epsilon}}, \qquad \text{if} \qquad 4\sigma_u^2\epsilon \leq 1.$$

Since $\sigma_u^2 \le \bar{\sigma}^2$, then any $\epsilon < \frac{1}{4}\bar{\sigma}^{-2}$ satisfies $4\sigma_u^2\epsilon < 1$ because

$$4\sigma_u^2\epsilon \le 4\bar{\sigma}^2\epsilon < 1. \tag{3.4.11}$$

Fix $\epsilon < \frac{1}{4}\bar{\sigma}^{-2}$; if we prove that $E\Big[\exp\Big(\epsilon Y_u^2\Big)\Big]$ is a continuous function, uniformly bounded in t on the interval $[t, t+\epsilon]$, for the ϵ chosen above, then its integral is finite and it is enough to define the sequence $t_m := m\epsilon$. Equation (3.4.11) implies $1 - 4\sigma_u^2\epsilon \ge 1 - 4\bar{\sigma}^2\epsilon$, and both terms are between 0 and 1 because of the choice of ϵ. Thus, defining $\kappa_\epsilon = \frac{1}{1-4\bar{\sigma}^2\epsilon}$, it follows that

$$\frac{1}{1-4\sigma_u^2\epsilon} \le \kappa_\epsilon \qquad \text{and} \qquad \frac{1}{\sqrt{1-4\sigma_u^2\epsilon}} \le \kappa_\epsilon.$$

As a consequence

$$E\Big[\exp\Big(2\epsilon Y_u^2\Big)\Big] \le \kappa_\epsilon \exp\Big(2\kappa_\epsilon\epsilon\bar{\mu}^2\Big) < +\infty.$$

$E\Big[\exp\Big(2\epsilon Y_u^2\Big)\Big]$ is a continuous and bounded function on the interval $[t, t+\epsilon]$ and so for every $\epsilon > 0$ and every $t \ge 0$

$$E\left[\frac{1}{\epsilon}\int_t^{t+\epsilon}\exp\Big(2\epsilon Y_u^2\Big)du\right] = \frac{1}{\epsilon}\int_t^{t+\epsilon}E\Big[\exp\Big(2\epsilon Y_u^2\Big)\Big]du < +\infty.$$

The same reasoning shows also that

$$E\left[\frac{1}{\epsilon}\int_t^{t+\epsilon}\exp\Big(2\epsilon Z_u^2\Big)du\right], E\left[\frac{1}{\epsilon}\int_t^{t+\epsilon}\exp\Big(2\epsilon K_u^2\Big)du\right] < +\infty$$

and (B) follows from (3.4.9). □

Definition 3.4.1. We introduce the following constants,

$$\begin{aligned}
A_{MD} &= o_M\sigma_D^{-1}\Delta_D^i + o_M\epsilon_i\sigma_i\iota_D^i + o_M\epsilon_{i\perp}\rho_D^i, & A_{DD} &= \sigma_D\Delta_D^i - k,\\
A_{MM} &= o_M\sigma_D^{-1}\Delta_M^i + o_M\epsilon_i\sigma_i\iota_M^i + o_M\epsilon_{i\perp}\rho_M^i - a - o_M\nu, & A_{DM} &= \sigma_D\Delta_M^i,\\
A_{Mi} &= o_M\sigma_D^{-1}\Delta_i + o_M\epsilon_i\sigma_i\iota_i + o_M\epsilon_{i\perp}\rho_i + o_M\nu, & A_{Di} &= \sigma_D\Delta_i + 1,\\
b_M &= o_M\sigma_D^{-1}\Delta_0^i + o_M\epsilon_i\sigma_i\iota_0^i + o_M\epsilon_{i\perp}\rho_0^i + a\bar{\pi}, & b_D &= \sigma_D\Delta_0^i,
\end{aligned}$$

$$\begin{aligned}
A_{iD} &= o_i\sigma_D^{-1}\Delta_D^i + o_i\sigma_i^{-1}\iota_D^i + o_i\sigma_{i\perp}^{-1}\rho_D^i,\\
A_{iM} &= o_i\sigma_D^{-1}\Delta_M^i + o_i\sigma_i^{-1}\iota_M^i + o_i\sigma_{i\perp}^{-1}\rho_M^i,\\
A_{ii} &= o_i\sigma_D^{-1}\Delta_i + o_i\sigma_i^{-1}\iota_i + o_i\sigma_{i\perp}^{-1}\rho_i - a,\\
b_i &= o_i\sigma_D^{-1}\Delta_0^i + o_i\sigma_i^{-1}\iota_0^i + o_i\sigma_{i\perp}^{-1}\rho_0^i + a\bar{\pi},
\end{aligned}$$

where

$$\Delta^i = \begin{pmatrix} \Delta^i_D \\ \Delta^i_M \\ \Delta^i_i \\ \Delta^i_0 \end{pmatrix} = o_i \begin{pmatrix} \delta_{Di} \\ \delta_{Mi} \\ 2\delta_{ii} \\ \delta_i \end{pmatrix} \sigma_D^{-1} + \frac{o_M}{M} \begin{pmatrix} M\delta^i_{DM} - M^i_D r\epsilon_\pi \\ 2M\delta^i_{MM} - M^i_M r\epsilon_\pi \\ M\delta_{Mi} - M_i r\epsilon_\pi \\ M\delta^i_M - M^i_0 r\epsilon_\pi \end{pmatrix} \sigma_D^{-1} + \frac{1}{M} \begin{pmatrix} 2\delta^i_{DD}M - M^i_D r\epsilon_D \\ M\delta^i_{DM} - M^i_M r\epsilon_D \\ M\delta_{Di} - M_i r\epsilon_D \\ M\delta^i_D - M^i_0 r\epsilon_D \end{pmatrix} \sigma_D,$$

$$\iota^i = \begin{pmatrix} \iota^i_D \\ \iota^i_M \\ \iota_i \\ \iota^i_0 \end{pmatrix} = o_i \begin{pmatrix} \delta_{Di} \\ \delta_{Mi} \\ 2\delta_{ii} \\ \delta_i \end{pmatrix} \sigma_i^{-1} + \frac{o_M}{M} \begin{pmatrix} M\delta^i_{DM} - M^i_D r\epsilon_\pi \\ 2M\delta^i_{MM} - M^i_M r\epsilon_\pi \\ M\delta_{Mi} - M_i r\epsilon_\pi \\ M\delta^i_M - M^i_0 r\epsilon_\pi \end{pmatrix} \epsilon_i\sigma_i,$$

$$\rho^i = \begin{pmatrix} \rho^i_D \\ \rho_M \\ \rho_i \\ \rho_0 \end{pmatrix} = o_i \begin{pmatrix} \delta_{Di} \\ \delta_{Mi} \\ 2\delta_{ii} \\ \delta_i \end{pmatrix} \sigma_{i\perp}^{-1} + \frac{o_M}{M} \begin{pmatrix} M\delta^i_{DM} - M^i_D r\epsilon_\pi \\ 2M\delta^i_{MM} - M^i_M r\epsilon_\pi \\ M\delta_{Mi} - M_i r\epsilon_\pi \\ M\delta^i_M - M^i_0 r\epsilon_\pi \end{pmatrix} \epsilon_{i\perp},$$

and

$$\begin{aligned} M &= r\left[\epsilon_D(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + \epsilon_\pi o_M(\epsilon_D + \epsilon_\pi o_M \nu)\right], \\ M^i_D &= 1 - \epsilon_D(k+r) + 2\delta^i_{DD}(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + (\delta_{Di}o_i + \delta^i_{DM}o_M)(\epsilon_D + \epsilon_\pi o_M\nu), \\ M^i_M &= -\epsilon_\pi(a + r + o_M\nu) + \delta^i_{DM}(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + (o_i\delta_{Mi} + 2o_M\delta^i_{MM})(\epsilon_D + \epsilon_\pi o_M\nu), \\ M_i &= \delta_{Di}(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + (o_M\delta_{Mi} + 2o_i\delta_{ii} + 1)(\epsilon_D + \epsilon_\pi o_M\nu), \\ M^i_0 &= (a\bar{\pi}\epsilon_\pi - rC) + \delta^i_D(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + (\delta^i_M o_M + \delta_i o_i)(\epsilon_D + \epsilon_\pi o_M\nu). \end{aligned}$$

The constants $\delta^i_{DD}, \delta^i_{MM}, \delta_{ii}, \delta^i_{DM}, \delta_{Di}, \delta_{Mi}, \delta^i_D, \delta^i_M, \delta_i, \delta^i_0$ are those of Theorem 3.4.1 and the process $(v^i_t)_{t\geq 0} = (D_t, \hat{\pi}^M_t, \hat{\pi}^i_t, 1)^T_{t\geq 0}$.

Corollary 3.4.1. *The process $(\mathcal{E}^i_t)_{t\geq 0} = (e^{rt}N^i_t)_{t\geq 0}$, in (3.1.6), is a $\mathbb{P}-$martingale.*

As $(\mathcal{E}^i_t)_{t\geq 0}$ is a $\mathbb{P}-$martingale, Girsanov's Theorem [22, Theorem 5.1] holds. In particular, $(\mathcal{E}^i_t)_{t\geq 0}$ defines a probability measure $\bar{\mathbb{P}}^i := \bar{\mathbb{P}}^{(\Delta^i,\iota^i,\rho^i)}$, such that $\mathcal{E}^i = d\bar{\mathbb{P}}^i/d\mathbb{P}^i$. We denote by $\bar{E}^i[\cdot]$ and $\bar{\mathrm{Var}}^i[\cdot]$ the conditional expectation and variance under the measure $\bar{\mathbb{P}}^i$. Any equality or inequality between random variables is understood $\mathbb{P}$ and $\bar{\mathbb{P}}^i-$almost surely. The process

$$(\bar{B}^{iD}_t, \bar{B}^i_t, \bar{B}^{i\perp}_t)_{t\geq 0} = \left(B^{iD}_t - \int_0^t \Delta^i \cdot v_u du, B^i_t - \int_0^t \iota^i \cdot v_u du, B^{i\perp}_t - \int_0^t \rho^i \cdot v_u du\right)_{t\geq 0} \tag{3.4.12}$$

is a $\bar{\mathbb{P}}^i-$Brownian motion and furthermore Bayes' formula [22, Lemma 5.3] applies: for every $\mathcal{F}^i_t-$measurable random variable X satisfying $\bar{E}^i[|X|] < +\infty$ and for every

$0 \leq s \leq t$

$$\bar{E}^i[X|\mathcal{F}_s^i] = \frac{1}{\mathcal{E}_s^i} E[X\mathcal{E}_t^i|\mathcal{F}_s^i].$$

The next lemma describes the processes $(D_t)_{t\geq 0}, (\hat{\pi}_t^M)_{t\geq 0}$, and $(\hat{\pi}_t^i)_{t\geq 0}$ under the new measure $\bar{\mathbb{P}}^i$.

Lemma 3.4.5 (Joint dynamics). Suppose that Assumption 3.1.1 holds. Then there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$, all the following hold. The process $\chi_t := (D_t, \hat{\pi}_t^M, \hat{\pi}_t^i)_{t\geq 0}$ satisfies the stochastic differential equation

$$d\chi_t = (b + A\chi_t)dt + \Sigma d\bar{B}_t; \tag{3.4.13}$$

where

$$A = \begin{pmatrix} A_{DD} & A_{DM} & A_{Di} \\ A_{MD} & A_{MM} & A_{Mi} \\ A_{iD} & A_{iM} & A_{ii} \end{pmatrix}, \quad b = \begin{pmatrix} b_D \\ b_M \\ b_i \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_D & 0 & 0 \\ o_M\sigma_D^{-1} & o_M\epsilon_i\sigma_i & o_M\epsilon_{i\perp} \\ o_i\sigma_D^{-1} & o_i\sigma_i^{-1} & o_i\sigma_{i\perp}^{-1} \end{pmatrix}, \quad \bar{B}_t = \begin{pmatrix} \bar{B}_t^{iD} \\ \bar{B}_t^i \\ \bar{B}_t^{i\perp} \end{pmatrix},$$

and the entries of matrix A and of the vector b are described in Definition 3.4.1. The matrix A is invertible, and the unique solution of (3.4.13) is

$$\chi_t = e^{A(t-s)}\chi_s + A^{-1}(e^{A(t-s)} - I_3)b + e^{At}\int_s^t e^{-Au}\Sigma d\bar{B}_u. \tag{3.4.14}$$

For every $0 \leq s \leq t$, there exists an $\mathcal{F}_s^i-$measurable random variable $\eta_s \geq 0$ and a positive constant η such that

$$\begin{aligned} \|\bar{E}^i\left[\chi_t|\mathcal{F}_s^i\right]\| &\leq \eta_s e^{\|A\|t} + \eta, && (i) \\ \|\bar{\text{Var}}^i\left[\chi_t|\mathcal{F}_s^i\right]\| &\leq \eta e^{(\|A\|+\|A^T\|)t}, && (ii) \\ \|\bar{E}^i[\chi_t|\mathcal{F}_s^i] \otimes \bar{E}^i[\chi_t|\mathcal{F}_s^i]\| &\leq \eta_s e^{(\|A\|+\|A^T\|)t} + \eta, && (iii) \\ \|\bar{E}^i\left[\chi_t \otimes \chi_t|\mathcal{F}_s^i\right]\| &\leq \eta_s e^{(\|A\|+\|A^T\|)t} + \eta. && (iv) \end{aligned} \tag{3.4.15}$$

$$\begin{aligned} \|\bar{E}^i\left[\chi_t\right]\| &\leq \eta e^{\|A\|t} + \eta, && (i) \\ \|\bar{\text{Var}}^i\left[\chi_t\right]\| &\leq \eta e^{(\|A\|+\|A^T\|)t} + \eta, && (ii) \\ \|\bar{E}^i[\chi_t] \otimes \bar{E}_t^i[\chi_t]\| &\leq \eta e^{(\|A\|+\|A^T\|)t} + \eta, && (iii) \\ \|\bar{E}^i\left[\chi_t \otimes \chi_t\right]\| &\leq \eta e^{(\|A\|+\|A^T\|)t} + \eta, && (iv) \\ \bar{E}^i\left[\|\chi_t\|\right] &< +\infty. && (v) \end{aligned} \tag{3.4.16}$$

For every $s \geq 0$ and for every $\eta_0, \ldots, \eta_{11} \in \mathbb{R}$

(a)

$$E\left[N_t^i(C+\epsilon_D D_t+\epsilon_\pi\hat{\pi}_t^M)+\int_s^t N_u^i D_u du\Big|\mathcal{F}_s^i\right]=$$
$$N_s^i\bar{E}^i\left[e^{-r(t-s)}(C+\epsilon_D D_t+\epsilon_\pi\hat{\pi}_t^M)+\int_s^t e^{-r(u-s)}D_u du\Big|\mathcal{F}_s^i\right];\quad(3.4.17)$$

(b)

$$-\infty<\bar{E}^i\Big[\int_s^t \eta_9 D_u^2+\eta_8(\hat{\pi}_u^M)^2+\eta_7(\hat{\pi}_u^i)^2+\eta_6 D_u\hat{\pi}_u^M+\eta_5 D_u\hat{\pi}_u^i+$$
$$+\eta_4\hat{\pi}_u^M\pi_u+\eta_3 D_u+\eta_2\hat{\pi}_u^M+\eta_1\hat{\pi}_u^i+\eta_0 du\Big]<+\infty;$$

(c)

$$\int_s^t(\eta_{11}D_u+\eta_{10}\hat{\pi}_u^M+\eta_9\hat{\pi}_u^i+\eta_8)d\bar{B}_u^{iD}+\int_s^t(\eta_7 D_u+\eta_6\hat{\pi}_u^M+\eta_5\hat{\pi}_u^i+\eta_4)d\bar{B}_u^i+$$
$$+\int_s^t(\eta_3 D_u+\eta_2\hat{\pi}_u^M+\eta_1\hat{\pi}_u^i+\eta_0)d\bar{B}_u^{i\perp}\qquad\text{is }\bar{\mathbb{P}}^i\text{−martingale;}$$

(d) for every $s\geq 0$

$$\lim_{t\to+\infty}e^{-rt}\bar{E}^i\left[\chi_t\Big|\mathcal{F}_s^i\right]=\lim_{t\to+\infty}e^{-rt}\bar{E}^i\left[\chi_t\otimes\chi_t\Big|\mathcal{F}_s^i\right]=$$
$$=\lim_{t\to+\infty}e^{-rt}\bar{E}^i\left[\int_s^t\chi_u du\Big|\mathcal{F}_s^i\right]=\lim_{t\to+\infty}e^{-rt}\bar{E}^i\left[\int_s^t\chi_u\otimes\chi_u du\Big|\mathcal{F}_s^i\right]=0;$$

(e) for every $s\geq 0$

$$\frac{1}{2}\lim_{t\to+\infty}\bar{E}^i\Big[\int_s^t e^{-r(u-s)}\Big((\Delta^i\cdot v_u^i)^2+(\iota^i\cdot v_u^i)^2+(\rho^i\cdot v_u^i)^2\Big)du\Big|\mathcal{F}_s^i\Big]$$
$$=-\delta_{DD}^i D_s^2-\delta_{MM}^i(\hat{\pi}_s^M)^2-\delta_{ii}(\hat{\pi}_s^i)^2-\delta_{DM}^i D_s\hat{\pi}_s^M-\delta_{Di}D_s\hat{\pi}_s^i-\delta_{Mi}\hat{\pi}_s^M\hat{\pi}_s^i+$$
$$-\delta_D^i D_s-\delta_M^i\hat{\pi}_s^M-\delta_i\hat{\pi}_s^i-\delta_0^i-\frac{\beta-r}{r}.$$

Remark 3.4.1. All the above are local results for σ_π in a right neighbourhood of 0.

Proof. (3.4.13) is a direct consequence of (3.4.2), (3.4.3) and (3.4.12). For $\sigma_\pi=0$, the matrix A becomes

$$\begin{pmatrix} r-\frac{1}{\epsilon_D} & \frac{(a+r)\epsilon_\pi}{\epsilon_D} & 0\\ 0 & -a & 0\\ 0 & 0 & -a\end{pmatrix},$$

whose determinant is $\frac{a^2(-1+r\epsilon_D)}{\epsilon_D}\neq 0$. Due to the continuity of the determinant, there exists $\bar{\sigma_\pi}(\epsilon_D,\epsilon_\pi,C)>0$ such that for all $0\leq\sigma_\pi\leq\bar{\sigma_\pi}(\epsilon_D,\epsilon_\pi,C)$, A is invertible. Because A is invertible, the unique solution of (3.4.13) is (3.4.14). Equation (3.4.14), Lemma

A.0.1 and the triangle inequality imply that $\|\bar{E}^i[\chi_u|\mathcal{G}_s]\| \leq \eta_s e^{\|A\|t} + \eta$, $\|\bar{\text{Var}}^i[\chi_u|\mathcal{G}_s]\| \leq \eta e^{(\|A\|+\|A^T\|)t}$ and that $\|\bar{E}^i[\chi_u|\mathcal{G}_s] \otimes \bar{E}^i[\chi_u|\mathcal{G}_s]\| \leq \eta_s e^{(\|A\|+\|A^T\|)t} + \eta$. The definition of the conditional variance yields $\|\bar{E}^i[\chi_u \otimes \chi_u|\mathcal{G}_s]\| \leq \eta_s e^{(\|A\|+\|A^T\|)t} + \eta$. The unconditional inequalities follow similarly.
(a) is true thanks to (3.4.16) (v) and to Fubini's Theorem. Likewise, Fubini's Theorem and (3.4.16), yields to equation (b) and hence (c).
Proof of (d)
We proceed in several steps.
Claim: There exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$ and for every $s \geq 0$,

$$\lim_{t\to+\infty} e^{-rt} e^{A(t-s)} = 0$$

If $\epsilon_D \neq \frac{1}{a+r}$ and $\sigma_\pi = 0$, then

$$A = \begin{pmatrix} r - \frac{1}{\epsilon_D} & \frac{\epsilon_\pi(a+r)}{\epsilon_D} & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}$$

is diagonalizable with two different eigenvalues $-a$ and $r - \frac{1}{\epsilon_D}$. If A is diagonalizable in a right neighbourhood of $\sigma_\pi = 0$, thanks to the continuity of the eigenvalues [2, Remark 3.4] and Lemma A.0.4, there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$

$$e^{-rt} e^{A(t-s)} = e^{-rt} H \begin{pmatrix} e^{\lambda_1(t-s)} & 0 & 0 \\ 0 & e^{\lambda_2(t-s)} & 0 \\ 0 & 0 & e^{\lambda_3(t-s)} \end{pmatrix} H^{-1}, \tag{3.4.18}$$

with

$$\max(Re\{\lambda_1, \lambda_2, \lambda_3, 2\lambda_1, 2\lambda_2, 2\lambda_3, \lambda_1+\lambda_2, \lambda_1+\lambda_3, \lambda_2+\lambda_3\}) < r. \tag{3.4.19}$$

The real parts of all exponentials on every entry are negative, therefore

$$\lim_{t\to+\infty} e^{-rt} e^{A(t-s)} = 0. \tag{3.4.20}$$

Even if A is not diagonalizable in a right neighbourhood of $\sigma_\pi = 0$, Lemma A.0.4 implies that there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$ (3.4.19) and (3.4.20) hold. If $\epsilon_D = \frac{1}{a+r}$ and $\sigma_\pi = 0$, then the only eigenvalue of A is $-a$. Lemma A.0.4 implies implies that there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$ (3.4.19) and (3.4.20) hold.
Claim: $\lim_{t\to+\infty} e^{-rt} \bar{E}^i\left[\int_s^t \chi_u du \middle| \mathcal{F}_s^i\right] = 0 \iff \lim_{t\to+\infty} e^{-rt} \bar{E}^i\left[\chi_t|\mathcal{F}_s^i\right] = 0$

