Complexity Studies of Firm Dynamics

by

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ABSTRACT

This thesis consists of three projects employing complexity economics methods to explore firm dynamics. The first is the Firm Ecosystem Model, which addresses the institutional conditions of capital access and entrenched competitive advantage. Larger firms will be more competitive than smaller firms due to efficiencies of scale, but the persistence of larger firms is also supported institutionally through mechanisms such as tax policy, capital access mechanisms and industry-favorable legislation. At the same time, evidence suggests that small firms innovate more than larger firms, and an aggressive firm-as-value perspective incentivizes early investment in new firms in an attempt to capture that value. The Ecological Firm Model explores the effects of the differences in innovation and investment patterns and persistence rates between large and small firms.

The second project is the Structural Inertia Model, which is intended to build theory around why larger firms may be less successful in capturing new marketshare than smaller firms, as well as to advance fitness landscape methods. The model explores the possibility that firms with larger scopes may be less effective in mitigating the costs of cooperation because conditions may arise that cause intrafirm conflicts. The model is implemented on structured fitness landscapes derived using the maximal order of interaction (NM) formulation and described using local optima networks (LONs), thus integrating these novel techniques.

Finally, firm dynamics can serve as a proxy for the ease at which people can voluntarily enter into the legal cooperative agreements that constitute firms. The third project, the Emergent Firm model, is an exploration of how this dynamic of voluntary association may be affected by differing capital institutions, and explores the macroeconomic implications of the economies that emerge out of the various resulting firm populations.

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Many individuals contributed to the conceptualization, implementation and interpretation of the models presented in this dissertation.

The Firm Ecosystem Model arose out of an assignment for Adam Lampert's Modeling with Game Theory course. Glenn Hoetker helped establish the theoretical foundations, and both Adam and Mohamed Moustaoui were instrumental in model development.

The Structural Inertia Model is the culmination of several semesters of work with Glenn Hoetker on fitness landscapes in the context of strategic management theory.

The Emergent Firm Model is a reimplementation of the Endogenous Model of Multi-Agent Firms by Rob Axtell, a pioneer of complexity economics. This project was initially conceived in Michael Barton's Introduction to Complex Adaptive Systems Science seminar, and was developed under the guidance of Marco Janssen. Galina Vereshchagina thoughtfully entertained numerous discussions about economic theory pertaining to labor, firms and borrowing behaviors.

Each member of my committee (Marco Janssen representing Applied Mathematics, Glenn Hoetker representing Strategic Management, Erik Johnston representing Policy Informatics and Shade Shutters representing Ecology) understood and supported the interdisciplinary nature of this work.

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Chapter 1

INTRODUCTION

> The very nature of the economy is to some extent defined in terms of the kind of firms that compose it, their size, the way in which they are established and grow, their methods of doing business, and the relationships between them.
>
> – Edith Penrose, *The Theory of the Growth of the Firm*, 1959

1.1 Contemporary Firm Dynamics

Recent work has highlighted some curious and concerning trends in US firm behavior for both small and large firms. On the small side, evidence shows declining business dynamism and a decrease in the formation and persistence of new firms (Hathaway and Litan 2014b; Alon et al. 2018). New firms are entering the economy at a lower rate than they are exiting the economy, while those firms that do enter do not typically persist beyond a handful of years. Fewer firms are scaling up, and this trend is evident across industries as well as across geographic regions (Hathaway and Litan 2014a).

Another multidecade trend is affecting the other end of the firm spectrum. We see large increases in profits for large firms that are not accompanied by the expected benefits rolling across the entire economy, such as higher wages and lower prices (Economist 2016a). This lack of dispersion of benefits doesn't appear to be a matter of temporary periods of imperfect competition, but rather a shift in entrepreneurial efforts from productive activity such as innovation, toward unproductive activity such as rent-seeking (Baumol 1996; Furman and Orszag 2015; Barkai 2016; Bessen 2016; Litan and Hathaway 2017). Incumbent firms are using their market power to entrench competitive advantage and monopolistic positions rather than to improve existing products and services and offer them at competitive prices.

In general, we see larger, established firms controlling more of the economic activity, at the same time as there is a shift toward unproductive activity on the part of those same firms. This is troubling for a myriad of reasons, and I'll briefly describe three. The first is that econometric evidence suggests net gains in employment are to be found in medium size firms (Haltiwanger, Jarmin, and Miranda 2013). Small, young firms tend to shed jobs at roughly the same rate as they create them, and larger, established firms ultimately shed jobs as they continue to cut productive activity. If small firms are not growing into medium sized firms, the US economy may not be making employment gains.[1] Second, econometric evidence also suggests that younger, smaller firms serve as innovation engines (Acs and Audretsch 1987; Hathaway and Litan 2014b). The degree of innovation determines cultural progress in technology and practice, as well as in the variety of goods and services available. If small firms are not entering the economy and larger firms are choosing not to step in and perform innovation services, we could be running an innovation deficit. Which brings us to the third issue, which is that there is an overall reduction in firm diversity whether measured by structure, size, age or activity. Diversity within an ecosystem, provided it produces a diversity in responses, promotes both stability and resilience (Peterson, Allen, and Holling 1998; Elmqvist et al. 2003; Gunderson 2008). A lack of appropriate response diversity within a complex system such as an economy implies a brittle and fragile economy, one less able to sustain shocks whether they arise from economic crashes, sociopolitical unrest or climate change (Beinhocker 2006).

Why is the path to long-term viability obstructed such that smaller firms are not scaling up as expected? What is causing incumbent firms to shift their focus away from producing value and toward building monopolies? Are these issues a matter of institutional incentives, or do they reflect some fundamental failing in our practice of capitalism?[2] Are these trends the cause or consequence of the larger problematic trend of increasing wealth inequality?

Significant efforts have been made toward unraveling this tangled skein. With regard to the decline in firms scaling-up, Israel is experiencing a similar dynamic where there is a flurry of startup activity but nearly all new firms are purchased by a handful of large, incumbent

[1] The quantity of jobs is not the only issue. Increased firm consolidation limits the variety of available work, making the economy less able to meet the needs of a diverse and multi-interest workforce.

[2] Here and throughout this thesis we apply Douglass North's (1991) definition of institution, meaning the constraints that structure political, economic and social interaction, both formal and informal.

firms rather than scaling up (Economist 2014). One simply needs to pay attention to business news in the United States to recognize a similar story where successful tech startups are purchased by Amazon, Microsoft or Google. In the case of Israel, the government is attempting to solve the problem by limiting the extent to which an incumbent firm can diversify. Another possible policy lever is to address the exploitative nature of venture capital funding that claims a controlling ownership position and demands a five to ten-year exit strategy, thus making such purchases attractive (Cumming 2008).

Capital behavior and the resulting incentives may be driving the disinterest in competition as well. Recent studies draw a link between large institutional investors such as BlackRock and Vanguard who hold ownership in multiple companies within the same industry and decreased competition in those industries (Azar, Schmalz, and Tecu 2018). These institutional investors use subtle, quasilegal methods to incentivize companies toward cabalistic behavior such as price setting (Posner and Weyl 2018). Will institutional reforms address these issues (Baumol and Strom 2007; Litan and Hathaway 2017) or do we need to reform capitalism itself (Economist 2018)? Perhaps we should first better understand how capitalist firms operate and respond to institutional conditions.

1.2 Why Complexity Studies of Firms?

Conventional economic theory has little to say about the actual workings of firms and their relationships to each other and the institutional environment, a lack which has been recognized even by economists. Oliver Hart in 1989 wrote an essay attempting to explain this fact to contract lawyers:

> Firms are the engines of growth of modern capitalistic economies, and so economists must surely have fairly sophisticated views of how they behave. In fact, little could be further from the truth. Most formal models of the firm are extremely rudimentary, capable only of portraying hypothetical firms that bear little relation to the complex organizations we see in the world. (Hart 1989)

The need for a more informative economic treatment of firms is again described almost ten years later by Harold Demsetz:

> Within the [neoclassical] model there is a hypothetical construct called the firm. This construct consists of a single decision criterion and an ability to get information from an external world called the market. The market information determines the behavior of the so called firm. None of the problems of real firms can find a home within this special construct. (Demsetz 1997)

Economic historian Alfred Chandler came to the conclusion that meaningful economic theories about firms must be ecological in nature (Chandler 1992). In *Scope and Scale*, he provides an illustrative example of how this ecological perspective can be meaningfully applied to understanding firm dynamics and the resultant economy. Chandler describes the emergence of the management-oriented organizational structure (known as M-form) as a result of the confluence of technological advances in communication and transportation and existing cultural institutions. Advances in communication and technology provided the opportunity for economies of scale in production provided the inputs and outputs could be adequately coordinated. In the United States, with its cultural aversion to cartels and focus on competition, this resulted in the M-form corporation funded by publicly traded stock, and these corporations were incentivized to consolidate with competitors. Germany, on the other hand, created credit banks instead of a stock exchange, and allowed for cooperation between firms such that there was no incentive for mergers. The same technological advances manifested different organizational structures in Germany and the United States based on the institutional conditions in both countries (Chandler 1990). More importantly, the United States chose to fund large enterprises such as railroad construction through a stock market, while Germany opted for the formation of capital banks, which did not confer ownership rights to the firm. The stock market creates incentives to ensure that firms persist because the American firm came to be considered a store of value. This persistence architecture comes at the expense of creative destruction and innovation and is a significant factor in how people managing firms decide to form, grow, consolidate and abandon these structures. This narrative could provide insights into why noncompetitive behavior is currently tolerated, perhaps by considering the *de facto* power institutional investors exercise over regulatory bodies (Acemoglu and Robinson 2006).

Ecological perspectives on firm dynamics are not new, having been advocated by Thorstein Veblen as early as 1898 (Veblen 1898), and applied to institutional and organizational relationships (Hannan and Freeman 1993), descriptions of industry dynamics (Moore 1993), finance (May, Levin, and Sugihara 2008), innovation (Durst and Poutanen 2013; Auerswald and Dani 2017) and the economy as a whole (Beinhocker 2006). An ecosystem is an instance of a complex adaptive system involving the interdependence between many actors and a contextual environment, where the actors and environment co-evolve. Applied to an economy, the actors are firms, governments and consumers, and the environment includes both the institutional context and the physical environment in which the actors perform their activity. Therefore moving from an ecosystem understanding of firm dynamics to a complex systems understanding is merely a shift in label (Ritala and Almpanopoulou 2017).

Complex adaptive systems science implies a set of quantitative methods such as agent-based modeling, dynamical modeling, network analytics and machine learning, which adds rigor and clarity to qualitative descriptions. The nascent field of computational economics supports interdisciplinary efforts to address complex economic questions via complexity science methods. In response to the continued calls for reimagining the "compendium of dead ideas" (Economist 2016b) that broadly describe the canonical theories of firm dynamics, this thesis continues to extend the application of complexity science methods to treat firm dynamics and the resulting economy "not as a system in equilibrium, but as one in motion, perpetually constructing itself anew" (Arthur 1999). The intent is to contribute to a more complete understanding of how firms respond to aspects of the institutional environment in order to better inform efforts toward evolving an economy that is progressive, creative and just.

1.3 The Projects

This thesis consists of three projects employing complexity economics methods to explore firm dynamics. The first is the Firm Ecosystem Model, which addresses the institutional conditions of capital access and entrenched competitive advantage. Larger firms will be more competitive than smaller firms because of the fact of efficiencies of scale, but the persistence of larger firms is supported institutionally through mechanisms such as tax policy, capital access mechanisms and industry-favorable legislation. At the same time, evidence suggests that small firms innovate more than larger firms, and an aggressive firm-as-value perspective incentivizes early investment in new firms in an attempt to capture that value. The Ecological Firm Model explores the effects of the differences in innovation and investment patterns and persistence rates between large and small firms.

The second project is the Structural Inertia Model, which is intended to build theory around why larger firms may be less successful in capturing new marketshare than smaller firms, as well as to advance fitness landscape methods. The model explores the possibility that firms with larger scopes may be less effective is mitigating the costs of cooperation because conditions may arise that cause intrafirm conflicts. The model is implemented on structured fitness landscapes derived using the maximal order of interaction (NM) formulation and described using local optima networks (LONs), thus integrating these novel techniques.

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1.3.1 The Firm Ecosystem Model

The Firm Ecosystem Model is a dynamical model based on the empirical finding that firm characteristics, such as the tendency to innovate and competitive advantages, vary according to firm size. Firm dynamics leading to various population distributions are considered as a competition-colonization scenario in a spatially defined market, where firms of differing sizes are treated as separate species with different competition and colonization characteristics. Smaller firms, given adequate investment funds, are able to colonize available space more quickly than larger firms, and larger firms are assumed to have stronger competition characteristics and are able to outcompete smaller firms for occupied space. With startup and mortality parameters determined empirically, firm populations reach equilibriums dependent on the values of the capital investment parameters. The model predictions provide a good qualitative fit to empirical data from the Business Dynamics Statistics database. Finally, we explore how alternative mortality or investment conditions affect the firm size distributions.

1.3.2 The Structural Inertia Model

The Structural Inertial Model (SIM) is a demonstration model that showcases the power of combining two recent developments in fitness landscapes. We first introduce the two innovations: the NM parametric formulation of fitness landscapes, and the LON mapping of landscapes. The maximal order of interaction specification of a landscape (known as NM) greatly simplifies the landscape computational requirements as well as provides a straightforward and transparent method of defining structure within a landscape. The local optima network (LON) mapping of a landscape provides insights into landscape structure as well as actual searchability metrics that allow for relevant comparisons between landscapes. By combining these two innovations, we can easily structure landscapes and then describe how this structure affects such a landscape's searchability characteristics. We demonstrate that these searchability characteristics are different for structured and random landscapes, and furthermore that these characteristics actually represent differences in ease of search with a simple iterated search algorithm. Finally, we motivate and develop the SIM model describing a possible driver for the dynamics behind structural inertia, which is the internal and external constraints on an organization adopting change. We compare the improvement over a short time period between those firms with larger and smaller scopes, postulating that firms with larger scope will be more inertial. The SIM model is build on a variety of NM structured landscapes, and we employ LON measures to describe the comparative landscape searchability. Thus the SIM combines both the NM and LON methods into a theoretically meaningful demonstration model.

1.3.3 The Emergent Firm Model

The Emergent Firm (EF) model is an agent-based model that evolves firms through an endogenous dynamic engine and captures the complexities of firm formation under capital constraints. The EF model is a scaled extension of Axtell's Endogenous Dynamics of Multi-Agent Firms model (Axtell 1999; Axtell 2015; Axtell 2018). Firm formation and dissolution are driven by agent preferences for income and variances in production characteristics. Agents decide whether to stay with current employers, join another firm employing an agent in their social network, or start their own firms based on a maximization of utility. The EF model assumes employment changes incur costs for agents, and imposes a cash-in-advance constraint on agents who wish to make a firm change. A universal credit-creating lender supplies loans to agents who wish to make a change but lack the funds to do so. The EF model therefore generates three scenarios for exploration: costless changes, costly changes that must be paid in advance, and costly changes that can be paid via borrowing. Firms are characterized as star subgraphs with owners as the central node and employees as neighbors. The collection of these star graphs comprise the economy, and the macroeconomic manifestations of microeconomic parameters can be explored by a variety of metrics, including firm sizes distributions, mobility, wages, debt, wealth and productivity. Simulations with the parameters described in this paper demonstrate that with a cash-in-advance constraint and a universal credit-creating lender, mobility, wages and productivity are all lower than without costs or lending. In addition, with a positive lending rate, aggregate loan values are super-linear in time, net wealth is increasingly negative and wealth inequality manifests.

Chapter 2

THE FIRM ECOSYSTEM MODEL[1]

2.1 Introduction

Firms are the medium through which individuals participate in the productive aspect of an economy, and in aggregate compose the domain of business that is one of the three pillars of macroeconomic theory. Despite the central role in a myriad of economic questions, an individual firm is classically modeled as a black box that essentially serves as a vehicle for a production function. This formulation is unsatisfying to anyone interested in understanding the macroeconomic implications of firm dynamics where heterogeneity in firm characteristics is important, firms interact with each other and dynamics are endogenous.

Firm dynamics determine the relative proportions of different size firm populations, and these proportions are linked to macroeconomic questions, such as what types of firms provide the most employment (Birch 1981; Haltiwanger, Jarmin, and Miranda 2013), which types innovate (Acs and Audretsch 1987; Hathaway and Litan 2014b) and whether a large variety of firm types promotes social well-being (Hannan and Freeman 1993). Understanding the drivers of firm dynamics is therefore of importance not only to academics but also to management professionals and policy makers.

Early models exploring firm dynamics focused on empirical firm size distributions and the equations that described them. In 1931, Robert Gibrat examined firm plant sizes across France and derived a logarithmic relationship between firm population and size where firm growth was in proportion to its size. This became popularly known as Gibrat's Law of Proportional Effect. Later work built on this law and produced variants that modified birth, growth and mortality parameters (Kalecki 1945; Simon and Bonini 1958; Mansfield 1962). The Gibrat distribution and its variants fit the larger end of the firm size spectrum well but failed to adequately predict the distribution of small firm sizes, as demonstrated

[1]Many thanks to Adam Lampert for guidance with model development.

in Figure 2.1 where the Gibrat distribution is mapped against empirical firm size data from the Business Dynamics Statistics (BDS) database. The BDS dataset is an aggregated longitudinal dataset with complete representation across all firm size categories.[2] Firm sizes are organized into 12 categories based on numbers of employees: 1 to 4, 5 to 9, 10 to 19, 20 to 49, 50 to 99, 100 to 249, 250 to 499, 500 to 999, 1000 to 2499, 2500 to 4999, 5000 to 9999 and 10000+.

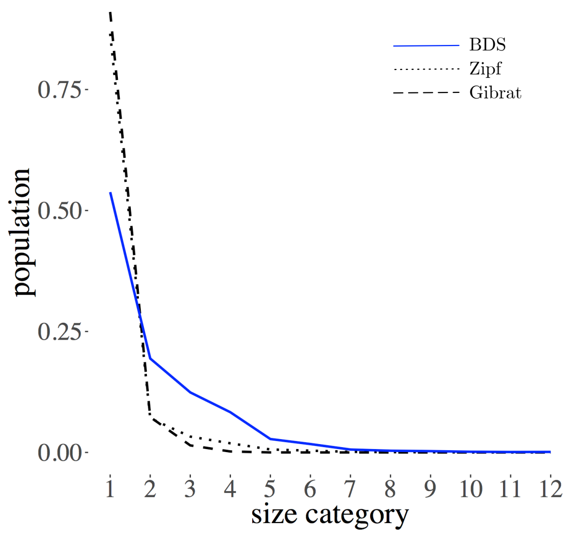


Figure 2.1: BDS Statistics for Average Size

Business Dynamics Database (BDS) statistics for average size distribution averaged over all industries and all years from 1977 to 2014 is described by the solid line. The dashed line describes the classic Gibrat distribution, and the dotted line the Zipf distribution.

Econometric work attempting to clarify these breakdowns in the Gibrat assumption (Hall 1986; Dunne, Roberts, and Samuelson 1988) found statistical regularities in the empirical data, namely that survival increases with size and growth decreases with increasing size. Smaller firms fail more often and grow faster than larger firms. These results appear to hold regardless of how size is defined, whether by output, plant size or employees (Sutton 1997; deWit 2005).

[2]The BDS database is explained in Appendix A and discussed extensively in Section 2.3.

More recent size distribution work proposes alternative distributions such as the Zipf (Axtell 2001; Bottazzi, Pirino, and Tamagni 2015), Rank (Podobnik et al. 2010), log-log OLS (Di Giovanni, Levchenko, and Ranciere 2011) and Hill (Gabaix 2009). These distributions are considered valid only above a minimum firm and don't attempt to explain the distributions at the smaller end of the size spectrum, in part due to data collection methods that focus on larger firms. Our use of the BDS dataset with representation across all firm sizes should mitigate this issue, as well as the fact that our main goal is not to seek a statistical fit to the empirical data.

A more descriptive exploration of firm dynamics arises out of the management literature, where populations of firms are considered in the context of industries and firm dynamics considered as industry life cycles. This industry life cycle work identified regularities in behavior within populations of various sized firms such as shakeouts, where a large number of entrants falls to a small number of persistent firms, explained as the process of winnowing excess capacity and settling on a minimum efficient scale, as well as a positive correlation between firm entry and exit rates within an industry, known as turbulence[3] (Klepper and Miller 1995; Klepper 1996; Klepper 1997; Haltiwanger, Jarmin, and Miranda 2013). This industrial dynamics theory considers competition as monopolistic, an imperfect form of competition as understood classically, whereby firm offerings are differentiated and surviving firms' products and services address the needs of a particular niche. But there also exists a heterogenous institutional[4] competitive advantage enjoyed by established firms by way of supply chain relationships, established brands and legal protections, which is not explicitly addressed in these studies. These institutional advantages create barriers to entry for new firms (Bain 1954; Stigler 1964; Caves and Porter 1977; Demsetz 1982).

Meanwhile, organizational ecologists viewed firm dynamics as determined by processes of competition (Sleuwaegen and Goedhuys 1998). Hannan and Freeman (1977), in particular, considered questions of firm dynamics to be "fundamentally ecological in nature".

[3]This turbulence was previously described by Joseph Schumpeter in 1942 as *creative destruction*.

[4]Here we employ Douglass North's (1991) definition of an institution as any form of constraint that shapes interactions, which include formal institutions such as laws and policies as well as informal institutions such as cultural norms.

Organizational ecologists paid considerable attention to the institutional setting in which firms complete (Nugent and Nabli 1992; Hannan and Carroll 1992; Hannan and Freeman 1993). Firm growth and survival depend on the competitive characteristics of other firms as well as the institutional settings that determine such factors as access to capital and the nature of competitive advantage. Generally speaking, "if processes generating variation and retention are present in a system and that system is subject to selection processes, evolution will occur" (Aldrich 1999). In this view, firms are adaptive entities that respond, according to their unique characteristics, to the environments in which they operate (Nelson and Winter 1982; Beinhocker 2006; Ebeling and Feistel 2011). Firms that don't adapt will fail due to institutional and competitive selection pressures.[5] Hannan and Freeman theorized that firms don't necessarily adapt to changing conditions due to the difficulties for large established organizations to change quickly, a quality described as *structural inertia* (Hannan and Freeman 1984).

Larger firms may enjoy institutional competitive advantage, but smaller firms are more agile when it comes to adapting to new market opportunities because they are less inertial. Small firms also tend to have less available capital than larger firms to pursue innovations, but they also enjoy more attention from external investors, and the degree to which small firms can take advantage of their agility depends on the availability of this external investment. Venture capital typically funds startup firms for five years, at which point investment falls off dramatically as investors turn their attention toward exit strategies and realizing returns on that investment (Hall and Lerner 2010; Feld and Mendelson 2013; Gompers and Lerner 2001).[6] Venture capital chases startups because they have potential for higher growth through innovation and market capture than incumbent firms.[7]

[5] A unique twist to modeling economic considerations through population ecology is that we have a great deal of agency in determining the evolutionary selection pressures and responses in an economic ecosystem (Jones and Breslin 2012).

[6] By *investment* we refer to money a firm attracts from outside sources for the development of new products and services, and not exchanges of existing shares or money used as leverage to obtain operating efficiencies.

[7] Alternatively, venture capitalists can obtain a much larger ownership stake in startups than incumbent firms for relatively little investment.

Putting this ecological picture together with the observed size-specific characteristics previously described, we could consider firms of various sizes as unique species with size-specific behaviors and characteristics, all competing within an institutional context. The outcome of these competitive dynamics produces a particular firm population distribution. Specifically, larger firms have a stronger competitive advantage, both through economies of scale and institutional barriers to entry, but are less able to adapt to changing environments due to structural inertia. They have lower mortality and growth rates than smaller firms. Smaller firms are better at adapting to change because they are less inertial, but the degree to which they can innovate is dependent on external investment. In summary, a brief inventory of the empirical regularities described gives us a list of seven stylized facts:

1. Firm survival increases with size (Dunne, Roberts, and Samuelson 1988)
2. Firm growth decreases with size (Hall 1986)
3. A new market will initially generate a large numbers of small firms that fail (shakeout) (Klepper and Miller 1995)
4. When mortality increases, more firms enter the market (turbulence) (Klepper 1997)
5. Larger firms enjoy a competitive advantage over smaller firms (Demsetz 1982)
6. Smaller firms attract more outside investment than larger firms (Feld and Mendelson 2013)
7. Smaller firms are more agile than larger firms because they have less structural inertia (Hannan and Freeman 1984)

The last two items are of particular interest, because taken together they suggest multiple flavors of competition at play in firm dynamics: institutional competitive advantage in established markets and agility in capturing new markets.