Writing the explicit dynamics for (3.4.13), reordering and multiplying by e^{-rt} yields

$$e^{-rt}\bar{E}_t^i\left[\int_s^t \chi_u du \Big| \mathcal{F}_s^i\right] = A^{-1}e^{-rt}\bar{E}_t^i\left[\chi_t|\mathcal{F}_s^i\right] - e^{-rt}A^{-1}\chi_s - e^{-rt}A^{-1}b(t-s).$$

Taking $\lim_{t\to+\infty}$ of both sides, we get

$$\lim_{t\to+\infty} e^{-rt}\bar{E}_t^i\left[\int_s^t \chi_u du \Big| \mathcal{F}_s^i\right] = 0 \text{ if and only if } \lim_{t\to+\infty} e^{-rt}\bar{E}_t^i\left[\chi_t|\mathcal{F}_s^i\right] = 0.$$

$$\text{Claim: } \lim_{t\to+\infty} e^{-rt}\bar{E}^i\left[\chi_t|\mathcal{F}_s^i\right] = 0$$

Apply the conditional expectation to (3.4.14) and multiply by e^{-rt} to get

$$e^{-rt}\bar{E}_t^i\left[\chi_t|\mathcal{F}_s^i\right] = e^{-rt}e^{A(t-s)}\chi_s + A^{-1}(e^{-rt}e^{A(t-s)} - I_2e^{-rt})b.$$

From (3.4.20) it follows that $\lim_{t\to+\infty} e^{-rt}\bar{E}_t^i\left[\chi_t|\mathcal{F}_s^i\right] = 0$.

$$\text{Claim: } \lim_{t\to+\infty} e^{-rt}\bar{E}^i\left[\int_s^t \chi_u \otimes \chi_u \Big| \mathcal{G}_s\right] du = 0$$

Integrating the definition of the conditional variance [5, Definition 11.23], and multiplying both sides by e^{-rt},

$$e^{-rt}\int_s^t \bar{E}^i\left[\chi_u \otimes \chi_u|\mathcal{F}_s^i\right] du = e^{-rt}\int_s^t \bar{\mathrm{Var}}_t^i[\chi_u|\mathcal{F}_s^i]du + e^{-rt}\int_s^t \bar{E}^i[\chi_u|\mathcal{F}_s^i] \otimes \bar{E}^i[\chi_u|\mathcal{F}_s^i]du. \tag{3.4.21}$$

By dint of (3.4.14) we get

$$\bar{\mathrm{Var}}^i[\chi_t|\mathcal{F}_s^i] = \int_s^t e^{A(t-u)}\Sigma\Sigma^T\left(e^{A(t-u)}\right)^T du. \tag{3.4.22}$$

A is triangularizable in $\mathbb{C}$, therefore there exist an invertible matrix H and a nilpotent matrix N (cfr. [20, 3.2.7 and 3.2.8 page 181], [16, Proposition A.6]) such that

$$\begin{aligned} e^{A(t-u)}\Sigma\Sigma^T\left(e^{A(t-u)}\right)^T = H &\begin{pmatrix} e^{\lambda_1(t-s)} & 0 & 0 \\ 0 & e^{\lambda_2(t-s)} & 0 \\ 0 & 0 & e^{\lambda_3(t-s)} \end{pmatrix} H^{-1}\left(\sum_{h=0}^3 \frac{N^h}{h!}(t-u)^h\right) \\ \cdot\Sigma\Sigma^T\cdot &\left[H\begin{pmatrix} e^{\lambda_1(t-s)} & 0 & 0 \\ 0 & e^{\lambda_2(t-s)} & 0 \\ 0 & 0 & e^{\lambda_3(t-s)} \end{pmatrix} H^{-1}\left(\sum_{h=0}^3 \frac{N^h}{h!}(t-u)^h\right)\right]^T. \end{aligned}$$

Due to (3.4.19), each entry of $e^{A(t-u)}\Sigma\Sigma^T\left(e^{A(t-u)}\right)^T$ is a linear combination of powers of

t smaller than 6, multiplied by exponentials with real part of the coefficient in t smaller than r. As a consequence, the same holds for $\int_s^t \bar{\text{Var}}^i[\chi_u|\mathcal{F}_s^i]du$, and

$$\lim_{t\to+\infty} e^{-rt}\int_s^t \bar{\text{Var}}^i[\chi_u|\mathcal{F}_s^i]du = 0.$$

In the same way $\lim_{t\to+\infty} e^{-rt}\int_s^t \bar{E}^i[\chi_u|\mathcal{F}_s^i]\otimes\bar{E}^i[\chi_u|\mathcal{F}_s^i]du = 0$.
The proof of $\lim_{t\to+\infty} e^{-rt}\bar{E}^i\left[\chi_t\otimes\chi_t\middle|\mathcal{F}_s^i\right]du = 0$ is the same as that of $\lim_{t\to+\infty} e^{-rt}\bar{E}^i\left[\int_s^t\chi_u\otimes\chi_u\middle|\mathcal{F}_s^i\right]du = 0$, skipping the step of integration in (3.4.21). It follows that (d) holds.
Proof of (e):

$$\frac{1}{2}\lim_{t\to+\infty}\bar{E}^i\Big[\int_s^t e^{-r(u-s)}\left((\Delta^i\cdot v_u^i)^2+(\iota^i\cdot v_u^i)^2+(\rho^i\cdot v_u^i)^2\right)du\Big|\mathcal{F}_s^i\Big]$$
$$= -\delta_{DD}^i D_s^2-\delta_{MM}^i(\hat{\pi}_s^M)^2-\delta_{ii}(\hat{\pi}_s^i)^2-\delta_{DM}^i D_s\hat{\pi}_s^M-\delta_{Di}D_s\hat{\pi}_s^i-\delta_{Mi}\hat{\pi}_s^M\hat{\pi}_s^i+$$
$$-\delta_D^i D_s-\delta_M^i\hat{\pi}_s^M-\delta_i\hat{\pi}_s^i-\delta_0^i-\frac{\beta-r}{r}.$$

Let $0\le s\le t$ and define $v^i=(D,\hat{\pi}^M,\hat{\pi}^i)^T$. The function $W:[0,t]\times\mathbb{R}^3\to\mathbb{R}$

$$W(s,D,\hat{\pi}^M,\hat{\pi}^i) = -\delta_{DD}^i D_s^2-\delta_{MM}^i(\hat{\pi}_s^M)^2-\delta_{ii}(\hat{\pi}_s^i)^2-\delta_{DM}^i D_s\hat{\pi}_s^M-\delta_{Di}D_s\hat{\pi}_s^i+$$
$$-\delta_{Mi}\hat{\pi}_s^M\hat{\pi}_s^i-\delta_D^i D_s-\delta_M^i\hat{\pi}_s^M-\delta_i\hat{\pi}_s^i-\delta_0^i-\frac{\beta-r}{r},$$

is the solution of the Cauchy problem in $[0,t]$

$$0 = W_s+\left(\nabla_{(D,\hat{\pi}^M,\hat{\pi}^i)}W\right)\cdot\left(A(D,\hat{\pi}^M,\hat{\pi}^i)^T+b\right)+\frac{1}{2}tr\left(\left(\text{He}_{(D,\hat{\pi}^M,\hat{\pi}^i)}W\right)\Sigma\Sigma^T\right)-rW+$$
$$+\frac{1}{2}\left((\Delta^i\cdot v^i)^2+(\iota^i\cdot v^i)^2+(\rho^i\cdot v^i)^2\right), \tag{3.4.23}$$

$$W(t,D,\hat{\pi}^M,\hat{\pi}^i) = -\delta_{DD}^i(D_s)^2-\delta_{MM}^i(\hat{\pi}_s^M)^2-\delta_{ii}(\hat{\pi}_s^i)^2-\delta_{DM}^i D_s\hat{\pi}_s^M-\delta_{Di}D_s\hat{\pi}_s^i+$$
$$-\delta_{Mi}\hat{\pi}_s^M\hat{\pi}_s^i-\delta_D^i D_s-\delta_M^i\hat{\pi}_s^M-\delta_i\hat{\pi}_s^i-\delta_0^i-\frac{\beta-r}{r}.$$

In view of [22, Theorem 7.6],

$$W(s,D_s,\hat{\pi}_s^M,\hat{\pi}_s^i) = \bar{E}_t^i\Big[\int_s^t e^{-r(u-s)}\frac{1}{2}\left((\Delta^i\cdot v_u^i)^2+(\iota^i\cdot v_u^i)^2+(\rho^i\cdot v_u^i)^2\right)du+$$
$$+e^{-r(t-s)}\Big(-\delta_{DD}^i D_t^2-\delta_{MM}^i(\hat{\pi}_t^M)^2-\delta_{ii}(\hat{\pi}_t^i)^2-\delta_{DM}^i D_t\hat{\pi}_t^M-\delta_{Di}D_t\hat{\pi}_t^i+$$
$$-\delta_{Mi}\hat{\pi}_t^M\hat{\pi}_t^i-\delta_D^i D_t-\delta_M^i\hat{\pi}_t^M-\delta_i\hat{\pi}_t^i-\delta_0^i-\frac{\beta-r}{r}\Big)\Big|\mathcal{F}_s^i\Big].$$

Since W does not depend by t, for every $t > 0$

$$\bar{E}_t^i\Big[\int_s^t e^{-r(u-s)}\frac{1}{2}\left((\Delta^i\cdot v_u^i)^2+(\iota^i\cdot v_u^i)^2+(\rho^i\cdot v_u^i)^2\right)du+e^{-r(t-s)}\Big(-\delta_{DD}^i D_t^2-\delta_{MM}^i(\hat{\pi}_t^M)^2+$$
$$-\delta_{ii}(\hat{\pi}_t^i)^2-\delta_{DM}^i D_t\hat{\pi}_t^M-\delta_{Di}D_t\hat{\pi}_t^i+-\delta_{Mi}\hat{\pi}_t^M\hat{\pi}_t^i-\delta_D^i D_t-\delta_M^i\hat{\pi}_t^M-\delta_i\hat{\pi}_t^i-\delta_0^i-\frac{\beta-r}{r}\Big)\Big|\mathcal{G}_s\Big]$$
$$=-\delta_{DD}^i D_s^2-\delta_{MM}^i(\hat{\pi}_s^M)^2-\delta_{ii}(\hat{\pi}_s^i)^2-\delta_{DM}^i D_s\hat{\pi}_s^M-\delta_{Di}D_s\hat{\pi}_s^i-\delta_{Mi}\hat{\pi}_s^M\hat{\pi}_s^i+$$
$$-\delta_D^i D_s-\delta_M^i\hat{\pi}_s^M-\delta_i\hat{\pi}_s^i-\delta_0^i-\frac{\beta-r}{r}.$$

Take $\lim_{t\to+\infty}$ of both sides and apply (d) to conclude. □

Remark 3.4.2. If $\epsilon_D = 1/r$ the matrix A is not invertible for $\sigma_\pi = 0$ and (3.4.14) no longer holds. In this case we conjecture the existence of a solution for (3.4.13) but we would need a different way of proving the result since the direct calculations become more difficult.

With the properties of $(\chi_t)_{t\geq 0}$ shown in Lemma 3.4.5, we prove that $(N_t^i)_{t\geq 0}$ of (3.1.6) is a stochastic discount factor.

Theorem 3.4.2. *Under Assumption 3.1.1 the process $(N_t^i)_{t\geq 0}$ of (3.1.6) is a normalized stochastic discount factor. The dynamics of the process $(\log\mathcal{E}_t^i)_{t\geq 0}$ can be written as*

$$\log\mathcal{E}_t^i=\log\mathcal{E}_s^i-\frac{1}{2}\int_s^t\left[(\Delta^i\cdot v_u^i)^2+(\iota^i\cdot v_u^i)^2+(\rho^i\cdot v_u^i)^2\right]du+$$
$$+\int_s^t\Delta^i\cdot v_u^i dB_u^{iD}+\int_s^t\iota^i\cdot v_u^i dB_u^i+\int_s^t\rho^i\cdot v_u^i dB_u^{i\perp},\quad(3.4.24)$$

or as

$$\log\mathcal{E}_t^i=\log\mathcal{E}_s^i+\frac{1}{2}\int_s^t\left[(\Delta^i\cdot v_u^i)^2+(\iota^i\cdot v_u^i)^2+(\rho^i\cdot v_u^i)^2\right]du+$$
$$+\int_s^t\Delta^i\cdot v_u^i d\bar{B}_u^{iD}+\int_s^t\iota^i\cdot v_u^i d\bar{B}_u^i+\int_s^t\rho^i\cdot v_u^i d\bar{B}_u^{i\perp}.\quad(3.4.25)$$

For every $t\geq 0$

$$\bar{E}^i[|\log\mathcal{E}_t^i|]\leq\eta\left(e^{(\|A\|+\|A^T\|)t}+t+1\right).\quad(3.4.26)$$

Proof. The process $(N_t^i)_{t\geq 0}$ needs to satisfy conditions (3.2.4) and (3.2.5) of Definition 3.2.2 to be a stochastic discount factor. Property (3.2.4) is a direct calculation. The definition of $\mathcal{E}_t^i = e^{rt}N_t^i$ and Lemma 3.4.5 (a) imply that

$$E\left[N_t^iP_t+\int_0^t N_u^iD_udu\Big|\mathcal{F}_s^i\right]=\int_0^s N_u^iD_udu+E\left[N_t^i(C+\epsilon_DD_t+\hat{\pi}_t^M)+\int_s^t N_u^iD_udu\Big|\mathcal{F}_s^i\right]$$
$$=\int_0^s N_u^iD_udu+N_s^i\bar{E}^i\left[e^{-r(t-s)}(C+\epsilon_DD_t+\epsilon_\pi\hat{\pi}_t^M)+\int_s^t e^{-r(u-s)}D_udu\Big|\mathcal{F}_s^i\right].\quad(3.4.27)$$

The function $W(s, D, \hat{\pi}^M \hat{\pi}^i) = C + \epsilon_D D + \epsilon_\pi \hat{\pi}^M$ solves the Cauchy problem on $[0, t]$

$$0 = W_s + \left(\nabla_{(D,\hat{\pi}^M\hat{\pi}^i)} W\right) \cdot \left(A(D, \hat{\pi}^M \hat{\pi}^i)^T + b\right) + \frac{1}{2} tr\left(\left(\mathrm{He}_{(D,\hat{\pi}^M\hat{\pi}^i)}\, W\right) \Sigma\Sigma^T\right) - rW + D$$
$$W(t, D, \hat{\pi}^M \hat{\pi}^i) = C + \epsilon_D D + \epsilon_\pi \hat{\pi}^M,$$

where A, b and Σ are in Lemma 3.4.5. By [22, Theorem 7.6], for every $0 \leq s \leq t$

$$W(s, D, \hat{\pi}^M \hat{\pi}^i) = \bar{E}\left[\int_s^t e^{-r(u-s)} D_u du + e^{-r(t-s)}(C + \epsilon_D D_t + \epsilon_\pi \hat{\pi}_t^M) \middle| \mathcal{F}_s^i\right] = C + \epsilon_D D_s + \epsilon_\pi \hat{\pi}_s^M.$$

Plugging W into (3.4.27) proves (3.2.5), hence $(N_t^i)_{t\geq 0}$ is a stochastic discount factor. The stochastic process $(N_t^i)_{t\geq 0}$ of (3.1.6) solves the initial value problem

$$\frac{dN_t^i}{N_t^i} = -rdt + (\Delta^i \cdot v_t^i) dB_t^{iD} + (\iota^i \cdot v_t^i) dB_t^i + (\rho^i \cdot v_t^i) dB_t^{i\perp}, \qquad N_0^i = 1,$$

thus the process $(\mathcal{E}_t^i)_{t\geq 0}$ solves the initial value problem

$$\frac{d\mathcal{E}_t^i}{\mathcal{E}_t^i} = (\Delta^i \cdot v_t^i) dB_t^{iD} + (\iota^i \cdot v_t^i) dB_t^i + (\rho^i \cdot v_t^i) dB_t^{i\perp}, \qquad \mathcal{E}_0^i = 1,$$

by virtue of its definition $\mathcal{E}_t^i = e^{rt} N_t^i$. Applying Itô's formula to $f(\mathcal{E}_t^i) = \log \mathcal{E}_t^i$ we get (3.4.24) and because of (3.4.12) we get (3.4.25). Thanks to (3.4.25) and to the triangle inequality

$$\bar{E}^i[|\log \mathcal{E}_u^i|] \leq \frac{1}{2} \bar{E}^i \left[\int_0^u \left((\Delta^i \cdot v_h^i)^2 + (\iota^i \cdot v_h^i)^2 + (\rho^i \cdot v_h^i)^2\right) dh\right] +$$
$$+ \bar{E}^i \left[\left|\int_0^u (\Delta^i \cdot v_h^i) d\bar{B}_h^{iD}\right|\right] + \bar{E}^i \left[\left|\int_0^u (\iota^i \cdot v_h^i) d\bar{B}_h^i\right|\right] + \bar{E}^i \left[\left|\int_0^u (\rho^i \cdot v_h^i) d\bar{B}_h^{i\perp}\right|\right].$$

$\int_0^u (\Delta^i \cdot v_h^i) d\bar{B}_h^{iD}$ is a $\bar{\mathbb{P}}^i-$normal random variable with mean $\mu_u = 0$ and variance

$$\sigma_u^2 = \int_0^u \bar{E}^i[(\Delta^i \cdot v_h^i)^2] dh \leq \eta(e^{(\|A\|+\|A^T\|)u} + u + 1). \tag{3.4.28}$$

(3.4.16) (iv) implies the last inequality, where η is a positive constant. In view of Lemma A.0.1 (IX) and (X),

$$\bar{E}^i \left[\left|\int_0^u (\Delta^i \cdot v_h^i) d\bar{B}_h^{iD}\right|\right] \leq \sigma_u \sqrt{\frac{2}{\pi}} \leq \sigma_u^2 + 1 \leq \eta(e^{(\|A\|+\|A^T\|)u} + u + 1). \tag{3.4.29}$$

Because of (3.4.28), the right side of (3.4.29) is a bound also for

$$\bar{E}^i \left[\left|\int_0^u (\iota^i \cdot v_h^i) d\bar{B}_h^i\right|\right], \bar{E}^i \left[\left|\int_0^u (\rho^i \cdot v_h^i) d\bar{B}_h^{i\perp}\right|\right]$$

and for $\bar{E}^i \left[\int_0^u \left((\Delta^i \cdot v_h^i)^2 + (\iota^i \cdot v_h^i)^2 + (\rho^i \cdot v_h^i)^2\right) dh\right]$, thus (3.4.26) follows. □

Theorem 3.4.3 (Admissibility and utility)**.** *Define*

$$y^{i*} = e^{-r\alpha_i x_0^i + \delta_{DD}^i (D_0)^2 + \delta_{MM}^i (\hat{\pi}_0^M)^2 + \delta_{ii} (\hat{\pi}_0^i)^2 + \delta_{DM}^i D_0 \hat{\pi}_0^M + \delta_{Di} D_0 \hat{\pi}_0^i + \delta_{Mi} \hat{\pi}_0^M \hat{\pi}_0^i + \delta_D^i D_0 + \delta_M^i \hat{\pi}_0^M + \delta_i \hat{\pi}_0^i + \delta_0^i},$$

the processes $(c_t^{i*}, \theta_t^{i*})_{t \geq 0}$ *as*

$$c_t^{i*} = rX_t^{i*} - \frac{\delta_{DD}^i}{\alpha_i}(D_t)^2 - \frac{\delta_{MM}^i}{\alpha_i}(\hat{\pi}_t^M)^2 - \frac{\delta_{ii}}{\alpha_i}(\hat{\pi}_t^i)^2 - \frac{\delta_{DM}^i}{\alpha_i} D_t \hat{\pi}_t^M - \frac{\delta_{Di}}{\alpha_i} D_t \hat{\pi}_t^i + \\ - \frac{\delta_{Mi}}{\alpha_i} \hat{\pi}_t^M \hat{\pi}_t^i - \frac{\delta_D^i}{\alpha_i} D_t - \frac{\delta_M^i}{\alpha_i} \hat{\pi}_t^M - \frac{\delta_i}{\alpha_i} \hat{\pi}_t^i - \frac{\delta_0^i}{\alpha_i},$$

$$\theta_t^{i*} = \frac{M_D^i D_t + M_M^i \hat{\pi}_t^M + M_i \hat{\pi}_t^i + M_0^i}{\alpha_i M},$$

and the process $(X_t^{i*})_{t \geq 0}$ *as*

$$\begin{aligned} X_t^{i*} = x_0^i + \frac{1}{M\alpha_i} \Big\{ & \Big[M\delta_{DD}^i + M_D^i \Big(1 - \epsilon_D(k+r)\Big) \Big] \int_0^t D_u^2 du + \\ & + \Big[M\delta_{MM}^i + M_M^i \Big(- \epsilon_\pi(a+r) - \epsilon_\pi o_M \nu \Big) \Big] \int_0^t (\hat{\pi}_u^M)^2 du + \\ & + \Big[M\delta_{ii} + M_i(\epsilon_D + \epsilon_\pi o_M \nu) \Big] \int_0^t (\hat{\pi}_u^i)^2 du + \\ & + \Big[M\delta_{DM}^i + M_M^i \Big(1 - \epsilon_D(k+r)\Big) + M_D^i \Big(- \epsilon_\pi(a+r) - \epsilon_\pi o_M \nu \Big) \Big] \int_0^t D_u \hat{\pi}_u^M du + \\ & + \Big[M\delta_{Di} + M_i \Big(1 - \epsilon_D(k+r)\Big) + M_D^i(\epsilon_D + \epsilon_\pi o_M \nu) \Big] \int_0^t D_u \hat{\pi}_u^i du + \\ & + \Big[M\delta_{Mi} + M_i \Big(- \epsilon_\pi(a+r) - \epsilon_\pi o_M \nu \Big) + M_M^i(\epsilon_D + \epsilon_\pi o_M \nu) \Big] \int_0^t \hat{\pi}_u^M \hat{\pi}_u^i du + \\ & + \Big[M\delta_D^i + M_D^i(\epsilon_\pi a \bar{\pi} - rC) + M_0^i \Big(1 - \epsilon_D(k+r)\Big) \Big] \int_0^t D_u du + \\ & + \Big[M\delta_M^i + M_M^i(\epsilon_\pi a \bar{\pi} - rC) + M_0^i \Big(- \epsilon_\pi(a+r) - \epsilon_\pi o_M \nu \Big) \Big] \int_0^t \hat{\pi}_u^M du + \\ & + \Big[M\delta_i + M_i(\epsilon_\pi a \bar{\pi} - rC) + M_0^i(\epsilon_D + \epsilon_\pi o_M \nu) \Big] \int_0^t \hat{\pi}_u^i du + \\ & + \Big[M\delta_0^i + M_0^i(\epsilon_\pi a \bar{\pi} - rC) \Big] t + \\ & + (\epsilon_D \sigma_D + \epsilon_\pi o_M \sigma_D^{-1}) \int_0^t (M_D^i D_u + M_M^i \hat{\pi}_u^M + M_i \hat{\pi}_u^i + M_0^i) dB_u^{iD} + \\ & + o_M \epsilon_\pi \epsilon_i \sigma_i \int_0^t (M_D^i D_u + M_M^i \hat{\pi}_u^M + M_i \hat{\pi}_u^i + M_0^i) dB_u^i + \\ & + o_M \epsilon_\pi \epsilon_{i\perp} \int_0^t (M_D^i D_u + M_M^i \hat{\pi}_u^M + M_i \hat{\pi}_u^i + M_0^i) dB_u^{i\perp} \Big\}. \end{aligned} \tag{3.4.30}$$

Under Assumption 3.1.1, there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ *such that for every* $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$*, the following holds.*

(A) (First order condition)

$$-\alpha_i c_t^{i*} = \log(y^{i*}) + (\beta - r)t + \log(\mathcal{E}_t^i); \tag{3.4.31}$$

(B) (Budget equation) $N_t^i X_t^{i*} + \int_0^t N_u^i c_u^{i*} du$ *is a* $\mathbb{P}-$*martingale;*

(C) (Saturation) for every $s \geq 0$, $\lim_{t\to+\infty} E[N_t^i X_t^{i*} | \mathcal{F}_s^i] = 0$;

(D) (Admissibility) for every $i \in \{1, \ldots, n\}$, $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ *is an admissible strategy with wealth process* $(X_t^{i*})_{t\geq 0}$*. The utility of the strategy is*

$$E\left[\int_0^{+\infty} e^{-\beta u} U(c_u^{i*}) du \middle| \mathcal{F}_0^i\right] = -\frac{y^{i*}}{r\alpha_i}$$

Proof. Let $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$ be the minimum between the constants (with the same name) in Theorem 3.4.1 and Lemma 3.4.5. We proceed in several steps.

Proof of (A): First order condition

The equality $-\alpha_i c_0^{i*} = -r\alpha_i x_0^i + \delta_{DD}^i (D_0)^2 + \delta_{MM}^i (\hat{\pi}_0^M)^2 + \delta_{ii} (\hat{\pi}_0^i)^2 + \delta_{DM}^i D_0 \hat{\pi}_0^M + \delta_{Di} D_0 \hat{\pi}_0^i + \delta_{Mi} \hat{\pi}_0^M \hat{\pi}_0^i + \delta_D^i D_0 + \delta_M^i \hat{\pi}_0^M + \delta_i \hat{\pi}_0^i + \delta_0^i$ holds. Apply Itô's formula to both sides of (3.4.31) and check that they are equal.

Proof of the equality $\mathcal{E}_s^i \bar{E}^i \left[\int_s^t e^{-ru} c_u^{i*} du \middle| \mathcal{F}_s^i\right] = E\left[\int_s^t e^{-ru} \mathcal{E}_u^i c_u^{i*} du \middle| \mathcal{F}_s^i\right]$

Due to (3.4.31) and to the triangle inequality, there exists $\eta > 0$ such that

$$|c_u^{i*}| \leq \eta |-r\alpha_i x_0^i + \delta_{DD}^i (D_0)^2 + \delta_{MM}^i (\hat{\pi}_0^M)^2 + \delta_{ii} (\hat{\pi}_0^i)^2 + \delta_{DM}^i D_0 \hat{\pi}_0^M + \delta_{Di} D_0 \hat{\pi}_0^i + \delta_{Mi} \hat{\pi}_0^M \hat{\pi}_0^i + \\ + \delta_D^i D_0 + \delta_M^i \hat{\pi}_0^M + \delta_i \hat{\pi}_0^i + \delta_0^i| + \eta u + \eta |\log \mathcal{E}_u^i|. \tag{3.4.32}$$

Applying the conditional expectation to both sides of (3.4.32), the properties of normal random variables and (3.4.26) imply that

$$\bar{E}^i[|c_u^{i*}|] \leq \eta \left(e^{(\|A\| + \|A^T\|)t} + t + 1\right).$$

Fubini's Theorem [4, Theorem 1.1.7] yields to

$$\int_s^t e^{-ru} \bar{E}^i \left[c_u^{i*}\right] du = \bar{E}^i \left[\int_s^t e^{-ru} |c_u^{i*}| du\right] < +\infty$$

and by Bayes' formula and the conditional version of Fubini's Theorem, we get

$$\mathcal{E}_s^i \bar{E}^i \left[\int_s^t e^{-ru} c_u^{i*} du \middle| \mathcal{F}_s^i\right] = E\left[\int_s^t e^{-ru} \mathcal{E}_u^i c_u^{i*} du \middle| \mathcal{F}_s^i\right]. \tag{3.4.33}$$

Proof of (B): $N_t^i X_t^{i*} + \int_0^t N_u^i c_u^{i*}$ is a martingale

Direct calculations show that $(X_t^{i*})_{t\geq 0}$ is the wealth process of the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$

and they satisfy equality (3.4.6), equivalent to the self-financing condition. The equalities

$$\begin{aligned}
\Big(\epsilon_D\sigma_D+\epsilon_\pi o_M\sigma_D^{-1}\Big)\Delta_D^i+\Big(\epsilon_\pi\epsilon_i o_M\sigma_i\Big)\iota_D^i+\Big(\epsilon_\pi\epsilon_{i\perp}o_M\Big)\rho_D^i&=-1+\epsilon_D(k+r),\\
\Big(\epsilon_D\sigma_D+\epsilon_\pi o_M\sigma_D^{-1}\Big)\Delta_M^i+\Big(\epsilon_\pi\epsilon_i o_M\sigma_i\Big)\iota_M^i+\Big(\epsilon_\pi\epsilon_{i\perp}o_M\Big)\rho_M^i&=\epsilon_\pi(a+r)+\epsilon_\pi o_M\nu,\\
\Big(\epsilon_D\sigma_D+\epsilon_\pi o_M\sigma_D^{-1}\Big)\Delta_i+\Big(\epsilon_\pi\epsilon_i o_M\sigma_i\Big)\iota_i+\Big(\epsilon_\pi\epsilon_{i\perp}o_M\Big)\rho_i&=-\epsilon_D-\epsilon_\pi o_M\nu,\\
\Big(\epsilon_D\sigma_D+\epsilon_\pi o_M\sigma_D^{-1}\Big)\Delta_0^i+\Big(\epsilon_\pi\epsilon_i o_M\sigma_i\Big)\iota_0^i+\Big(\epsilon_\pi\epsilon_{i\perp}o_M\Big)\rho_0^i&=-\epsilon_\pi a\bar{\pi}+rC,
\end{aligned}$$

and (3.4.12) imply that

$$dX_t^{i*}=(-c_t^{i*}+rX_t^{i*})dt+\Big(\epsilon_D\sigma_D+\epsilon_\pi o_M\sigma_D^{-1}\Big)\,\theta_t^{i*}d\bar{B}_t^{iD}+\epsilon_\pi\epsilon_i o_M\sigma_i\theta_t^{i*}d\bar{B}_t^{i}+\epsilon_\pi\epsilon_{i\perp}o_M\theta_t^{i*}d\bar{B}_t^{i\perp}.$$

Applying Itô's formula to the function $f(t,X_t^{i*})=e^{-rt}X_t^{i*}$ we get

$$\begin{aligned}
e^{-rt}X_t^{i*}=e^{-rs}X_s^{i*}+\int_s^t -e^{-ru}c_u^{i*}du+\Big(\epsilon_D\sigma_D+\epsilon_\pi o_M\sigma_D^{-1}\Big)\int_s^t e^{-ru}\theta_u^{i*}d\bar{B}_u^{iD}+\\
+\epsilon_\pi\epsilon_i o_M\sigma_i\int_s^t e^{-ru}\theta_u^{i*}d\bar{B}_u^{i}+\epsilon_\pi\epsilon_{i\perp}o_M\int_s^t e^{-ru}\theta_u^{i*}d\bar{B}_u^{i\perp}.
\end{aligned}$$

Multiply both sides by $\mathcal{E}_t^i$, add $\int_0^t N_u^i c_u^{i*}du$, take the conditional expectation and use Bayes' formula to get

$$\begin{aligned}
&E\left[N_t^iX_t^{i*}+\int_0^t N_u^ic_u^{i*}du\middle|\mathcal{F}_s^i\right]=N_s^iX_s^{i*}+\int_0^s N_u^ic_u^{i*}du+\\
&+\mathcal{E}_s^i\Big(\epsilon_D\sigma_D+\epsilon_\pi o_M\sigma_D^{-1}\Big)\bar{E}^i\left[\int_s^t e^{-ru}\theta_u^{i*}d\bar{B}_u^{iD}\middle|\mathcal{F}_s^i\right]+\mathcal{E}_s^i\epsilon_\pi\epsilon_i o_M\sigma_i\bar{E}^i\left[\int_s^t e^{-ru}\theta_u^{i*}d\bar{B}_u^{i}\middle|\mathcal{F}_s^i\right]\\
&+\mathcal{E}_s^i\epsilon_\pi\epsilon_{i\perp}o_M\bar{E}^i\left[\int_s^t e^{-ru}\theta_u^{i*}d\bar{B}_u^{i\perp}\middle|\mathcal{F}_s^i\right]+\mathcal{E}_s^i\bar{E}^i\left[\int_s^t -e^{-ru}c_u^{i*}du\middle|\mathcal{F}_s^i\right]+E\left[\int_s^t N_u^ic_u^{i*}du\middle|\mathcal{F}_s^i\right].
\end{aligned}$$

The Brownian terms are martingales because of Lemma 3.4.5 (c) and since (3.4.33) holds, then

$$E\left[N_t^iX_t^{i*}+\int_0^t N_u^ic_u^{i*}du\middle|\mathcal{F}_s^i\right]=\int_0^s N_u^ic_u^{i*}du+N_s^iX_s^{i*}. \tag{3.4.34}$$

Proof of (C): $\lim_{t\to+\infty}E[N_t^iX_t^{i*}|\mathcal{F}_s^i]=0$

Because of (3.4.12), for the process X_t^{i*} of (3.4.30), there exist $\eta_1, \dots, \eta_{22} \in \mathbb{R}$ such that

$$N_t^i X_t^{i*} = e^{-rt}\mathcal{E}_t^i X_s^{i*} + \eta_1 e^{-rt}\mathcal{E}_t^i \int_s^t D_u^2 du + \eta_2 e^{-rt}\mathcal{E}_t^i \int_s^t (\hat{\pi}_u^M)^2 du + \eta_3 e^{-rt}\mathcal{E}_t^i \int_s^t (\hat{\pi}_u^i)^2 du +$$
$$+\eta_4 e^{-rt}\mathcal{E}_t^i \int_s^t D_u \hat{\pi}_u^M du + \eta_5 e^{-rt}\mathcal{E}_t^i \int_s^t D_u \hat{\pi}_u^i du + \eta_6 e^{-rt}\mathcal{E}_t^i \int_s^t \hat{\pi}_u^M \hat{\pi}_u^i du + \eta_7 e^{-rt}\mathcal{E}_t^i \int_s^t D_u du +$$
$$+ \eta_8 e^{-rt}\mathcal{E}_t^i \int_s^t \hat{\pi}_u^M du + \eta_9 e^{-rt}\mathcal{E}_t^i \int_s^t \hat{\pi}_u^i du + \eta_{10} e^{-rt}\mathcal{E}_t^i (t-s) +$$
$$+\eta_{11} e^{-rt}\mathcal{E}_t^i \int_s^t D_u d\bar{B}_u^{iD} + \eta_{12} e^{-rt}\mathcal{E}_t^i \int_s^t \hat{\pi}_u^M d\bar{B}_u^{iD} + \eta_{13} e^{-rt}\mathcal{E}_t^i \int_s^t \hat{\pi}_u^i d\bar{B}_u^{iD} + \eta_{14} e^{-rt}\mathcal{E}_t^i \int_s^t d\bar{B}_u^{iD} +$$
$$+ \eta_{15} e^{-rt}\mathcal{E}_t^i \int_s^t D_u d\bar{B}_u^i + \eta_{16} e^{-rt}\mathcal{E}_t^i \int_s^t \hat{\pi}_u^M d\bar{B}_u^i + \eta_{17} e^{-rt}\mathcal{E}_t^i \int_s^t \hat{\pi}_u^i d\bar{B}_u^i + \eta_{18} e^{-rt}\mathcal{E}_t^i \int_s^t d\bar{B}_u^i +$$
$$+\eta_{19} e^{-rt}\mathcal{E}_t^i \int_s^t D_u d\bar{B}_u^{i\perp} + \eta_{20} e^{-rt}\mathcal{E}_t^i \int_s^t \hat{\pi}_u^M d\bar{B}_u^{i\perp} + \eta_{21} e^{-rt}\mathcal{E}_t^i \int_s^t \hat{\pi}_u^i d\bar{B}_u^{i\perp} + \eta_{22} e^{-rt}\mathcal{E}_t^i \int_s^t d\bar{B}_u^{i\perp}.$$

Taking the conditional expectation and using Bayes' formula yields

$$E[N_t^i X_t^{i*} | \mathcal{F}_s^i] = e^{-rt}\mathcal{E}_s^i X_s^{i*} + \eta_1 e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t D_u^2 du \middle| \mathcal{F}_s^i \right] + \eta_2 e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t (\hat{\pi}_u^M)^2 du \middle| \mathcal{F}_s^i \right] +$$
$$+\eta_3 e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t (\hat{\pi}_u^i)^2 du \middle| \mathcal{F}_s^i \right] + \eta_4 e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t D_u \hat{\pi}_u^M du \middle| \mathcal{F}_s^i \right] + \eta_5 e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t D_u \hat{\pi}_u^i du \middle| \mathcal{F}_s^i \right] +$$
$$+ \eta_6 e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t \hat{\pi}_u^M \hat{\pi}_u^i du \middle| \mathcal{F}_s^i \right] + \eta_7 e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t D_u du \middle| \mathcal{F}_s^i \right] + \eta_8 e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t \hat{\pi}_u^M du \middle| \mathcal{F}_s^i \right] +$$
$$+ \eta_9 e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t \hat{\pi}_u^i du \middle| \mathcal{F}_s^i \right] + \eta_{10} e^{-rt}\mathcal{E}_s^i (t-s) + \eta_{11} e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t D_u d\bar{B}_u^{iD} \middle| \mathcal{F}_s^i \right] +$$
$$+\eta_{12} e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t \hat{\pi}_u^M d\bar{B}_u^{iD} \middle| \mathcal{F}_s^i \right] + \eta_{13} e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t \hat{\pi}_u^i d\bar{B}_u^{iD} \middle| \mathcal{F}_s^i \right] + \eta_{14} e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t d\bar{B}_u^{iD} \middle| \mathcal{F}_s^i \right] +$$
$$+\eta_{15} e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t D_u d\bar{B}_u^i \middle| \mathcal{F}_s^i \right] + \eta_{16} e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t \hat{\pi}_u^M d\bar{B}_u^i \middle| \mathcal{F}_s^i \right] + \eta_{17} e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t \hat{\pi}_u^i d\bar{B}_u^i \middle| \mathcal{F}_s^i \right] +$$
$$+\eta_{18} e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t d\bar{B}_u^i \middle| \mathcal{F}_s^i \right] + \eta_{19} e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t D_u d\bar{B}_u^{i\perp} \middle| \mathcal{F}_s^i \right] + \eta_{20} e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t \hat{\pi}_u^M d\bar{B}_u^{i\perp} \middle| \mathcal{F}_s^i \right] +$$
$$+ \eta_{21} e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t \hat{\pi}_u^i d\bar{B}_u^{i\perp} \middle| \mathcal{F}_s^i \right] + \eta_{22} e^{-rt}\mathcal{E}_s^i \bar{E}^i \left[\int_s^t d\bar{B}_u^{i\perp} \middle| \mathcal{F}_s^i \right].$$

All the Brownian terms are $\bar{\mathbb{P}}^i-$martingales by virtue of Lemma 3.4.5 (c). Thanks to Lemma 3.4.5 (d), $\lim_{t\to+\infty} E[N_t^i X_t^{i*} | \mathcal{F}_s^i] = 0$.

Proof of (D): Admissibility and utility

Property (i) of Definition 3.1.1 is clear and proving that $(X_t^{i*})_{t\geq 0}$ is the wealth process of the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ is a direct calculation. Take $\lim_{t\to+\infty}$ to both sides of (3.4.34) and use (C) to prove (3.1.4) and thus the admissibility of the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$. (3.4.31) implies

$$E\left[\int_0^{+\infty} e^{-\beta u} U^i(c_u^{i*}) du \middle| \mathcal{F}_0^i \right] = -\frac{1}{\alpha_i} E\left[\int_0^{+\infty} e^{\log y^{i*} - ru + \log \mathcal{E}_u^i} du \middle| \mathcal{F}_0^i \right] = -\frac{y^{i*}}{r\alpha_i}.$$

□

Theorem 3.4.4 (Duality Theorem). *Let $(c_t, \theta_t)_{t\geq 0}$ be an admissible strategy for the*

$i-$th investor and let $(N^i_t)_{t\geq 0}$ be the process of (3.1.6); then

$$\begin{aligned} \lim_{t\to+\infty} E\left[\int_0^t e^{-\beta u}U^i(c_u)du\Big|\mathcal{F}^i_0\right] &= E\left[\int_0^{+\infty} e^{-\beta u}U^i(c_u)du\Big|\mathcal{F}^i_0\right], \\ \lim_{t\to+\infty} E\left[\int_0^t e^{-\beta u}\tilde{U}^i(ye^{\beta u}N^i_u)du\Big|\mathcal{F}^i_0\right] &= E\left[\int_0^{+\infty} e^{-\beta u}\tilde{U}^i(ye^{\beta u}N^i_u)du\Big|\mathcal{F}^i_0\right]. \end{aligned} \tag{3.4.35}$$

Furthermore

$$E\left[\int_0^{+\infty} e^{-\beta u}U^i(c_u)du\Big|\mathcal{F}^i_0\right] \leq \inf_{y>0}\left\{E\left[\int_0^{+\infty} e^{-\beta u}\tilde{U}^i(ye^{\beta u}N^i_u)du\Big|\mathcal{F}^i_0\right] + x^i_0 y\right\}, \tag{3.4.36}$$

where

$$\tilde{U}^i(y) = \begin{cases} \frac{y}{\alpha_i}(\log y - 1) & y > 0 \\ 0 & y = 0. \end{cases} \tag{3.4.37}$$

If there exist $y^{i*} > 0$ and an admissible strategy $(c^*_t, \theta^*_t)_{t\geq 0}$ for which

$$E\left[\int_0^{+\infty} e^{-\beta u}U^i(c^*_u)du\Big|\mathcal{F}^i_0\right] = E\left[\int_0^{+\infty} e^{-\beta u}\tilde{U}^i(y^{i*}e^{\beta u}N^i_u)du\Big|\mathcal{F}^i_0\right] + x^i_0 y^{i*}, \tag{3.4.38}$$

then $(c^*_t, \theta^*_t)_{t\geq 0}$ is optimal.