We postulate that smaller firms are particularly susceptible to institutional effects, such as barriers to entry and investment incentives, and a model that directly addresses these realities will provide useful insights into firm dynamics, better explain how those dynamics affect the distribution of firms at the small end of the spectrum, and serve as a policy experiment tool to explore how modifications to the institutional context that modulate the selection pressures may affect firm size distributions. Given the emphasis on competition in the standard narratives of firm dynamics, we believe ecological modeling of firm dynamics is underutilized. Is there an ecological analogy that would apply to the multilevel competition description of firm population dynamics described above?

We believe we have found such an analogy in David Tilman's (1994) spatially structured competition-colonization dynamics. Tilman describes a Wisconsin prairie populated by different species of grass. One particular species has superior nitrogen-fixing ability, so tends to overrun areas populated by species with lesser ability. But all grass organisms have a mortality rate so regions of empty patches are continuously emerging, which can be populated by lesser grass species. The population dynamics on the prairie therefore consist of empty patches colonized by lesser fixating species that were eventually overrun by superior fixating species, while new space regularly becomes available for colonization through the death of individual plants.

In the context of firm dynamics, a market could be analogous to a prairie, and larger firms with superior competitive advantage will take over the marketshare populated by smaller firms. Meanwhile, smaller firms will populate new marketshare (empty prairie in our analogy) before larger species because they are more agile. The degree to which they can populate empty space is governed by investment. Large firms excel at competition, while small firms excel at colonization, and we make use of the distinction between the types of competitive dynamics identified previously.

The remainder of this paper will further develop the competition-colonization firm dynamics model with appropriate modifications. We will then propose a parameterization scheme based on empirical US firm size data and demonstrate that our model fit is superior in the small firm region than other distribution fits. We will then explore how understanding the endogenous firm dynamics allows us to conduct meaningful policy experiments by altering the institutional parameters.

2.2 The Model

In building the model, we assume that firms of differing sizes have different competition, mortality and investment profiles. Firms of various sizes compete over marketshare, conceived as a spatial entity and henceforth referred to as *marketspace*, and are considered analogous to different species competing over any bounded resource, such as prairie grasses competing for space in a field (Tilman 1994).[8] Competition describes firms vying for space in populated marketspace, and differences in competitive ability are decided by disparities in economies of scale and barriers to entry. Colonization, on the other hand, describes firms vying for empty marketspace by innovating to develop new offerings. Each size category of firms is considered a species with different effective competition and colonization characteristics, where size is defined by number of employees. However, firms are not completely analogous to species of grass because a firm of a given size can either grow or contract into a firm of another size. We account for this additional dynamic by allowing for a given size-species to mutate into an adjacent size-species.

[8] There are myriad reasons for firms to fail, and in this analogy we consider failures as resulting from changing market conditions, therefore colonization implies innovation to address these new conditions.

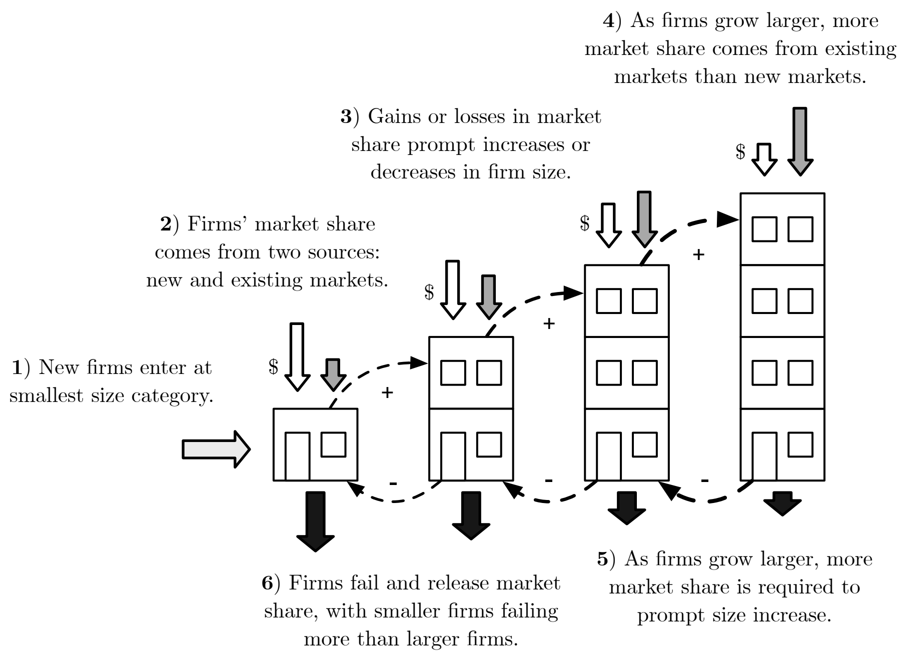


Figure 2.2: Schematic of the Model Dynamics

Firm populations for each size-species i are specified by μ_i. Larger firms will outperform smaller firms in acquiring occupied marketspace because of efficiencies of scale and institutional competitive advantage. Smaller firms will outperform larger firms in populating empty marketspace because they are more agile since they have less structural inertia; however, the degree to which a small firm can take advantage of this agility is mitigated by its available capital. All firms face a mortality rate, which decreases with size, so at every time step a portion of firms will fail and their marketspace will become empty and available for colonization.

The rate of change of marketspace that firms occupy in each size-species is the sum of the marketspace change acquired through colonization and that acquired through competition, minus the proportion of firms that fail due to mortality processes.

Empty marketspace is all the space not currently filled by firms, modeled as $1 - \sum_1^n \mu_j$ where j indexes all the size species categories. Species colonize this empty space at a rate proportional to their current population, μ_i, and an investment parameter ν_i, and the colonized marketspace is represented as

$$\mu_i \nu_i \left(1 - \sum_1^n \mu_j\right). \tag{2.1}$$

Larger firms win populated marketspace from smaller firms due to economies of scale, so species will win marketspace from smaller species and lose marketspace to larger species. This process is modeled as

$$\mu_i \left(\sum_{j<i} \mu_j - \sum_{j>i} \mu_j\right). \tag{2.2}$$

The change in marketspace, s_i, for a given size-species i is therefore

$$s_i = \mu_i \nu_i \left(1 - \sum_1^n \mu_j\right) + \mu_i \left(\sum_{j<i} \mu_j - \sum_{j>i} \mu_j\right) - m_i \mu_i. \tag{2.3}$$

We can compare this competition-colonization firm dynamics model with the original Tilman model, which was[9]

$$\frac{dp_i}{dt} = c_i p_i \left(1 - \sum_{j=1}^{i} p_j\right) - m_i p_i - \left(\sum_{j=1}^{i-1} c_j p_j p_i\right) \tag{2.4}$$

Aside from notational differences (p_i in the Tilman model is the population of species i and c_i is that species competitive value) we see that Equations 2.3 and 2.4 are structurally similar, both consisting of a colonization term, a competition term and a mortality term. The mortality terms are identical, and differences in the remaining terms are due to the specifics of modeling firm dynamics where competition and colonization are governed by different parameters and firms grow or shrink into neighboring size-species.

In our model, mortality is modeled linearly and described by

$$m_i = i^a + e^b \tag{2.5}$$

[9]Equation 6 (Tilman 1994).

where a is the slope and b is the intercept. Investment is modeled as a logistic function with three parameters, K, p and q, which respectively control the height, position and steepness of the curve. This innovation investment curve describes the tendency of smaller firms to innovate more than larger firms due to structural inertia, and the parameterization of the curve describes the availability and conditions on external investment which makes small firm innovation activity possible.

$$\nu_i = \frac{K}{1 + e^{\frac{i-p}{q}}}. \tag{2.6}$$

A detailed discussion of the parameterizations for m and ν can be found in Appendices B.1 and B.2.

Growth and decline are considered to be instantaneous so any change in quantity of marketspace for a size species will result in that quantity moving to an adjacent species category, either a smaller or larger category depending on the sign of the change. Thus any net change in the marketspace of a given size-species will come from firms of the smaller category growing and firms from the larger category declining. In other words, since firms of a given size-species can grow or decline into another size-species according to changes in marketspace, any if the marketspace for a size-species increases or decreases, that marketspace should be attributed to either the larger or smaller size-species.

According to our definition of size, as a firm grows it obtains more employees. Ultimately, the model will need to describe a population distribution of size-species in order to compare with empirical distributions, so we need to transform marketshare into a number of firms. For example, consider x dollars of marketspace populated by small firms. If that same x dollars were populated by larger firms, the number of larger firms would necessarily be less than the number of smaller firms populating that marketspace. To resolve this logical inconsistency, we assume a fixed proportional relationship between an employee and marketshare, and then scale the marketshare transferred from growing or declining size-species such that marketshare effectively contracts as it moves up to larger firms and expands as it moves down to smaller firms by a growth scaling factor, γ. This growth scaling allows us to speak of marketspace and number of firms synonymously.

The growth scaling factor gamma is modeled as

$$\gamma = g^{-\frac{1}{N}} \tag{2.7}$$

where N is the number of size-species categories. A discussion of the derivation and parameterization of γ can be found in Appendix B.3, and is essentially describing a power law between the number of employees and marketshare.

The total change in size-species population over time will involve competition-colonization dynamics along with growth and decline dynamics. Therefore

$$\frac{d\mu_i}{dt} = \gamma s_{i-1}\theta(s_{i-1}) - \frac{1}{\gamma} s_{i+1}\theta(-s_{i+1}) - |s_i| \tag{2.8}$$

where s_i is given by Equation 2.3 and θ is the step function defined by

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases} . \tag{2.9}$$

The boundary conditions for the largest and smallest firms size-species, $i = [1, n]$, are handled as exceptions. For $i = 1$, new firms occupy empty marketspace at a startup rate σ and this smallest species population also grows through declines in $i = 2$ species.

$$\frac{d\mu_1}{dt} = \sigma \left(1 - \sum_{1}^{n} \mu_j\right) - \frac{1}{\gamma} s_2\theta(-s_2) - |s_1|. \tag{2.10}$$

The largest size category retains the species that otherwise would have grown into larger species, as well as through growth in the $n-1$ size species.

$$\frac{d\mu_n}{dt} = \gamma s_{n-1}\theta(s_{n-1}) + \gamma s_n\theta(s_n) - |s_n|. \tag{2.11}$$

2.3 Results

The model in its full representation is analytically intractable,[10] yet analysis is straightforward computationally using an Euler method. The model dynamics produce distributions of either one, two or three peaks in the equilibrium population distribution, depending on parameter value combinations for K, p, q, a, b, g, σ and m_i. Of particular importance is the ratio between maximum investment K and startup rate σ. Both ν and σ produce gains related to empty space. Too high a value of σ leaves too little space for other firms to obtain investment gains. Too low a value of σ with respect to mortality, m_i, and investment, ν, produces too few startups to allow for sustained growth, and the equilibrium state has all firms in the smallest size category, a zero populations in larger categories.

The single peak occurs at either the first size category or in the size category just above the inflection point of the investment curve, specified by p. Two peaks will occur at the smallest size and the inflection size $(p+1)$ for certain ratios of K and σ, and the dynamics are very sensitive to this ratio. In these scenarios, startup dynamics continually populate the smallest size and investment encourages growth up to the inflection category p. The growth dynamics slow down for larger firm sizes so the larger categories have small but non-zero populations. Three peaks, in the first, last and inflection categories, manifest in cases where the growth parameter γ is greater than 1. This three peak parameterization allows for a significant number of firms to grow through to the largest size category, but is not consistent with the firms and marketspace paradigm described in Section 2.2.

We began our exploration with the model by recreating the observed distribution of US firms. We modeled mortality, m_i, as a negatively sloped line, with values parameterized from the Business Dynamics Statistics (BDS) database. We also modeled investment, ν_i, as a logistic function with inflection point, p, and gradient, q, set to represent a steep drop at the second size-species to mimic observed venture investment behavior, where small firms

[10] With a grossly simplified two-dimensional version of the model, neglecting growth dynamics, fixing four out of six parameters and using only two firm sizes (the smaller being x, the larger y), we could demonstrate 1) a stable node at (0, 0) when mortality rate exceeded investment rate for both sizes, 2) a stable x-axis with $m_x < \nu_x$ and $m_y > \nu_y$, 3) a stable y-axis if $m_x > \nu_x$and $m_y < \nu_y$ and 4) nonzero populations for x and y which under some parameter conditions was a stable spiral, under others a linear center.

are funded for short time periods. These parameterizations account for observations 1, 6 and 7 of our inventory, namely that survival increases with size and that the will to innovate is higher for smaller firms. The growth scaling, γ, is also parameterized from the BDS data.[11]

The BDS data gives a mean entry rate over the last thirty years as 10% of the existing population each year. This metric is not directly applicable to the model since σ multiplies the empty space in order to determine the entry into the first size category. Therefore using a σ value of .1 would significantly underestimate the startup rate. Our intent is to derive a baseline distribution that recreates the observed US firm size distributions in order to explore how changes in competition and investment conditions may affect that distribution. We thus chose the reasonable value of .4 to represent 40% of empty space each year populated by startups. As previously demonstrated, the ratio of σ to K is critical to model dynamics, and while model results are robust to changes in the absolute values of σ and K, they are less robust to changes in their ratio. We therefore chose a corresponding value of $K = 2.5$ to represent the maximum investment over all small firms to be 2.5 times the empty marketshare, thus suggesting anything greater than a 250% growth in firms would provide a return on that investment, roughly corresponding to a firm of 7 employees growing to a firm of 1750 employees, again a reasonable assumption. Similar K and σ ratio combinations yield similar results.

Figure 2.3 shows the resulting population distribution as a histogram for investment parameterized by $K = 2.5$, $p = 2$ and $q = .1$, mortality parameterized by $a = -1.8$ and $b = -1.8$, and with growth scaling factor $\gamma = .5$ and the startup rate $\sigma = .4$. Competitive advantage due to economies of scale is built into the model dynamics as described by Equation 2.2, accounting for observation 5, that larger firms enjoy a competitive advantage over smaller firms.

[11] Details about this database and our use of it for the parameterization of m, ν and γ are explained in Appendix B.

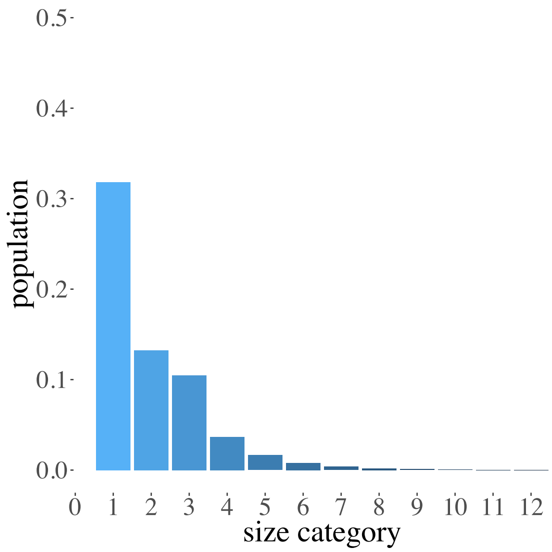


Figure 2.3: Histogram of Equilibrium Firm Sizes

Investment is parameterized by $K = 2.5$, $p = 2$ and $q = .1$. Mortality is parameterized by $a = -1.8$ and $b = -1.8$. The growth scaling factor $\gamma = .5$ and the startup rate $\sigma = .4$.

Regardless of parameterization, the model's equilibrium values are independent of starting conditions so the same equilibrium distribution emerges regardless of the initial conditions. Figure 2.4 demonstrates this by comparing the results of a simulation starting from empty marketspace with a simulation starting with equal populations across all size-species. We also see in the left hand plot Figure 2.4 the differences in growth rates between firm sizes, with smaller firms having steeper slopes than larger firms, thus accounting for observation 2 in our inventory, that of smaller firms having faster growth rates than larger firms. We also see in the right hand plot the expected shakeout in small firm populations where initially the population grows then falls, thus accounting for observation 3, shakeout.

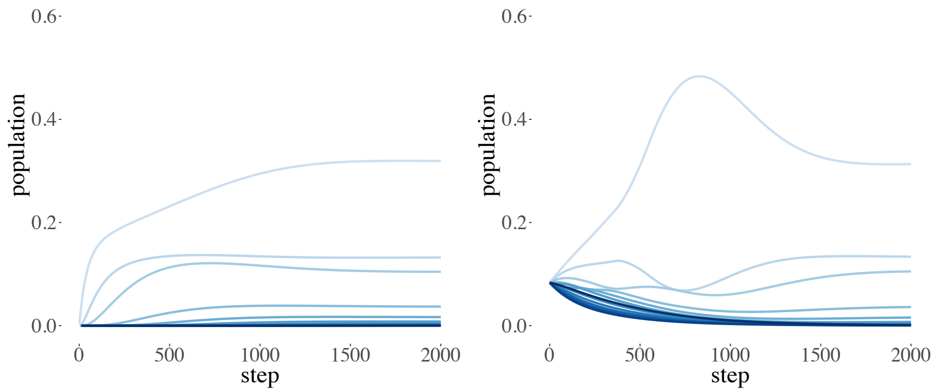


Figure 2.4: Evolution of Equilibrium State from Different Initial Conditions

The figure on the left shows the model results starting with all populations equal to zero, and the figure on the right shows the results starting with equal populations. Investment is parameterized by $K = 2.5$, $p = 2$ and $q = .1$. Mortality is parameterized by $a = -1.8$ and $b = -1.8$. The growth scaling factor $\gamma = .5$ and the startup rate $\sigma = .4$.

Also of note is that the model populations do not add to one, suggesting that there is always empty marketspace and opportunity for expansion and growth within the economic ecosystem.

In Figure 2.5 we show the model results plotted against empirical data from the BDS dataset. We also include the classic Gibrat and Zipf distributions for comparison. We see that the model prediction qualitatively follows the shape of the empirical curve for smaller firm sizes compared to the Gibrat or Zipf curves. Table 2.1 shows the results for Kolmogorov-Smirnov tests comparing each theoretical distribution with the empirical distribution, and though the Zipf distribution is a better fit than the Gibrat distribution, we see our model fit is superior to that of the Zipf. Therefore, as postulated, competition and investment dynamics could indeed be important drivers of firm size distributions at the lower end of the spectrum.

Table 2.1: Kolmogorov-Smirnov Test Results for Theoretical Fits to the Empirical Distribution.

Distribution	D statistic	p-value
model	0.25	0.869
Gibrat	0.667	0.008
Zipf	0.417	0.256

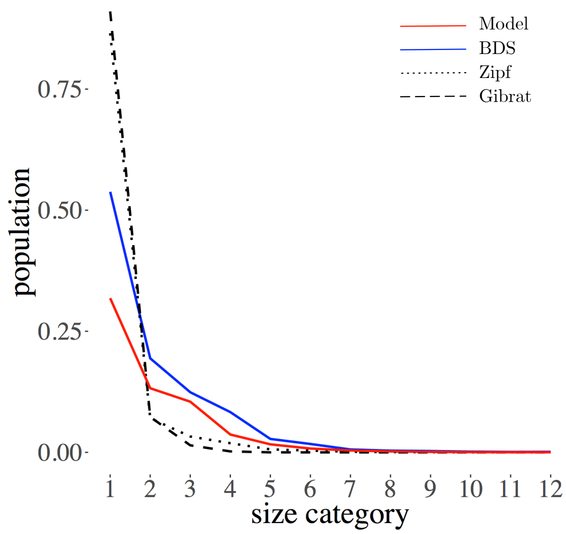


Figure 2.5: BDS Statistics for Average Size with Model Results

Business Dynamics Database (BDS) statistics for average size distribution averaged over all industries and all years from 1977 to 2014 is described by the blue line. The model-predicted size distribution for the described parameter set is shown by the red line. The dashed line describes the classic Gibrat distribution and the dotted line the Zipf distribution.

Despite the greatly improved fit in small firm sizes, the model predicts significantly lower populations of the smallest size category than indicated by the data. A possible explanation could be that an idiosyncrasy of American firm size distributions is the large number of single-proprietorship firms. The average firm size in Germany is 700, while that in the US is 12 (Hart 2008). These singleton firms form in response to institutional facts such as tax benefits that accrue to firms as well as the legal protections such a designation provides, and are not founded with an intent to grow, therefore do not fully participate in the described dynamics. The model results show that indeed both the predicted population of smallest size firms is smaller than actually found in data, though the strength of this finding is somewhat dependent on the choice of σ.

Next we conduct some experiments with the model whereby we modify the selection pressures by changing the mortality and investment parameters, singly and in combination in order to explore how modifications in the institutional conditions controlling competitive advantage and innovation investment affect the distributions of smaller firms. We first modified the mortality parameter to represent a reduction in the institutional competitive advantage enjoyed by larger firms, meaning that more larger firms will fail. We then modified the external investment parameter such that investment is available to firms in small to medium size categories, which represents a lengthening in the investment timelines for venture capital. Next we combined both these modifications, and finally we combined both modifications with a smoothing of the investment curve, which represents larger firms innovating more. Table 2.2 summarizes the population for each size category for the different experimental parameter configurations.

To see how the dynamics change with an increase in mortality for larger firms, corresponding to a lessening of competitive advantage, we increased the mortality rate across larger size-species by flattening the slope of the mortality line, setting $a = -.01$, and produced the distribution described in Figure 2.6. This figure demonstrates that, compared to the BDS parameterized fit, an increase in mortality rates does indeed result in an increase in the population of the smallest size category, thus accounting for observation 4, that turbulence increases firm entry. (Thus we have now accounted for all seven of the size-specific firm observations in our inventory.) There is also a slight increase in the second size category and decreases in the populations for the remaining size categories.

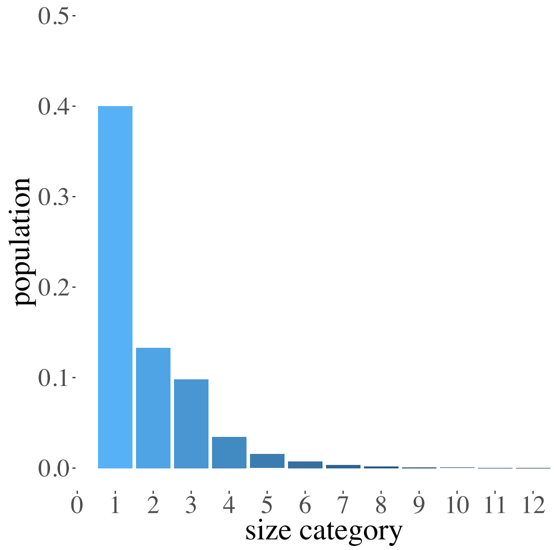


Figure 2.6: Model Results for the Long-Run Equilibrium Size Distribution with Increased Mortality

Investment is parameterized by $K = 2.5$, $p = 2$ and $q = .1$. Mortality is parameterized by $a = -.01$ and $b = -1.8$. The growth scaling factor $\gamma = .5$ and the startup rate $\sigma = .4$.

If we increase the length of time a venture capital investment is made, then investment would be available to intermediate firms sizes. If we effect this change by moving the inflection point of our investment logistic to $p = 4$ we obtain the equilibrium shown in Figure 2.7. Notice the emergence of a second peak in the fifth size category. Compared to the BDS parameterized fit, the populations decrease for the first four size categories and increase for the remaining categories.

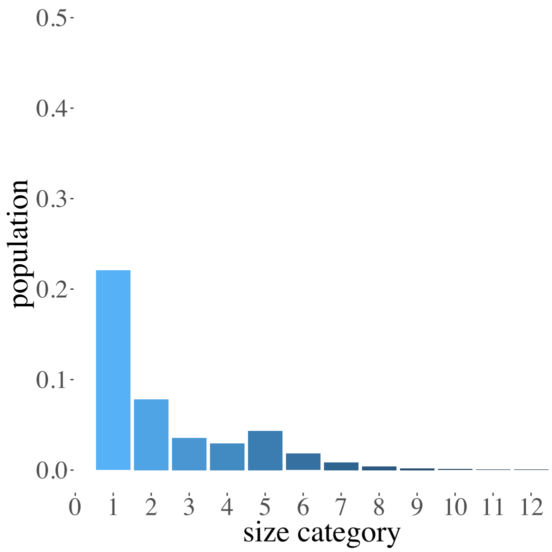


Figure 2.7: Model Results for the Long-Run Equilibrium Size Distribution with Longer Term Investment

Investment is parameterized by $K = 2.5$, $p = 4$ and $q = .1$. Mortality is parameterized by $a = -1.8$ and $b = -1.8$. The growth scaling factor $\gamma = .5$ and the startup rate $\sigma = .4$.

Modifying both mortality and investment results by $a = -.01$ and $p = 4$, we arrive at the equilibrium shown in Figure 2.8. Now there is less of a decrease in the first four size categories than with just an investment change.