Proof. Define the random variables

$$\lambda^m = \int_0^m e^{-\beta u - \alpha_i c_u}du, \qquad \lambda = \int_0^{+\infty} e^{-\beta u - \alpha_i c_u}du,$$

on the probability space $(\Omega, \mathcal{F}^i, (\mathcal{F}^i_t)_{t\geq 0}, \mathbb{P})$. Then $\lambda^m \geq 0$ for every $m \in \mathbb{N}$ and $(\lambda^m)_{m\in\mathbb{N}}$ is an increasing sequence of random variables such that $\lim_{m\to+\infty}\lambda^m = \lambda$. The Conditional Monotone Convergence Theorem yields to

$$\lim_{m\to+\infty} E[\lambda^m|\mathcal{F}^i_0] = E[\lambda|\mathcal{F}^i_0],$$

which implies the first equality in (3.4.35). The function $\tilde{U}^i$ defined in (3.4.37) has a global minimum at $y = 1$; apply the Conditional Monotone Convergence Theorem to the random variables

$$\lambda^m = \int_0^m e^{-\beta u}\left(\tilde{U}^i(ye^{\beta u}N^i_u) + \frac{1}{\alpha_i}\right)du, \qquad \lambda = \int_0^{+\infty} e^{-\beta u}\left(\tilde{U}^i(ye^{\beta u}N^i_u) + \frac{1}{\alpha_i}\right)du,$$

to conclude the second equality in (3.4.35). For the proof of (3.4.36) apply (A.0.1) to the random variables c_u and $Y_u = ye^{\beta u}N^i_u$; for every $y > 0$

$$U^i(c_u) \leq \tilde{U}^i(ye^{\beta u}N^i_u) + c_u ye^{\beta u}N^i_u.$$

Multiply both sides by $e^{-\beta u}$, integrate in $[0, t]$ and take conditional expectations; for every $y > 0$

$$E\left[\int_0^t e^{-\beta u}U^i(c_u)du\Big|\mathcal{F}_0^i\right] \leq E\left[\int_0^t e^{-\beta u}\tilde{U}^i(ye^{\beta u}N_u^i)du\Big|\mathcal{F}_0^i\right] + yE\left[\int_0^t c_u N_u^i du\Big|\mathcal{F}_0^i\right].$$

Take $\limsup_{t\to+\infty}$ of both sides and use (3.4.35) and (3.1.4); for every $y > 0$

$$E\left[\int_0^{+\infty} e^{-\beta u}U^i(c_u)du\Big|\mathcal{F}_0^i\right] \leq E\left[\int_0^{+\infty} e^{-\beta u}\tilde{U}^i(ye^{\beta u}N_u^i)du\Big|\mathcal{F}_0^i\right] + x_0^i y.$$

Take $\inf_{y>0}$ to obtain (3.4.36). If there exist $y^{i*} > 0$ and an admissible strategy $(c_t^*, \theta_t^*)_{t\geq 0}$ for which (3.4.38) holds, then

$$\begin{aligned} E\left[\int_0^{+\infty} e^{-\beta u}U^i(c_u^*)du\Big|\mathcal{F}_0^i\right] &\leq \inf_{y>0}\left\{E\left[\int_0^{+\infty} e^{-\beta u}\tilde{U}^i(ye^{\beta u}N_u^i)du\Big|\mathcal{F}_0^i\right] + x_0^i y\right\} \\ &\leq E\left[\int_0^{+\infty} e^{-\beta u}\tilde{U}^i(y^{i*}e^{\beta u}N_u^i)du\Big|\mathcal{F}_0^i\right] + x_0^i y^{i*} = E\left[\int_0^{+\infty} e^{-\beta u}U^i(c_u^*)du\Big|\mathcal{F}_0^i\right]. \end{aligned}$$

All the above are equalities therefore $(c_t^*, \theta_t^*)_{t\geq 0}$ is optimal. □

Theorem 3.4.5 (Existence). *Under Assumption 3.1.1, there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$ the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ in Theorem 3.4.3 is optimal for the $i-$th investor for every $i \in \{1, \ldots, n\}$. The function V^i of Theorem 3.4.1 is the value function of the $i-$th investor.*

Proof. Fix $0 \leq s \leq t$ and $y > 0$; thanks to the definition of $\tilde{U}(\cdot)$ in (3.4.37)

$$\begin{aligned} E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u^i)du\Big|\mathcal{F}_s^i\right] &= \frac{y}{\alpha_i}\Big\{ (\log y - 1)\, E\left[\int_s^t N_u^i du\Big|\mathcal{F}_s^i\right] + \\ &\quad + \beta E\left[\int_s^t uN_u^i du\Big|\mathcal{G}_s\right] + E\left[\int_s^t N_u^i \log N_u^i du\Big|\mathcal{F}_s^i\right]\Big\}. \end{aligned}$$

The following integrability conditions hold:

$$\begin{aligned} \int_s^t E\left[|N_u^i|\right] du &= \int_s^t E\left[N_u^i\right] du = \int_s^t e^{-ru}du = \frac{(e^{-rs} - e^{-rt})}{r} < +\infty, \\ \int_s^t E\left[|uN_u^i|\right] du &= \int_s^t uE\left[N_u^i\right] du = \int_s^t ue^{-ru}du = \frac{e^{-rs}(1+rs) - e^{-rt}(1+rt)}{r^2} < +\infty. \end{aligned}$$

The conditional version of Fubini's Theorem [4, Theorem 1.1.8] applies and yields

$$\begin{aligned} E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u^i)du\Big|\mathcal{F}_s^i\right] &= \frac{y}{\alpha_i}\Big\{ (\log y - 1)\, \mathcal{E}_s^i \int_s^t e^{-ru}du + \\ &\quad + (\beta - r)\mathcal{E}_s^i \int_s^t ue^{-ru}du + E\left[\int_s^t e^{-ru}\mathcal{E}_u^i \log \mathcal{E}_u^i du\Big|\mathcal{F}_s^i\right]\Big\}. \end{aligned}$$

(3.4.26) implies that

$$\int_s^t e^{-ru}\bar{E}^i\left[|\log \mathcal{E}_u^i|\right] du \leq \eta(t-s)\left(e^{(\|A\|+\|A^T\|)t} + t + 1\right) < +\infty.$$

Fubini's Theorem and Bayes' formula yield to

$$E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u^i)du\Big|\mathcal{F}_s^i\right] = \frac{y}{\alpha_i}\Big\{ (\log y-1)\,\mathcal{E}_s^i\int_s^t e^{-ru}du+ \\ +(\beta-r)\mathcal{E}_s^i\int_s^t ue^{-ru}du+\mathcal{E}_s^i\int_s^t e^{-ru}\bar{E}^i\left[\log\mathcal{E}_u^i|\mathcal{F}_s^i\right]du\Big\}$$

and computing the integrals we get

$$E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u^i)du\Big|\mathcal{F}_s^i\right] = \frac{y}{r\alpha_i}\mathcal{E}_s^i\Big\{ (\log y-1)\,(e^{-rs}-e^{-rt})+ \\ +(\beta-r)\frac{e^{-rs}(1+rs)-e^{-rt}(1+rt)}{r}+r\int_s^t e^{-ru}\bar{E}^i\left[\log\mathcal{E}_u^i|\mathcal{F}_s^i\right]du\Big\}.$$

By virtue of (3.4.25) and Lemma 3.4.5 (c),

$$\bar{E}^i\left[\log\mathcal{E}_u^i|\mathcal{F}_s^i\right]=\log\mathcal{E}_s^i+\frac{1}{2}\bar{E}^i\Big[\int_s^u\Big((\Delta^i\cdot v_u^i)^2+(\iota^i\cdot v_u^i)^2+(\rho^i\cdot v_u^i)^2\Big)du\Big|\mathcal{F}_s^i\Big].$$

Defining $Y_t=\int_s^t[(\Delta^i\cdot v_u^i)^2+(\iota^i\cdot v_u^i)^2+(\rho^i\cdot v_u^i)^2]\,du$ it follows that

$$r\int_s^t e^{-ru}\bar{E}^i\left[\log\mathcal{E}_u^i|\mathcal{F}_s^i\right]du = r\log\mathcal{E}_s^i\int_s^t e^{-ru}du+\frac{r}{2}\int_s^t e^{-ru}\bar{E}^i[Y_u|\mathcal{F}_s^i]du$$

and thanks to Lemma 3.4.5 (b) and to Fubini's Theorem we get

$$r\int_s^t e^{-ru}\bar{E}^i\left[\log\mathcal{E}_u^i|\mathcal{F}_s^i\right]du = r\log\mathcal{E}_s^i\int_s^t e^{-ru}du+\frac{r}{2}\bar{E}^i\left[\int_s^t e^{-ru}Y_u du\Big|\mathcal{F}_s^i\right].$$

As a consequence,

$$E\left[\int_s^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u^i)du\Big|\mathcal{F}_s^i\right] = \frac{y}{r\alpha_i}\mathcal{E}_s^i\Big\{ (\log y-1)\,(e^{-rs}-e^{-rt})+ \\ +(\beta-r)\frac{e^{-rs}(1+rs)-e^{-rt}(1+rt)}{r}+r\log\mathcal{E}_s^i\int_s^t e^{-ru}du+\frac{r}{2}\bar{E}^i\left[\int_s^t e^{-ru}Y_u du\Big|\mathcal{F}_s^i\right]\Big\}. \tag{3.4.39}$$

Applying Itô's formula to the function $e^{-rt}Y_t$ and taking the conditional expectation yields

$$r\bar{E}^i\left[\int_s^t e^{-ru}Y_u du\Big|\mathcal{F}_s^i\right]=e^{-rs}Y_s-e^{-rt}\bar{E}^i\Big[\int_s^t\Big((\Delta^i\cdot v_u^i)^2+(\iota^i\cdot v_u^i)^2+(\rho^i\cdot v_u^i)^2\Big)du\Big|\mathcal{F}_s^i\Big]+ \\ +\bar{E}^i\Big[\int_s^t e^{-ru}\Big((\Delta^i\cdot v_u^i)^2+(\iota^i\cdot v_u^i)^2+(\rho^i\cdot v_u^i)^2\Big)du\Big|\mathcal{F}_s^i\Big]. \tag{3.4.40}$$

Plug (3.4.40) into (3.4.39), fix $s=0$, take $\lim_{t\to+\infty}$ of both sides and add $x_0^i y$ to get

$$\lim_{t\to+\infty} E\left[\int_0^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u^i)du\Big|\mathcal{F}_0^i\right] + x_0^i y = \frac{y}{r\alpha_i}\Bigg\{(\log y - 1) + \frac{\beta - r}{r} + \\ -\frac{1}{2}\lim_{t\to+\infty} e^{-rt}\bar{E}^i\left[\int_0^t \Big((\Delta^i\cdot v_u^i)^2 + (\iota^i\cdot v_u^i)^2 + (\rho^i\cdot v_u^i)^2\Big)du\Big|\mathcal{F}_0^i\right] + \\ +\frac{1}{2}\lim_{t\to+\infty}\bar{E}^i\left[\int_0^t e^{-ru}\Big((\Delta^i\cdot v_u^i)^2 + (\iota^i\cdot v_u^i)^2 + (\rho^i\cdot v_u^i)^2\Big)du\Big|\mathcal{F}_0^i\right]\Bigg\} + x_0^i y.$$

Choosing $y = y^{i*}$ and using Lemma 3.4.5 (d) it follows that

$$\lim_{t\to+\infty} E\left[\int_0^t e^{-\beta u}\tilde{U}(ye^{\beta u}N_u^i)du\Big|\mathcal{F}_0^i\right] + x_0^i y^{i*} = \frac{y^{i*}}{r\alpha_i}\Big\{(-r\alpha_i x_0^i + \delta_{DD}^i(D_0)^2 + \delta_{MM}^i(\hat{\pi}_0^M)^2 + \delta_{ii}(\hat{\pi}_0^i)^2 + \\ + \delta_{DM}^i D_0\hat{\pi}_0^M + \delta_{Di}D_0\hat{\pi}_0^i + \delta_{Mi}\hat{\pi}_0^M\hat{\pi}_0^i + \delta_D^i D_0 + \delta_M^i\hat{\pi}_0^M + \delta_i\hat{\pi}_0^i + \delta_0^i - 1) + \frac{\beta - r}{r} + \\ + \frac{1}{2}\lim_{t\to+\infty}\bar{E}_t^i\Big[\int_0^t e^{-ru}\Big((\Delta^i\cdot v_u^i)^2 + (\iota^i\cdot v_u^i)^2 + (\rho^i\cdot v_u^i)^2\Big)du\Big|\mathcal{F}_0^i\Big]\Big\} + x_0^i y^{i*}.$$

Lemma 3.4.5 (e) and (3.4.35) imply

$$E\left[\int_0^{+\infty} e^{-\beta u}\tilde{U}(ye^{\beta u}N_u^i)du\Big|\mathcal{F}_0^i\right] = -\frac{y^{i*}}{r\alpha_i}.$$

The conclusion follows from Theorem 3.4.3 (D) and from Theorem 3.4.4. □

Uniqueness of the optimal strategy

Lemma 3.4.6. Let $i\in\{1,\dots,n\}$ and let $(c_t,\theta_t)_{t\geq 0}$ be an optimal strategy for the $i-$th agent with wealth process $(X_t)_{t\geq 0}$, then for every $s\geq 0$

$$\limsup_{t\to+\infty} E\left[\int_s^t N_u^i c_u du\Big|\mathcal{F}_s^i\right] = N_s^i X_s. \tag{3.4.41}$$

Proof. Suppose, for a contradiction, that there exist $i\in\{1,\dots,n\}$, $s\geq 0$, $S\in\mathcal{F}_s^i$ with $\mathbb{P}(S)>0$ and an optimal strategy such that

$$\limsup_{t\to+\infty} E\left[\int_s^t N_u^i c_u du\Big|\mathcal{F}_s^i\right] < N_s^i X_s \quad \text{on } S.$$

Let η_s be a $\mathcal{F}_s^i-$adapted random variable and define the new strategy $(\bar{c}_t,\theta_t)_{t\geq 0}$ as $(\bar{c}_t)_{t\geq 0} = (c_t)_{t\geq 0} + \eta_s\mathbf{1}_{t\geq s}$ and its wealth process

$$\bar{X}_t = X_t\mathbf{1}_{t\leq s} + \mathbf{1}_{t>s}\Big\{X_s + \int_s^t\Big[-\bar{c}_u + r\bar{X}_u + \theta_u^i(\epsilon_\pi a\bar{\pi} - rC) + \theta_u^i D_u\Big(1 - \epsilon_D(k+r)\Big) + \\ + \theta_t^i\hat{\pi}_t^M\Big(-\epsilon_\pi(a+r) - \epsilon_\pi o_M\nu\Big) + \theta_t^i\hat{\pi}_t^i(\epsilon_D + \epsilon_\pi o_M\nu)\Big]du + \int_s^t \theta_u^i(\epsilon_D\sigma_D + \epsilon_\pi o_M\sigma_D^{-1})dB_u^{iD} + \\ + \int_s^t \theta_u^i o_M\epsilon_\pi\epsilon_i\sigma_i dB_u^i + \int_s^t \theta_u^i o_M\epsilon_\pi\epsilon_{i\perp}dB_u^{i\perp}\Big\}.$$

If $\limsup_{t\to+\infty} E\left[\int_s^t N_u^i c_u du | \mathcal{F}_s^i\right] = -\infty$ the claim follows because $\eta_s = 1$ makes $(\bar{c}_t)_{t\geq 0}$ a better strategy, still admissible. Otherwise, if $\limsup_{t\to+\infty} E\left[\int_s^t N_u^i c_u du | \mathcal{F}_s^i\right] > -\infty$, define $\epsilon = X_s N_s^i - \limsup_{t\to+\infty} E\left[\int_s^t N_u^i c_u du \Big| \mathcal{F}_s^i\right] > 0$. Choose $\eta_s = \epsilon r (\mathcal{E}_s^i)^{-1} e^{rs}$ to obtain a better strategy, which is still admissible because

$$X_s N_s^i - \limsup_{t\to+\infty} E\left[\int_s^t N_u^i (c_u + \eta_s) du \Big| \mathcal{F}_s^i\right] = \epsilon - \eta_s \frac{\mathcal{E}_s^i}{r} e^{-rs} = 0.$$

□

Theorem 3.4.6 (Uniqueness)**.** *Under Assumption 3.1.1, there exists $\bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C) > 0$ such that for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}(\epsilon_D, \epsilon_\pi, C)$ the strategy $(c_t^{i*}, \theta_t^{i*})_{t\geq 0}$ in Theorem 3.4.3 is the unique optimal strategy for the $i-$th investor for all $i \in \{1, \ldots, n\}$.*

Proof. Claim: The consumption process is unique.
Suppose there exist optimal strategies for the $i-$th investor $(c_t^A, \theta_t^A)_{t\geq 0}$ and $(c_t^B, \theta_t^B)_{t\geq 0}$ and suppose, for a contradiction, that there exists $S \in \mathcal{B} \otimes \mathcal{F}^i$ such that $(\lambda_{|[0,+\infty[} \otimes \mathbb{P})(S) > 0$ and $c_t^A \mathbf{1}_S \neq c_t^B \mathbf{1}_S$. The wealth process of the strategy $\frac{1}{2}(c_t^A + c_t^B, \theta_t^A + \theta_t^B)_{t\geq 0}$ is the process $\frac{1}{2}(X_t^A + X_t^B)_{t\geq 0}$, with dynamics

$$\frac{1}{2} d(X_t^A + X_t^B) = \frac{1}{2}\Big[- c_t^A - c_t^B + r(X_t^A + X_t^B) + (\theta_t^A + \theta_t^B)(\epsilon_\pi a \bar{\pi} - rC) +$$
$$+ (\theta_t^A + \theta_t^B) D_t \Big(1 - \epsilon_D (k+r)\Big) + (\theta_t^A + \theta_t^B) \hat{\pi}_t^M \Big(-\epsilon_\pi (a+r) - \epsilon_\pi o_M \nu\Big) + (\theta_t^A + \theta_t^B) \hat{\pi}_t^i \Big(\epsilon_D + \epsilon_\pi o_M \nu\Big)\Big] dt +$$
$$+ \frac{1}{2}(\theta_t^A + \theta_t^B)\left(\epsilon_D \sigma_D + \epsilon_\pi o_M \sigma_D^{-1}\right) dB_t^{iD} + \frac{1}{2}(\theta_t^A + \theta_t^B) \epsilon_\pi \epsilon_i o_M \sigma_i dB_t^i + \frac{1}{2}(\theta_t^A + \theta_t^B) \epsilon_\pi \epsilon_{i\perp} o_M dB_t^{i\perp}.$$

The new strategy has initial wealth x_0^i and is admissible because $(c_t^A, \theta_t^A)_{t\geq 0}$ and $(c_t^B, \theta_t^B)_{t\geq 0}$ are. Since the utility function is strictly concave, $c_t^A \neq c_t^B$ on S implies

$$U\left(\frac{c_t^A + c_t^B}{2}\right) > \frac{1}{2} U(c_t^A) + \frac{1}{2} U(c_t^B) \qquad \text{on } S.$$

Define $H := \Big\{ w \in \Omega : \lambda_{|[0,+\infty[}(\{t \geq 0 : (t,w) \in S\}) \Big\} \in \mathcal{G}$, then $\mathbb{P}(H) > 0$,

$$\int_0^{+\infty} e^{-\beta t} U\left(\frac{c_t^A + c_t^B}{2}\right) dt \geq \frac{1}{2}\int_0^{+\infty} e^{-\beta t} [U(c_t^A) + U(c_t^B)] dt \qquad \text{a.s.}$$

and

$$\int_0^{+\infty} e^{-\beta t} U\left(\frac{c_t^A + c_t^B}{2}\right) dt > \frac{1}{2}\int_0^{+\infty} e^{-\beta t} [U(c_t^A) + U(c_t^B)] dt \qquad \text{on } H.$$

By Lemma A.0.2 (II) it follows that

$$E\left[\int_0^{+\infty} e^{-\beta t} \left(\frac{c_t^A + c_t^B}{2}\right) dt \Big| \mathcal{F}_0^i\right] > E\left[\int_0^{+\infty} e^{-\beta t} U(c_t^{i*}) dt \Big| \mathcal{F}_0^i\right]$$

on a positive probability set, thus contradicting the optimality of the consumption processes $(c_t^A)_{t\geq 0}$ and $(c_t^B)_{t\geq 0}$.
Claim: Investment and wealth processes are unique.
Thanks to (3.4.41), it follows that

$$X_s^{i*} = (N_s^i)^{-1} \limsup_{t\to+\infty} E\left[\int_s^t N_u^i c_u^{i*} du \middle| \mathcal{F}_s^i\right],$$

which proves the uniqueness of the optimal wealth process. From (3.4.6) it follows that

$$dX_t^{i*} + c_t^{i*}dt - rX_t^{i*}dt = \theta_t^i\Big[(\epsilon_\pi a\bar{\pi} - rC) + D_t\Big(1-\epsilon_D(k+r)\Big) + \hat{\pi}_t^M\Big(-\epsilon_\pi(a+r) - \epsilon_\pi o_M \nu\Big) +$$
$$+\hat{\pi}_t^i\Big(\epsilon_D + \epsilon_\pi o_M \nu\Big)\Big]dt + \theta_t^{i*}\left(\epsilon_D\sigma_D + \epsilon_\pi o_M \sigma_D^{-1}\right) dB_t^{iD} + \theta_t^{i*}\epsilon_\pi\epsilon_i o_M \sigma_i dB_t^i + \theta_t^{i*}\epsilon_\pi\epsilon_{i\perp} o_M dB_t^{i\perp}.$$

If there exist two strategies with wealth process $(X_t^{i*})_{t\geq 0}$ and consumption $(c_t^{i*})_{t\geq 0}$, then drifts and volatilities must be the same. This implies the uniqueness of the optimal investment strategy. □

3.4.1 Market clearing and proof of Theorem 3.2.1

The economy has one risky asset, i.e. for every $t \geq 0$

$$\sum_{i=1}^n \theta_t^{i*} = \sum_{i=1}^n \frac{M_D^i D_t + M_M^i \hat{\pi}_t^M + M_i \hat{\pi}_t^i + M_0^i}{M\alpha_i} = 1, \tag{3.4.42}$$

where M_D^i, M_M^i, M_i, M_0^i are given in Definition 3.4.1.