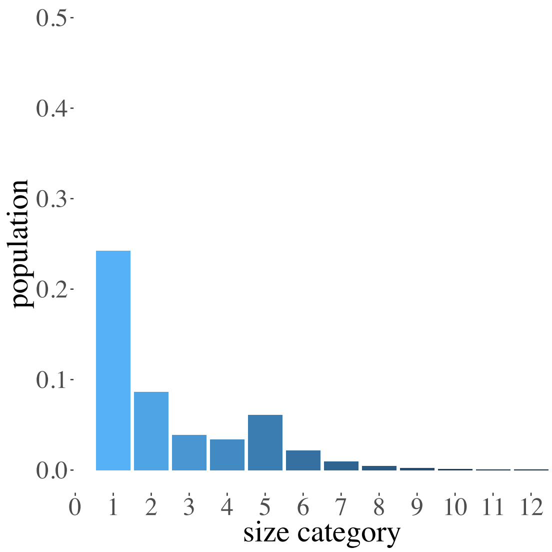


Figure 2.8: Model Results for the Long-Run Equilibrium Size Distribution with Both Increased Mortality and Longer Term Investment

Investment is parameterized by $K = 2.5$, $p = 4$ and $q = .1$. Mortality is parameterized by $a = -.01$ and $b = -1.8$. The growth scaling factor $\gamma = .5$ and the startup rate $\sigma = .4$.

In Figure 2.8 we moved the inflection point of the investment curve, but left the investment curve steep, which means larger firms spend little investment funds on innovation. We can smooth the investment curve so that the drop off is more gradual, which represents larger firms being more willing and able to innovate. Combining all three modification, mortality slope and investment inflection and steepness changes, we obtain the equilibrium given in Figure 2.9. With the smoothed curve, the middle peak is not as prominent and size categories two, three, and eight through ten see slight population increases.

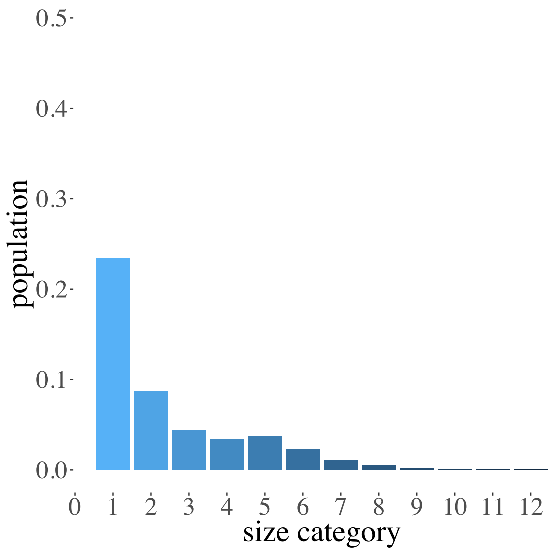


Figure 2.9: Model Results for the Long-Run Equilibrium Size Distribution with Increased Mortality and Longer-Term Investment with a Smooth Investment Curve

Investment is parameterized by $K = 2.5$, $p = 4$ and $q = .5$. Mortality is parameterized by $a = -.01$ and $b = -1.8$. The growth scaling factor $\gamma = .5$ and the startup rate $\sigma = .4$.

Table 2.2: Population Values for Selected Model Configurations.

Size Category	BDS fit	Investment	Mortality	Both	Smooth
1	0.3183	0.4002	0.2206	0.2425	0.2415
2	0.1324	0.1329	0.0783	0.0864	0.0934
3	0.1046	0.0982	0.0357	0.0389	0.0490
4	0.0369	0.0343	0.0297	0.0341	0.0320
5	0.0166	0.0154	0.0435	0.0611	0.0240
6	0.0079	0.0074	0.0181	0.0217	0.0168
7	0.0039	0.0036	0.0084	0.0098	0.0098
8	0.0019	0.0018	0.0041	0.0047	0.0051
9	0.0009	0.0009	0.0020	0.0023	0.0026
10	0.0005	0.0004	0.0010	0.0011	0.0013
11	0.0002	0.0002	0.0005	0.0006	0.0006
12	0.0002	0.0002	0.0005	0.0006	0.0006

In summary, Figure 2.10 shows all five model configurations together: the BDS parameterization with investment only in small firms with a sharp investment drop off, an increased mortality rate for larger firms, increased investment for middle-size firms, both increased mortality for larger firms and increased investment for smaller firms, and increased mortality and smoothed increase in investment. Note that the smoothest increase in middle-sized firms emerges from the combination of parameter modifications for m, p and q. This suggests that increasing available marketspace by decreasing the competitive advantage is not enough on its own, and innovation investment by both small and large firms is also required for a vibrant middle-size firm population.

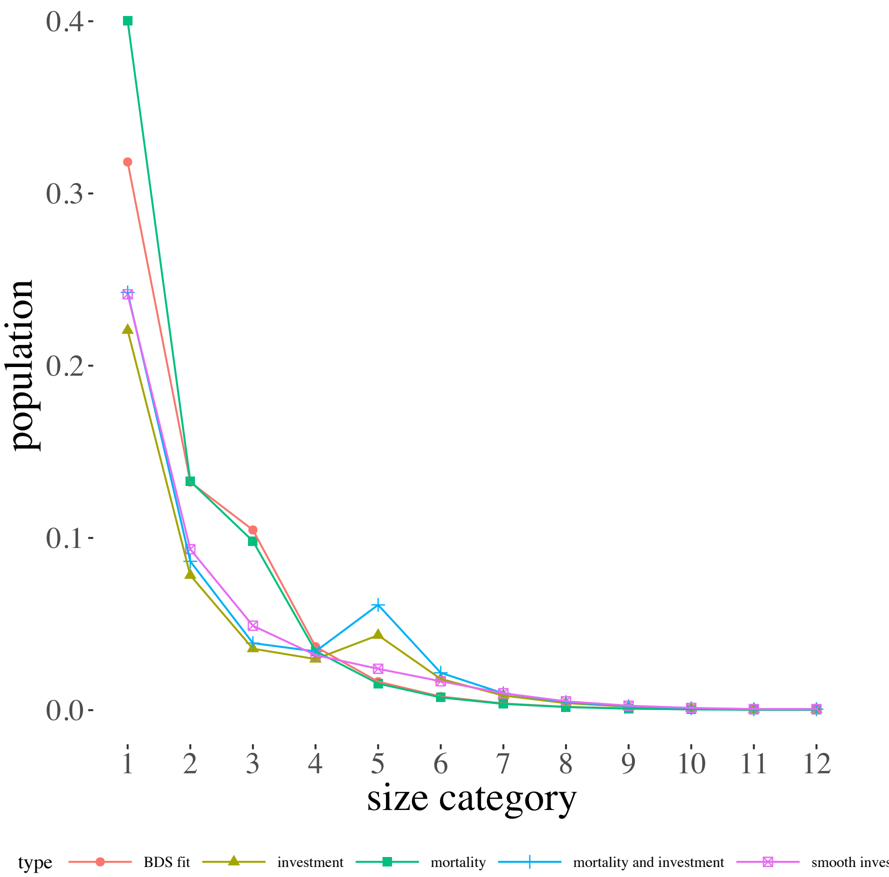


Figure 2.10: Comparison of Distributions for Selected Model Configurations.

Model predictions of firm-size population distributions for five different parameter scenarios: BDS parameterization with small firm investment and sharp drop off, same investment with increased mortality, same mortality with increased investment, with both increased mortality and investment, and both increased investment and mortality with smoothed investment curve.

2.4 Discussion

Several model assumptions warrant specific consideration. We have modeled all startup firms as populating the smallest size category, which is mostly accurate but not always true, notably in the case of spin-offs. Also, real firms grow or contract in a sticky manner. Thus the instantaneous growth assumption would not necessarily hold in the short term but are reasonable over the long term. Finally, we have been using interchangeably size-specific and age-specific characteristics, which are correlated but don't always hold in specific instances

(Haltiwanger, Jarmin, and Miranda 2013). For example, we have used investment cutoffs according to size, when actually the investment cutoff is determined by time, therefore a function of age rather than size. Further model exploration could examine if and where these size/age equivalency assumptions break down.

Because this model is dynamic and explanatory, it offers an opportunity to explore policy implications on firm-size distributions. For example, there is a noted decline in the scale-up of young American firms (Hathaway and Litan 2014b). Yet the intermediate-aged firms are also the biggest contributors to net employment growth, and it is suggested that firms under ten years of age may require institutional support in order to persist into the middle age ranges (Haltiwanger, Jarmin, and Miranda 2013). Given that this middle phase of a firm life-cycle is vital to employment, how could this support be best implemented?

This model proposes three possible levers: longer-term venture investment, reducing the effects of institutional competitive advantage, and encouraging larger firms to innovate. Venture capital exits could be curtailed such that investments extend over longer periods of time than the typical five to ten year window. This doesn't directly mitigate structural inertia for older, larger firms, but it does supply investment for R&D and innovation to a broader swath of more agile firms. This continued investment allows intermediate size firms to participate more aggressively in colonization activity.

Creative destruction theoretically should ensure a failure rate for older, larger firms, but there is evidence that mortality for larger firms is mitigated through institutional mechanisms such as non-competitive consolidation and legislated industry protections, all of which encourage the persistence of larger firms and construct barriers to entry for smaller firms. If larger firms were to fail more often, more marketspace would become available for smaller and even intermediate firms to populate.

Increasing turbulence could also incentivize larger firms to participate in colonization activity. If the reward for seeking institutional competitive advantage were to diminish, and firms were required to actively compete in order to survive, larger firms may be willing to overcome structural inertia issues and innovate (Baumol 1996).

In conclusion, we have shown that modeling heterogenous competition as an endogenous driver of firm dynamics not only offers a viable theoretical explanation for those dynamics, but also provides an improved fit to empirical observations for US firm data. The model also is demonstrably useful as a policy exploration tool to explore modifications to institutional investment and competition conditions, suggesting that working the levers of competition and investment could indeed alter distributions of firm sizes, but that they need to work in conjunction with each other.

Chapter 3

STRUCTURED FITNESS LANDSCAPES

3.1 A Brief History of Fitness Landscapes and the Firm

Evolutionary fitness landscapes were first proposed by Sewell Wright in 1932 as a topological map analogy to fitness values mapped to various combinations of genes. Organisms were imagined to traverse this landscape through evolutionary processes that modified their genetic strings, ending up on different peaks in the fitness terrain. Sixty-five years later, Levinthal (1997) introduced a class of fitness landscapes, known as NK landscapes, into the strategy domain to describe how different organizations adapt to *selection pressures* in changing environments. The binary string now represents a firm's strategy instead of a genome with each bit a single decision component of that strategy. This strategy is mapped to a performance payoff and the fitness landscape is imagined as the collection of all possible strategy strings, points on a multidimensional map, with values equal to the payoffs associated with those points.

Firms roam the landscape by modifying their strategy strings via different algorithms such as random bit-flips, hill-climbing, imitation and numerous combinations thereof.

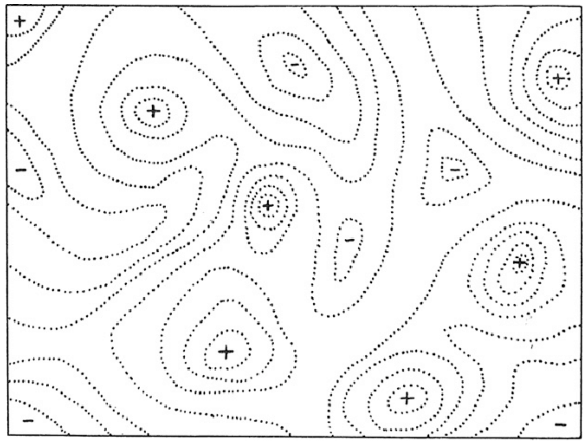


Figure 3.1: Sewell Wright's Evolutionary Fitness Landscape Diagram

Sewell Wright's topological map depiction of an evolutionary fitness landscape from his 1932 paper, "The roles of mutation, inbreeding, crossbreeding, and selection in evolution."

The varying levels of decision interdependence is denoted by K, which denotes the number of other decisions to which any given decision is connected. As K increases, so does the landscape complexity, meaning that the global optimum is harder to find and there are more local optima on which fitness-seeking firms could become stuck. Levinthal demonstrated that organizations with a higher degree of complexity experienced higher failure rates in changing environments.

Since Levinthal's innovation, NK models have been used in a variety of strategy applications, notably by Rivkin (2000) to explore imitative search algorithms, Rivkin and Siggelkow (2002) to explore organizations and decision making, Ethiraj and Levinthal (2004) exploring modularity and innovation, Siggelkow and Rivkin (2005) again exploring optimal organizational designs for turbulent environments, Ethiraj (2008) again continuing with modularity and innovation, and Marengo (2012) exploring agency and incentives.

Fitness landscapes provide a comprehensive exogenous fitness function that that allows the researcher to focus on the results of various search algorithms and strategy structures, as illustrated by this varied list of applications.[1] However, fitness landscapes have fallen

[1]It is possible for fitness landscapes to be modified by the agents searching the landscape through niche construction, in which case the landscape would become an endogenous model element. The landscapes we describe in this paper are static and exogenous.

out of favor as a tool to study firm and strategy questions. We think there are two primary reasons for this loss of interest and one secondary reason.

First, NK landscapes as N-dimensional hypercubes are impossible to visualize and have a level of complexity somewhat mysteriously defined as *ruggedness*, which is a description of the qualitative degree of spikiness in a random walk, demonstrated in Figure 3.2. Ruggedness as a descriptor yields no information about the structure of the landscape, such as how many optima there are or if some regions of a landscape are more rugged than others. Yet fully understanding the results of search algorithms requires understanding something about the structure of fitness landscapes (Van Cleve and Weissman 2015).

The second primary issue is that while search algorithms are innovative and plentiful, the structure of NK landscapes remains limited, coarse and highly randomized. However, most of the issues researchers care about, such as interactions across an organization, or interactions between technologies in an industry, are not at all random and unstructured. For example, some decisions (bit-flips) could have more impact than others for all firms, or some set of decisions could be more tightly interconnected than others. Rivkin and Siggelkow (2007) explore the consequences of a variety of patterns and find that a given K could represent different levels of complexity if patterning is taken into account, and therefore something beyond the NK specification is required to address the need for structured landscapes.[2]

Finally, the lesser issue is that constructing and searching NK landscapes are computationally expensive. For $N = 20$, not unusual for divided string algorithms, a 20 bit strategy string yields $2^{20} = 1,048,576$ potential strategies, which certainly exceeds the requirements for any imaginable theory of the firm question. One has the impression of wasted resources and information. Furthermore, it is computationally difficult to identify the maximum value in the landscape, a useful metric for understanding the range of values as well as for any normalization procedures (Ganco and Hoetker 2009).

[2]Rivkin and Sigglekow (2007) expand their use of K beyond its original definition, and match various patterned landscapes through the metric $N(K + 1)$ in order to include a variety of decision patterns found in industry.

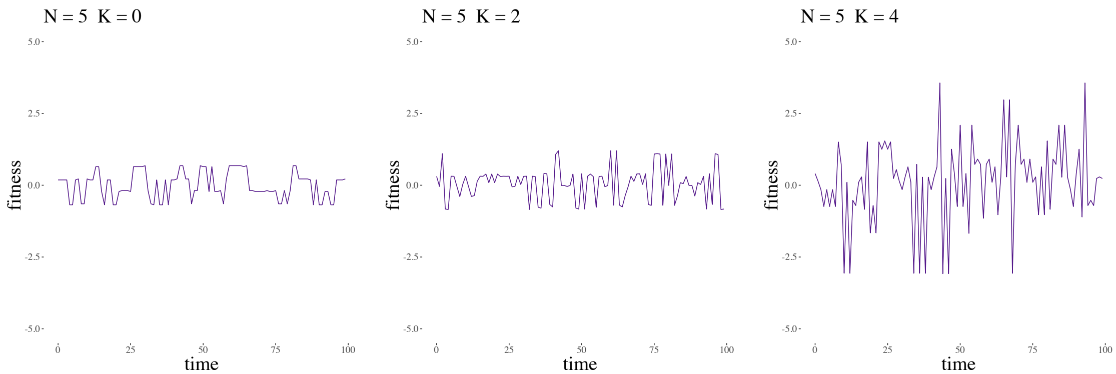


Figure 3.2: Random Walks on $N = 5$ Landscapes with Increasing K

Fitness values for a random walk of a single firm along landscapes with differing K values, demonstrating increasing landscape complexity, evidenced by an increase in size and frequency of spikes, or *ruggedness*, as the value of K increases.

Recent developments in fitness landscape methods offer solutions to these issues, described in the following section. We will then combine these solutions and demonstrate how the resulting method expands the frontier of possibilities for fitness landscapes and studies of firms.

3.2 Recent Developments in Fitness Landscapes

3.2.1 Local Optima Networks

If we imagine a fitness landscape as a series of peaks and valleys, then each point on the landscape is either a peak or in a basin of attraction for a peak. Every point on a fitness landscape can be mapped to a local optima's basin of attraction and the landscape can therefore be considered the collection of these optima and their associated basins of attraction. A firm that only accepts improved positions through single bit flips, known as a simple hill-climbing or greedy acceptance search algorithm, will eventually land on the local optima associated with its original position. Ochoa (2014) exploits this feature to describe a fitness landscape as a network of connected basins of attraction, known as a Local Optima Network (LON), where the nodes represent the optima and their basins, the size of the nodes represent the number of strategies in that basin, and the node color represents the fitness value of the basin peaks. The edges in a LON are directed links with weights describing the transition probability of moving from one basin to another via a single random bit-flip.[3]

LONs distill the essential structure of a fitness landscape into a compact form. For example, an $N = 20$ landscape with over a million location points may have only a few hundred local optima, and the 20-dimensional hypercubic landscape can be visualized as a meaningful 2D representation. The LON maps for the NK combinations in Figure 3.2 are shown in Figure 3.3. Comparing the two representations, we see the degree to which random walks fail to communicate landscape structure and the additional information readily available in the network representation. Each landscape will be different due to the random selection of coefficients and interactions, but for these three specific landscapes we see immediately that for $N = 5, K = 0$ there is a single optimum and all strategies fall in the same basin of attraction. For $N = 5, K = 2$ we see that the 32 possible strategies are arranged into five basins, the smallest basin corresponds to the highest fitness peak, and the lower fitness peaks have the largest basins of attraction. For $N = 5, K = 4$ we see seven basins of attraction, the optimum value has the second largest basin of attraction, and lower fitness values have smaller basins. We also see that the probability of moving into the global optima basin from the penultimate (and largest) basin with a random bit flip is lower than that of moving to the 3rd or 4th ranked optimum.

[3] A class of fitness landscapes known as neutral landscapes consist of flat regions, or plateaus, such that a bit-flip would change the location of the firm but not its fitness value. In the case of landscapes with plateaus there may not be a clear basin of attraction. Verel et al. (2011) describe how neutral landscapes can also be represented as a LON by modifying the definition of a basin of attraction.

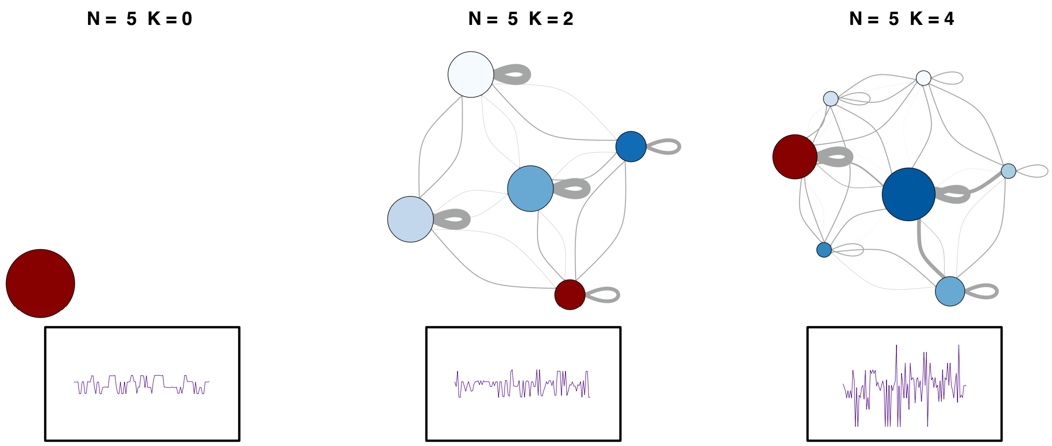


Figure 3.3: LON Maps Compared with Random Walks

Local optima network graphs for $N = 5$ and $K = 0$ (left), $K = 2$ (middle) and $K = 4$ (right). In accordance with Ochoa (2014) the size of the nodes represents the size of the basins, and the intensity of the blue color represents the relative value of that basin's optimum. Red nodes represent the global optima. Links between basins are weighted on the probability of a random bit flip moving a searching agent from one basin to another. The insets show the random walk representations for each of the landscapes.

More importantly, Ochoa (2014) identifies network measures that describe normally hidden characteristics of fitness landscapes, such as number and size of nodes, node strength, path lengths, transitivity, disparity and community affiliation. This important paper concludes the LON analysis of NK landscapes with a demonstration that the time for an agent to find the global maximum on an NK landscape is indeed positively correlated with an example measure, in this case the average shortest path length to the global optima on the LON (Ochoa et al. 2014; Daolio et al. 2012).[4] Not only do LONs allow us to visualize a landscape, but their network measures also tell us something qualitative about the metrics of a landscape's searchability.

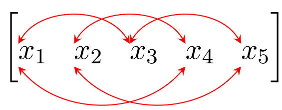
Thus LONs solve our first problem regarding the opacity of NK landscapes and allow for a more complete understanding and description of these landscapes. We'll come back to them after we explore a solution to our second problem: the coarse and undifferentiated nature of NK landscapes.

[4] The agent used an iterated local search algorithm where it first finds the local optima through single-bit flips, then performs a two-bit flip perturbation and the process starts again until the agent reaches the global optima. These search experiments were conducted on $N = 18$ and a range of K landscapes.

3.2.2 NM Landscapes

Buzas and Dinitz (2013) reconceptualized NK landscapes as parametric equations which they originally termed generalized NK landscapes, and in subsequent work (Manukyan, Eppstein, and Buzas 2014) renamed NM landscapes where M stands for *maximum order of interaction*. The NM landscape formulation essentially allows the straightforward implementation of variable K values (Santana, Mendiburu, and Lozano 2015). An example will clarify the distinction and demonstrate the possibilities opened up by the NM formulation.

With $N = 5$ and $K = 1$ and a "next plus one" neighbor linkage algorithm,[5] the bits in the strategy string could be linked as follows:



(3.1)

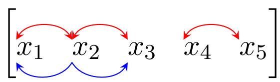
The first order terms are called the main effects and there are five of those, as well as five second degree terms out of a maximum possible number of 10 combinations (5 choose 2). The maximum order of specification, M, is a vector describing the ordered list of the numbers of terms for each degree, and in this case $M = \begin{bmatrix} 5, & 5 \end{bmatrix}$. To specify particular interactions rather than five random second degree terms, we need additional information, and we can specify this information by translating Equation 3.1 into the following parametric equation that defines the strategy payoff Π, for the point on the landscape specified by the coordinates $(x_1, x_2, x_3, x_4, x_5)$:

$$\Pi = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5 + \beta_{1,3} x_1 x_3 + \beta_{2,4} x_2 x_4 + \beta_{3,5} x_3 x_5 + \beta_{1,4} x_1 x_4 + \beta_{2,5} x_2 x_5. \tag{3.2}$$

[5]Interaction terms for NK landscapes could be algorithmically specified, such as "link to next neighbor" for $K = 1$, or randomly specified. Algorithmic and random specifications have been found to manifest similar landscapes (Weinberger 1991). Algorithmic landscapes usually yield N distinct $(K + 1)$th order interaction terms, one for each bit, but randomly generated interactions, "link to any other bit," landscapes may involve fewer distinct terms if duplicated terms are allowed.

We still have five first degree terms and five second degree terms, but unlike the M vector specification we've now indicated the specific second degree interactions we want the landscape to model.

The landscape described by the set of payoffs determined by Equation 3.2 could also be generated via a classic NK formulation by manually specifying an appropriate interaction matrix. But the following linkage with three second degree and a single third degree interaction,



(3.3)

represented generally by the maximal order vector as $M = \begin{bmatrix} 5, & 3, & 1 \end{bmatrix}$, and specifically by

$$\Pi = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5 + \beta_{1,2} x_1 x_2 + \beta_{2,3} x_2 x_3 + \gamma_{1,2,3} x_1 x_2 x_3 + \beta_{4,5} x_4 x_5, \quad (3.4)$$

does not have a simple NK analog. While it is possible to pick and choose the specific $(K + 1)$th order terms in an NK formulation by manually defining the interaction matrix, the lower order interactions are only the subsets of the highest order interaction terms. In Equation 3.4 the highest order interaction term is $x_1 x_2 x_3$, so the only other allowed terms would be $x_1 x_2, x_1 x_3, x_2 x_3$ along with the first order terms x_1, x_2, x_3. The NM formulation, on the other hand, allows for the independent selection of the terms of each order.[6]

Manukyan et al. (2016) explicitly describe the gradual increase in ruggedness as the number of parametric terms increases, thus allowing for finer graduations of complexity than classic NK representations. Since any NK landscape can be represented by a maximal order of interaction vector, with structure in the interactions made explicit through the selection of terms, the NM landscape algorithm encompasses all NK landscapes. Therefore, going forward we will consider fitness landscapes in the NM formulation.