Definition 3.4.2. We introduce the following constants

$$\bar{\epsilon_D}^* = \frac{1}{k+r}, \qquad \bar{\epsilon_\pi}^* = \frac{1}{(a+r)(k+r)}, \qquad \bar{C}^* = \frac{a\bar{\pi}}{r(a+r)(k+r)} - \frac{\bar{\alpha}\sigma_D^2}{(k+r)^2},$$
$$\bar{\delta}_{DD}^* = 0, \qquad \bar{\delta}_{MM}^* = -\frac{1}{2(2a+r)\sigma_D^2}, \qquad \bar{\delta}_{ii}^* = -\frac{1}{2(2a+r)\sigma_D^2},$$
$$\bar{\delta}_{DM}^* = 0, \qquad \bar{\delta}_{Di}^* = 0, \qquad \bar{\delta}_{Mi}^* = \frac{1}{(2a+r)\sigma_D^2},$$
$$\bar{\delta}_D^* = 0, \qquad \bar{\delta}_M^* = \frac{r\bar{\alpha}}{(a+r)(k+r)}, \qquad \bar{\delta}_i^* = -\frac{r\bar{\alpha}}{(a+r)(k+r)},$$
$$\bar{\delta}_0^* = \frac{r-\beta}{r} - \frac{r\bar{\alpha}^2\sigma_D^2}{2(k+r)^2}$$

and $\bar{\delta}^* = \bar{\delta}(\bar{\epsilon_D}^*, \bar{\epsilon_\pi}^*, \bar{C}^*) = (\bar{\delta}_{DD}^*, \bar{\delta}_{MM}^*, \bar{\delta}_{ii}^*, \bar{\delta}_{DM}^*, \bar{\delta}_{Di}^*, \bar{\delta}_{Mi}^*, \bar{\delta}_D^*, \bar{\delta}_M^*, \bar{\delta}_i^*, \bar{\delta}_0^*)$. Consider the following functions $[0,+\infty[\longrightarrow \mathbb{R}$

$$\epsilon_D{}^*(\sigma_\pi)=\frac{1}{k+r},\qquad \epsilon_\pi^*(\sigma_\pi)=\frac{1}{(a+r)(k+r)},$$
$$C^*(\sigma_\pi)=\frac{a\bar{\pi}}{r(a+r)(k+r)}-\bar{\alpha}\left(\frac{\sigma_D^2}{(k+r)^2}+\frac{\sigma_\pi^2}{(a+r)^2(k+r)^2}\left(1+\frac{2r}{a+\sqrt{a^2+\sigma_\pi^2(\sigma_D^{-2}+\sum_{i=1}^n\sigma_i^{-2})}}\right)\right)$$

$$\delta_{MM}^*(\sigma_\pi)=\delta_{ii}^*(\sigma_\pi)=-\frac{(a+r+o_M^*\nu^*)^2}{2(2a+r+2o_M^*\nu^*)\left((a+r)^2\sigma_D^2+o_M^*(2(a+r)+o_M^*\nu^*)\right)},$$
$$\delta_{Mi}^*(\sigma_\pi)=\frac{(a+r+o_M^*\nu^*)^2}{(2a+r+2o_M^*\nu^*)\left((a+r)^2\sigma_D^2+o_M^*(2(a+r)+o_M^*\nu^*)\right)},$$

$$\delta_{DD}^*(\sigma_\pi)=0,\qquad \delta_{DM}^*(\sigma_\pi)=0,\qquad \delta_{Di}^*(\sigma_\pi)=0,$$
$$\delta_D^*(\sigma_\pi)=0,\qquad \delta_M^*(\sigma_\pi)=\frac{r\bar{\alpha}}{(a+r)(k+r)},\qquad \delta_i^*(\sigma_\pi)=-\frac{r\bar{\alpha}}{(a+r)(k+r)},$$

$$\delta_0^*(\sigma_\pi)=\frac{1}{2r(a+r)^2(k+r)^2}\Big[a^2(2(k+r)^2(r-\beta)-\bar{\alpha}^2r^2\sigma_D^2)+r^2\Big(-2\bar{\alpha}^2o_M^*r+2r^3+$$
$$+2k^2(r-\beta)+4kr(r-\beta)-\bar{\alpha}^2(o_M^*)^2\nu-r^2(2\beta+\bar{\alpha}^2\sigma_D^2)\Big)+2ar(2k^2(r-\beta)+4kr(r-\beta)+$$
$$+r(-\bar{\alpha}^2o_M^*+r(2r-2\beta-\bar{\alpha}^2\sigma_D^2)))\Big],$$

$\delta^*(\sigma_\pi)=(\delta_{DD}^*,\delta_{MM}^*,\delta_{ii}^*,\delta_{DM}^*,\delta_{Di}^*,\delta_{Mi}^*,\delta_D^*,\delta_M^*,\delta_i^*,\delta_0^*)^T$, where $o_M^*=\frac{-a+\sqrt{a^2+\sigma_\pi^2(\sigma_D^{-2}+\sum_{i=1}^n\sigma_i^{-2})}}{\sigma_D^{-2}+\sum_{i=1}^n\sigma_i^{-2}}$ and $\nu^*=\sigma_D^{-2}+\sum_{i=1}^n\sigma_i^{-2}$.

Proof of Theorem 3.2.1. We proceed in several steps.
Claim: If there exists $J\subseteq\{1,\dots,n\}$, such that $J\neq\emptyset$ and $\epsilon_i\neq\sigma_i^{-2}$ for every $i\in J$, then there is no linear equilibrium.
Because of (3.1.1),(3.4.4),(3.4.5) and using Itô's rule on $e^{(a+o_M\nu)t}\hat{\pi}_t^M$, $e^{(a+o_i\nu_i)t}\hat{\pi}_t^i$ and $e^{kt}D_t$ respectively, we get

$$\hat{\pi}_t^M=e^{-(a+o_M\nu)(t-s)}\hat{\pi}_s^M+\frac{a\bar{\pi}}{a+o_M\nu}\left(1-e^{-(a+o_M\nu)(t-s)}\right)+o_M\nu\int_s^t e^{-(a+o_M\nu)(t-u)}\pi_u du+$$
$$+o_M\sigma_D^{-1}\int_s^t e^{-(a+o_M\nu)(t-u)}dW_u^D+o_M\sum_{i=1}^n\epsilon_i\sigma_i\int_s^t e^{-(a+o_M\nu)(t-u)}dW_u^i,$$

$$\hat{\pi}_t^i = e^{-(a+o_i\nu_i)(t-s)}\hat{\pi}_s^i + \frac{a\bar{\pi}}{a+o_i\nu_i}\left(1-e^{-(a+o_i\nu_i)(t-s)}\right) + o_i\nu_i\int_s^t e^{-(a+o_i\nu_i)(t-u)}\pi_u du +$$
$$+ o_i\sigma_D^{-1}\int_s^t e^{-(a+o_i\nu_i)(t-u)}dW_u^D + o_i\sigma_i^{-1}\int_s^t e^{-(a+o_i\nu_i)(t-u)}dW_u^i +$$
$$+ o_i(\epsilon_{i\perp}\sigma_{i\perp})^{-1}\sum_{j\neq i}\epsilon_j\sigma_j\int_s^t e^{-(a+o_i\nu_i)(t-u)}dW_u^j,$$

and

$$D_t = e^{-k(t-s)}D_s + \int_s^t e^{-k(t-u)}\pi_u du + \sigma_D\int_s^t e^{-k(t-u)}dW_u^D.$$

For the market clearing condition (3.4.42) to be satisfied, in particular, the sum of the terms in $(W_t^D)_{t\geq 0}$ has to be identically 0. It follows that for every $0\leq u\leq t$

$$\sigma_D\left(\sum_{i=1}^n\frac{M_D^i}{\alpha_i}\right)e^{-k(t-u)} + \sigma_D^{-1}o_M\left(\sum_{i=1}^n\frac{M_M^i}{\alpha_i}\right)e^{-(a+o_M\nu)(t-u)} + \sigma_D^{-1}\left(\sum_{i=1}^n\frac{M_io_i}{\alpha_i}e^{-(a+o_i\nu_i)(t-u)}\right) = 0. \tag{3.4.43}$$

Since $k\neq a$, there exists $\bar{\sigma_\pi}>0$ such that for every $0\leq\sigma_\pi\leq\bar{\sigma_\pi}$, $a+o_M\nu$ and $a+o_i\nu_i$ are different from k for every $i\in\{1,\dots,n\}$. By Lemma 3.4.2 and since $J\neq\emptyset$, we can write (3.4.43) as

$$\sigma_D\left(\sum_{i=1}^n\frac{M_D^i}{\alpha_i}\right)e^{-k(t-u)} + \sigma_D^{-1}o_M\left(\sum_{i=1}^n\frac{M_M^i}{\alpha_i}\right)e^{-(a+o_M\nu)(t-u)} +$$
$$+\sigma_D^{-1}o_M\left(\sum_{j\notin J}\frac{M_j}{\alpha_j}\right)e^{-(a+o_M\nu)(t-u)} + \sigma_D^{-1}\left(\sum_{j\in J}\frac{M_jo_j}{\alpha_j}e^{-(a+o_j\nu_j)(t-u)}\right) = 0.$$

It follows that there exist $m\in\mathbb{N}, m>0$ and a partition $H_1,\dots,H_m$ of J such that

$$\sum_{i=1}^n\frac{M_D^i}{\alpha_i} = 0 \qquad \text{and} \qquad \sum_{i\in H_1}\frac{M_i}{\alpha_i} = 0.$$

By Definition 3.4.1, it follows that

$$\sum_{i=1}^n\frac{1}{\alpha_i}\left(1-\epsilon_D(k+r)+2\delta_{DD}^i(\epsilon_D\sigma_D^2+\epsilon_\pi o_M)+(\delta_{Di}o_i+\delta_{DM}^io_M)(\epsilon_D+\epsilon_\pi o_M\nu)\right) = 0,$$
$$\sum_{i\in H_1}\frac{1}{\alpha_i}\left(\delta_{Di}(\epsilon_D\sigma_D^2+\epsilon_\pi o_M)+(o_M\delta_{Mi}+2o_i\delta_{ii}+1)(\epsilon_D+\epsilon_\pi o_M\nu)\right) = 0. \tag{3.4.44}$$

$(\epsilon_D^0,\epsilon_\pi^0,C^0) := \lim_{\sigma_\pi\to 0^+}(\epsilon_D^{\sigma_\pi},\epsilon_\pi^{\sigma_\pi},C^{\sigma_\pi})$ exists and it is finite because of (3.1.11); thanks to Lemma D.0.1, the solution of the system $f^i(0,\epsilon_D^0,\epsilon_\pi^0,C^0,\delta^i)$ is $\bar{\delta}(\epsilon_D^0,\epsilon_\pi^0,C^0)$, given in Definition D.0.1 for every $i\in\{1,\dots,n\}$. It follows that $(\epsilon_D^0,\epsilon_\pi^0,C^0)$ identifies uniquely

$\delta^i_{DD}, \delta^i_{DM}, \delta_{Di}, \delta_{Mi}, \delta_{ii}$ and taking $\lim_{\sigma_\pi \to 0}$ of both sides of (3.4.44) we get

$$1 - \epsilon^0_D(k+r) + 2\bar{\delta}_{DD}(\epsilon^0_D, \epsilon^0_\pi, C^0)\epsilon_D\sigma^2_D = 0,$$
$$\bar{\delta}_{Di}(\epsilon^0_D, \epsilon^0_\pi, C^0)\epsilon_D\sigma^2_D + \epsilon_D = 0.$$

Solving both equation for ϵ^0_D shows that no solution exists.
Claim: The equilibrium is unique for $\sigma_\pi = 0$.
For $\sigma_\pi = 0$ and for every $i \in \{1, \ldots, n\}$ there exists only one solution of the system $F^i(0, \epsilon^0_D, \epsilon^0_\pi, C^0, \delta^i)$ given in Lemma D.0.2. Such solution is $\epsilon^0_D = \bar{\epsilon_D}^*, \epsilon^0_\pi = \bar{\epsilon_\pi}^*, C^0 = \bar{C}^*$ and $\delta^i = \bar{\delta}(\epsilon^0_D, \epsilon^0_\pi, C^0) = \bar{\delta}^*$ (cf. Definition 3.4.2 and Definition D.0.1).
Claim: If $\epsilon_i = \sigma_i^{-2}$ for every $i \in \{1, \ldots, n\}$, then a continuous equilibrium exists.
$\epsilon_i = \sigma_i^{-2}$ implies $o_i = o_M, \nu_i = \nu$ and $(\hat{\pi}^i_t)_{t\geq 0} = (\hat{\pi}^M_t)_{t\geq 0}$ for every $i \in \{1, \ldots, n\}$ thanks to Lemma 3.4.2. The choice $(\epsilon^{\sigma_\pi}_D, \epsilon^{\sigma_\pi}_\pi, C^{\sigma_\pi}) = (\epsilon^*_D(\sigma_\pi), \epsilon^*_\pi(\sigma_\pi), C^*(\sigma_\pi))$ and $\delta^i(\sigma_\pi) = \delta^*(\sigma_\pi)$ (cf. Definition 3.4.2) implies that the system $F^i(\sigma_\pi, \epsilon^{\sigma_\pi}_D, \epsilon^{\sigma_\pi}_\pi, C^{\sigma_\pi}, \delta^i)$, (cf. (D.0.3)), is equal to 0 for every $i \in \{1, \ldots, n\}$. Because of Theorem 3.4.5, there exists $\bar{\sigma_\pi} > 0$ such that the choice of $(\epsilon^*_D(\sigma_\pi), \epsilon^*_\pi(\sigma_\pi), C^*(\sigma_\pi))$ and of $\delta^*(\sigma_\pi)$ identifies also the optimal strategy for all the investors for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}$ and thus a continuous equilibrium.
Claim: If $\epsilon_i = \sigma_i^{-2}$ for every $i \in \{1, \ldots, n\}$, then the continuous equilibrium is unique.
Due to Lemma D.0.2, for every $i \in \{1, \ldots, n\}$, there exists a unique solution of the system $F^i(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) = 0$ in U neighbourhood of $(0, \bar{\epsilon_D}^*, \bar{\epsilon_\pi}^*, \bar{C}^*, \bar{\delta}^*)$. Since we found a solution $(\sigma_\pi, \epsilon^*_D, \epsilon^*_\pi, C^*, \delta^*(\sigma_\pi))$ such that

$$\lim_{\sigma_\pi \to 0}(\sigma_\pi, \epsilon^*_D(\sigma_\pi), \epsilon^*_\pi(\sigma_\pi), C^*(\sigma_\pi), \delta^*(\sigma_\pi)) = (0, \bar{\epsilon_D}^*, \bar{\epsilon_\pi}^*, \bar{C}^*, \bar{\delta}^*),$$

then such solution has to be unique. □

Appendices

Appendix A

Auxiliary results

Lemma A.0.1. If $A, B \in M_n(\mathbb{R}), x, y \in \mathbb{R}^n, \lambda > 0, z \in \mathbb{R}$ and f is a Riemann integrable function, then the following hold:

(I) $\|AB\| \leq \|A\|\|B\|$;

(II) $\|Ax\| \leq \|A\|\|x\|$;

(III) $\|e^A\| \leq e^{\|A\|}$;

(IV) $\|e^{Ax}\| \leq e^{\|A\|\|x\|}$;

(V) $\|(e^A)^T\| \leq e^{\|A^T\|}$;

(VI) $\|\int_s^t f(u)du\| \leq \int_s^t \|f(u)\|du$;

(VII) $\int_s^t e^{\lambda(t-u)}du \leq \frac{e^{\lambda t}}{\lambda}$;

(VIII) $\|x \otimes y\| \leq \|x\|_\infty \|y\|_1$;

(IX) if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $E[|X|] \leq 3|\mu| + \sigma$;

(X) $|x| \leq x^2 + 1$;

(XI) suppose $J \subseteq \mathbb{N}$ with $|J| < +\infty$, $\mathcal{F}_s$ is a $\sigma-$algebra, η_s^j are positive $\mathcal{F}_s-$measurable random variables and α_j are positive constants for every $j \in J$. For every $0 \leq s \leq t < +\infty$ there exists a positive $\mathcal{F}_s-$measurable random variable η_s^0 such that $\sum_{j \in J} \eta_s^j e^{\alpha_j t} \leq \eta_s^0 e^{\max_{j \in J}\{\alpha_j\} t}$.

Proof. (I), and (II) are true because of [12, Lemma 1.7] while (III),(IV),(VI),(VII),(VIII),(X) and (XI) are direct calculations. (IX) is a property of folded normal random variables and (V) follows from [16, Proposition 2.3] □

Lemma A.0.2. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\mathcal{F}_0 \subseteq \mathcal{F}, H \in \mathcal{F}$ such that $\mathbb{P}(H) > 0$ and X, Y are real valued random variables. If $X \geq Y$ a.s. and $X > Y$ on H, then

(I) $E[X] > E[Y]$;

(II) $E[X|\mathcal{F}_0] > E[Y|\mathcal{F}_0]$ on a positive probability set.

Lemma A.0.3. For every $x \in \mathbb{R}, y > 0, i \in \{1, \dots, n\}$ and for every $\alpha_i > 0$

$$U^i(x) \leq \tilde{U}^i(y) + xy; \tag{A.0.1}$$

where $U^i(\cdot) = -e^{-\alpha_i \cdot}/\alpha_i$ is the utility function of the $i-$th agent and $\tilde{U}^i(\cdot)$ is

$$\tilde{U}^i(y) = \begin{cases} \frac{y}{\alpha_i}(\log y - 1) & y > 0 \\ 0 & y = 0. \end{cases} \tag{A.0.2}$$

Proof. Use the definition of Fenchel conjugate in [21, Subsection 4.4.1]. Table 4.1 and 4.2 show that the conjugate of the function $f(x) = \frac{1}{\alpha_i} e^{\alpha_i x}$ is the function

$$g(y) = \begin{cases} \frac{y}{\alpha_i}(\log y - 1) & y > 0 \\ 0 & y = 0. \end{cases}$$

Thanks to Fenchel-Young Inequality [21, Proposition 4.4.1] for every $x \in \mathbb{R}, y > 0$

$$\frac{1}{\alpha_i} e^{\alpha_i x} \geq -\frac{y}{\alpha_i} (\log y - 1) + xy.$$

Since $U(x)$ is defined in $\mathbb{R}$, the same inequality is true substituting x for $-x$. Conclude multiplying both sides by -1. □

Lemma A.0.4. If a real squared matrix A has all (complex) eigenvalues with strictly negative real part then

$$\lim_{t \to +\infty} e^{At} = 0 \tag{A.0.3}$$

Proof. If A is diagonalizable, then there exist an invertible matrix H and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ with negative real part such that

$$e^{At} = H \begin{pmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n t} \end{pmatrix} H^{-1}$$

and (A.0.3) follows. If A is similar to a Jordan block, then there exist an invertible matrix H, a nilpotent matrix U with only 1 on the upper diagonal and $\lambda \in \mathbb{C}$ with negative real part such that

$$e^{At} = e^{H(\lambda Id+U)H^{-1}t} = He^{(\lambda Id)t}H^{-1}e^{Nt} = H \begin{pmatrix} e^{\lambda t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda t} \end{pmatrix} H^{-1} \left(\sum_{h=0}^{n} \frac{N^h}{h!} t^h \right), \quad \text{(A.0.4)}$$

where $N := HUH^{-1}$ and $H(\lambda Id)H^{-1}$ commute [20, 3.2.7 and 3.2.8 page 181], [16, Proposition A.6]. (A.0.3) is true because, in view of (A.0.4), every entry of e^{At} is a linear combination of powers of t and exponentials with exponent λ with negative real part. Every matrix is triangularizable in $\mathbb{C}$ and the above reasoning can be done for every Jordan block.

□

Appendix B

Appendix to Baseline Model

Definition B.0.1. Fix $T > 0$, define $M = r\epsilon_D^2\sigma_D^2$ and the following function $[0,T] \to \mathbb{R}$.

$$\mathbf{M_D}(\tau) = \Big(1 - e^{-r\tau}\Big)\Big(1 - \epsilon_D(k+r) + 2\epsilon_D\sigma_D^2\delta_{\mathbf{DD}}(\tau)\Big),$$
$$\mathbf{M_0}(\tau) = \Big(1 - e^{-r\tau}\Big)\Big(\epsilon_D\bar{\pi} - rC + \epsilon_D\sigma_D^2\delta_{\mathbf{D}}(\tau)\Big),$$

where

$$\delta_{\mathbf{DD}}(\tau) = \frac{(-1+\epsilon_D(k+r))^2}{2\sigma_D^2\epsilon_D(\epsilon_D r - 2)(e^{r\tau}-1)}\Big[\frac{2 - r\epsilon_D(1 + e^{2(r-\frac{1}{\epsilon_D})\tau})}{2(r\epsilon_D - 1)} + e^{r\tau}\Big],$$

$$\delta_{\mathbf{D}}(\tau) = \frac{1}{2\sigma_D^2(1-e^{r\tau})}\Bigg\{\frac{2e^{(r-\frac{1}{\epsilon_D})\tau}r\epsilon_D\Big(-1+\epsilon_D(k+r)\Big)\Big(krC + \bar{\pi}(r\epsilon_D - 1)\Big)}{(r\epsilon_D-1)^2} +$$
$$+\frac{e^{2(r-\frac{1}{\epsilon_D})\tau}r^2C\Big(-1+\epsilon_D(k+r)\Big)^2}{(r\epsilon_D-2)(r\epsilon_D-1)^2} - \frac{2e^{r\tau}\Big(-1+\epsilon_D(k+r)\Big)\Big(\epsilon_D\bar{\pi}(r\epsilon_D-2) + rC(1+k\epsilon_D)\Big)}{\epsilon_D(r\epsilon_D-2)} +$$
$$+\frac{\Big(-1+\epsilon_D(k+r)\Big)\Big(-2\bar{\pi}\epsilon_D(r\epsilon_D-1) + rC(-1-k\epsilon_D+r\epsilon_D)\Big)}{\epsilon_D(r\epsilon_D-1)^2}\Bigg\},$$

$$
\begin{aligned}
\delta_{\mathbf{0}}(\tau) = \frac{1}{8\sigma_D^2(1-e^{r\tau})}\Bigg\{ & \frac{e^{2(r-\frac{1}{\epsilon_D})\tau} r\Big(-1+\epsilon_D(k+r)\Big)^2\Big(2r^2C^2+\epsilon_D\sigma_D^2(r\epsilon_D-1)\Big)}{(r\epsilon_D-2)(r\epsilon_D-1)^3} + \\
& + \frac{8e^{(r-\frac{1}{\epsilon_D})\tau} r^2 C\epsilon_D\Big(-1+\epsilon_D(k+r)\Big)(rCk+\bar{\pi}(r\epsilon_D-1)}{(-1+r\epsilon_D)^3} + \\
& - \frac{4e^{r\tau}}{r\epsilon_D(r\epsilon_D-2)}\Big[\bar{\pi}^2\epsilon_D(2-r\epsilon_D) + 2rC\bar{\pi}\epsilon_D(k+r)(r\epsilon_D-2) + r^2C^2\Big(r+2k\epsilon_D(k+r)\Big) + \\
& + \sigma_D^2\Big(1+k^2\epsilon_D^2+2k\epsilon_D(r\epsilon_D-1)+\epsilon_D(r\epsilon_D-2)(-2\beta+3r-2r^2\tau)\Big)\Big] + \\
& - \frac{2}{\epsilon_D(r\epsilon_D-1)^2}\Big[2C^2k^2r^2\epsilon_D\tau + 4Ckr\bar{\pi}\epsilon_D(r\epsilon_D-1)\tau + 2\bar{\pi}^2\epsilon_D(r\epsilon_D-1)^2\tau + \\
& + \sigma_D^2(r\epsilon_D-1)\Big(\tau\Big(-1+2\epsilon_D(k+r-2\beta)-\epsilon_D^2((k+r)^2-4r\beta)\Big) + \frac{1}{2r\sigma_D^2(r\epsilon_D-1)^2}\Big(8\bar{\pi}\epsilon_D rC(r\epsilon_D + \\
& -1)(r(r\epsilon_D-1)+k(2r\epsilon_D-1)) - 2r^2C^2\Big(2k^2\epsilon_D+r^3\epsilon_D^2-2r^2\epsilon_D(k\epsilon_D+1)+r(1+2k\epsilon_D-5k^2\epsilon_D^2) + \\
& + (r\epsilon_D-1)\Big(4\bar{\pi}^2\epsilon_D(r\epsilon_D-1)^2 + \sigma_D^2(k^2\epsilon_D^2(2-3r\epsilon_D) - (r\epsilon_D-1)^2(-2+11r\epsilon_D-8\beta\epsilon_D) + \\
& - 2k\epsilon_D(2-5r\epsilon_D+3r^2\epsilon_D^2))\Big)\Big)\Big)\Big)\Big]\Bigg\}.
\end{aligned}
$$

Lemma B.0.1. Fix $T > 0$, for every $0 \le t \le T$, there exists a constant $\eta(T)$ such that

(a) $|\delta_{\mathbf{DD}}(T-t)|, |\delta_{\mathbf{D}}(T-t)|, |\delta_{\mathbf{0}}(T-t)| \le \eta(T)$;

(b) $\left|\frac{1}{1-e^{-r(T-t)}}\delta_{\mathbf{DD}}(T-t)\right|, \left|\frac{1}{1-e^{-r(T-t)}}\delta_{\mathbf{D}}(T-t)\right|, \left|\frac{1}{1-e^{-r(T-t)}}\delta_{\mathbf{0}}(T-t)\right| \le \eta(T)$;

(c) $|\mathbf{M_D}(T-t)|, |\mathbf{M_0}(T-t)| \le \eta(T)$;

(d) $\left|\frac{1}{1-e^{-r(T-t)}}\mathbf{M_D}(T-t)\right|, \left|\frac{1}{1-e^{-r(T-t)}}\mathbf{M_0}(T-t)\right| \le \eta(T)$;

(e) for every $i \in \{1, \dots, n\}$, $\bar{E}\left[\left|\epsilon_D\sigma_D r\int_0^t \frac{\theta_u^{iT}}{1-e^{-r(T-u)}}d\bar{W}_u^D\right|\right] \le \eta(T)$, where $(\theta_u^{iT})_{u\ge0}$ is in Theorem 1.4.7.