[6]While it is theoretically possible to construct any NM landscape through an NK formulation by generating a fully connected NK landscape with the highest possible value of K, and then zeroing out the coefficients for those terms one by one until you obtained the NM specification, one would be torturing the original definition of K. Once the meaning of K is altered it ceases to become a relevant description of complexity (Rivkin and Siggelkow 2007).

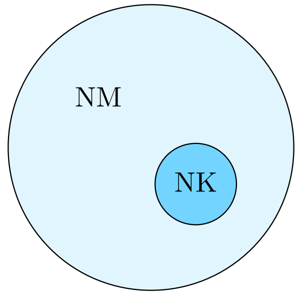


Figure 3.4: NK Landscapes Are an Algorithmic Subset NM Landscapes

3.3 Structured Landscapes

The NM formulation provides a clear mechanism for structuring the relationships between bits with variable levels of interaction. Suppose a research question based on a six-bit strategy string requires a strong linkage between the first three and the last three strategy bits.

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix} \tag{3.5}$$

This landscape is represented generally by the interaction vector $M = [6, 6, 2]$ and specifically by the parametric payoff

$$\Pi = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5 + \alpha_6 x_6 + \beta_{1,2} x_1 x_2 + \beta_{2,3} x_2 x_3 + \beta_{1,3} x_1 x_3 + \gamma_{1,2,3} x_1 x_2 x_3 + \beta_{4,5} x_4 x_5 + \beta_{5,6} x_5 x_6 + \beta_{4,6} x_4 x_6 + \gamma_{4,5,6} x_4 x_5 x_6. \tag{3.6}$$

An unstructured version of the same degree of interaction would consist of six random second degree combinations of the six bits and two random third degree combinations.[7] The following sections explore whether or not landscape structure makes a material difference to search results.

[7] There are $\binom{6}{2} = 15$ possible second degree interactions $\binom{6}{3} = 20$ and $\binom{6}{3} = 20$ possible third degree interactions.

3.3.1 Landscape Searchability Measures

To demonstrate that structuring a landscape produces meaningful effects on search results, we constructed 20 pairs of structured and random landscapes for $M = [6, 6, 2]$. The structured landscapes are specified by Equation 3.6 with specific combinations of bits and 20 different sets of random values for the α, β and γ coefficients. The unstructured counterparts have the same 20 sets of coefficient values, but random interaction terms. The unstructured landscapes will have all six first order terms just as the structured landscapes, but six randomly selected second order interaction terms and two randomly selected third order interaction terms. Therefore, for each pair, the values of α, β and γ are the same, but the second and third order terms should be different. Thus we can generate 20 comparisons between a structured landscape and its degree equivalent random landscape, and any differences between the landscapes will be a consequence of the structuring.

But how are we going to tell whether landscapes are different or not? Here's where the network measures for LONs are indispensable. We identified network measures directly related to the searchability of a landscape; number and size of basins, number of edges, and the shortest paths to the optimum (Ochoa et al. 2014; Daolio et al. 2012). The number of basins is the number of landscape optima and their size is the number of strategies that fall within each optima's basin of attraction. If a basin is large, it is difficult to leave, and numerous basins make the optimum harder to locate. The number of edges represent the number of basins that are connected by a single bit-flip.[8] The more edges, the easier it is to move from one basin to another. The shortest path to optima is the least number of bit-flips possible to move from the current position to the global optima, so a shorter path describes an easier search.

[8]A single-bit flip is a hamming distance of 1.

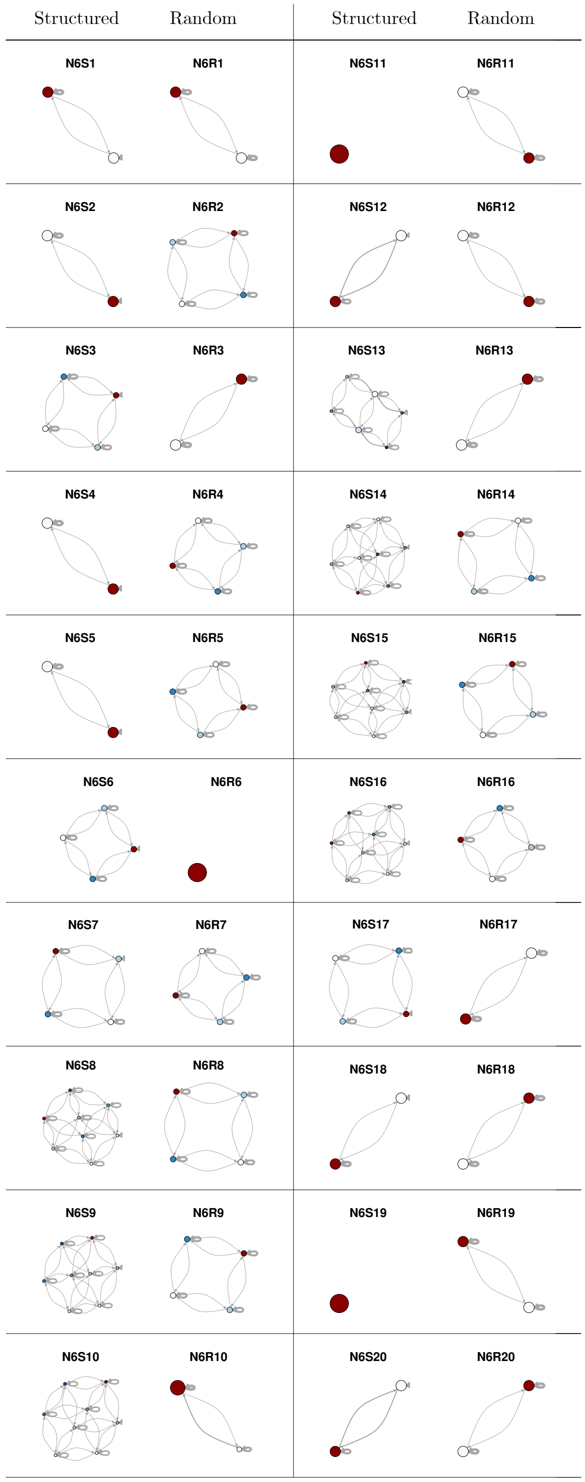


Figure 3.5: LON Maps for Pairs of Structured and Random $N = 6, M = [6, 6, 2]$ with Identical Coefficients

Figure 3.5 shows the LON graphs for all 20 pairs of landscapes. There is variation in the structured landscapes because the coefficients of the terms are random draws.[9] The unstructured landscapes have the same coefficients as their structured counterparts, but random terms of the given degrees. Table 3.1 shows the results of paired t-tests across the 20 landscape pairs for each of the network measures.

Table 3.1: Statistical Test Results for Comparisons of LON Network Statistics Between Random and Structured Landscapes

measure	means (S, R)	statistic	p-value	test
number of basins	4.30, 2.85	2.5701	0.0187*	Paired t-test
number of edges	14.4, 7.4	2.699	0.0142*	Paired t-test
mean basin size	32, 32	-0.6352	0.5329	Paired t-test
variance in basin size	.4, 1.6	16	1.218e-07****	F test
shortest path to optimum	0.1639, 0.1153	2.1586	0.0439*	Paired t-test

Statistical test results for comparisons on LON network statistics between random and structured $N = 6, M = [6, 6, 2]$. An S indicates a structured landscape and an R indicates a random landscape. We used the paired T-test for most measures and an F-test for the variance measure.

We see that indeed, there are statistical differences in the searchability measures between structured and unstructured landscapes, except for mean basin size. Figure 3.6 demonstrates that though the mean sizes are identical, the sizes are not distributed equally, and Table 3.1 shows that the within-pair size variance for random landscapes is four times that of structured landscapes. As mentioned in Section 3.2.1 differences in these network measures translate into meaningful differences in search results, which we will demonstrate next.

[9] The coefficients in the NM formulation we adopted from Manyukan (2014) are randomly drawn from $e^{-|\mathcal{N}(0,\sigma)|}$ with $\sigma = 10$.

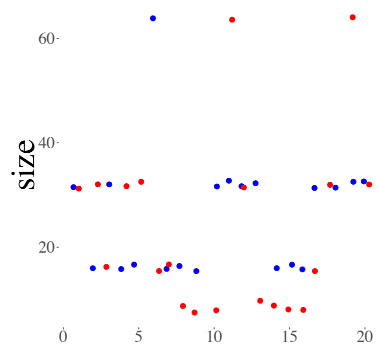


Figure 3.6: Basin Sizes for Landscape Pairs

Jittered scatterplots of basin size values by landscape pair. Random landscapes are blue and structured landscapes are red.

3.3.2 Iterated Search Results on Random and Structured Landscapes

To verify that the differences in searchability measures identified in Section 3.3.1 actually describe differences in search results, we placed a firm on each of the possible strategies in the landscape and tracked the number of strategy changes it took for those firms to reach the global optimum for each of the 20 structured-random landscape pairs described in Section 3.3. We used the iterated search algorithm illustrated in Figure 3.7, a variant of the algorithm employed by Daolio (2012). The algorithm consists of two parts, a greedy hill climbing phase followed by a random multi-bit flip phase. These phases are repeated until the firm has found the global optimum. Essentially, the first phase finds the local optimum attached to the basin of attraction associated with the firm's current position, and the next phase is a jump from one basin to another with a higher optimum value.

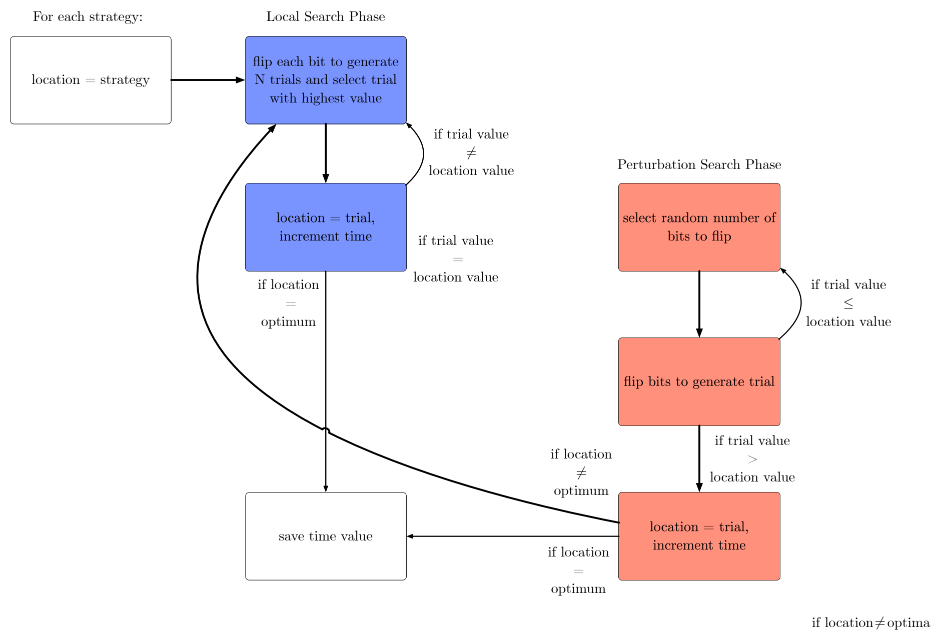


Figure 3.7: Iterated Search Algorithm

A firm on the structured landscape requires an average of 3.17 steps to find the global optimum, while a firm on the random variant will only require on average 2.59 steps. A paired t-test reveals a statistically significant difference in these means, with a p-value of 0.0359*. As anticipated, the structured landscapes requires more search steps, corresponding to the longer path length identified by the LON network statistics.

In Appendix 3.5 we demonstrate an application of the combined NM and LON method by modeling structural inertia on a set of $N = 20$ structured and unstructured landscapes. We again find that the search results on the structured landscape differ statistically and materially from those on a random landscape.

3.4 Computational Issues

The computational gains from using a parametric (NM) landscape formulation are significant. A non-parametric construction of an $N = 20, K = 9$ landscape requires storing and manipulating a matrix consisting of 21,474,836,480 elements $((2^{10})(20) \times 2^{20})$. This same landscape formulated parametrically as an NM landscape requires two one-dimensional arrays of 5,451 elements each to identify the terms and coefficients, and the 1,048,576 strategies are binary representations of numeric values so don't need to be stored.

More computational gains are obtained by structuring the landscapes. When using randomly generated landscapes, a researcher will tend to ramp up the K value in order to become comfortable that the landscape contains an appropriate number and degree of interactions. Structured landscapes allow a researcher to specify where exactly those interactions occur, therefore creating similarly interesting models with effectively smaller K values.[10] In Appendix 3.5 we demonstrate a K-means clustering analysis to match searchability measures across random and structured landscapes, and demonstrate that a structured landscape with $N = 20$ and strategically placed fifth order interactions is searchably more similar to a $N = 20, K = 2$ landscape than an $N = 20, K = 5$ landscape.

Lastly, if the value of the global optimum is required for landscape normalization, this is trivially found in the NM formulation as the sum of all the coefficients, as opposed to finding the maximum of 2^N strategy values, which is computationally intensive for large values of N.

3.5 An Example: The Structural Inertia Model

The Structural Inertia Model (SIM) was created to test the theoretical concept of structural inertia by exploring tradeoffs between the scope of a firm and its ability adopt new strategies. The basic tension in evolutionary theory is adaption versus selection. To what degree do species genetically adapt to an environment, and to what degree are poorly adapted species removed through natural selection before they can adapt? Applying evolutionary concepts to firm dynamics, in *Evolutionary Theory of Economic Change*, Nelson and Winter suggest that firms evolve in order to adapt to changing environmental conditions, and that adaptation is the key evolutionary consideration in firm dynamics (Nelson and Winter 1982). Hannan and Freeman instead argue that firms don't change rapidly enough to adapt to changing environments, and that certain organizational forms are selected over

[10]Structured landscapes produce even more interesting models since they better represent the question under consideration. See Appendix 3.5 for an example of this structuring.

other forms, and thus selection pressures are the key consideration (Hannan and Freeman 1977). They define structural inertia as the "hierarchical layers that vary in their ability to respond to change" and older firms are generally found to have more inertia than younger firms. A firm with a large degree of structural inertia will be resistant to change and, therefore, less likely to adapt and more likely to fail due to selection pressures. The evidence is less clear as to whether small firms are conversely more flexible than larger firms (Hannan and Freeman 1984). The SIM explores this adaptation-selection dynamic by modeling firms of varying degrees of scope, which directly represent varying degrees of structural inertia.

The SIM describes a firm as a collection of decisions that compose a strategy that is organized into divisions, some of which are under the control of the firm, the remainder of which are determined by the environment. Using $N = 20$ we specify a division as four consecutive bits, thus each strategy is a collection of five four-bit divisions. Firms can control one to five divisions and the number of divisions under a firm's control is the firm's *scope*. If a firm has $scope = 1$, it controls only one division and the remaining bits are pulled from an ambient string. These ambient bits could represent components not produced in house that the firm purchases or activity it outsources. The firm's fitness is determined by its location on the landscape, described by all 20 bits. The SIM explores how a firm's scope affects its overall improvement within a window of time.

3.5.1 The SIM Landscape

The SIM is built on an NM landscape structured such that each decision within a division is fully connected to all the other decisions and groups of decisions within that division, and only partially connected to decisions in other divisions. This structuring describes the effect of division scope more meaningfully than purely random bit connections. As an illustration, the simplified case of $N = 6$ with fully connected 3-bit divisions and two random second order linkages across divisions could look like

$$\left[x_1 \; x_2 \; x_3 \qquad x_4 \; x_5 \; x_6. \right] \tag{3.7}$$

We generated 10 instances each of a series of $N = 20$ random and structured landscapes described in Table 3.2.

Table 3.2: Structural Inertia Model Landscape Descriptions

Name	Class	Description
N20M1p5	random	all first order interaction terms and half of all second order terms, M = [20, 95]
N20M2	random	all first and all second order terms, M = [20, 190]
N20M2p5	random	all first order, second order and half of all third order terms, M = [20, 190, 570]
N20M3	random	all first, second and third order terms, M = [20, 190, 1140]
N20D5inter2h	structured	fully connected divisions and half of all second order intradivision terms
N20D5inter2f	structured	fully connected divisions and all second order intradivision terms
N20D5inter2h3h	structured	fully connected divisions and half of all second and third order intradivision terms
N20D5inter2f3f	structured	fully connected divisions and all second and third order intradivision terms.

Names and descriptions for the four cases of each landscape class, random and structured. For random landscapes, there are 20 possible first order terms, 190 possible second order terms and 1140 possible third order terms. For structured landscapes, there are 160 possible second order intradivision terms and 640 possible third order intradivision terms.

3.5.2 Search Results

The SIM allows various search and evaluation algorithms, but for this study we used the `by_firm` search and the `veto` evaluation algorithm.[11] Firms produce a single test strategy each time step and all owned divisions must benefit in order for the firm to implement the new strategy. If any in-scope division would experience a decrease in fitness value, the test strategy will not be accepted. We ran 100 searches on each of 10 versions of the random and structured landscapes described above for 28 steps. We assume firms have limited time to make changes before they fail, and the 28 time steps correspond roughly to quarterly changes over seven years.

[11] The full SIM functionality is described in Appendix E.

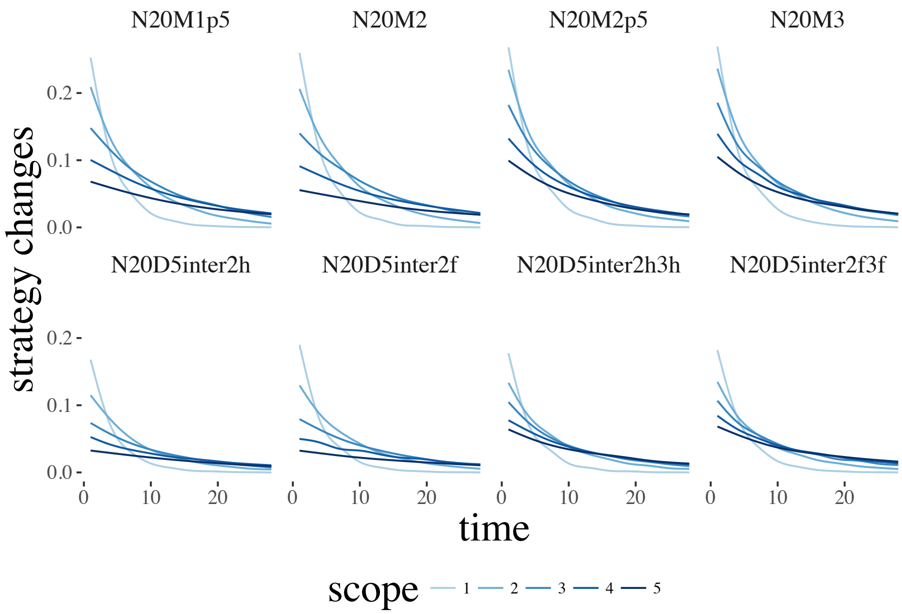


Figure 3.8: Changes over Time by Scope and Landscape

Changes over time by scope over 28 time steps fitted over 100 runs on 10 different versions of the landscape for `by_firm` exploration with `veto` evaluation scenario. The top row are random landscapes and the bottom row are structured landscapes.

In Figure 4.6 we see the number of changes, or accepted strategies, by scope over the 28 steps. Structural inertia would dictate that firms with smaller scope would make more changes than firms with larger scope, which is the result we see over all landscapes in the early stages. But the narrower the scope, the smaller the search space, so narrowly scoped firms exhaust their search space sooner than more broadly scoped firms. Comparing the top and bottom rows, we notice that random landscapes produce more overall strategy changes than structured landscapes. There are fewer terms in the calculations for division performance for random landscapes than for structured landscapes, so it is more difficult to find a structured improvement than a random improvement.[12]

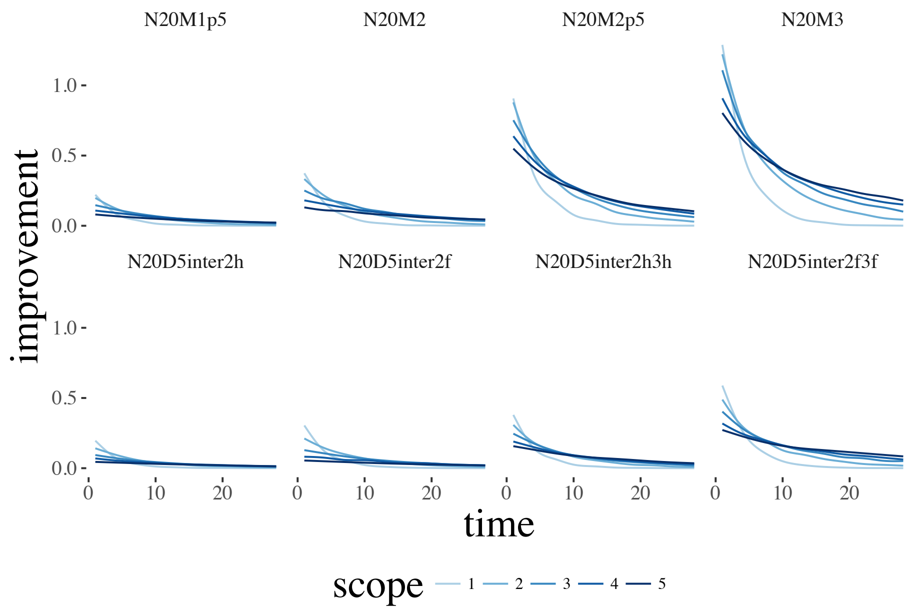


Figure 3.9: Performance Improvement over Time by Scope and Landscape

Performance improvement over time by scope over 28 time steps fitted over 100 runs on 10 different versions of the landscape for `by_firm` exploration with `veto` evaluation scenario. The top row are random landscapes and the bottom row are structured landscapes.

In Figure 3.9 we see the share of total industry improvement for firms of each scope over the course of 28 steps. Again comparing the top and bottom rows, we see that structured landscapes produce more variation in improvement between firms of different scope than

[12]For a random landscape with specification M = [20, 190] a single division's performance consists of ten intradivision terms (four first order and six second order) and 64 second order interdivision terms. A similar structured landscape with fully connected divisions and all second order interdivision connections (N20D5inter2f) involves the same second order interdivision terms but has a total of 15 interdivision terms (one first order, six second order, four third order and one fourth order).

random landscapes. As the number of parametric terms increase, we expect the performance value to decrease because the random nature of the coefficients approximates a simple random walk.[13]

3.5.3 Landscape Metrics and Comparisons

In order to make a meaningful comparison between random and structured landscapes, we need to be able to compare landscapes with similar searchability characteristics in order to rule out differences in search results being a consequence of one landscape being much more complex than another. Ochoa (2014) defines numerous network measures of interest to characterize fitness landscapes, and for this search we choose three; number of optima, size of basin and shortest path to maximum value. We used a K-means cluster analysis to reveal similarities in metrics describing searchability between landscapes, and identified five clusters. We identified cluster five as containing the best mix of random and structured landscapes, as demonstrated in Figure 3.10

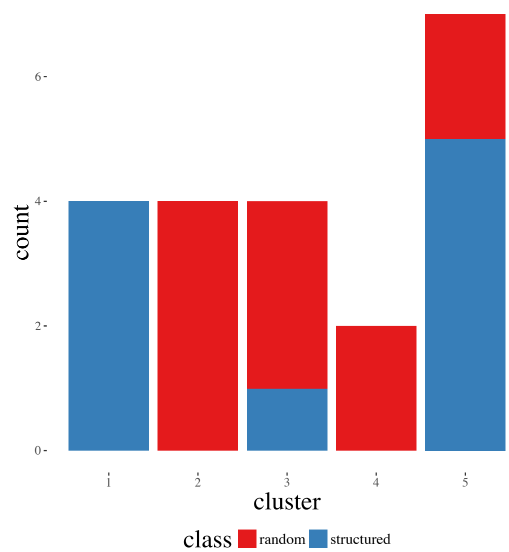


Figure 3.10: Cluster Locations for Random and Structured Landscapes

Count of random and custom landscapes found within each of the five identified clusters. We see that cluster 5 contains the most mixed landscapes. The landscapes in cluster 5 are: N20M1p5, N20M2, N20D5inter2h and N20D5inter2f.

[13]Assuming the interaction terms have equal probability of 1 or −1.