Proof. Since $|\delta_{\mathbf{DD}}(T)| < +\infty$ and $\lim_{t\to T}|\delta_{\mathbf{DD}}(T-t)| < +\infty$, then the function $\delta_{\mathbf{DD}}(T-\cdot) \in \mathscr{C}^0[0,T[$. The function $\delta_{\mathbf{DD}}(T-\cdot)$ has finite extremal points and, because of Weierstrass' Extreme Value Theorem, its absolute value is bounded by a positive constant. The same proof holds for all the inequalities in (a) and (b). Since

$$
\begin{aligned}
|\mathbf{M_D}(T-t)| &\le |1-e^{-r(T-t)}||1-\epsilon_D(k+r)+2\epsilon_D\sigma_D^2\delta_{\mathbf{DD}}(T-t)|,\\
|\mathbf{M_0}(T-t)| &\le |1-e^{-r(T-t)}||\epsilon_D\bar{\pi}-rC+\epsilon_D\sigma_D^2\delta_{\mathbf{D}}(T-t)|,
\end{aligned}
$$

and all the terms are bounded by a constant, then (c) follows. Equation (d) follows from (a) and (c). Lemma A.0.1 (X) leads to

$$
\bar{E}\left[\left|\int_0^t \frac{\theta_u^{iT}}{1-e^{-r(T-u)}}d\bar{W}_u^D\right|\right] \le \bar{E}\left[\left(\int_0^t \frac{\theta_u^{iT}}{1-e^{-r(T-u)}}d\bar{W}_u^D\right)^2 + 1\right]
$$

and applying Itô's isometry it follows that

$$\bar{E}\left[\left|\int_0^t \frac{\theta_u^{iT}}{1-e^{-r(T-u)}} d\bar{W}_u^D\right|\right] \leq 1 + \int_0^t \bar{E}\left[\left(\frac{\theta_u^{iT}}{1-e^{-r(T-u)}}\right)^2\right] du.$$

Thanks to the definition of $(\theta_u^{iT})_{u\geq 0}$ in Theorem 1.4.7 we get

$$\left|\frac{\theta_u^{iT}}{1-e^{-r(T-u)}}\right| \leq \frac{1}{|M\alpha_i|}\left|\frac{\mathbf{M_D}(T-u)}{1-e^{-r(T-u)}}\right| |D_u| + \frac{1}{|M\alpha_i|}\left|\frac{\mathbf{M_0}(T-u)}{1-e^{-r(T-u)}}\right|$$

which implies

$$\bar{E}\left[\left|\int_0^t \frac{\theta_u^{iT}}{1-e^{-r(T-u)}} d\bar{W}_u^D\right|\right] \leq 1 + \int_0^t \eta(T)(\bar{E}[D_u^2]+1) du \leq \eta(T)$$

because of (1.4.11) (iii). □

Appendix C

Appendix to Perfect Information

Definition C.0.1. Define the function

$$\begin{aligned} \bar{\delta} : \mathbb{B} \times \mathbb{R}^* \times \mathbb{R} &\longrightarrow \mathbb{R}^{10} \\ (\epsilon_D, \epsilon_\pi, C) &\longrightarrow (\bar{\delta}_{DD}, \bar{\delta}_{\pi\pi}, \bar{\delta}_{D\pi}, \bar{\delta}_D, \bar{\delta}_\pi, \bar{\delta}_0) \end{aligned} \tag{C.0.1}$$

where

$$\bar{\delta}_{DD} = \frac{\Big(1-\epsilon_D(k+r)\Big)^2}{2\epsilon_D(r\epsilon_D-2)\sigma_D^2},\ \bar{\delta}_{D\pi} = -\frac{(1-\epsilon_D(k+r)\Big)(\epsilon_D(r\epsilon_D-2)+\epsilon_\pi(a+r)(1+k\epsilon_D)\Big)}{\epsilon_D\sigma_D^2(1+a\epsilon_D)(r\epsilon_D-2)},$$

$$\bar{\delta}_{\pi\pi} = -\frac{1}{2\epsilon_D\sigma_D^2(2a+r)(1+a\epsilon_D)(r\epsilon_D-2)}\Big\{\epsilon_D(1+a\epsilon_D)(r\epsilon_D-2)+$$
$$-2\epsilon_D\epsilon_\pi(a+r)(a+k+r)(r\epsilon_D-2)+\epsilon_\pi^2(a+r)^2(a(r\epsilon_D-2)-r-2k(k+r))\Big\},$$

$$\bar{\delta}_D = \frac{-\Big(1-\epsilon_D(k+r)\Big)\Big(rC(1+a\epsilon_D)(1+k\epsilon_D)+a\bar{\pi}(\epsilon_D^2(r\epsilon_D-2)+\epsilon_\pi(1+k\epsilon_D)(r\epsilon_D-1))\Big)}{\epsilon_D\sigma_D^2(1+a\epsilon_D)(r\epsilon_D-2)},$$

$$\bar{\delta}_\pi = \frac{1}{\epsilon_D\sigma_D^2(r\epsilon_D-2)(1+a\epsilon_D)(a+r)(2a+r)}\Big\{-\epsilon_D(-2+\epsilon_D r)(-Cr^2(k+r)+a^2(\epsilon_D\bar{\pi}-2Cr)+$$
$$+a(\bar{\pi}-Cr(2k+3r)))+\epsilon_\pi(Cr^3(r+2\epsilon_D k(k+r))+2a^3(\epsilon_D\bar{\pi}(-2+\epsilon_D r)(-1+\epsilon_D(k+r))+$$
$$+Cr(1+\epsilon_D^2 k(k+r)))+ar(\epsilon_D\bar{\pi}(-2+\epsilon_D r)(k+\epsilon_D r(k+r))+Cr(4r+6\epsilon_D k(k+r)+\epsilon_D^2 kr(k+r)))+$$
$$+a^2 r(\epsilon_D\bar{\pi}(-2+\epsilon_D r)(-2+3\epsilon_D(k+r))+C(5r+4\epsilon_D k(k+r)+3\epsilon_D^2 kr(k+r))))+a\epsilon_\pi^2\bar{\pi}(a+r)(a^2(-2+\epsilon_D r)+$$
$$+r^2(-1+\epsilon_D(r+\epsilon_D k(k+r)))+a(-2\epsilon_D k^2+r(-3+2\epsilon_D(-k+r+\epsilon_D k(k+r)))))\Big\},$$

$$\bar{\delta}_0 = \frac{1}{2r\epsilon_D\sigma_D^2(r\epsilon_D-2)(a+r)(2a+r)(1+a\epsilon_D)}\Big\{a^4\epsilon_\pi^2\bar{\pi}^2r(2-\epsilon_Dr)+C^2r^4(r+2\epsilon_Dk(k+r))+$$
$$+a^3(2C^2\epsilon_Dr^2(r+2\epsilon_Dk(k+r))+4C\bar{\pi}r(\epsilon_D^2(k+r)(-2+\epsilon_Dr)+\epsilon_\pi(1+\epsilon_Dk)r(-1+\epsilon_D(k+r)))+$$
$$+\bar{\pi}^2(-2\epsilon_D^2(-2+\epsilon_Dr)+2\epsilon_D\epsilon_\pi r(-2+\epsilon_Dr)+\epsilon_\pi^2r(2\epsilon_Dk^2+r(3+2\epsilon_Dk-\epsilon_Dr)))+$$
$$+2\epsilon_D\sigma_D^2(1+\epsilon_D(\epsilon_Dk^2+2k(-1+\epsilon_Dr)+(-2+\epsilon_Dr)(3r-2\beta))))+ar(Cr(Cr(3+\epsilon_Dr)(r+2\epsilon_Dk(k+r))+$$
$$+2\bar{\pi}(\epsilon_\pi(1+\epsilon_Dk)r^2(-1+\epsilon_D(k+r))+\epsilon_D(-2+\epsilon_Dr)(k+\epsilon_Dr(k+r))))+(3+\epsilon_Dr)\sigma_D^2(1+\epsilon_D(\epsilon_Dk^2+$$
$$+2k(-1+\epsilon_Dr)+(-2+\epsilon_Dr)(3r-2\beta))))+a^2(C^2r^2(2+3\epsilon_Dr)(r+2\epsilon_Dk(k+r))+$$
$$+2C\bar{\pi}r(3\epsilon_\pi(1+\epsilon_Dk)r^2(-1+\epsilon_D(k+r))+\epsilon_D(-2+\epsilon_Dr)(2k+3\epsilon_Dr(k+r)))+\bar{\pi}^2(\epsilon_\pi^2r^3+$$
$$+2\epsilon_D^2r(-1+\epsilon_\pi r(k+r))+2\epsilon_D(2+\epsilon_\pi r(k+r)(-2+\epsilon_\pi kr)))+(2+3\epsilon_Dr)\sigma_D^2(1+\epsilon_D(\epsilon_Dk^2+2k(-1+\epsilon_Dr)+$$
$$+(-2+\epsilon_Dr)(3r-2\beta))))+r^2\sigma_D^2(1+\epsilon_D(\epsilon_Dk^2+2k(-1+\epsilon_Dr)+(-2+\epsilon_Dr)(3r-2\beta)))\Big\}.$$

Lemma C.0.1. Let $\delta = (\delta_{DD}, \delta_{\pi\pi}, \delta_{D\pi}, \delta_D, \delta_\pi, \delta_0)$ and define the function

$$f : \mathbb{R}^+ \times \mathbb{B} \times \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^6 \longrightarrow \mathbb{R}^6$$
$$(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) \longrightarrow \begin{pmatrix} f_1(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta) \\ \vdots \\ f_6(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta) \end{pmatrix}, \qquad \text{(C.0.2)}$$

where $f_1(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta), \ldots, f_6(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta)$ are stated at the end of the lemma. For every $\epsilon_D^0 \in \mathbb{B}, \epsilon_\pi^0 \neq 0$ and $C^0 \in \mathbb{R}$ fixed, $\bar{\delta}(\epsilon_D^0, \epsilon_\pi^0, C^0)$ (cf. Definition C.0.1) is the unique solution of the system $f(0, \epsilon_D^0, \epsilon_\pi^0, C^0, \delta) = 0$ and there exist

- $U(\epsilon_D^0, \epsilon_\pi^0, C^0) \subseteq \mathbb{R}^4 \times \mathbb{R}^6$ open neighbourhood of $\left(0, \epsilon_D^0, \epsilon_\pi^0, C^0, \bar{\delta}(\epsilon_D^0, \epsilon_\pi^0, C^0)\right)$;
- $W(\epsilon_D^0, \epsilon_\pi^0, C^0) \subseteq \mathbb{R}^4$ open neighbourhood of $(0, \epsilon_D^0, \epsilon_\pi^0, C^0)$;

such that

- for every $(\sigma_\pi, \epsilon_D, \epsilon_\pi, C) \in W(\epsilon_D^0, \epsilon_\pi^0, C^0)$, there exists a unique δ such that

$$(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta) \in U(\epsilon_D^0, \epsilon_\pi^0, C^0) \qquad \text{and} \qquad f(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta) = 0.$$

- If this δ is defined to be $g(\sigma_\pi, \epsilon_D, \epsilon_\pi, C)$, then $g \in \mathscr{C}^1(W, U)$ and $g(0, \epsilon_D^0, \epsilon_\pi^0, C^0) = \bar{\delta}(\epsilon_D^0, \epsilon_\pi^0, C^0)$.

$$f_1 = -\sigma_\pi^2\Big(\sigma_D^2\left(\epsilon_D\delta_{D\pi} - 2\epsilon_\pi\delta_{DD}\right)^2 + 2\epsilon_\pi\left(\epsilon_D(k+r)\delta_{D\pi} - \epsilon_\pi(2k+r)\delta_{DD}\right)\Big) - 2\sigma_D^2\epsilon_D\delta_{DD}\left(r\epsilon_D - 2\right) +$$
$$+2\epsilon_\pi\sigma_\pi^2\delta_{D\pi} + \epsilon_D(k+r)\left(\epsilon_D(k+r) - 2\right) + 1,$$
$$f_2 = \epsilon_D^2\left(2\delta_{\pi\pi}\sigma_D^2\left(2a - 2\delta_{\pi\pi}\sigma_\pi^2 + r\right) + 1\right) - 2\epsilon_\pi\epsilon_D\left(\sigma_D^2\delta_{D\pi} + 1\right)\left(a - 2\delta_{\pi\pi}\sigma_\pi^2 + r\right) +$$
$$+\epsilon_\pi^2\left((a+r)^2 - \sigma_\pi^2\left(\delta_{D\pi}\left(\sigma_D^2\delta_{D\pi} + 2\right) + 2r\delta_{\pi\pi}\right)\right),$$

$$
\begin{aligned}
f_3 &= 2(\epsilon_D(\epsilon_\pi(a+r)(-2\sigma_D^2\delta_{DD}+k+r)+\epsilon_\pi\sigma_\pi^2(\sigma_D^2(4\delta_{\pi\pi}\delta_{DD}+\delta_{D\pi}^2)+\delta_{D\pi}-2\delta_{\pi\pi}(k+r))+\\
&\quad+\sigma_D^2\delta_{D\pi}+1)+\epsilon_D^2(-(\sigma_D^2(-\delta_{D\pi})(a-2\delta_{\pi\pi}\sigma_\pi^2)+k+r))-\epsilon_\pi(a+r)+\\
&\quad+\epsilon_\pi\sigma_\pi^2(\epsilon_\pi(k\delta_{D\pi}-2(\sigma_D^2\delta_{DD}\delta_{D\pi}+\delta_{DD}))+2\delta_{\pi\pi})),\\
f_4 &= 4a\bar{\pi}\epsilon_\pi\sigma_D^2\epsilon_D\delta_{DD}-2a\bar{\pi}\sigma_D^2\epsilon_D^2\delta_{D\pi}-2a\bar{\pi}k\epsilon_\pi\epsilon_D-2a\bar{\pi}r\epsilon_\pi\epsilon_D+2a\bar{\pi}\epsilon_\pi-4Cr\sigma_D^2\epsilon_D\delta_{DD}+\\
&\quad+2Ckr\epsilon_D+2Cr^2\epsilon_D-2Cr\epsilon_\pi\sigma_\pi^2\delta_{D\pi}-2Cr-4\epsilon_\pi^2\delta_D\sigma_D^2\sigma_\pi^2\delta_{DD}+4\epsilon_\pi\delta_\pi\sigma_D^2\epsilon_D\sigma_\pi^2\delta_{DD}+\\
&\quad+2\epsilon_\pi\delta_\pi\sigma_\pi^2-2\delta_\pi\sigma_D^2\epsilon_D^2\sigma_\pi^2\delta_{D\pi}+2\epsilon_\pi\delta_D\sigma_D^2\epsilon_D\sigma_\pi^2\delta_{D\pi}+2\delta_D\sigma_D^2\epsilon_D+2k\epsilon_\pi^2\delta_D\sigma_\pi^2-2k\epsilon_\pi\delta_\pi\epsilon_D\sigma_\pi^2+\\
&\quad+2r\epsilon_\pi^2\delta_D\sigma_\pi^2-2r\epsilon_\pi\delta_\pi\epsilon_D\sigma_\pi^2,\\
f_5 &= -2a^2\bar{\pi}\epsilon_\pi^2+2aCr\epsilon_\pi+2a\delta_\pi\sigma_D^2\epsilon_D^2-2a\epsilon_\pi\delta_D\sigma_D^2\epsilon_D+2a\bar{\pi}\epsilon_\pi\sigma_D^2\epsilon_D\delta_{D\pi}-4a\bar{\pi}\delta_{\pi\pi}\sigma_D^2\epsilon_D^2+\\
&\quad+2a\bar{\pi}\epsilon_\pi\epsilon_D-2a\bar{\pi}r\epsilon_\pi^2-2Cr\sigma_D^2\epsilon_D\delta_{D\pi}-2Cr\epsilon_D+2Cr^2\epsilon_\pi-4Cr\epsilon_\pi\delta_{\pi\pi}\sigma_\pi^2-4\delta_\pi\delta_{\pi\pi}\sigma_D^2\epsilon_D^2\sigma_\pi^2+\\
&\quad+4\epsilon_\pi\delta_{\pi\pi}\delta_D\sigma_D^2\epsilon_D\sigma_\pi^2-2\epsilon_\pi^2\delta_D\sigma_\pi^2-2\epsilon_\pi^2\delta_D\sigma_D^2\sigma_\pi^2\delta_{D\pi}+2\epsilon_\pi\delta_\pi\sigma_D^2\epsilon_D\sigma_\pi^2\delta_{D\pi}+2\epsilon_\pi\delta_\pi\epsilon_D\sigma_\pi^2+\\
&\quad+2r\delta_\pi\sigma_D^2\epsilon_D^2-2r\epsilon_\pi\delta_D\sigma_D^2\epsilon_D,\\
f_6 &= a^2\bar{\pi}^2\epsilon_\pi^2-2Cr\left(a\bar{\pi}\epsilon_\pi+\epsilon_\pi\delta_\pi\sigma_\pi^2+\delta_D\sigma_D^2\epsilon_D\right)+2a\bar{\pi}\sigma_D^2\epsilon_D\left(\epsilon_\pi\delta_D-\delta_\pi\epsilon_D\right)+2\beta\epsilon_\pi^2\sigma_\pi^2+C^2r^2+\\
&\quad-2\epsilon_\pi^2\sigma_D^2\sigma_\pi^2\delta_{DD}-2\sigma_D^4\epsilon_D^2\delta_{DD}-2\epsilon_\pi^2\delta_{\pi\pi}\sigma_\pi^4+2\beta\sigma_D^2\epsilon_D^2-2\delta_{\pi\pi}\sigma_D^2\epsilon_D^2\sigma_\pi^2-\delta_\pi^2\sigma_D^2\epsilon_D^2\sigma_\pi^2-\epsilon_\pi^2\delta_D^2\sigma_D^2\sigma_\pi^2+\\
&\quad+2\epsilon_\pi\delta_\pi\delta_D\sigma_D^2\epsilon_D\sigma_\pi^2+2\delta_0r\sigma_D^2\epsilon_D^2-2r\sigma_D^2\epsilon_D^2+2\delta_0r\epsilon_\pi^2\sigma_\pi^2-2r\epsilon_\pi^2\sigma_\pi^2.
\end{aligned}
$$

Proof. $f(0,\delta_{DD},\delta_{D\pi},\delta_{\pi\pi},\delta_D,\delta_\pi,\delta_0)=0$ is a linear system with 6 unknown and 6 equation which admits as unique solution $(\bar{\delta}_{DD},\bar{\delta}_{D\pi},\bar{\delta}_{\pi\pi},\bar{\delta}_D,\bar{\delta}_\pi,\bar{\delta}_0)$. Direct calculations lead to

$$
\begin{aligned}
&\det\left(\nabla_{(\delta_{DD},\delta_{D\pi},\delta_{\pi\pi},\delta_D,\delta_\pi,\delta_0)}f(0,\delta_{DD},\delta_{D\pi},\delta_{\pi\pi},\delta_D,\delta_\pi,\delta_0)\right)\\
&\qquad\qquad\qquad\qquad = 64r(a+r)(2a+r)\epsilon_D^9(1+a\epsilon_D)(-2+r\epsilon_D)\sigma_D^{12}.
\end{aligned}
$$

The Implicit Function Theorem [32, Theorem 9.28] concludes the proof. □

Appendix D

Appendix to Heterogeneous information

Since Theorem 10.3 in [25] is fundamental for the solution of the filtering problem of Lemma 3.4.1, we write it here with a more convenient notation.