The intradivision structure gets lost as the degree interdivision interactions increase. We therefore identified M1.5, M2, 2h and 2f as searchably similar, yet having enough of a difference between interdivision and intradivision structure such that we would expect the structure to affect the search results.

3.5.4 Structure Matters!

In Figure 3.11 we see the share of total industry improvement for firms of each scope at the final time step presented as a bar chart.

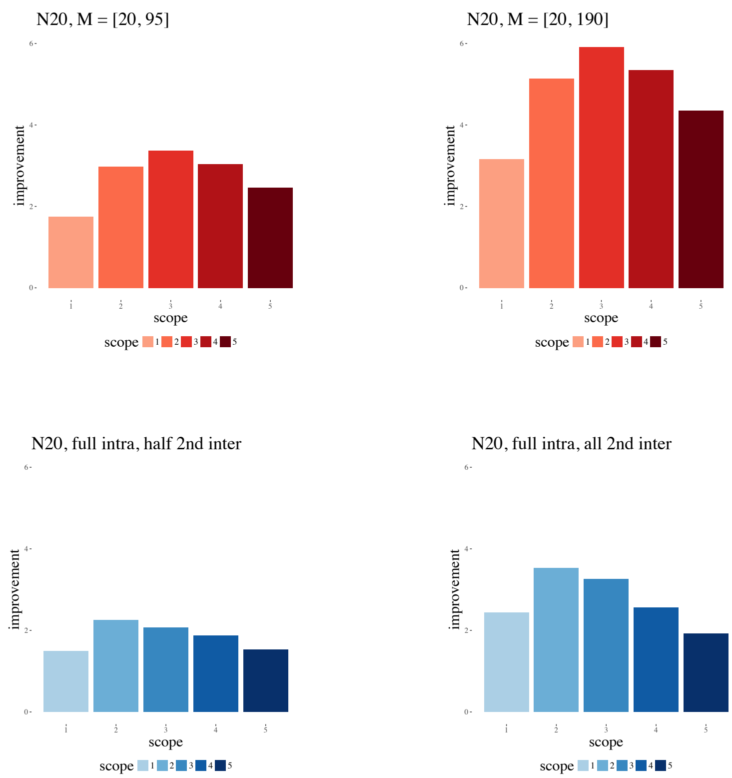


Figure 3.11: Final Performance Improvement for Similar Landscapes

Histograms of mean performance improvement at time step 28 by scope for cluster 5 landscapes and for `by_firm` exploration with `veto` evaluation scenario.

Table 3.3 provides the results of ANOVA test comparing the improvement results shown in Figure 3.11 for random versus structured results, as well as the results within each landscape class. The first comparison tests for three-way interactions between class, scope and landscape. We see that improvement differs across scope as expected, and that the pattern of deviations from the mean across scopes differs by class. Thus random landscapes produce statistically different improvement by scope results than structured landscapes. We also see that once class is accounted for, there is no meaningful variation between landscapes.

Table 3.3: ANOVA Results for Performance Improvement Across Landscapes

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Random vs Structured Landscapes					
class	1	10584.28	10584.28	175.44	0.0000
scope	4	6006.25	1501.56	24.89	0.0000
landscape	2	12668.81	6334.41	105.00	0.0000
class:scope	4	1601.66	400.41	6.64	0.0001
scope:landscape	8	633.75	79.22	1.31	0.2395
Residuals	180	10859.30	60.33		
Random Landscapes					
scope	4	5744.80	1436.20	16.45	0.0000
landscape	1	10631.05	10631.05	121.77	0.0000
scope:landscape	4	371.09	92.77	1.06	0.3797
Residuals	90	7857.07	87.30		
Structured Landscapes					
scope	4	1863.11	465.78	13.96	0.0000
landscape	1	2037.76	2037.76	61.09	0.0000
scope:landscape	4	262.65	65.66	1.97	0.1061
Residuals	90	3002.22	33.36		

ANOVA results for three different comparisons: cross class data with improve scope + class + landscapes + scope * class * landscape, random landscape data with improve scope + landscapes + scope * landscape, and structured landscape data with improve scope + landscapes + scope * landscape.

When comparing improvement values within landscape classes, we looked for interactions between scope and landscape, and found that the distribution of improvement across scopes doesn't differ significantly across landscapes of the same class. Thus we find evidence that the structure we build into a search landscape via the NM formulation has a meaningful effect on the search results.

3.5.5 Results for Voting Evaluation

The results for the SIM presented in Section 3.5.2 were based on the conservative veto algorithm, where if any proposed strategy change caused an in-scope division to lose performance value the change was rejected. In these scenarios, firms with broader scopes experienced less improvement than firms with narrower scope due to structural inertia.

One could imagine a broadly scoped firm with a less conservative evaluation algorithm, such as voting, where each in-scope division has a single vote and the majority prevails. In the case of a three-division firm considering a strategy change, if a single division were to experience a loss in performance while the other two experienced gains, the veto evaluation algorithm would reject the change while the vote algorithm would accept the change. Figure 3.12 demonstrates that with the voting evaluation algorithm, broadly scoped firms experience more improvement than narrowly scoped firms and the effects of structural inertia are mitigated through the decision process.

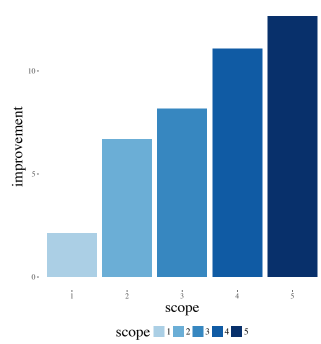


Figure 3.12: Performance Improvement over Time by Scope and Landscape

Performance improvement over time by scope over 28 time steps fitted over 100 runs on 10 different versions of the N20D5inter2f landscape for `by_firm` exploration with `vote` evaluation scenario. The top row are random landscapes and the bottom row are structured landscapes. Compare with the lower right plot in Figure 3.11.

3.6 Summary

In this chapter, we first introduced LON maps to the management literature. LONs allow us to both visualize and measure searchability of a given landscape. Our second contribution is the introduction of the NM method for generating landscapes, by which we are able to greatly reduce the burden of using fitness landscapes in management research. Our most meaningful contribution, however, is demonstrating how these two methods can be combined to produce and describe structured fitness landscapes that are understandable and produce meaningful search outcomes, and the development of a relevant model demonstrating these advantages.

We then applied this hybrid method to model structural inertia in firms with variable scope. The structured landscape enabled decisions within divisions to be more tightly connected than decisions between divisions. We demonstrated that broadly scoped firms under a conservative evaluation algorithm experience less improvement within a period of time than firms with narrower scope.

Chapter 4

THE EMERGENT FIRM MODEL

4.1 Background

Academia has produced many sophisticated definitions of a firm, such as a collection of resources (Wernerfelt 1984), dynamic capacities (Teece, Pisano, and Shuen 1997), routines (Nelson and Winter 1982) or bundles of contractual obligations (Coase 2012; O. E. Williamson 1985), and we can easily forget that a firm is fundamentally a collection of individuals voluntarily engaging in some form of team production (Gavetti and Levinthal 2000; Blair and Stout 1998; Ray 2007). The Emergent Firm Model (henceforth referred to as the *EF* model) is based on the premise that firms arise out of individuals choosing to work together to advantage themselves of the benefits of returns-to-scale and coordination. The EF model is a new implementation and extension of Axtell's Endogenous Dynamics of Multi-Agent Firms Model (henceforth referred to as the *Axtell* model), where individual agents chose to work with a given firm, move to another firm, or start their own firms based on their opportunities and preferences for work (Axtell 1999; Axtell 2015; Axtell 2018).

The EF model provides three levels of analyses: 1) agents with varying preferences for work and leisure and with varying innate capabilities, 2) the firms they create by joining together into teams, and 3) the overall economy, which is the emergent collection of firms that results through the individual agent decisions.

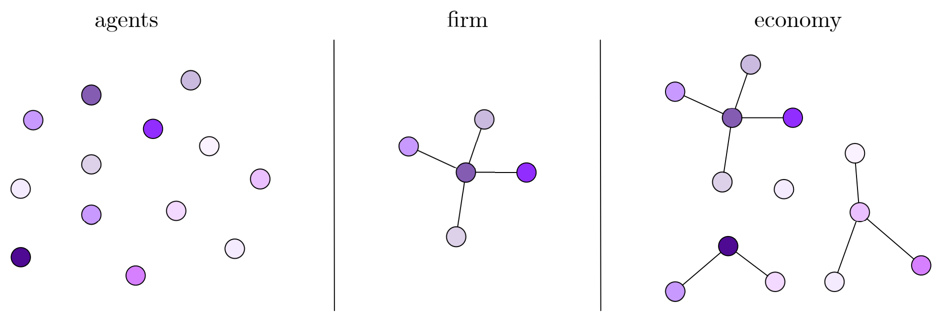


Figure 4.1: Three Levels of Model Entitites

Hierarchical depiction of the three levels of model entities: individuals, firms and an economy. Individuals compose firms, and firms compose the economy.

The EF model extends the Axtell model to address macroeconomic questions pertaining to job mobility, debt and wealth, and explores how institutional conditions affecting microeconomic behavior modify emergent macroeconomic outcomes. Conventional theory regarding lending and borrowing behavior (Bewley 1986; Huggett 1993; Aiyagari 1994) suggest poorer agents save more than wealthier agents, likely due to lack of access to credit. However Getter (1996) describes how the consumer credit market has loosened restrictions and made credit more widely available to poorer agents, resulting in more debt and less savings. Likewise, Pollin describes a sharp increase in the use of consumer credit applied to necessitous spending, where agents borrow to make regular purchases, which in turn may lead to liquidity-traps that make future saving difficult (Pollin 1988; T. A. Sullivan, Warren, and Westbrook 2001; Eggertsson and Krugman 2012). Alternatively, Steindl explicitly modeled household saving and debt, and proposed that consumer credit could act as an economic stimulus (Steindl 1990), whereas Dutt, on the other hand, finds evidence that consumer debt results in economic contraction (Dutt 2006). Clearly there are open questions regarding how credit affects an economy, and an endogenously-driven agent-based model may be helpful in understanding these dynamics.

I make the assumption that changing jobs and creating new firms incur costs above those that could be regarded as general household expenditures, and that these expenses are not smoothed over a period of time and are funded via savings or borrowing (J. X.

Sullivan 2008; Clark and Davies Withers 1999). The ability to make a change will therefore be dependent on an individual's savings and access to credit, and I have thus added a constraint to the free movement of individuals in the Axtell model. In addition, I have borrowed and implemented two models from monetary policy and finance literature. The first is a cash-in-advance model, where an agent must have available funds to make an employment change (Lucas and Stokey 1985). Agents save a portion of their wage each time step, the quantity dependent on their individual saving rate, and savings accrue until agents spend all or a portion on making an employment change. I have also implemented a basic credit-creation model (Werner 2014; Jakab and Kumhof 2015) by adding a universal lender who makes funds available to agents without current loans upon demand.[1]

4.1.1 Axtell's Endogenous Dynamics of Multi-Agent Firms Model

The US economy involves a great deal of frictional employment activity (Davis, Faberman, and Haltiwanger 2006; Davis, Faberman, and Haltiwanger 2012). People at various times move from one job to another, and as a result firms are established, grow, shrink and dissolve. Any general equilibrium model of firm dynamics requires, by definition, exogenous shocks in order to perturb the stable system, with the assumption that firms are differentially affected by these exogenous shocks (Hopenhayn 1992a; Hopenhayn 1992b). However, the persistent and continual nature of firm dynamics contradicts the description of these dynamics as driven by exogenous shocks. Axtell (2015; 2018) has developed an alternative to the general equilibrium model of firm dynamics, in which firm dynamics are driven by a microeconomic level *perpetual adaptation* on the part of agents who "regularly adjust their work effort, and periodically seek better jobs or start new teams when it is in their self-interest." Firm dynamics are therefore entirely endogenous, driven by the engine of individuals continuously adjusting their work effort and firm affiliation in the quest for

[1] Two of principles behind "New Monetarist Economics" are that theories should be based in microeconomic foundations and that money should be explicitly handled rather than implicitly handled through utility functions (S. Williamson and R. Wright 2010). Both these principles lend themselves well to agent-based modeling.

utility improvement. The modeled economy, as a result, is perpetually in flux, rather than in stasis until perturbed by an external force.

Agents in the Axtell model explore options for changing firms, forming a startup, or remaining in their current position based on maximizing a Cobb-Douglas utility function with their individual preference set for income and leisure,

$$U = \left(\frac{O}{n}\right)^{\theta} (\omega - e)^{1-\theta}, \tag{4.1}$$

where O is total firm output, n the number of persons in the firm, such that $\frac{O}{n}$ is the individual's income in the current firm configuration. The individual's preference for income is given by θ, therefore preference for leisure is $1-\theta$. The individual's total time endowment is ω and e is the individual's work effort, thus the individual's leisure is $\omega - e$.

Each firm has unique parameters a, b and β that characterize the returns to scale in its production function

$$O = aE + bE^{\beta}, \tag{4.2}$$

where E is the sum of all the firm members' efforts.

Agents are connected via an underlying social network, modeled as an Erdös-Renyi network, and can chose to join a firm that employs a neighbor in this social network (Montgomery 1991).

These basic microeconomic principles embodied by a group of utility-seeking agents create macroeconomic conditions of "fluctuating effort and sustainable cooperation" (Huberman and Glance 1996). Individuals choose to remain with their current firm, change firms, or create a new firm based on these utility and output functions under dynamic conditions, and drive the evolution of an economy composed of various sized firms.

For large numbers of agents approximating the working population of the United States, the Axtell model realizes an economy that matches key macroeconomic statistics for the US economy such as firm size, job tenure, employment, wage distribution and productivity distributions.

4.2 Emergent Firms Model: The Axtell Model Reimagined and Extended

The Emergent Firms (EF) model uses the same individual utility-seeking engine as the Axtell model to drive firm formation, growth and dissolution. I've used the Axtell values for parameters governing the utility maximization and output calculations, as well as for the structure and degree distribution of the social network. I've also adopted the Axtell starting condition where all agents are in their own firms, designated *singleton* firms, as well as the equal compensation rule for distribution of outcome. These Axtell parameter values are given in Table 4.1.

Table 4.1: Axtell Model Parameter Values Retained in the EF Model

Attribute	Description	Value
`a`	effort multiplier in output formula	$\mathcal{U}(0, .5)$
`b`	exponential effort multiplier	$\mathcal{U}(.75, 1.25)$
β	returns to scale exponent	$\mathcal{U}(1.5, 2)$
θ	preference for income	$\mathcal{U}(0, 1)$
ω	time endowment	1
ν	number of social network links	$\mathcal{U}(2, 6)$
	compensation rule	equal shares
	initial condition	all singleton firms

I've modified the original Axtell implementation such that each firm has an owner, and if that owner decides to pursue another opportunity and has employees, a random employee assumes firm ownership. Agents are characterized by both preferences for income and savings rates, as well as by the production function parameters that determine firm output levels. The owner of a firm determines the production function parameter values for that firm.

In addition, firms are implemented as individual star subgraphs within the full collection of graphs that describe the entire economy of firms (see Figure 4.1), thus the new implementation of the Axtell model contains two networks: the static information exchange social network and dynamic network describing the firm structure. I've retained the Erdös-Renyi structure for the social network to maintain consistency with the original model, though social networks could also be described by small world or preferential attachment structures (Watts and Strogatz 1998; Barabási and Albert 1999). The firm structure network emerges out of the model dynamics and is a collection of star graph components and single nodes.

The EF model not only replicates the Axtell model functionality, but also extends the original model with the addition of two major functional elements. The first addition is costs to changing jobs or forming a startup, with a cash-in-advance constraint to movement. Each time step, all agents save a portion of their income dependent on their individual savings rate. I assume living costs are covered by wage and any residual goes into savings, so the varied savings rates are a proxy for varied levels of consumption. This savings is used to pay the costs of accepting a better employment opportunity, where better is defined as providing a higher utility value, or for starting a new firm if that option provides the highest utility.

The second addition is a credit-creating universal lender providing loans to agents who wish to make an employment change but have insufficient savings. If the agent has an opportunity for increasing her utility and does not have the savings to pay the costs of the change, she can take out a loan with interest compounded each time step at a constant rate. Loans are paid with a borrowing agent's full savings each step until repaid. An agent cannot take out a loan if she already has as unpaid loan.

The EF model is significantly scaled down from the Axtell model, and rather than attempting to replicate quantitative macroeconomic statistics I am looking for qualitative changes in firm size population distributions under different cost and lending conditions, as well as emergent patterns in macroeconomic characteristics such as per capita wealth, wages, productivity and debt. Experiments with the EF model were made over 20 runs for 600 agents over 500 steps with an activation rate, or *churn*, of 10%. Therefore an average of 60 agents explore alternative employment options each step, for a total of 30,000 explorations for each of the 20 simulation runs.[2] Table 4.2 describes the additional parameters in the EF model and their values.

Table 4.2: Parameters Specific to the EF Model

Attribute	Description	Value
`N`	number of agents	600
`churn`	agent activation rate	.1
`tmax`	number of steps	500
`move`	job change cost, multiplies last wage	1
`startup`	startup cost, multiplies last wage	2
`rate`	multiplies wage each time step	$\mathcal{N}(.03, .01)$, truncated at 0
`lendingrate`	cost of loan each time step	.03

The full EF model functionality is illustrated as a flowchart in Figure E.3. Cost and lending functionality can be toggled independently so I can explore three distinct scenarios:

Scenario 1: free movement, the original Axtell model which serves as a baseline

Scenario 2: cash-in-advance constraint to movement, or costs

Scenario 3: cash-in-advance constraint to movement with credit-creating universal lender, or costs with credit.

[2] The Axtell model was run with 120 million agents, activated at a rate of 4% per turn, running for 300 steps. This means 4.8 million firms explored alternatives each step, for a total of 1.44 million explorations.

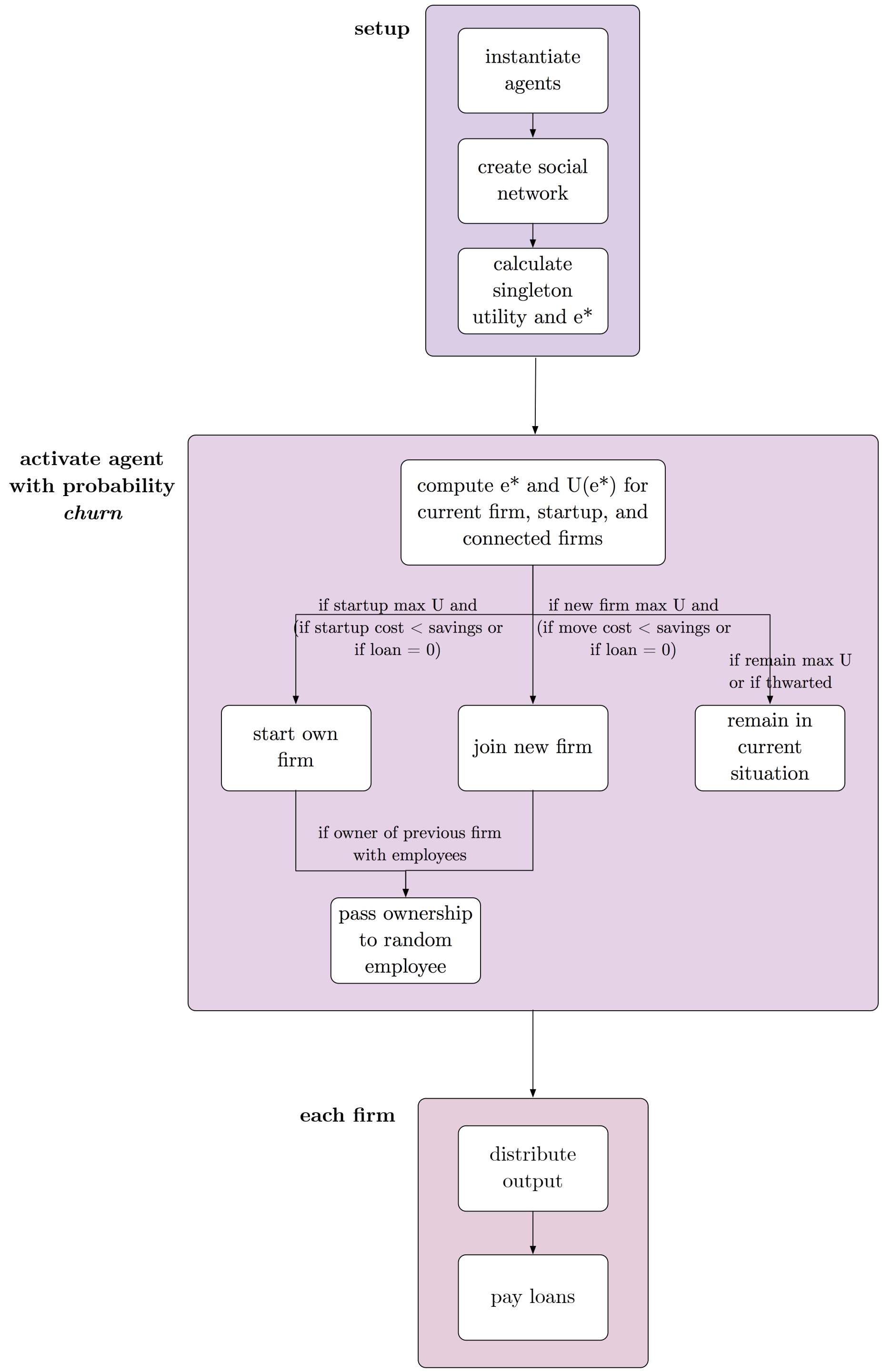


Figure 4.2: Algorithmic Flow for the Emergent Firm Model

4.3 Results

For the three scenarios described in Section 4.2, I explore the number of firms and their size distributions, mean and maximum firm size, numbers of changes and lost opportunities for change (described as *thwarts*), per capita wealth, wages, productivity, as well as loans and debt. I will demonstrate that, across the board, the institutional conditions represented by the cash-in-advance constraint and the credit-creation lender implemented at the microeconomic level have statistically significant effects on these macroeconomic measures in the various emergent economies. Unless otherwise indicated, all simulations were run 20 times with the parameters and settings described in Tables 4.1 and 4.2.

Despite the statistical differences in measures across the scenarios, the EF model exhibits the same overall behavior as the original Axtell model. Regions of steady-state population stability emerge at the macroeconomic level even though the composition and size of any given firm is in flux. For all scenarios, an equilibrium region emerges after roughly 300 time steps and the number of firms oscillate within this region, as demonstrated in the 20-run spaghetti time series plots for each scenario in Figure 4.3.

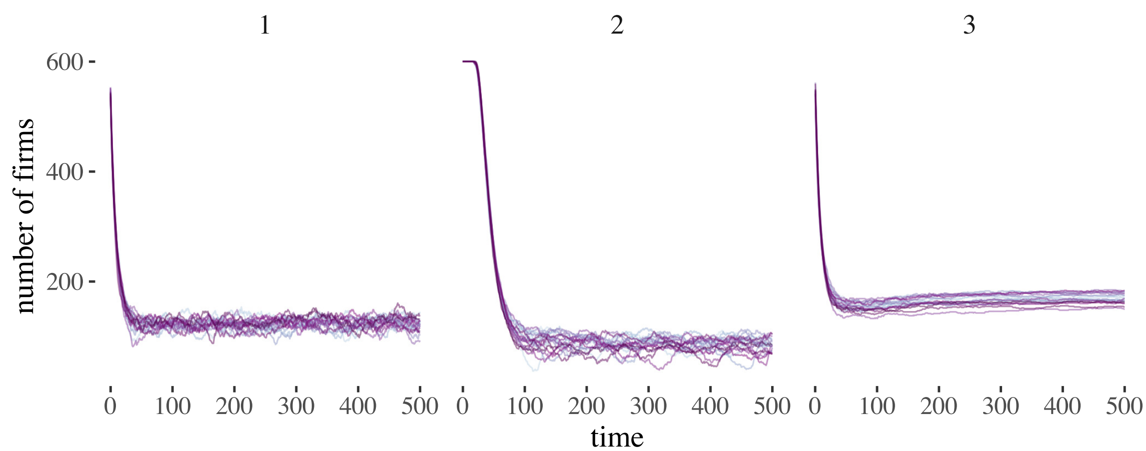


Figure 4.3: Macroeconomic Convergence of Total Number of Firms

Spaghetti plot of number of firms over time showing firm population results for 20 distinct runs for all three scenarios, demonstrating the macroeconomic convergence into a steady-state equilibrium band.

4.3.1 Number and Sizes of Firms

The first measures we'll explore are the total number of firms in the macroeconomic steady-state and the sizes of those firms as described by the number of agents within the firm. The mean number of firms for the three scenarios at the final time step are 122, 85 and 171 respectively. The mean sizes for the three scenarios are 17, 20 and 10 respectively, and maximum sizes are 78, 123 and 31. More firms implies smaller firms as measured by number of employees. It follows that the maximum number of firms in an economy with 600 agents is 600 firms, which occurs when each agent is in a singleton (this is the initial condition). Scenario 3, the cost and credit scenario, produces the most firms, therefore the smallest firms. Scenario 2, the cost scenario, produces the fewest, conversely the largest, firms. The number of firms in scenario 1, the baseline scenario, falls between those produced by scenarios 2 and 3.

Evidence of the differences in number of firms and mean and maximum sizes across the scenarios is presented visually by the violin plots in Figure 4.4 and statistically in Table 4.3. All differences between scenario results for numbers of firms have four-star significance. All comparison differences for both mean size and maximum size are statistically significant, most pronounced for comparisons between scenarios 1 or 2 with scenario 3. We also see from the Figure 4.4 that scenario 2 produces the most variation in firm size, with the largest maximum size, and scenario 3 produces the least variation in size and the smallest maximum.

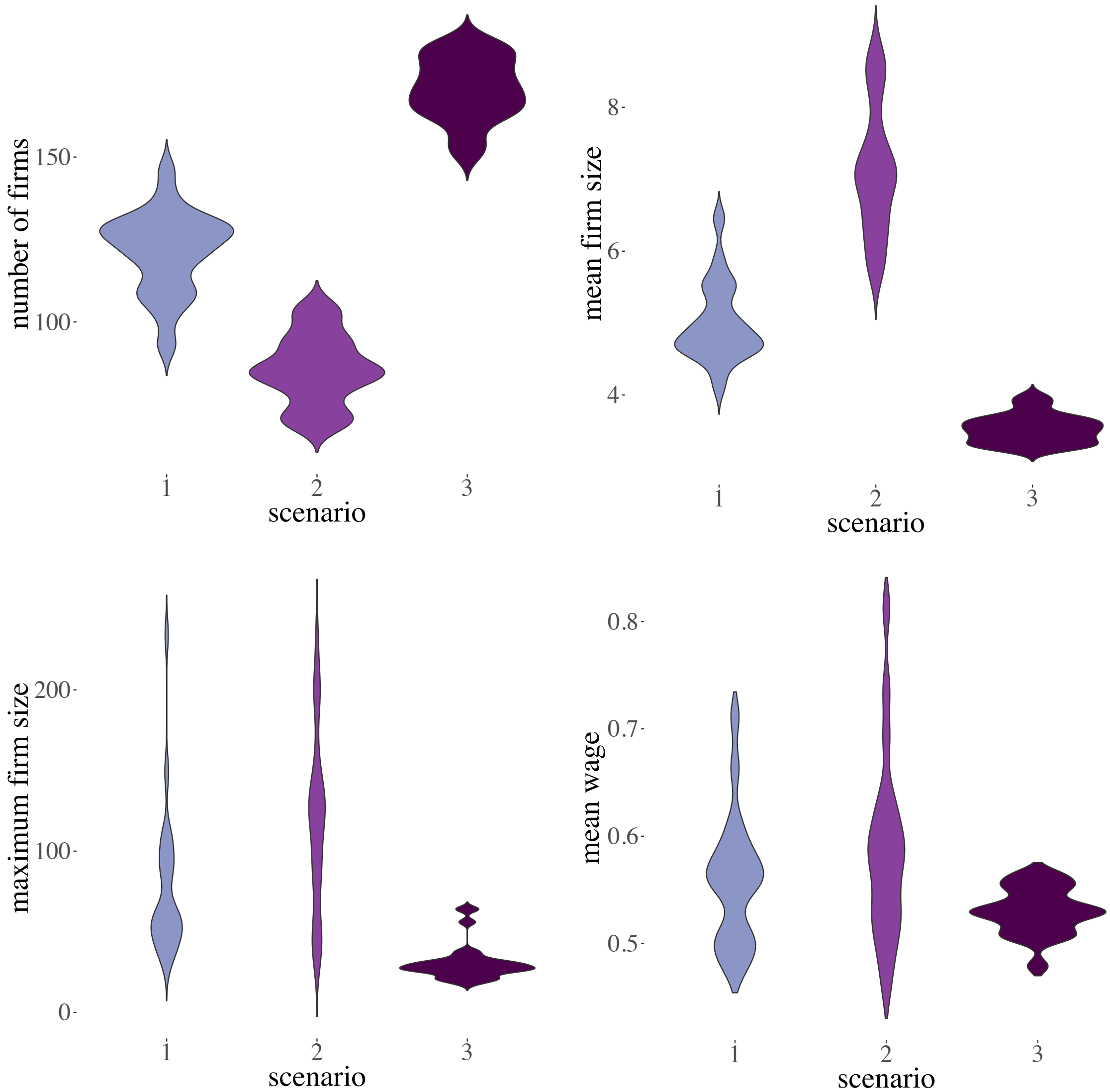


Figure 4.4: Violin Plots for Number of Firms, Mean and Maximum Firm Size and Mean Wage

Violin plots showing the data distribution over 20 runs of the total number of firms in the final time step for number of firms, mean and maximum firm size, and mean wage.

Table 4.3: Statistical Test Results for Macroeconomic Measures

Scenarios 1 & 2: Baseline vs. Costs

Measure	Means (1, 2)	SDs (1, 2)	Statistic	df	p-value
number of firms	122, 85.1	12.8, 11.0	9.7	37.2	1.13e-11****
firm size	17.0, 20.2	3.8, 4.6	-2.4	36.8	.0225*
maximum firm size	78.7, 123	47.2, 56.5	-2.7	36.8	.0116*
changes	44.3, 24.3	7.0, 4.6	10.6	33.0	3.21e-12****
thwarts	0 , 17.9	0, 4.0	-20.2	38	2.2e-16****
wage	0.56, 0.59	0.06, 0.09	-1.2	33.9	.2364
wealth	5206, 667	304, 102	63.3	23.2	2.2e-16****
productivity	2.78, 4.24	0.41, 1.08	-5.66	24.3	7.6e-6****

Scenarios 1 & 3: Baseline vs. Costs with Credit

comparison	means (1, 3)	SDs (1, 3)	statistic	df	p-value
number of firms	122, 171	12.8, 9.7	-13.8	35.5	7.8e-16****
firm size	17.0, 10.3	3.8, 1.4	7.4	24.3	1.0e-07****
maximum firm size	78.7, 30.7	47.2, 11.2	4.4	21.1	0.0002***
changes	44.3, 3.45	7.0, 1.2	25.7	20.2	2.2e-16****
thwarts	0, 38.1	0, 6.1	-27.8	38	2.2e-16****
wage	0.56, 0.53	0.06, 0.02	1.9	24.1	0.066
wealth	5206, 793	304, 117	.6	24.5	2.2e-16****
productivity	2.78, 1.87	0.41, 0.15	9.44	23.9	1.6e-9****

Scenarios 2 & 3: Costs vs. Costs with Credit

comparison	means (2, 3)	SDs (2, 3)	statistic	df	p-value
number of firms	85.1, 171	11.0, 9.7	-26.2	37.4	2.2e-16****
firm size	20.2, 10.3	4.6, 1.4	9.3	22.7	3.5e-09****
maximum firm size	123, 30.7	56.5, 11.2	7.1	20.5	5.9e-07****
changes	24.3, 3.45	4.6, 1.2	19.4	21.7	3.4e-15****
thwarts	17.9, 38.1	4.0, 6.1	-12.4	38	7.0e-15****
wage	0.59, 0.53	0.09, 0.02	2.8	21.5	0.0103*
wealth	667, 793	102, 117	-3.6	37.3	0.00088***
productivity	4.24, 1.88	1.08, 0.15	9.75	19.7	5.6e-9****

Statistical test results for scenario measures at the final time step (`time` = 500). Equality of means tests for each combination were conducted via the Welch Two Sample t-test.

4.3.2 *Size Distributions and Power Law Functions*

We've established that the numbers and sizes of firms are different from one scenario to another, but what about the distributions of those sizes in the macroeconomic steady-state? Numerous works dealing with firm size distributions claim that such distributions are power law functions with a shape parameter, α, close to 1 (Axtell 2001; Rossi-Hansberg and M. L. Wright 2007; Stanley et al. 1996).

If the firm size distributions follow a power law function, $p(x) = Cx^{-\alpha}$, then the logarithmic plots of size vs population density will yield a straight line. The leftmost plot in Figure 4.5 is the logarithmic plot of firm size distributions for all three scenarios. Note the spreading in the region of the largest firms, suggesting the sample size of 600 agents is too small to generate a full representation of consecutive firm sizes because some larger sizes are missing completely, and the data contains many zero values. A rudimentary power law function analysis could be obtained through a linear fit to this data, but the results will suffer from this spread in larger firm sizes.

A cumulative density function mitigates this incompleteness problem because every size value in the cumulative distribution will have a non-zero value. If the probability density function is a power law function, the cumulative density function is also a power law. This CDF is shown in the model plot in Figure 4.5 and doesn't appear to fit a power law function across the whole distribution, but does suggest a power law function could apply to the region of the distribution with a complete set of consecutive size values, roughly up to firms with 30 employees. Figure 4.5 shows a CDF of firm size distributions for all three scenarios truncated at a size of 30 employees and Table 4.4 shows the results of fitting linear models to this truncated data.[3]

[3] Following the formal power law validation protocol described in Chauset et al. (2009), I obtained results that the EF firm size distributions with 600 agents could indeed be represented by power law functions for the range up to 30 employees.

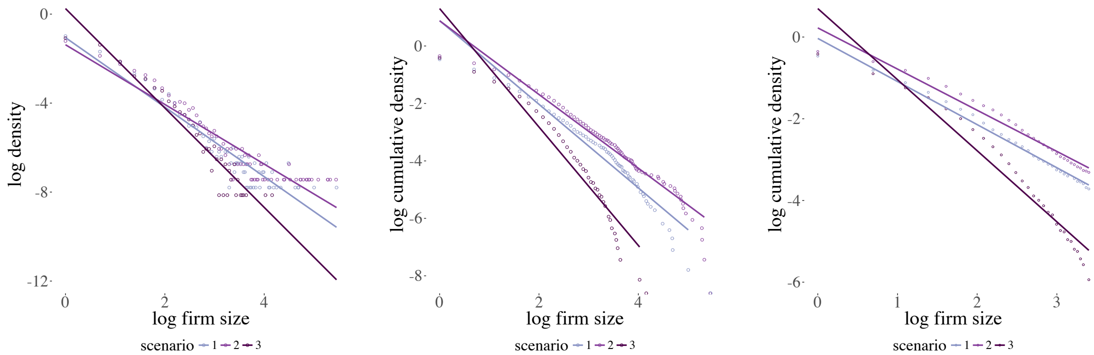


Figure 4.5: Firm Population Distributions for the Three Scenarios

Linear regression fits to the logarithmic probability density function (left) and cumulative density function (middle) using the population values for each size aggregated over all 20 runs. On the right is are linear regression fits to the cumulative density function from firms up to 30 employees using the population values for each size aggregated over all 20 runs.

Axtell found a power law function fitting the CDF with a shape parameter α = -1.06. Notice the near exact agreement with our baseline scenario 1 result for alpha in Table 4.4.[4]

Table 4.4: Results of Linear Fit for the CDF for Firm Size Distributions for Firms with 30 Employees or Fewer

	α	SE	R-squared	95% CI
Scenario 1: Baseline	-1.055	0 .026	0.983	[-1.109, -1.001]
Scenario 2: Costs	-1.008	0.036	0.964	[-1.082, -0.934]
Scenario 3: Costs with Credit	-1.740	0.086	0.936	[-1.916, -1.563]

4.3.3 Mobility: Changes and Thwarts

In the EF model, an agent's mobility describes its ability to make a desired change. Agents with lower mobility are thwarted more often than agents with higher mobility. Figure 4.6 shows a generalized additive model fit of the total number of employment changes and missed opportunities (thwarts) across 20 runs for each scenario. Mean numbers of

[4]This agreement is likely determined since we've used the Axtell model parameters for the utility and production parameters. I'll demonstrate later in Section 4.5 that seemingly slight modifications to either the preferences for income, θ, or the returns to scale exponent, β, change the α values. These parameters could be adjusted such that alternative scenarios also provide the expected value for the power law function shape factor.

changes and thwarts for the three scenarios in the steady state are 44 changes and 0 thwarts, 24 changes and 18 thwarts, and 18 changes and 38 thwarts, respectively. There are no restrictions on making changes in scenario 1 so any utility improving opportunity can be acted upon, thus the number of thwarts is 0 and scenario 1 produces the highest number of changes.

The number of changes decreases across the scenarios and conversely the number of thwarts increases. Since scenario 2 contains more changes than scenario 3, we find the interesting result that borrowing to make a move results in fewer moves than overcoming the cash-in-advance constraint via savings alone, likely due to agents with loans waiting longer to acquire savings or paying off loans rather than simply saving without loans.

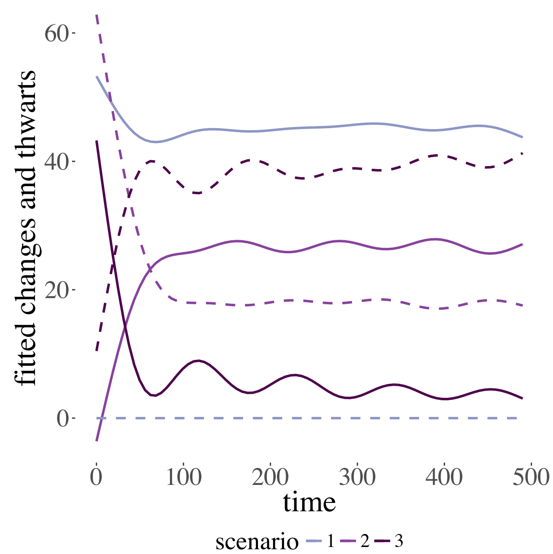


Figure 4.6: Mobility Measures over Time for Three Scenarios

Generalized additive model fits for the two mobility descriptors: changes and thwarts, with total changes (solid lines) and thwarts (dashed lines) over time at every 10th step for the three scenarios, indicated by color.

Statistical evidence of differences in changes and thwarts across the scenarios are given as Welch t-test results in Table 4.3.

4.3.4 Wages and Productivity

Wages in the EF model are an employees share of firm output, as defined in Equation 4.2, distributed each time step, and a firm's productivity is synonymous with the firm's output. Mean wages for the three scenarios are on average .56, .59 and .53, respectively. Per capita wage differences between scenarios are only significant when comparing scenarios 2 and 3, as demonstrated by the Welch t-test results in Table 4.3. As can be observed in the violin plots in Figure 4.4, scenario 2 demonstrates the greatest variation in mean wages, while scenario 3 demonstrates the least variation.

Empirical evidence suggests that wages are exponentially distributed (Yakovenko and Rosser 2009), and Axtell found a linear relationship in a semi-log plot of the wage distribution for larger firms, which characterizes an exponential relationship. The left hand plot in Figure 4.7 shows the results for a similar analysis with scenario 1 simulation results, which also demonstrates a linear relationship for larger firms, suggesting the baseline EF also produces exponentially distributed wages.

The mean productivity values for the three scenarios are 2.78, 4.24 and 1.88. Statistical evidence of differences in productivity across the scenarios are given as Welch t-test results in Table 4.3. Just as with wages, scenario 2 not only yields the greatest productivity values but also the greatest variance in productivity. Labor productivity is the relationship between the increase in firm productivity by adding an employee, and previous empirical work suggests that macroeconomic labor productivity can be described as constant returns to scale (Basu and Fernald 1997). Axtell demonstrated constant returns to scale for labor productivity at the macro-level despite increasing returns to scale at the agent level. I conducted a similar analysis with scenario 1 results, and produce constant returns to scale for labor productivity in agreement with Axtell's results (Figure 4.7 right).

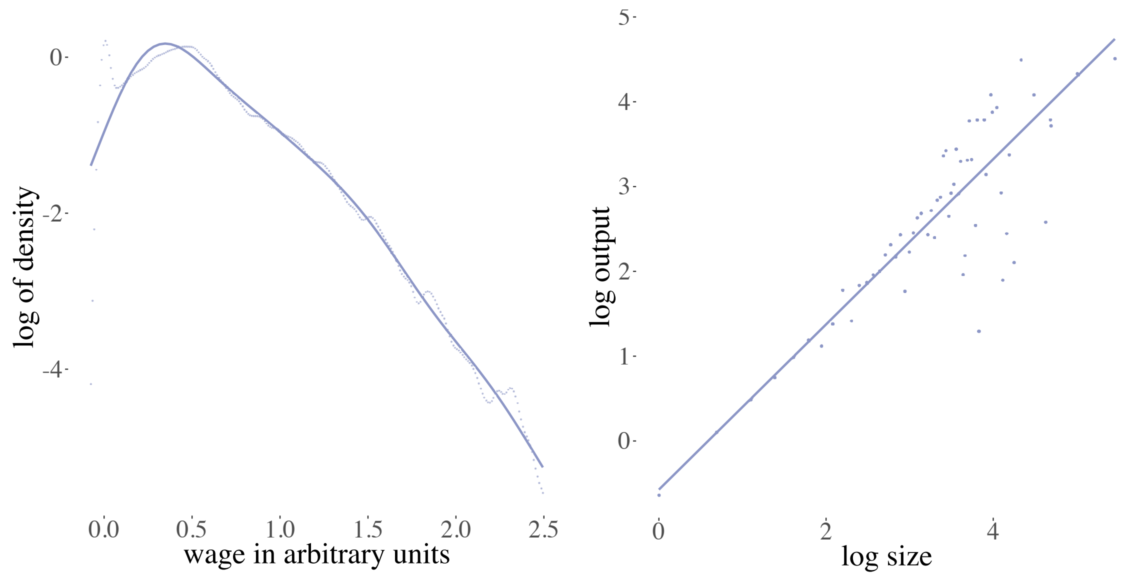


Figure 4.7: Wage and Productivity vs. Size

A linear semi-log wage distribution indicating an exponential wage distribution (left), and a constant return to scale for labor productivity in a logarithmic plot of size vs productivity (right).

Larger firms produce higher output, but larger firms do not necessarily produce higher wages, as demonstrated by the correlation plots in Figure 4.8 and Table 4.5. In fact, there is a slight but significant negative correlation between wage and firm size for scenarios 1 and 3. [5]

[5] An interesting microeconomic observation is the correlation between an individual's wage and θ value, which is preference for income. Individuals with the highest wages also have the highest values for θ. In addition, individuals who are firm owners have, on average, higher θ values as well (at $t = 500$ in scenario 1, $\bar{\theta} = .67$). Entrepreneurs have a predisposition toward starting firms that other individuals may not share, represented in the model as a high θ value.

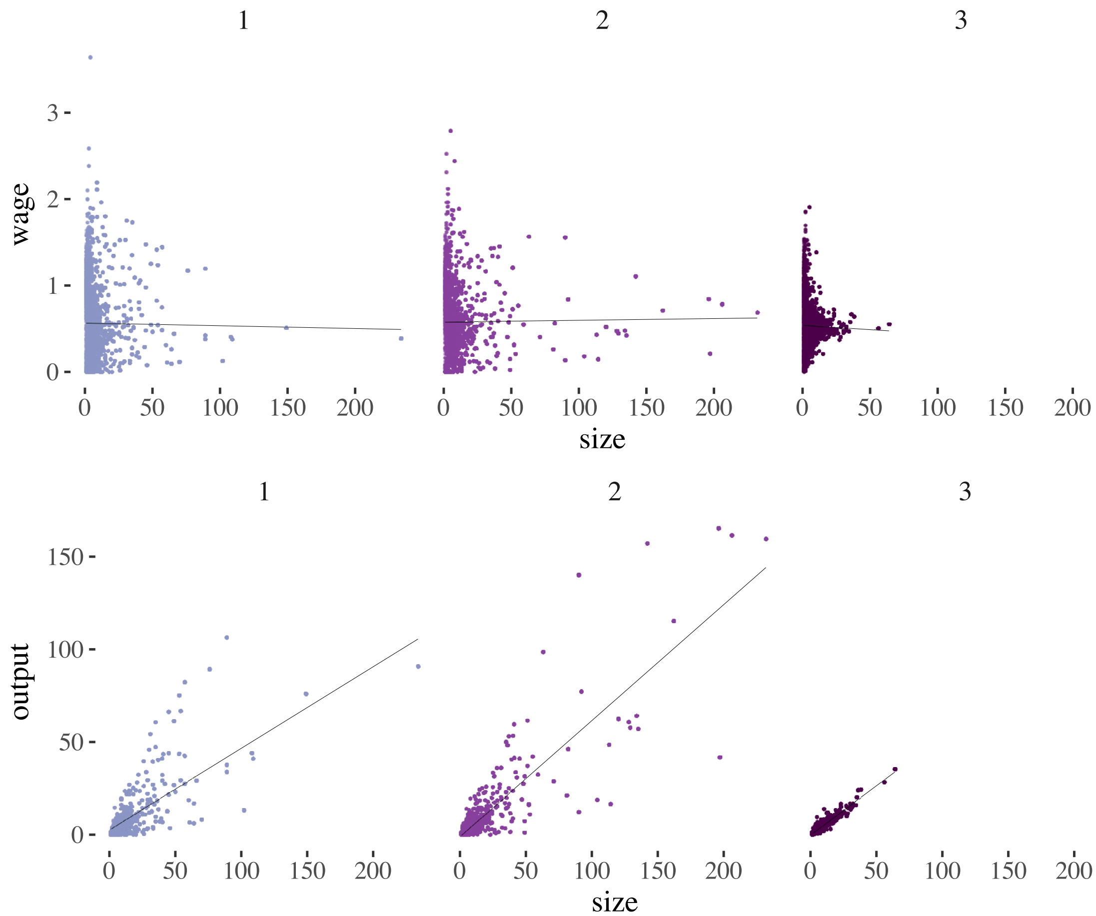


Figure 4.8: Wage and Productivity Correlated with Size

Correlation plot for size and wage (top) and size and productivity (bottom).

Table 4.5: Correlation Values Between Wage, Productivity and Size

	Scenario 1 Baseline	Scenario 2 Costs	Scenario 3 Costs with Credit
Size and Productivity	0.809	0.871	0.973
Size and Wage	-0.031	0.031	-0.044

Pearson correlation values between size and productivity and between size and wage for the three scenarios at time = 500, all values are statistically significant.

Figure 4.9 shows the macroeconomic measures presented thus far as a parallel plot with each line representing an individual run and each color representing a scenario. We see from this plot many of the relationships discussed, such as population inversely related to size, the variation in maximum size, the inverse relationship between changes and thwarts, and the similarity in wages.

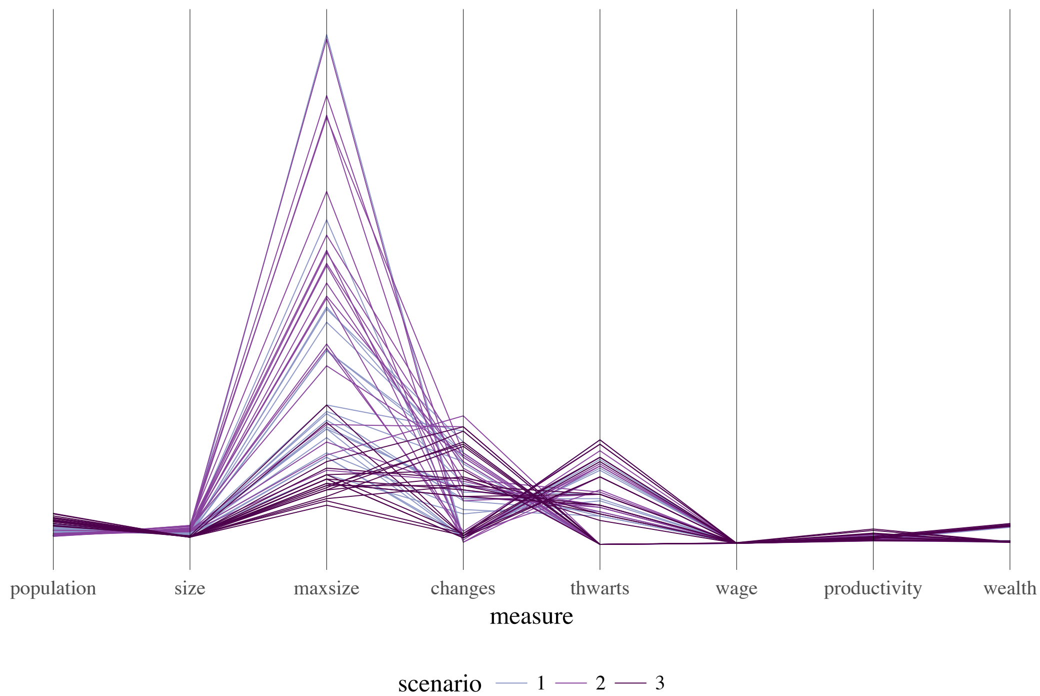


Figure 4.9: Parallel Plot of Mean Values for Selected Macroeconomic Measures

Parallel plot showing mean values for the measures discussed above. Lines are shown for each of 20 runs of the three different scenarios. Units are arbitrary.