Theorem D.0.1. *$(W_t^\pi, W_t^D, W_t^A, W_t^B)^T$ is a 4−dimensional Brownian motion, $a_0, a_1, b \in \mathbb{R}$, $A_1 \in \mathbb{R}^3$ and $A_2, B \in M_3(\mathbb{R})$. The real process $(\Pi_t)_{t\geq 0}$ and the process $(\Psi_t)_{t\geq 0}$, with values in $\mathbb{R}^3$, have dynamics*

$$\begin{aligned} d\Pi_t &= (a_0 + a_1\Pi_t)dt + bdW_t^\pi \\ d\Psi_t &= (A_1\Pi_t + A_2\Psi_t)dt + B(dW_t^D, dW_t^A, dW_t^B)^T. \end{aligned}$$

The stationary Kalman-Bucy filter for the process $(\Pi_t)_{t\geq 0}$, with signal $(\Psi_t)_{t\geq 0}$, is the process $(\hat{\Pi}_t)_{t\geq 0}$, solution of the stochastic differential equation

$$d\hat{\Pi}_t = (a_0 + a_1\hat{\Pi}_t)dt + oA_1^T(BB^T)^{-1}[d\Psi_t - (A_1\Pi_t + A_2\Psi_t)dt],$$

where $o \in \mathbb{R}$ is the only positive solution of the quadratic equation

$$0 = 2a_1 o + b^2 - o^2(A_1^T B^2 A_1)$$

and $(BB^T)^{-1/2}[d\Psi_t - (A_1\Pi_t + A_2\Psi_t)dt]$ is a Brownian motion, adapted to the filtration generated by $(\Psi_t)_{t\geq 0}$.

Definition D.0.1. Define the function

$$\begin{aligned} \bar{\delta} : \mathbb{B} \times \mathbb{R}^* \times \mathbb{R} &\longrightarrow \mathbb{R}^{10} \\ (\epsilon_D, \epsilon_\pi, C) &\longrightarrow (\bar{\delta}_{DD}, \bar{\delta}_{MM}, \bar{\delta}_{ii}, \bar{\delta}_{DM}, \bar{\delta}_{Di}, \bar{\delta}_{Mi}, \bar{\delta}_D, \bar{\delta}_M, \bar{\delta}_i, \bar{\delta}_0) \end{aligned} \tag{D.0.1}$$

where

$$\bar{\delta}_{MM} = -\frac{\epsilon_\pi^2(a+r)^2(-r-2\epsilon_D k(k+r)+a(-2+\epsilon_D r))}{2\epsilon_D(1+a\epsilon_D)(2a+r)(-2+\epsilon_D r)\sigma_D^2}, \quad \bar{\delta}_{DD} = \frac{(-1+\epsilon_D(k+r))^2}{2\epsilon_D(r\epsilon_D-2)\sigma_D^2},$$

$$\bar{\delta}_{ii} = -\frac{1}{2(2a+r)\sigma_D^2}, \quad \bar{\delta}_{Mi} = \frac{\epsilon_\pi(a+r)(a+k+r)}{(1+a\epsilon_D)(2a+r)\sigma_D^2},$$

$$\bar{\delta}_{DM} = \frac{\epsilon_\pi(1+\epsilon_D k)(a+r)(-1+\epsilon_D(k+r))}{\epsilon_D(1+a\epsilon_D)(-2+\epsilon_D r)\sigma_D^2}, \quad \bar{\delta}_{Di} = \frac{-1+\epsilon_D(k+r)}{(1+a\epsilon_D)\sigma_D^2},$$

$$\bar{\delta}_D = \frac{(-1+\epsilon_D(k+r))(C(1+a\epsilon_D)(1+\epsilon_D k)r + a\bar{\pi}(\epsilon_D^2(-2+\epsilon_D r)+\epsilon_\pi(1+\epsilon_D k)(-1+\epsilon_D r)))}{\epsilon_D(1+a\epsilon_D)(-2+\epsilon_D r)\sigma_D^2},$$

$$\begin{aligned}\bar{\delta}_M = {} & \frac{1}{\epsilon_D(1+a\epsilon_D)(2a+r)(-2+\epsilon_D r)\sigma_D^2}\Big[\epsilon_\pi(a^3\epsilon_\pi\bar{\pi}(-2+\epsilon_D r) + Cr^2(r+2\epsilon_D k(k+r))+ \\ & + a(\epsilon_\pi(1+\epsilon_D k)\bar{\pi}r^2(-1+\epsilon_D(k+r)) + \epsilon_D\bar{\pi}(-2+\epsilon_D r)(k+\epsilon_D r(k+r)) + Cr(3r+ \\ & + 4\epsilon_D k(k+r) + \epsilon_D^2 kr(k+r))) + a^2(\epsilon_D\bar{\pi}(-2+\epsilon_D r)(-1+2\epsilon_D(k+r)) + 2Cr(1+ \\ & + \epsilon_D^2 k(k+r)) + \epsilon_\pi\bar{\pi}(-2\epsilon_D k^2 + r(-3+2\epsilon_D(-k+r+\epsilon_D k(k+r))))))\Big],\end{aligned}$$

$$\bar{\delta}_i = \frac{-a^3\epsilon_\pi\bar{\pi} + Cr^2(k+r) + a(-\bar{\pi}+Cr(2k+3r)) + a^2(2Cr-\bar{\pi}(\epsilon_D+\epsilon_\pi(k+r)))}{(1+a\epsilon_D)(a+r)(2a+r)\sigma_D^2},$$

$$\begin{aligned}\bar{\delta}_0 = {} & \frac{1}{2\epsilon_D(1+a\epsilon_D)r(a+r)(2a+r)(-2+\epsilon_D r)\sigma_D^2}\Big[a^4\epsilon_\pi^2\bar{\pi}^2 r(2-\epsilon_D r) + C^2r^4(r+2\epsilon_D k(k+r))+ \\ & + a^3(2C^2\epsilon_D r^2(r+2\epsilon_D k(k+r)) + 4C\bar{\pi}r(\epsilon_D^2(k+r)(-2+\epsilon_D r) + \epsilon_\pi(1+\epsilon_D k)r(-1+\epsilon_D(k+r)))+ \\ & + \bar{\pi}^2(-2\epsilon_D^2(-2+\epsilon_D r) + 2\epsilon_D\epsilon_\pi r(-2+\epsilon_D r) + \epsilon_\pi^2 r(2\epsilon_D k^2 + r(3+2\epsilon_D k-\epsilon_D r)))+ \\ & + 2\epsilon_D\sigma_D^2(1+\epsilon_D(\epsilon_D k^2 + 2k(-1+\epsilon_D r) + (-2+\epsilon_D r)(3r-2\beta)))) + ar(Cr(Cr(3+\epsilon_D r)(r+ \\ & + 2\epsilon_D k(k+r)) + 2\bar{\pi}(\epsilon_\pi(1+\epsilon_D k)r^2(-1+\epsilon_D(k+r)) + \epsilon_D(-2+\epsilon_D r)(k+\epsilon_D r(k+r))))+ \\ & + (3+\epsilon_D r)\sigma_D^2(1+\epsilon_D(\epsilon_D k^2 + 2k(-1+\epsilon_D r) + (-2+\epsilon_D r)(3r-2\beta)))) + a^2(C^2r^2(2+3\epsilon_D r)(r+ \\ & + 2\epsilon_D k(k+r)) + 2C\bar{\pi}r(3\epsilon_\pi(1+\epsilon_D k)r^2(-1+\epsilon_D(k+r)) + \epsilon_D(-2+\epsilon_D r)(2k+3\epsilon_D r(k+r)))+ \\ & + \bar{\pi}^2(\epsilon_\pi^2 r^3 + 2\epsilon_D^2 r(-1+\epsilon_\pi r(k+r)) + 2\epsilon_D(2+\epsilon_\pi r(k+r)(-2+\epsilon_\pi kr))) + (2+3\epsilon_D r)\sigma_D^2(1+ \\ & + \epsilon_D(\epsilon_D k^2 + 2k(-1+\epsilon_D r) + (-2+\epsilon_D r)(3r-2\beta)))) + r^2\sigma_D^2(1+\epsilon_D(\epsilon_D k^2 + 2k(-1+\epsilon_D r)+ \\ & + (-2+\epsilon_D r)(3r-2\beta)))\Big].\end{aligned}$$

Lemma D.0.1. Let $\delta^i = (\delta^i_{DD}, \delta^i_{MM}, \delta_{ii}, \delta^i_{DM}, \delta_{Di}, \delta_{Mi}, \delta^i_D, \delta^i_M, \delta_i, \delta^i_0) \in \mathbb{R}^{10}$ and for every

$i \in \{1, \dots, n\}$, define the function

$$
\begin{aligned}
f^i : \mathbb{R}^+ \times \mathbb{B} \times \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^{10} &\longrightarrow \mathbb{R}^{10} \\
(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) &\longrightarrow \begin{pmatrix} f_1^i(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) \\ \vdots \\ f_{10}^i(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) \end{pmatrix},
\end{aligned}
\tag{D.0.2}
$$

where $f_1^i(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i), \dots, f_{10}^i(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i)$ are stated at the end of the lemma. For every $\epsilon_D^0 \in \mathbb{B}, \epsilon_\pi^0 \neq 0$ and $C^0 \in \mathbb{R}$ fixed and for every $i \in \{1, \dots, n\}$, $\bar{\delta}(\epsilon_D^0, \epsilon_\pi^0, C^0)$ (cf. Definition D.0.1) is the unique solution of the system $f^i(0, \epsilon_D^0, \epsilon_\pi^0, C^0, \delta^i) = 0$ and there exist

- $U(\epsilon_D^0, \epsilon_\pi^0, C^0) \subseteq \mathbb{R}^4 \times \mathbb{R}^{10}$ open neighbourhood of $\left(0, \epsilon_D^0, \epsilon_\pi^0, C^0, \bar{\delta}(\epsilon_D^0, \epsilon_\pi^0, C^0)\right)$;

- $W(\epsilon_D^0, \epsilon_\pi^0, C^0) \subseteq \mathbb{R}^4$ open neighbourhood of $(0, \epsilon_D^0, \epsilon_\pi^0, C^0)$;

such that

- for every $(\sigma_\pi, \epsilon_D, \epsilon_\pi, C) \in W(\epsilon_D^0, \epsilon_\pi^0, C^0)$, there exists a unique δ^i such that

$$(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) \in U(\epsilon_D^0, \epsilon_\pi^0, C^0) \qquad \text{and} \qquad f^i(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) = 0.$$

- If this δ^i is defined to be $g^i(\sigma_\pi, \epsilon_D, \epsilon_\pi, C)$, then $g^i \in \mathscr{C}^1(W, U)$ and $g^i(0, \epsilon_D^0, \epsilon_\pi^0, C^0) = \bar{\delta}(\epsilon_D^0, \epsilon_\pi^0, C^0)$ for every $i \in \{1, \dots, n\}$.

$$
\begin{aligned}
f_1^i = &-2k\nu o_M \epsilon_\pi \left(-2o_M \epsilon_\pi \delta_{DD} + \epsilon_D o_i \delta_{Di} + \epsilon_D o_M \delta_{DM}\right) - 2k\epsilon_D(-2o_M \epsilon_\pi \delta_{DD} + \epsilon_D o_i \delta_{Di} + \\
&+ \epsilon_D o_M \delta_{DM} - r\epsilon_D + 1) - 4\epsilon_D o_i o_M \epsilon_\pi \delta_{DD} \delta_{Di} - \sigma_D^2 \Big(2\epsilon_D \delta_{DD} \Big(- 2\nu o_M \epsilon_\pi \Big(o_i \delta_{Di} + o_M \delta_{DM}\Big) + \\
&+ r\epsilon_D - 2\Big) + 4\nu o_M^2 \epsilon_\pi^2 \delta_{DD}^2 + \epsilon_D^2 \Big(\nu o_M \delta_{DM} \Big(2o_i \delta_{Di} + o_M \delta_{DM}\Big) + \nu_i o_i^2 \delta_{Di}^2\Big)\Big) + 4o_M^2 \epsilon_\pi^2 \delta_{DD}^2 + \\
&- 4\epsilon_D o_M^2 \epsilon_\pi \delta_{DD} \delta_{DM} + 4o_M \epsilon_\pi \delta_{DD} + 2\nu r o_M^2 \epsilon_\pi^2 \delta_{DD} + \nu^2 o_i^2 o_M^2 \epsilon_\pi^2 \delta_{Di}^2 - \nu \nu_i o_i^2 o_M^2 \epsilon_\pi^2 \delta_{Di}^2 + \\
&+ 2\nu o_i o_M \epsilon_\pi \delta_{Di} + 2\nu \epsilon_D o_i^2 o_M \epsilon_\pi \delta_{Di}^2 - 2\epsilon_D \nu_i o_i^2 o_M \epsilon_\pi \delta_{Di}^2 + 2\epsilon_D^2 o_i o_M \delta_{Di} \delta_{DM} + \\
&- 2r\epsilon_D \Big(\nu o_M \epsilon_\pi \Big(o_i \delta_{Di} + o_M \delta_{DM}\Big) + 1\Big) + \epsilon_D^2 o_i^2 \delta_{Di}^2 + 2\epsilon_D o_i \delta_{Di} - 2r\epsilon_D^2 o_i \delta_{Di} + k^2 \epsilon_D^2 + \\
&+ 2\nu o_M^2 \epsilon_\pi \delta_{DM} + \epsilon_D^2 o_M^2 \delta_{DM}^2 + 2\epsilon_D o_M \delta_{DM} - 2r\epsilon_D^2 o_M \delta_{DM} + r^2 \epsilon_D^2 + 1,
\end{aligned}
$$

$$
\begin{aligned}
f_2^i ={}& -2o_i\delta_{Mi}\Big(o_M\Big(2\epsilon_D^2\delta_{MM}\Big(\nu\sigma_D^2-1\Big)+\epsilon_\pi\Big(\nu\Big((a+r)\epsilon_\pi+\epsilon_D\Big)+\delta_{DM}\Big(\epsilon_D-\nu\sigma_D^2\epsilon_D\Big)\Big)\Big)+\\
&+(a+r)\epsilon_D\epsilon_\pi+\nu^2o_M^2\epsilon_\pi^2\Big)+2o_M\Big(\epsilon_\pi\Big(2a\epsilon_D\delta_{MM}-a\epsilon_\pi\delta_{DM}+\nu(a+r)\epsilon_\pi-r\epsilon_\pi\delta_{DM}\Big)+\\
&+\nu\sigma_D^2\epsilon_D\Big(2\epsilon_D\delta_{MM}-\epsilon_\pi\delta_{DM}\Big)\Big)+2\sigma_D^2\epsilon_D\Big((2a+r)\epsilon_D\delta_{MM}-(a+r)\epsilon_\pi\delta_{DM}\Big)+\\
&+o_i^2\delta_{Mi}^2\Big(2\epsilon_D\Big(\nu-\nu_i\Big)o_M\epsilon_\pi+\epsilon_D^2\Big(1-\sigma_D^2\nu_i\Big)+\nu\Big(\nu-\nu_i\Big)o_M^2\epsilon_\pi^2\Big)+\\
&+o_M^2\Big(-4\epsilon_D^2\delta_{MM}^2\Big(\nu\sigma_D^2-1\Big)-2\epsilon_\pi\delta_{MM}\Big(-2\nu\sigma_D^2\epsilon_D\delta_{DM}+2\epsilon_D\delta_{DM}-2\nu\epsilon_D+\nu r\epsilon_\pi\Big)+\\
&+\epsilon_\pi^2\Big((\delta_{DM}-\nu)^2-\nu\sigma_D^2\delta_{DM}^2\Big)\Big)+(a+r)^2\epsilon_\pi^2,\\
f_3^i ={}& 2\epsilon_Do_M\epsilon_\pi\Big(2\delta_{ii}\left(2o_i\left(\nu o_i\delta_{ii}-\nu_io_i\delta_{ii}+\nu\right)+2a+r\right)+\delta_{Di}\left(\nu\sigma_D^2-1\right)\left(o_M\delta_{Mi}+2o_i\delta_{ii}+\right.\\
&\left.+1\right)+\nu\Big)+o_M^2\epsilon_\pi^2\left(\nu\left(2\delta_{ii}\left(2o_i\left(\nu o_i\delta_{ii}-\nu_io_i\delta_{ii}+\nu\right)+2a+r\right)+\nu\right)+\delta_{Di}^2\left(1-\nu\sigma_D^2\right)\right)+\\
&+\epsilon_D^2\Big(\left(o_M\delta_{Mi}+2o_i\delta_{ii}+1\right)-\sigma_D^2\left(-4a\delta_{ii}+\nu o_M\delta_{Mi}\left(o_M\delta_{Mi}+4o_i\delta_{ii}+2\right)+4\nu_io_i^2\delta_{ii}^2-2r\delta_{ii}\right)\Big),\\
f_4^i ={}& -o_i(\delta_{Di}(o_M(2\epsilon_D^2\delta_{MM}(\nu\sigma_D^2-1)+\epsilon_\pi(\nu((a+r)\epsilon_\pi+\epsilon_D)+\delta_{DM}(\epsilon_D-\nu\sigma_D^2\epsilon_D)))+\\
&+(a+r)\epsilon_D\epsilon_\pi+\nu^2o_M^2\epsilon_\pi^2)+\delta_{Mi}(\epsilon_D(o_M\epsilon_\pi(\delta_{DD}(2-2\nu\sigma_D^2)+\nu(k+r))-1)+\\
&+\epsilon_D^2(o_M(\nu\sigma_D^2-1)\delta_{DM}+k+r)-\nu o_M\epsilon_\pi))+o_M(\epsilon_\pi(-2(a+r)\epsilon_\pi\delta_{DD}+(a+k)\epsilon_D\delta_{DM}+\\
&+\nu(\epsilon_D(k+r)-1)+\delta_{DM})-2\epsilon_D\delta_{MM}(\epsilon_D(k+r)-1)+\nu\sigma_D^2\epsilon_D(\epsilon_D\delta_{DM}-2\epsilon_\pi\delta_{DD}))+\\
&+\sigma_D^2\epsilon_D(-2(a+r)\epsilon_\pi\delta_{DD}+a\epsilon_D\delta_{DM}+\delta_{DM})+o_i^2\delta_{Di}\delta_{Mi}(2\epsilon_D(\nu-\nu_i)o_M\epsilon_\pi+\epsilon_D^2(1-\sigma_D^2\nu_i)+\\
&+\nu(\nu-\nu_i)o_M^2\epsilon_\pi^2)+o_M^2(\delta_{DM}(2\epsilon_\pi^2\delta_{DD}+\nu\epsilon_\pi(\epsilon_\pi(k-2\sigma_D^2\delta_{DD})+\epsilon_D)-2\epsilon_D^2\delta_{MM}(\nu\sigma_D^2-1))+\\
&-2\epsilon_\pi(\nu\delta_{MM}(\epsilon_D(k+r)-1)+\delta_{DD}(-2\nu\sigma_D^2\epsilon_D\delta_{MM}+2\epsilon_D\delta_{MM}+\nu\epsilon_\pi))+\epsilon_D\epsilon_\pi(\nu\sigma_D^2-1)\delta_{DM}^2)+\\
&+(a+r)\epsilon_\pi(\epsilon_D(k+r)-1),\\
f_5^i ={}& \epsilon_D(o_M(\epsilon_\pi(\delta_{Di}(2a-o_M\delta_{DM}+k+r)-2\delta_{DD})-2o_M\epsilon_\pi\delta_{DD}\delta_{Mi}-\nu(k+r)\epsilon_\pi(o_M\delta_{Mi}+\\
&+1)+\delta_{Mi})+o_i(o_M\epsilon_\pi\delta_{Di}(\delta_{Di}(\nu\sigma_D^2-1)+2\nu)-2\delta_{ii}(o_M\epsilon_\pi(\delta_{DD}(2-2\nu\sigma_D^2)+\nu(k+r))+\\
&-1))+\sigma_D^2(\nu o_M\epsilon_\pi(2\delta_{DD}(o_M\delta_{Mi}+1)+o_M\delta_{Di}\delta_{DM})+\delta_{Di})+4(\nu-\nu_i)o_i^2o_M\epsilon_\pi\delta_{ii}\delta_{Di}+1)+\\
&+o_M\epsilon_\pi(\delta_{Di}(o_M\epsilon_\pi(\nu(o_i(2\nu o_i\delta_{ii}-2\nu_io_i\delta_{ii}+\nu)+a+k+r)+\delta_{DD}(2-2\nu\sigma_D^2))+1)+\\
&+\nu o_M\delta_{Mi}+2\nu o_i\delta_{ii}+\nu)+\epsilon_D^2(\sigma_D^2(-\nu o_M(2o_i\delta_{ii}\delta_{DM}+o_M\delta_{DM}\delta_{Mi}+o_i\delta_{Di}\delta_{Mi}+\delta_{DM})+\\
&-2\nu_io_i^2\delta_{ii}\delta_{Di}+a\delta_{Di})-(o_M\delta_{Mi}+2o_i\delta_{ii}+1)(-o_i\delta_{Di}-o_M\delta_{DM}+k+r)),\\
f_6^i ={}& o_M^2(\delta_{Mi}(\epsilon_\pi(\nu(a\epsilon_\pi+\epsilon_D)+\epsilon_D(\nu\sigma_D^2-1)\delta_{DM})-2\epsilon_D^2\delta_{MM}(\nu\sigma_D^2-1))+\\
&-\epsilon_\pi(\delta_{Di}(\nu\sigma_D^2(\epsilon_\pi\delta_{DM}-2\epsilon_D\delta_{MM})+2\epsilon_D\delta_{MM}+\epsilon_\pi(\nu-\delta_{DM}))+\nu^2\epsilon_\pi))+\\
&+o_i(-2\delta_{ii}(\epsilon_D(\epsilon_\pi(a+o_M\delta_{DM}+r)-2\epsilon_Do_M\delta_{MM})+\nu o_M\epsilon_\pi((a+r)\epsilon_\pi+\epsilon_D)+\nu^2o_M^2\epsilon_\pi^2)+\\
&+\delta_{Mi}(\epsilon_Do_M\epsilon_\pi(\nu\sigma_D^2\delta_{Di}-\delta_{Di}+2\nu)+\epsilon_D^2+\nu^2o_M^2\epsilon_\pi^2)+2\nu\sigma_D^2\epsilon_Do_M\delta_{ii}(\epsilon_\pi\delta_{DM}-2\epsilon_D\delta_{MM})+\\
&+\epsilon_D^2o_M(1-\nu\sigma_D^2)\delta_{Mi}^2)+o_M(-\epsilon_\pi(-3a\epsilon_D\delta_{Mi}-r\epsilon_D\delta_{Mi}+a\epsilon_\pi\delta_{Di}+\nu((a+r)\epsilon_\pi+\epsilon_D)+\\
&+r\epsilon_\pi\delta_{Di}+\epsilon_D\delta_{DM})+\nu\sigma_D^2\epsilon_D(\epsilon_D\delta_{Mi}+\epsilon_\pi(\delta_{DM}-\delta_{Di}))-2\epsilon_D^2\delta_{MM}(\nu\sigma_D^2-1))+\\
&+\sigma_D^2\epsilon_D((2a+r)\epsilon_D\delta_{Mi}-(a+r)\epsilon_\pi\delta_{Di})-2o_i^2\delta_{ii}\delta_{Mi}(2\epsilon_D(\nu_i-\nu)o_M\epsilon_\pi+\epsilon_D^2(\sigma_D^2\nu_i-1)+\\
&+\nu(\nu_i-\nu)o_M^2\epsilon_\pi^2)-(a+r)\epsilon_D\epsilon_\pi,
\end{aligned}
$$