Scenario 3 allows agents with insufficient savings to pursue utility improving opportunities by taking out a loan. The total amount of loans in the simulated economies increases superlinearly in the equilibrium regions for lending rates greater than 0. Figure 4.10 demonstrates model results over 20 individual runs for total loan amounts, wages and wealth (the sum of agent savings) for lending rates of 0% and 1% and 3% with every agent having the exact saving rate of 3%. Cost multipliers are homogenous as well and equal to the current agent's wage. We see that with a lending rate of 0% there is no superlinearity in aggregate loan value, and the higher the lending rate the sooner the superlinearity appears. The colored regions indicate whether or not the difference between wealth and loans is positive (blue) or negative (red). Superlinear behavior in loans results in an economy with negative net wealth.

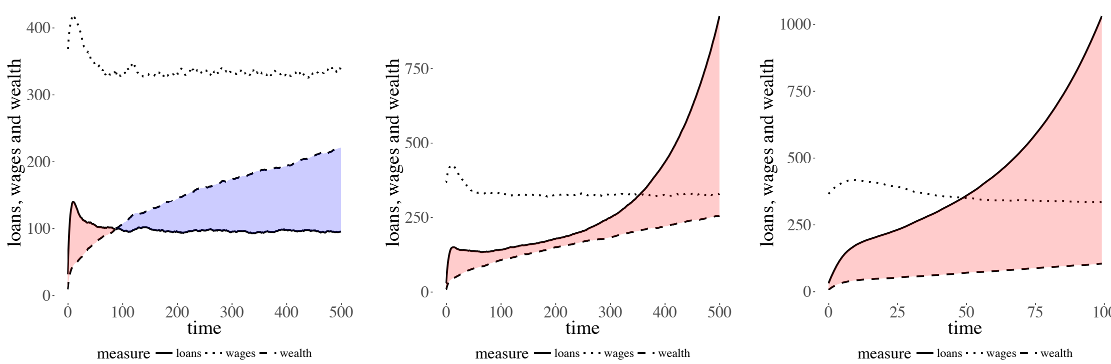


Figure 4.10: Loans, Wages and Wealth over Time for Different Savings Rates

Plots of loans, wages and wealth for values of lending rate 0%, 1% and 3%. Savings rates and cost multipliers are homogenous for all agents and types of moves. Net wealth, or savings minus loans, is indicated by the colored regions between the wealth and loan lines. Blue indicates positive net wealth and red is negative net wealth.

To explore the dynamics underlying this superlinearity in aggregate loan value, I consider λ_1 and λ_2 to be the quantity of loans at two consecutive time steps and if l is the lending rate, then $\lambda_2 = \lambda_1 + l\lambda_1 - \text{loan payments} + \text{new loans}$.

Loan payments are a function of the wages and savings rates of borrowers. If s is an agent's savings rate and w his wage, and b the set of agents with outstanding loans, then loan payments are

$$\sum_{b} s_b \omega_b. \tag{4.3}$$

New loans are a function of the number of start-up loans, company change loans, the costs for these two activities and the wages of the borrowers. If c_s and c_m are the wage multipliers to determine the costs for starting one's own company and changing company respectively, and ν_s and ν_m the instances of new loans made to facilitate start-ups and moves respectively, then the principle quantity of new loans are

$$\sum_{\nu_s} c_s \omega_{\nu_s} + \sum_{\nu_m} c_m \omega_{\nu_m}. \tag{4.4}$$

Assuming mean wage ω represents any given borrower's wage, mean savings rate s any given borrower's rate, and mean costs c represents both startup and move costs, the simplified total loan equation is

$$\lambda_2 = \lambda_1 + l\lambda_1 - sb\omega + c\nu\omega. \tag{4.5}$$

The superlinear behavior is described by an increasing difference in consecutive λ values. In the unusual case of interest-free loans, $l = 0$ and if $\lambda_2 - \lambda_1 > 0$ then

$$c\nu\omega > svb\omega \tag{4.6}$$

which reduces to

$$\begin{aligned} (\nu)(c) &> (b)(s) \\ \frac{\nu}{b} &> \frac{s}{c}. \end{aligned} \tag{4.7}$$

Equation 4.7 suggests that for homogenous costs and $c = 1$ loans will increase only if the ratio of new loans to existing borrowers exceeds the mean savings rate. Borrowers in this case will decrease over time because all borrowers will eventually pay back loans, loans will be repaid more quickly, and savings will be higher with a corresponding reduction in the need for loans. While loans may increase initially, macro-equilibrium will not exhibit superlinearity.

The superlinearity has two causes in a reasonably parameterized model. Savings rates are heterogenous in the EF model so there will be agents who make a loan and will not be able to repay that loan because their repayment rate is lower than the lending rate. In addition, a perpetually indebted agent may also have a saving rate equivalent to or higher than the lending rate, but may have chosen an opportunity that increased utility but decreased wage, again resulting in insufficient payments. Either way, the amount that borrowers owe will continue to grow over time.

Figure 4.11 shows the simulation values for the elements in Equation 4.7, costs, borrowers, wages and loans, with the lending rate 3% and homogenous savings and costs, the same values that produced the right most plot in Figure 4.10.

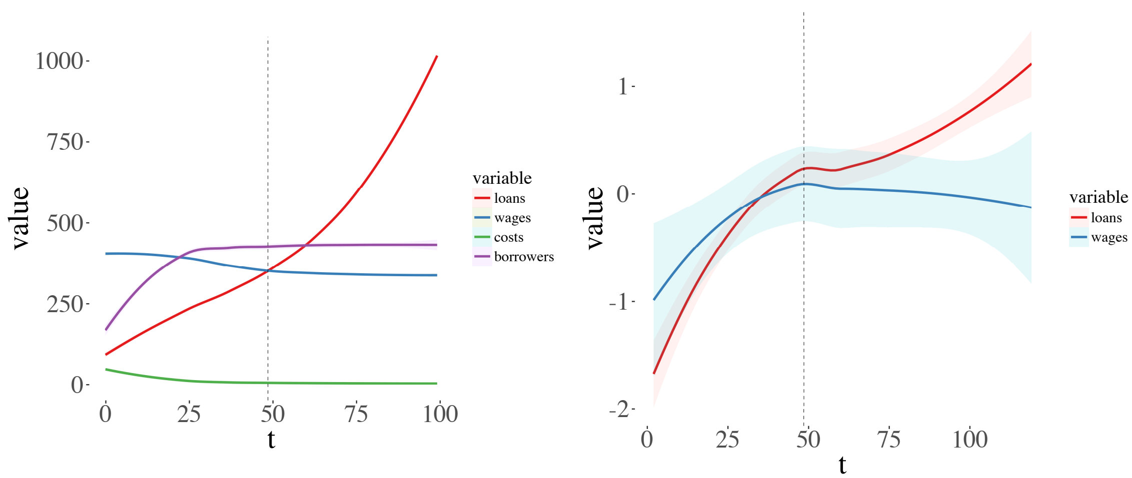


Figure 4.11: Loan Parameters Analysis

Simulation values averaged over 20 runs for the determinants of loan quantity with simulation parameters (left) and average discrete second derivatives of wages and loans (right). The cost multiplier for both startup and employer changes is 1, savings rate is homogenous at 3%.

The discrete second derivative of total loan and wage values for the two simulation are shown in the right hand plot in Figure 4.11, and notice the matching inflection points in both the loan and wage curves.

4.3.6 Wealth and Debt

Wealth in the EF model is modeled as the sum of all agents savings. Figure 4.12 shows the wealth values over time for each of the scenarios, and at macroeconomic equilibrium the mean values for the three scenarios are 8.68, 1.11 and 1.32, respectively. As expected, in scenario 1 agents incur no costs when making an employment change or starting a firm and their savings continually accrue, thus we see the highest and increasing wealth values. Wealth in scenario 2 is lower than scenario 1, again as expected, since there are costs to taking advantage of an opportunity for utility improvement and those costs deplete savings, thus lowering wealth. This scenario 2 wealth levels off over time, while scenario 3 wealth increases gradually, though at a lower rate than in scenario 1.

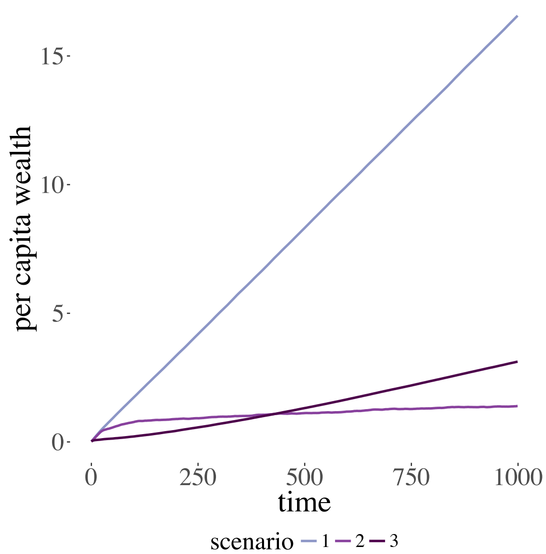


Figure 4.12: Per Capita Wealth for the Three Scenarios

But we noted in Section 4.3.5 that positive lending rates lead to super-linear loan behavior which results in rapidly expanding debt values. We'll therefore consider a variant of the wealth metric, net wealth, which is the difference between the sum of all agent savings and all agent loans. For scenario 3 the cumulative agent debt drives the net wealth into sub-linearity with respect to time. The shaded regions in Figure 4.10 indicate positive or negative net wealth.

We also find wealth inequality in scenario 3. Not only does scenario 3 have extreme negative net wealth values, but those agents who do have savings have higher individual savings values than those in scenarios 1 or 2. The maximum value for savings in scenario 1 is 28.1, for scenario 2 it is 17.1, but for scenario 3 the maximum amount of savings is 42.7. In scenarios 1 and 2 all 600 agents have some amount of savings, while in scenario 3 only 158 agents on average have savings greater than 0. The remaining 442 agents have debts. Thus we have produced a wealth inequality roughly characterized by agents with debt and agents without debt.

4.4 Network Graphs

In the EF model, I define an economy as a collection of firms, implemented as a collection of star graphs. The center node of each subgraph represents the agent owning the firm and the neighboring nodes are agents employed by the firm. These graphs provide an accessible visualization of qualitative characteristics of the various economies resulting from different cost and lending scenarios. Figure 4.13 shows the differences in equilibrium economies between the three scenarios, with node color indicating the agent's wage, the darker the blue the higher the wage, for a single simulation run at $t = 500$. We immediately notice our findings described in Section 4.3.1 that scenario 2 yields the largest firm and scenario 3 the smallest firms.

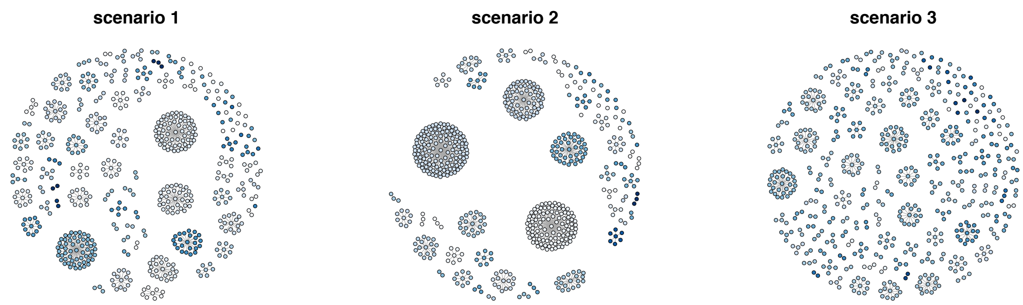


Figure 4.13: Network Graphs of Simulated Economies

Representative network graph for each of the scenarios for the final time step for a single run. Nodes are colored on wages with darker nodes representing higher wages.

The graphs can also be presented as an evolutionary time series. Figure 4.14 shows a scenario 3 economy in five snapshots in time, `time` = 100, 200, 300, 400 and 500 with color representing the value of an agent's wage.

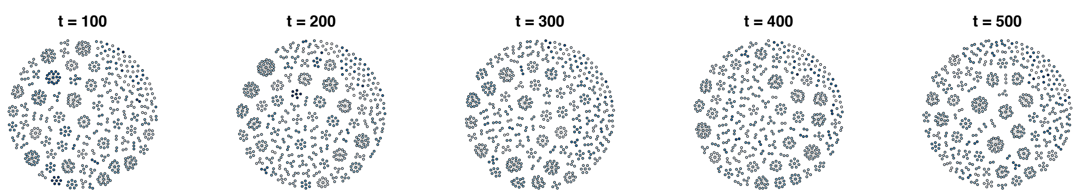


Figure 4.14: Network Graph Time Series of an Emerging Economy

Representative network graph for scenario 3 over time for a single run. Nodes are colored on wages with darker nodes representing higher wages.

By modeling the economy as a composite graph consisting of individual star graphs I can animate those graphs to demonstrate the evolution of the economy over time. Most macroeconomic measures described in the Section 4.3 could be described in terms of network measures, for example, firm size distributions can be described as component size distributions. Agent-level values such as wages and loans can be stored as node characteristics. The network graphs therefore provide a compact and comprehensive representation of model results, as well as compelling visualizations. While these characteristics are beneficial, I haven't found the network presentation to provide insights not available through traditional analysis of the model results.

4.5 Sensitivity Analysis

I conducted sensitivity analyses for number of agents, activation rate, cost ratios, savings and lending rates, initial conditions and utility and output parameters. The following sections are a sampling of the sensitivity finding that provide further evidence of model validity or explain model variations.

4.5.1 Numbers of Agents

The location and width of equilibrium regions similar to those that manifested in Figure 4.3 will depend of the number of agents. Table 4.6 shows how the number of firms and the maximum firm size, as well as their respective standard deviations, increase with the number of agents, N. The variability in the location and width of the bands suggests that the existence of the equilibrium region is a consequence of a fixed population.

Table 4.6: Mean Numbers of Firms and Sizes for Varying Numbers of Agents.

Agents	Number of Firms	Number SD	Maximum Firm Size	Maximum Size SD
200.00	3.68	4.32	36.77	20.80
300.00	4.60	5.97	41.97	21.08
400.00	5.38	7.82	55.08	27.73
500.00	6.17	9.48	64.63	36.46
600.00	6.65	10.86	69.68	39.66
700.00	7.17	12.14	77.82	43.31
800.00	7.74	13.56	84.43	42.32
1200.00	10.05	19.43	101.89	64.33

Mean numbers of firms and sizes of firms with standard deviations over the last 300 time steps for 20 runs in scenario 1 for increasing numbers of agents.

4.5.2 *Initial Conditions*

The EF model was run with the same starting condition as the Axtell model, where all agents are in singleton firms. I also ran the scenarios starting with all the agents in a single firm and starting with a typical 100 step firm configuration. We see in Figure 4.15 that the three starting conditions result in the same number of firms in the steady-state.

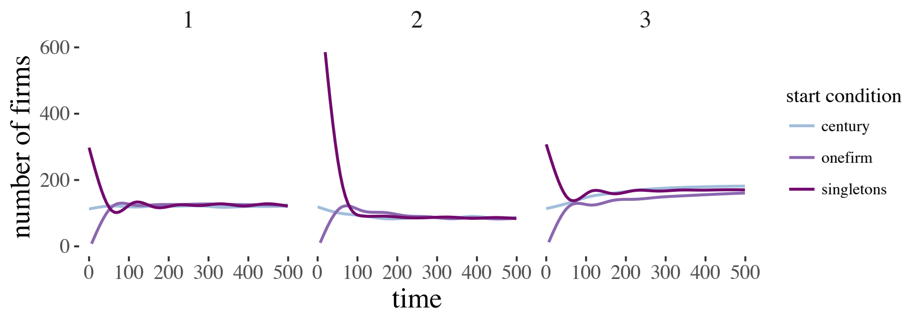


Figure 4.15: Firm Counts for Simulations with Different Initial Conditions

Fit of the total number of firms across 20 runs with the starting conditions of all singleton firms, a single firm and a mix of sizes (century start).

4.5.3 *Saving and Lending Rates*

Rates values for both savings and lending affect model mobility outcomes in predictable ways. As the savings rate increases, agents have more funds available to make desired moves so changes increase and thwarts decrease as the mean savings rate increases. Figure 4.16 shows a parallel plot with change and thwart values over rising mean savings rate values. Total possible changes are bounded by the `churn` value, so the number of changes levels off as the rate continues to increase, and the number of thwarts falls close to 0.

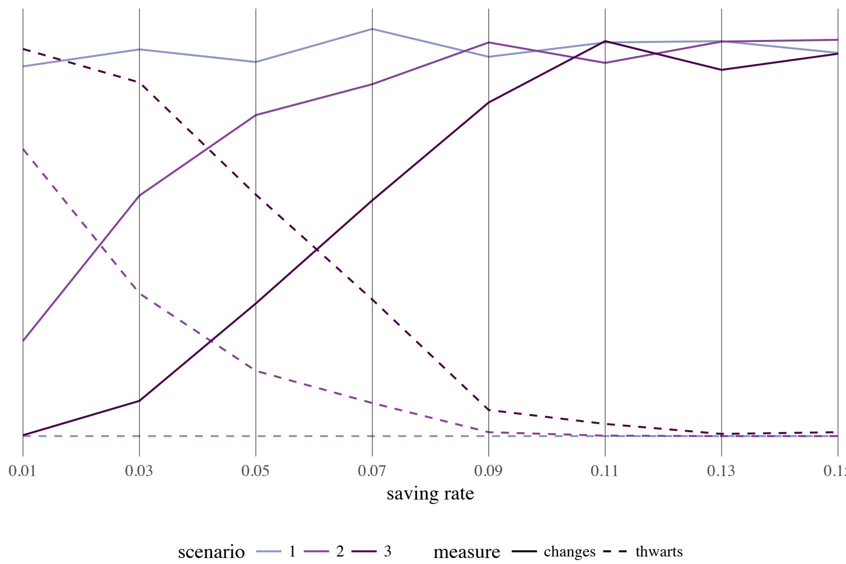


Figure 4.16: Mobility Measures for Different Savings Rates

Parallel plot of number of changes (solid line) and thwarts (dashed line) for increasing values of the mean savings rate.

Mean values for number of firms with lending rate zero is 148 and with positive lending rate of 3% was 171, and the mean number of firms for scenario 2 is 85. Scenario 3 with a lending rate of zero produces a final population of firms slightly lower than that with non-zero lending rates, and closer yet still significantly different from scenario 2 results.

4.5.4 Utility and Production Parameters: θ & β

I mentioned in Section 4.3.2 that the power law function shape parameter α is sensitive to the utility and output parameters. Table 4.7 shows three examples of parameter modifications and their effects on α values. Notice that something as seemingly trivial as choosing

θ from a normal as opposed to uniform distribution essentially doubles the value of α for scenario 1. We also see that higher values of the returns to scale exponent, β, increases the benefits of team production, thus resulting in a flattening of the power law function fit, meaning the emergent economy consists of larger firms, or a lower value of α.

Table 4.7: Power Law Function Fit Statistics for Representative θ and β Parameter Changes

parameter change	α	SE	R-squared
$\theta = \mathcal{N}(.5, .2)$	-2.000	0.1360	0.8889
$\beta = \mathcal{U}(1, 2)$	-1.286	0.0259	0.9891
$\beta = \mathcal{U}(1.75, 2.25)$	-0.727	0.0182	0.9888

Results of linear fit for the CDF for firm size distributions with firms up to 30 employees. As a reminder, Axtell parameter values were $\theta = \mathcal{U}[0, 1]$ and $\beta = \mathcal{U}[1.5, 2]$, which yielded a power law function shape parameter $\alpha = 1.055$.

4.6 Discussion

4.6.1 Overall Patterns

The EF model results suggest that constraints and lending essentially produce impedance effects in the Axtell model. Scenario 1 does not have costs and nothing impedes an agent from making a change to obtain a higher utility. In scenario 2 an agent must accrue enough savings to afford the costs of making a change, which adds a time delay in agent moves. Individuals who want to make a change need to wait for a period of time, roughly 30 time steps with a savings rate of 3%. By the time an agent can afford to move, their best utility options may have changed due to the movement of other agents and the constant reconfiguration of individual firms. This includes individuals who may want to leave a firm, thus firms in scenario 2 grow larger because agents are unable to leave. This impedance causes statistically significant changes in the resulting macroeconomic characteristics in the steady state from the free-movement scenario, as well as greater variances in firm sizes, output and wages.

In scenario 3 a subset of agents are further impeded in making changes because they have outstanding loans they must pay off before they could either begin saving for a future move or borrow again. Some of these agents pay off their loans quickly if their wages increase from the wage the loan is based on, and others never pay off loans because their wages decrease from the wage the loan was based on. Agents could also simply have a lower savings rate than the lending rate. Any unencumbered agent with insufficient savings will accept a loan to make an advantageous move so there are three groups of agents with different time impedances: 1) agents with sufficient savings who move at will, 2) agents with loans who will pay off that loan and either borrow again to make a move or accrue savings before an opportunity arises, and 3) agents who are hopelessly indebted and will never make a move.

Business dynamism is the degree of entry, growth and exit activity within a firm population. Scenario 1 has the most agents making changes, but scenario 2 with fewer total changes than scenario 1 results in larger variations in firm populations across time, and larger firms in general. Scenario 3 has the least changes and the smallest firms. This suggests the two different impedance dynamics, saving time and repayment time, result in different effects.

4.6.2 The Future of the EF Model

Two salient criticisms of conventional economic theory are that macro phenomena are assumed to correspond to micro phenomena and that models describe general equilibrium results. Kirman (2011) defines economic complexity as agent interactions generating phenomena at the microeconomic level that do not coincide with the microeconomic level, and Arthur (1999) argues that an economy is most appropriately considered "not as a system in equilibrium, but as one in motion, perpetually constructing itself anew." The Axtell model, and consequently the EF model, are attempts to address both criticisms by seeking emergent effects from micro foundations, as well as employing endogenous dynamics.

The EF model demonstrates that the dynamics in the Axtell model can be modified by institutional assumptions regarding cost constraints and borrowing opportunities, yet are robust enough to maintain the steady state macroeconomic result and observed patterns and relationships. The EF model's extensions with cash-in-advance and credit-creation lending models provide further insights into how capital constraints, in the form of available funds, affect agent mobility and therefore modify the emergent macro economy. The following are some possible further extensions of the EF model to explore questions of wages, productivity, loans and motivation.

Firm dynamism from a microeconomic perspective is theoretically supposed to shift employees to more productive positions and thus increase overall production (Caves 1998), thus optimizing production in equilibrium. The EF simulations suggest that productivity fluctuates within an equilibrium band, along with firm sizes and wages. Agents will move into positions that lower productivity because an increase in utility does not necessarily correspond to an increase in wage. Further exploration of this dynamic could improve our understanding of the relationship between micro and macro returns to scale. This dynamic could be augmented by the addition of merger and acquisition functionality, where firm owners would have the option to combine with another firm.

Another take on productivity is described by Solow growth models, which require either a population increase or a technology improvement in order to raise the level of macroeconomic productivity, and which in the context of the EF model would mean an increase in `N` or in the values of the output parameters in Equation 4.2 (Solow 1956). A future model extension could explore what happens to the emergent steady state productivity with such increases.

The superlinear behavior in aggregate loan values is an intriguing result as the total amount of loans in the model will exceed the total wealth in the model in the situations where individuals are able to repay loans. Future model versions could further explore this loan behavior. Pikkety (2014) provides evidence that in capitalist economic systems returns to capital will have a tendency to exceed economic growth. If the source of these inflated returns to capital are unbounded debt, the EF model provides an opportunity to understand the demand-driven endogenous system dynamics that might necessitate such an inflation. Future extensions would need to address the need for bounding conditions on lending dynamics, such as more restrictive lending and bankruptcy algorithms. Another example of a possible model extension would explore another of Pikkety's (1997) theories, whereby higher interest rates lead to lower median wealth, and lower interest rates lead to higher productivity.

Evidence suggests that an individual's social network has an important influence on any job search behaviour and results (Mouw 2003; Lin 2008; Smith and Rand 2017). The current implementation of the social network is a static Erdös-Renyi, and this or other network structures, such as preferential attachment, could be implemented as a dynamic network where an agent adds new neighbors when joining a firm, and gradually drops neighbors over time.