$$
\begin{aligned}
f_7^i &= o_i(\delta_i(\epsilon_D(o_M\epsilon_\pi(2\delta_{DD}(\nu\sigma_D^2-1)-\nu(k+r))+1)+\epsilon_D^2(-(o_M(\nu\sigma_D^2-1)\delta_{DM}+k+r))+\\
&+\nu o_M\epsilon_\pi)+\delta_{Di}(o_M(a\nu\bar{\pi}\epsilon_\pi^2-\delta_D\epsilon_D\epsilon_\pi+\nu\sigma_D^2\epsilon_D(\delta_D\epsilon_\pi-\epsilon_D\delta_M)+\epsilon_D^2\delta_M)+a\bar{\pi}\epsilon_D\epsilon_\pi))+\\
&+\sigma_D^2(\epsilon_D(a\bar{\pi}(2\epsilon_\pi\delta_{DD}-\epsilon_D(\delta_{Di}+\delta_{DM}))+\delta_D)+\nu o_M^2(\delta_D\epsilon_\pi-\epsilon_D\delta_M)(\epsilon_D\delta_{DM}-2\epsilon_\pi\delta_{DD}))+\\
&+2a\bar{\pi}o_M\epsilon_\pi^2\delta_{DD}+Cr(-2\sigma_D^2\epsilon_D\delta_{DD}-2o_M\epsilon_\pi\delta_{DD}-\nu o_io_M\epsilon_\pi\delta_{Di}-\epsilon_Do_i\delta_{Di}+k\epsilon_D+\\
&-\nu o_M^2\epsilon_\pi\delta_{DM}-\epsilon_Do_M\delta_{DM}+r\epsilon_D-1)+2\delta_Do_M^2\epsilon_\pi^2\delta_{DD}-2\epsilon_D\delta_Mo_M^2\epsilon_\pi\delta_{DD}-a\nu\bar{\pi}o_M^2\epsilon_\pi^2\delta_{Di}+\\
&-2a\bar{\pi}\epsilon_Do_M\epsilon_\pi\delta_{Di}-ak\bar{\pi}\epsilon_D\epsilon_\pi-a\bar{\pi}\epsilon_Do_M\epsilon_\pi\delta_{DM}-a\bar{\pi}r\epsilon_D\epsilon_\pi+a\bar{\pi}\epsilon_\pi+\delta_io_i^2\delta_{Di}(2\epsilon_D(\nu+\\
&-\nu_i)o_M\epsilon_\pi+\epsilon_D^2(1-\sigma_D^2\nu_i)+\nu(\nu-\nu_i)o_M^2\epsilon_\pi^2)+k\nu\delta_Do_M^2\epsilon_\pi^2-k\nu\epsilon_D\delta_Mo_M^2\epsilon_\pi+k\delta_D\epsilon_Do_M\epsilon_\pi+\\
&-k\epsilon_D^2\delta_Mo_M+\delta_Do_M\epsilon_\pi-\delta_D\epsilon_Do_M^2\epsilon_\pi\delta_{DM}+\epsilon_D^2\delta_Mo_M^2\delta_{DM}+\epsilon_D\delta_Mo_M+\nu r\delta_Do_M^2\epsilon_\pi^2+\\
&-\nu r\epsilon_D\delta_Mo_M^2\epsilon_\pi+r\delta_D\epsilon_Do_M\epsilon_\pi-r\epsilon_D^2\delta_Mo_M+\nu\delta_Mo_M^2\epsilon_\pi,\\
f_8^i &= Cr(-o_i\delta_{Mi}(\epsilon_D+\nu o_M\epsilon_\pi)-o_M(2\epsilon_D\delta_{MM}+\nu\epsilon_\pi(2o_M\delta_{MM}-1)+\epsilon_\pi\delta_{DM})+(a+r)\epsilon_\pi+\\
&+\sigma_D^2\epsilon_D(-\delta_{DM}))-2a\bar{\pi}\epsilon_Do_M\epsilon_\pi\delta_{Mi}+o_i(-\nu o_M\epsilon_\pi(\delta_i((a+r)\epsilon_\pi+\epsilon_D)-a\bar{\pi}\epsilon_\pi\delta_{Mi})+\\
&+\epsilon_D(\epsilon_Do_M(\delta_M\delta_{Mi}+2\delta_i\delta_{MM})-\epsilon_\pi(\delta_{Mi}(\delta_Do_M-a\bar{\pi})+\delta_i(a+o_M\delta_{DM}+r)))+\\
&+\nu\sigma_D^2\epsilon_Do_M(\delta_D\epsilon_\pi\delta_{Mi}+\epsilon_D(-\delta_M)\delta_{Mi}-2\epsilon_D\delta_i\delta_{MM}+\delta_i\epsilon_\pi\delta_{DM})-\nu^2\delta_io_M^2\epsilon_\pi^2)+\\
&+\sigma_D^2(a\epsilon_D(-\bar{\pi}\epsilon_D(\delta_{Mi}+2\delta_{MM})-\delta_D\epsilon_\pi+\epsilon_D\delta_M+\bar{\pi}\epsilon_\pi\delta_{DM})+(\epsilon_D\delta_M+\\
&-\delta_D\epsilon_\pi)(\nu o_M(-2\epsilon_Do_M\delta_{MM}+o_M\epsilon_\pi\delta_{DM}+\epsilon_D)+r\epsilon_D))-2a\bar{\pi}\epsilon_Do_M\epsilon_\pi\delta_{MM}-a\nu\bar{\pi}o_M^2\epsilon_\pi^2\delta_{Mi}+\\
&+\delta_io_i^2\delta_{Mi}(2\epsilon_D(\nu-\nu_i)o_M\epsilon_\pi+\epsilon_D^2(1-\sigma_D^2\nu_i)+\nu(\nu-\nu_i)o_M^2\epsilon_\pi^2)-2\delta_D\epsilon_Do_M^2\epsilon_\pi\delta_{MM}+\\
&+2\epsilon_D^2\delta_Mo_M^2\delta_{MM}-a^2\bar{\pi}\epsilon_\pi^2-a\delta_Do_M\epsilon_\pi^2+a\epsilon_D\delta_Mo_M\epsilon_\pi+a\bar{\pi}o_M\epsilon_\pi^2\delta_{DM}-a\nu\bar{\pi}o_M\epsilon_\pi^2-a\bar{\pi}r\epsilon_\pi^2+\\
&-\nu\delta_Do_M^2\epsilon_\pi^2+\nu\epsilon_D\delta_Mo_M^2\epsilon_\pi+\delta_Do_M^2\epsilon_\pi^2\delta_{DM}-\epsilon_D\delta_Mo_M^2\epsilon_\pi\delta_{DM}-r\delta_Do_M\epsilon_\pi^2+r\epsilon_D\delta_Mo_M\epsilon_\pi,\\
f_9^i &= o_i(2\delta_{ii}(o_M(a\nu\bar{\pi}\epsilon_\pi^2-\delta_D\epsilon_D\epsilon_\pi+\nu\sigma_D^2\epsilon_D(\delta_D\epsilon_\pi-\epsilon_D\delta_M)+\epsilon_D^2\delta_M)+a\bar{\pi}\epsilon_D\epsilon_\pi)+\\
&+\delta_i(\epsilon_D^2(-\nu\sigma_D^2o_M\delta_{Mi}+o_M\delta_{Mi}+1)+\epsilon_Do_M\epsilon_\pi(\nu\sigma_D^2\delta_{Di}-\delta_{Di}+2\nu)+\nu^2o_M^2\epsilon_\pi^2))+\\
&-4a\bar{\pi}\epsilon_Do_M\epsilon_\pi\delta_{ii}-a\bar{\pi}\epsilon_Do_M\epsilon_\pi\delta_{Mi}+\sigma_D^2(\epsilon_D(a\epsilon_D(\delta_i-\bar{\pi}(2\delta_{ii}+\delta_{Mi}))+a\bar{\pi}\epsilon_\pi\delta_{Di}+r\epsilon_D\delta_i)+\\
&-\nu o_M(\epsilon_D\delta_M-\delta_D\epsilon_\pi)(\epsilon_Do_M\delta_{Mi}-o_M\epsilon_\pi\delta_{Di}+\epsilon_D))-2a\nu\bar{\pi}o_M^2\epsilon_\pi^2\delta_{ii}-Cr(o_M\epsilon_\pi(\nu o_M\delta_{Mi}+\\
&+2\nu o_i\delta_{ii}+\delta_{Di}+\nu)+\epsilon_D(o_M\delta_{Mi}+2o_i\delta_{ii}+\sigma_D^2\delta_{Di}+1))+2\delta_io_i^2\delta_{ii}(2\epsilon_D(\nu-\nu_i)o_M\epsilon_\pi+\\
&+\epsilon_D^2(1-\sigma_D^2\nu_i)+\nu(\nu-\nu_i)o_M^2\epsilon_\pi^2)-\delta_D\epsilon_Do_M^2\epsilon_\pi\delta_{Mi}+\epsilon_D^2\delta_Mo_M^2\delta_{Mi}+2a\epsilon_D\delta_io_M\epsilon_\pi+\\
&+a\bar{\pi}o_M\epsilon_\pi^2\delta_{Di}+a\bar{\pi}\epsilon_D\epsilon_\pi+a\nu\delta_io_M^2\epsilon_\pi^2+a\nu\bar{\pi}o_M\epsilon_\pi^2+\delta_Do_M^2\epsilon_\pi^2\delta_{Di}-\epsilon_D\delta_Mo_M^2\epsilon_\pi\delta_{Di}+2r\epsilon_D\delta_io_M\epsilon_\pi+\\
&-\delta_D\epsilon_Do_M\epsilon_\pi+\epsilon_D^2\delta_Mo_M+\nu r\delta_io_M^2\epsilon_\pi^2,
\end{aligned}
$$

$$\begin{aligned}
f_{10}^i = & -2\nu^2\delta_{MM}\epsilon_\pi^2 o_M^4 - 2\nu\delta_{DM}\epsilon_\pi^2 o_M^3 - 4\nu\delta_{MM}\epsilon_D\epsilon_\pi o_M^3 + \delta_M^2\epsilon_D^2 o_M^2 + \delta_D^2\epsilon_\pi^2 o_M^2 - 2r\nu\epsilon_\pi^2 o_M^2 + \\
& + 2\beta\nu\epsilon_\pi^2 o_M^2 + 2r\nu\delta_0\epsilon_\pi^2 o_M^2 - 2a\bar{\pi}\nu\delta_i\epsilon_\pi^2 o_M^2 - 2\delta_D\delta_M\epsilon_D\epsilon_\pi o_M^2 - 4\delta_{DM}\epsilon_D\epsilon_\pi o_M^2 + 2a\bar{\pi}\delta_D\epsilon_\pi^2 o_M + \\
& - 4r\epsilon_D\epsilon_\pi o_M + 4\beta\epsilon_D\epsilon_\pi o_M + 4r\delta_0\epsilon_D\epsilon_\pi o_M - 4a\bar{\pi}\delta_i\epsilon_D\epsilon_\pi o_M - 2a\bar{\pi}\delta_M\epsilon_D\epsilon_\pi o_M - 2\delta_{DD}\epsilon_D^2\sigma_D^4 + \\
& + C^2r^2 + a^2\bar{\pi}^2\epsilon_\pi^2 + (2\epsilon_D((\beta + r(\delta_0 - 1) - a\bar{\pi}(\delta_i + \delta_M) - o_M\delta_{DM})\epsilon_D + (a\bar{\pi}\delta_D + \\
& - 2o_M\delta_{DD})\epsilon_\pi) - \nu o_M^2((\delta_M^2 + 2\delta_{MM})\epsilon_D^2 - 2\delta_D\delta_M\epsilon_\pi\epsilon_D + (\delta_D^2 + 2\delta_{DD})\epsilon_\pi^2))\sigma_D^2 + \\
& - 2Cr(\delta_D\epsilon_D\sigma_D^2 + a\bar{\pi}\epsilon_\pi + o_i\delta_i(\epsilon_D + \nu o_M\epsilon_\pi) + o_M(\delta_M\epsilon_D + \delta_D\epsilon_\pi + \nu o_M\delta_M\epsilon_\pi)) + \\
& + 2o_i(-\nu^2\delta_{Mi}\epsilon_\pi^2 o_M^3 - \nu\epsilon_\pi(2\delta_{Mi}\epsilon_D + \delta_{Di}\epsilon_\pi)o_M^2 + (\delta_i\delta_M\epsilon_D^2 - \nu(\delta_i\delta_M\epsilon_D + \delta_{Mi}\epsilon_D - \delta_D\delta_i\epsilon_\pi)\sigma_D^2\epsilon_D + \\
& - \delta_D\delta_i\epsilon_\pi\epsilon_D - 2\delta_{Di}\epsilon_\pi\epsilon_D + a\bar{\pi}\nu\delta_i\epsilon_\pi^2)o_M + \epsilon_D(a\bar{\pi}\delta_i\epsilon_\pi - \delta_{Di}\epsilon_D\sigma_D^2)) + o_i^2(\delta_i^2((1 - \nu_i\sigma_D^2)\epsilon_D^2 + \\
& + 2o_M\epsilon_\pi(\nu - \nu_i)\epsilon_D + \nu o_M^2\epsilon_\pi^2(\nu - \nu_i)) - 2\delta_{ii}\nu_i(\epsilon_D^2\sigma_D^2 + o_M\epsilon_\pi(2\epsilon_D + \nu o_M\epsilon_\pi))).
\end{aligned}$$

Proof. For every $i \in \{1, \ldots, n\}$, $f^i(0, \epsilon_D^0, \epsilon_\pi^0, C^0, \delta^i) = 0$ is a linear system whose unique solution is $\bar{\delta}(\epsilon_D^0, \epsilon_\pi^0, C^0)$. Direct calculations lead to

$$\det\left(\nabla_{\delta^i} f^i\left(0, \epsilon_D^0, \epsilon_\pi^0, C^0, \bar{\delta}(\epsilon_D^0, \epsilon_\pi^0, C^0)\right)\right) = \\ - 1024r(a+r)^2(2a+r)^3\epsilon_D^{16}(1 + a\epsilon_D)^2(-2 + r\epsilon_D)\sigma_D^{20} \neq 0$$

and the Implicit Function Theorem [32, Theorem 9.28] concludes the proof. □

Lemma D.0.2. Let $\delta^i = (\delta_{DD}^i, \delta_{MM}^i, \delta_{ii}, \delta_{DM}^i, \delta_{Di}, \delta_{Mi}, \delta_D^i, \delta_M^i, \delta_i, \delta_0^i) \in \mathbb{R}^{10}$ and for every $i \in \{1, \ldots, n\}$, define the function

$$\begin{aligned}
F^i : \mathbb{R}^+ \times \mathbb{B} \times \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^{10} & \longrightarrow \mathbb{R}^{13} \\
(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) & \longrightarrow \begin{pmatrix} F_1^i(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) \\ \vdots \\ F_{13}^i(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) \end{pmatrix};
\end{aligned} \tag{D.0.3}$$

where $F_j^i = f_j^i$ of Lemma D.0.1 (with $o_i = o_M$ and $\nu_i = \nu$) for every $i \in \{1, \ldots, n\}, j \in \{1, \ldots, 10\}$ and $F_{11}^i, F_{12}^i, F_{13}^i$ are stated at the end of this lemma.
Under Assumption 3.1.1, for every $i \in \{1, \ldots, n\}$, $(\bar{\epsilon_D}^*, \bar{\epsilon_\pi}^*, \bar{C}^*, \bar{\delta}^*)$ (cf. Definition 3.4.2) is the unique solution of the system $F^i(0, \epsilon_D, \epsilon_\pi, C, \delta^i) = 0$ and there exist

- $U \subseteq \mathbb{R}^4 \times \mathbb{R}^{10}$ open neighbourhood of $(0, \bar{\epsilon_D}^*, \bar{\epsilon_\pi}^*, \bar{C}^*, \bar{\delta}^*)$;
- $\bar{\sigma_\pi} > 0$;

such that

- for every $0 \leq \sigma_\pi \leq \bar{\sigma_\pi}$, there exists a unique $(\epsilon_D, \epsilon_\pi, C, \delta^i)$ such that

$$(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) \in U \qquad \text{and} \qquad F^i(\sigma_\pi, \epsilon_D, \epsilon_\pi, C, \delta^i) = 0.$$

- If this $(\epsilon_D, \epsilon_\pi, C, \delta^i)$ is defined to be $G^i(\sigma_\pi)$, then $G^i \in \mathscr{C}^1([0, \bar{\sigma_\pi}], U)$ and $G^i(0) = \left(\bar{\epsilon_D}^*, \bar{\epsilon_\pi}^*, \bar{C}^*, \bar{\delta}^*\right)$ for every $i \in \{1, \ldots, n\}$.

$$
\begin{aligned}
F^i_{11} &= 1 - \epsilon_D(k + r) + 2\delta^i_{DD}(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + (\delta_{Di} o_i + \delta^i_{DM} o_M)(\epsilon_D + \epsilon_\pi o_M \nu);\\
F^i_{12} &= (a\bar{\pi}\epsilon_\pi - rC) + \delta^i_D(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + (\delta^i_M o_M + \delta_i o_i)(\epsilon_D + \epsilon_\pi o_M \nu) +\\
&\qquad - r\left[\epsilon_D(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + \epsilon_\pi o_M(\epsilon_D + \epsilon_\pi o_M \nu)\right]\bar{\alpha};\\
F^i_{13} &= -\epsilon_\pi(a + r + o_M\nu) + \delta^i_{DM}(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + (o_i\delta_{Mi} + 2o_M\delta^i_{MM})(\epsilon_D + \epsilon_\pi o_M \nu) +\\
&\qquad + \delta_{Di}(\epsilon_D\sigma_D^2 + \epsilon_\pi o_M) + (o_M\delta_{Mi} + 2o_i\delta_{ii} + 1)(\epsilon_D + \epsilon_\pi o_M \nu).
\end{aligned}
$$

Proof. For every $i \in \{1, \ldots, n\}$, direct calculations show $(0, \bar{\epsilon_D}^*, \bar{\epsilon_\pi}^*, \bar{C}^*, \bar{\delta}^*)$ (cf. Definition 3.4.2) to be the unique acceptable ($\epsilon_D \in \mathbb{B}$) solution of the system $F^i(0, \epsilon_D, \epsilon_\pi, C, \delta^i) = 0$. Direct calculations lead to

$$
\det\left(\nabla_{(\epsilon_D,\epsilon_\pi,C,\delta^i)} F^i(0, \bar{\epsilon_D}^*, \bar{\epsilon_\pi}^*, \bar{C}^*, \bar{\delta}^*)\right) = -1024\frac{r^2(a+r)^3(2a+r)^3(a+k+r)^2(2k+r)\sigma_D^{20}}{(k+r)^{18}} \neq 0
$$

and the Implicit Function Theorem [32, Theorem 9.28] concludes the proof. □

Lemma D.0.3. The process (D_t, π_t) is stationary if and only if

$$
E\left[\begin{pmatrix} D_0 \\ \pi_0 \end{pmatrix}\right] = \begin{pmatrix} \bar{\pi}/k \\ \bar{\pi} \end{pmatrix}, \quad \text{Var}\left[\begin{pmatrix} D_0 \\ \pi_0 \end{pmatrix}\right] = \begin{pmatrix} \frac{\sigma_D^2}{2k} + \frac{\sigma_\pi^2}{(k-a)^2}\left(\frac{1}{2a} + \frac{1}{2k} - \frac{2}{a+k}\right) & \frac{\sigma_\pi^2}{2a(a+k)} \\ \frac{\sigma_\pi^2}{2a(a+k)} & \frac{\sigma_\pi^2}{2a} \end{pmatrix}. \tag{D.0.4}
$$

Proof. Define $m = E[(D_0, \pi_0)^T]$ and $V = \text{Var}[(D_0, \pi_0)^T]$; because of (3.1.1), (3.1.2) and [22, Problem 5.6.1], m and V are solutions of the equations

$$
\begin{pmatrix} -k & 1 \\ 0 & -a \end{pmatrix} m + \begin{pmatrix} 0 \\ a\bar{\pi} \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} -k & 1 \\ 0 & -a \end{pmatrix} V + V \begin{pmatrix} -k & 0 \\ 1 & -a \end{pmatrix} = 0. \tag{D.0.5}
$$

Direct calculations show that V and m of (D.0.4) solve (D.0.5); by Sylvester Theorem [20, Theorem 2.4.4.1] the solution of (D.0.5) is unique. □

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