The EF model, like the Axtell model before it, assumes agents are motivated solely by maximizing their utility with a preference for income, a formulation well in line with classic microeconomic theory. We've seen that high θ values correspond to firm ownership. Future versions of the model could explore other motivations to determine if they significantly modulate model outcomes. Science fiction writer and medieval historian Ada Palmer describes *vocateurs* in *Too Like the Lightening* (2016), individuals who work well beyond a required 20 hours a week for the sheer joy they take in practicing their professions. Such an agent would not exhibit shirking, and a possible model extension could explore whether such an agent would behave analogously to an EF agent with a high θ value, or if the actual decision rules for selecting employment opportunities would need modification. Another approach would be to assume such agents prefer to work with agents motivated by similar values, so one could imagine an Objectivist scenario where agents are heterogeneously typed and have preferences not only for income but also for collaborating with similarly typed agents, possibly modeled by an alternative social network structure such as preferential attachment.

Chapter 5

CONCLUSION

The three projects presented are intended to provide insights into complex economic dynamics with firms as the unit of analysis.

In the Firm Ecosystem Model a firm was defined as size-species with size-specific characteristics such as competition and innovation capabilities. Competition and colonization dynamics played out with various equilibrium population distributions resulting from differing institutional competition and investment conditions. We saw that the degree and conditions of investment in small firms have significant effects on the resultant population distributions. Large firms don't typically invest in innovation, even though they typically have cash reserves, and if this capital could be invested in innovation by large firms, we saw a rise in the middle range of firm sizes. Decreasing the returns to rent-seeking and increasing large firm failure rates could encourage such investment.

The Structural Inertia Model considered the inner dynamics of firms consisting of a variety of subdivisions and explored the assumption that structural inertia will reduce the performance improvement of large firms, suggesting that investment in innovation for large firms is not as efficient as that in small firms. But we also saw that with less conservative evaluation strategies, the effects of inertia can be overcome, which suggests that large firms can innovate successfully if incentivized to do so. Do large US firms choose not to innovate due to inertia or because they are incentivized to pursue rent-seeking rather than innovation activity?

The Emergent Firm Model modeled firms as structures emerging out of decisions on the part of individuals to engage in joint production. Model results demonstrated that capital constraints on firm formation imposed at the micro level have significant impacts on the resulting economy and macroeconomic measures such as wages and productivity. We also saw that the introduction of lending schemes to provide capital produced yet another variant of an economy and macroeconomic facts.

As described in Section 1, capital dynamics have been identified as significant contributors to economic issues. Results from these three models support this conclusion. Theoretically, capital is supposed to flow to its most productive use, practically considered to be where it obtains the highest rate of return. It is clear that that highest rate of return is not matching up to the most productive purpose, but less clear are the actual workings of contemporary finance, which remains largely opaque even to its practitioners. Therefore it is imperative to better understand how capital flows through an economic ecosystem, and how that flow can be directed toward achieving a just end.

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APPENDIX A

THE BDS DATABASE

We use the Business Dynamics Statistics (BDS) database produced by the US Census Bureau to explore firm size distribution empirically. The BDS is public database of anonymized and aggregated data from the Longitudinal Business Database (LBD), a research database developed by the Center for Economic Studies, which contains data from 1975 to present.

The LBD contains information on all U.S. business establishments that have paid employees and who are listed in the Census Bureaus business register. Data for the LBD is collected through the Standard Statistical Establishment List (SSEL) and is completed on a voluntary basis by firms. The SSEL collects information such as establishment size, payroll, age, industry, location, ownership, and legal form of organization as well as characteristics of the firms they belong to including firm age and firm size.

Despite numerous shortcomings and challenges arising from its aggregated nature, the BDS database is the most comprehensive and complete longitudinal picture of US firm dynamics publicly available.

This model uses BDS firm size data from the `bds_f_szsic_release.csv` dataset available at `https:www.census.govcesdataproductsbdsdata_firm.html`. Firm sizes are organized into 12 categories based on numbers of employees: 1 to 4, 5 to 9, 10 to 19, 20 to 49, 50 to 99, 100 to 249, 250 to 499, 500 to 999, 1000 to 2499, 2500 to 4999, 5000 to 9999 and 10000+. Firm size distributions described by this dataset are shown for all industries in Figure A.1 and for representative years in Figure A.2.

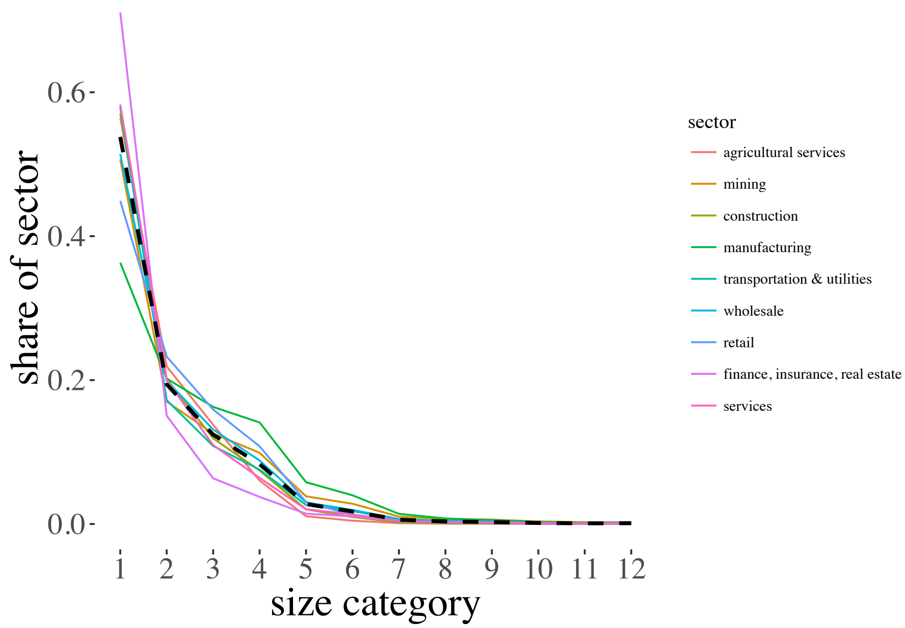

Figure A.1: BDS Firm Size Distributions by Industry

BDS firm size distributions broken out by industry averaged over all years from 1977 to 2014. The black line representing average size distribution across industries.

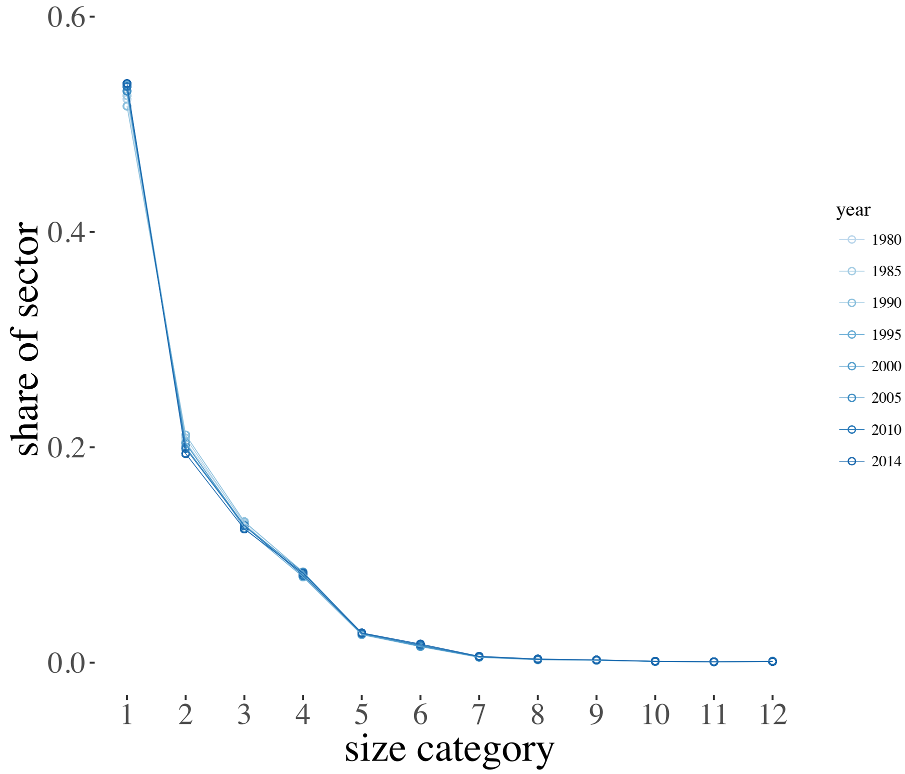

Figure A.2: BDS Average Size Distributions Across Industries for Representative Years

BDS firm size distributions for selected year averaged over all industries, showing the persistence of the general distribution trend over time.

APPENDIX B

MODEL PARAMETERIZATION

B.1 Parameterization of Mortality

The parameterization of mortality was obtained by fitting a straight line with slope a and intercept b to the logarithmic plot for average exit rates per year for each firm size over industries and years from the BDS data, shown in the left hand plot in Figure B.1. The right hand plot uses the same a and b parameters to predict the fit of the actual BDS exit data values with $x^a e^b$.

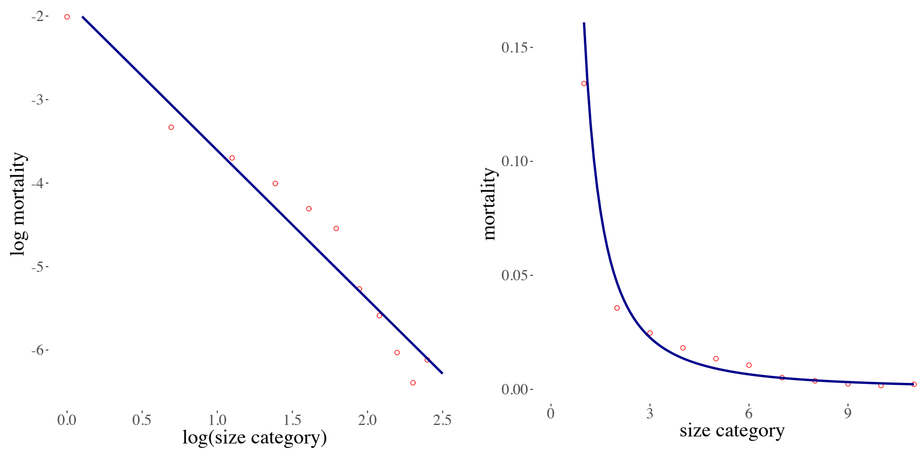

Figure B.1: Parameterized Fits of BDS Exit Data, both Logarithmic and Standard

On the left the circles are the BDS data graphed logarithmically, the line is the generalized linear model fit with slope $a = -1.7823$ and intercept $b = -1.8265$. The plot on the right shows the actual BDS data fitted to $x^a e^b$.

B.2 Parameterization of ν

Investment is modeling as a logistic curve with maximum investment K, inflection point p and steepness q. The model results are based on the three investment curves shown in Figure B.2.

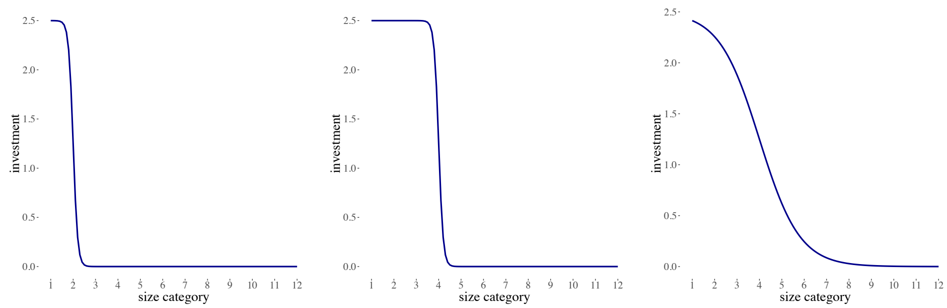

Figure B.2: Investment Curves Used in the Model

The plot on the left shows ν values with $p = 2$, $q = .1$ and $K = 2.5$, and is intended to mimic current investment behavior. The figure in the middle modifies this behavior by changing the inflection point parameter $p = 4$, thus extending the length of an investment. The right hand plot is a smoothed version of the middle plot, with $p = 4$ and $q = .5$.

B.3 Parameterization of γ

We assume a constant relationship p between marketshare s_i and a single employee such that $s_1 p \propto x_1$ and $s_2 p \propto x_2$. We then need to find γ such that $x_1 = \gamma x_2$ and less than 1 in order for marketspace to shrink as it moves to larger firms. Substituting for x_1 and x_2 we have

$$s_1 p = \gamma s_2 p \tag{B.1}$$
$$s_1 = \gamma s_2 \tag{B.2}$$
$$\therefore \gamma = \frac{s_1}{s_2} \tag{B.3}$$

Since the ratios of consecutive sizes are constant, we infer $s_n = A a_N^n$ where a_N is a function of N and

$$\gamma = \frac{A a_N^n}{A a_N^{n+1}} \tag{B.4}$$
$$\gamma = \frac{1}{a_N}. \tag{B.5}$$

The exponent needs to be normalized for any value of N so that the model will behave as intended for any number of size divisions. Therefore

$$a_N = g^{\frac{1}{N}}$$

and

$$\gamma = g^{-\frac{1}{N}}.$$

Figure B.3 shows a plot of the integer size categories against the geometric mean of each size category, and the fit of $s_n = A a_N^n$ where $a_N \approx 2$ therefore $g \approx 4096$ and $\gamma \approx .5$. The largest size category was omitted since the geometric mean of the size category 10,000+ is unknown.

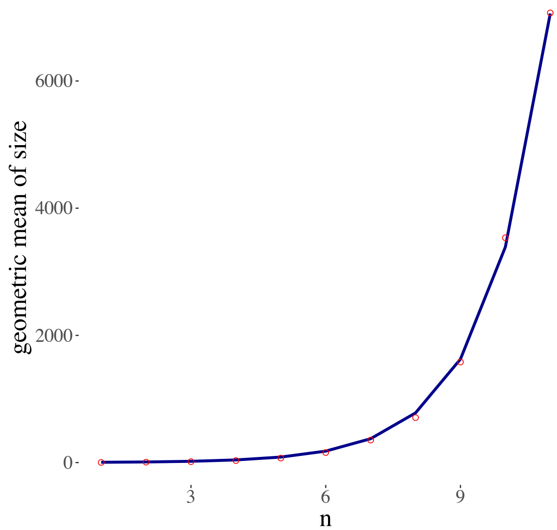

Figure B.3: Gamma Parameterization from BDS Data

Fit of the geometric mean by integer size category, $s_n = (2.2)(2)^n$, described by the blue line with the BDS data for categories 1 through 11 described by the red circles.

APPENDIX C

NK & NM COMPUTATIONAL DETAILS

C.1 Classic NK

Classic NK formulation consists of three components that come together to generate a series of values for strategies on a landscape: a performance function, a payoff matrix and an interaction matrix.

C.1.1 The Performance Function Π

Following Levinthal (1997) we define a firm's strategy s as a string of N decisions, d, which can be expressed as $s = (d_1, \ldots, d_N)$, with d taking on a binary value, such that $d \in \{0, 1\}$. K of those decisions interact with each decision d_i. Each decision, d_i and its related decisions, $d_1^k ... d_k^k$, together make a contribution to the overall performance of strategy $\mathbf{s_i}$. We can construct a normalized performance function $\Pi(s_i)$ equal to the sum of the contributions of each d_i and $K > 0$ additional related d values. Formally,

$$\Pi(s_i) = \frac{1}{N} \sum_{j=1}^{N} \pi(s) = \frac{1}{N} \sum_{j=1}^{N} \pi(d_i; d_1^k, \ldots, d_k^k). \tag{C.1}$$

Let's consider an example with $N = 4$ decisions d and $K = 1$ interactions between the decisions. Considering just the N decisions d that compose each strategy string $\mathbf{s}$, there will be $2^N = 16$ unique binary strings, each representing a strategy $\mathbf{s_i}$:

$$\begin{aligned}
\mathbf{s}_1 &= (0, 0, 0, 0) \\
\mathbf{s}_2 &= (0, 0, 0, 1) \\
\mathbf{s}_3 &= (0, 0, 1, 0) \\
\mathbf{s}_4 &= (0, 0, 1, 1) \\
\mathbf{s}_5 &= (0, 1, 0, 0) \\
\mathbf{s}_6 &= (0, 1, 0, 1) \\
\mathbf{s}_7 &= (0, 1, 1, 0) \\
\mathbf{s}_8 &= (0, 1, 1, 1) \\
\mathbf{s}_9 &= (1, 0, 0, 0) \\
\mathbf{s}_{10} &= (1, 0, 0, 1) \\
\mathbf{s}_{11} &= (1, 0, 1, 0) \\
\mathbf{s}_{12} &= (1, 0, 1, 1) \\
\mathbf{s}_{13} &= (1, 1, 0, 0) \\
\mathbf{s}_{14} &= (1, 1, 0, 1) \\
\mathbf{s}_{15} &= (1, 1, 1, 0) \\
\mathbf{s}_{16} &= (1, 1, 1, 1)
\end{aligned} \tag{C.2}$$

Now we'll consider K, the amount of interaction amongst the various decisions, and define $K = 1$ as representing an interaction between a decision bit d_i and its next neighbor d_{i+1}. Now, for each string s we can construct a Π function that is a sum of four π_i functions, each depending on the interaction for each d_i, described formally as d_i and d_{i+1}^k. We consider the string as cyclic such that if $i = 4$ then $i + 1 = 1$. For example, strategy s_4 is the bit string $(0, 0, 1, 1)$, so $d_1 = 0$ and its next neighbor $d_1^k = 0$, $d_2 = 0$ and its next neighbor $d_2^k = 2$, $d_3 = 1$ and its next neighbor $d_3^k = 1$, and $d_4 = 1$ and its next neighbor $d_4^k = 0$. Following this algorithm for all strategies, the following $\Pi(s_i)$ functions describe the fitness values for all the strategies with $N = 4$ decisions d interacting with $K = 1$ next neighbor elements within the string.

$$
\begin{aligned}
\mathbf{\Pi}(\mathbf{s}_1) &= \frac{1}{4}(\pi_1(0,0) + \pi_2(0,0) + \pi_3(0,0) + \pi_4(0,0)) \\
\mathbf{\Pi}(\mathbf{s}_2) &= \frac{1}{4}(\pi_1(0,0) + \pi_2(0,0) + \pi_3(0,1) + \pi_4(1,0)) \\
\mathbf{\Pi}(\mathbf{s}_3) &= \frac{1}{4}(\pi_1(0,0) + \pi_2(0,1) + \pi_3(1,0) + \pi_4(0,0)) \\
\mathbf{\Pi}(\mathbf{s}_4) &= \frac{1}{4}(\pi_1(0,0) + \pi_2(0,1) + \pi_3(1,1) + \pi_4(1,0)) \\
\mathbf{\Pi}(\mathbf{s}_5) &= \frac{1}{4}(\pi_1(0,1) + \pi_2(1,0) + \pi_3(0,0) + \pi_4(0,0)) \\
\mathbf{\Pi}(\mathbf{s}_6) &= \frac{1}{4}(\pi_1(0,1) + \pi_2(1,0) + \pi_3(0,1) + \pi_4(1,0)) \\
\mathbf{\Pi}(\mathbf{s}_7) &= \frac{1}{4}(\pi_1(0,1) + \pi_2(1,1) + \pi_3(1,0) + \pi_4(0,0)) \\
\mathbf{\Pi}(\mathbf{s}_8) &= \frac{1}{4}(\pi_1(0,1) + \pi_2(1,1) + \pi_3(1,1) + \pi_4(1,0)) \\
\mathbf{\Pi}(\mathbf{s}_9) &= \frac{1}{4}(\pi_1(1,0) + \pi_2(0,0) + \pi_3(0,0) + \pi_4(0,1)) \\
\mathbf{\Pi}(\mathbf{s}_{10}) &= \frac{1}{4}(\pi_1(1,0) + \pi_2(0,0) + \pi_3(0,1) + \pi_4(1,1)) \\
\mathbf{\Pi}(\mathbf{s}_{11}) &= \frac{1}{4}(\pi_1(1,0) + \pi_2(0,1) + \pi_3(1,0) + \pi_4(0,0)) \\
\mathbf{\Pi}(\mathbf{s}_{12}) &= \frac{1}{4}(\pi_1(1,0) + \pi_2(0,1) + \pi_3(1,1) + \pi_4(1,1)) \\
\mathbf{\Pi}(\mathbf{s}_{13}) &= \frac{1}{4}(\pi_1(1,1) + \pi_2(1,0) + \pi_3(0,0) + \pi_4(0,1)) \\
\mathbf{\Pi}(\mathbf{s}_{14}) &= \frac{1}{4}(\pi_1(1,1) + \pi_2(1,0) + \pi_3(0,1) + \pi_4(1,1)) \\
\mathbf{\Pi}(\mathbf{s}_{15}) &= \frac{1}{4}(\pi_1(1,1) + \pi_2(1,1) + \pi_3(1,0) + \pi_4(0,1)) \\
\mathbf{\Pi}(\mathbf{s}_{16}) &= \frac{1}{4}(\pi_1(1,1) + \pi_2(1,1) + \pi_3(1,1) + \pi_4(1,1))
\end{aligned}
\tag{C.3}
$$

Generally, each π_j function will depend upon a string of length $K + 1$ and will have 2^{K+1} possible values. In our case, each π_j will have $2^{(1+1)} = 4$ possible values and the possible π_j functions are given in Table C.1.

Table C.1: Possible Π_j Configurations for $N = 4$, $K = 1$

$$\begin{array}{ccc} 1 & : & \pi_j(0,0) \\ 2 & : & \pi_j(0,1) \\ 3 & : & \pi_j(1,0) \\ 4 & : & \pi_j(1,1). \end{array}$$

C.1.2 The Payoff Matrix

If each π_j function is represented by a random uniform value $[0, 1)$ we can construct a payoff matrix of size $2^{K+1} \times N$, here 4×4, with values for each of the possible π_j functions.

$$\begin{pmatrix} \pi_1(0,0) & \pi_2(0,0) & \pi_3(0,0) & \pi_4(0,0) \\ \pi_1(0,1) & \pi_2(0,1) & \pi_3(0,1) & \pi_4(0,1) \\ \pi_1(1,0) & \pi_2(1,0) & \pi_3(1,0) & \pi_4(1,0) \\ \pi_1(1,1) & \pi_2(1,1) & \pi_3(1,1) & \pi_4(1,1) \end{pmatrix} \tag{C.4}$$

Using these values for π_j in the $\Pi(s_i)$ functions above will yield fitness values for all possible strategies. For example, if $\pi_1(0, 0) = .2,\ \pi_2(0, 0) = .3,\ \pi_3(0, 0) = .4$ and $\pi_4(0, 0) = .5$ then the fitness value for strategy s_1 is

$$\begin{aligned} \Pi(s_1) &= \frac{1}{5}(\pi_1(0,0) + \pi_2(0,0) + \pi_3(0,0) + \pi_4(0,0)) \\ &= \frac{1}{4}(.2 + .3 + .4 + .5) \\ &= \frac{1.4}{5} \\ &= 0.28 \end{aligned} \tag{C.5}$$

The π_j values in the payoff matrix are usually obtained with a random uniform distribution $[0, 1)$. The payoff matrix, once generated, remains constant and all strategy values are solved using the values in this matrix. All strategies in the model are defined by this NK formulation and will have a fixed fitness value derived as above.

C.1.3 The Interaction Matrix

[1] In our demonstration example we are considering $N = 4$ decisions d, each of which are affected by $K = 1$ other decisions, defined as next neighbor (understood cyclically). This set of relationships can be described by an interaction matrix, an $N \times N$ matrix with 1s in the positions representing two decisions that are linked. The interaction matrix in this case will have two ones in each row and column, arranged symmetrically, as follows.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{C.6}$$

[1] Also known as the *influence matrix*.

Alternatively, each decision d_i could interact with one other randomly selected decision, and in this case the interaction matrix would have two ones in each row and column in random locations. Randomly assigning interactions does not turn out to be different from intentionally assigning interactions (Weinberger 1991) so interactions are usually randomly assigned such that the algorithm for determining the two d^k decisions don't end up referring to the original decision d_i. Formally stated, if i is the original decision index, and j is the index for the d^k decisions, then $i \neq j$. Each row and column will have two distinct ones, one of which will always be on the diagonal.

When $K = 0$ no decisions are linked and the interaction matrix is simply an $N \times N$ identity matrix. When $K = N - 1$ every decision affects every other decision and the interaction matrix is an $N \times N$ matrix of ones.

C.2 NK in Single Matrix Formulation

Buzas and Dinitz were the first to describe a matrix multiplication formulation of the classic NK landscape (2013) which is more efficient computationally. In their matrix multiplication formulation the interaction matrix is now construed as ordered interaction sets such that there are N interaction sets each of size $K + 1$. In our previous example where $N = 4, K = 1$ and each N is related to its next neighbor, the ordered interaction sets are:

$$\{1,2\},\ \{2,3\},\ \{3,4\},\ \{1,4\}. \tag{C.7}$$

It should be clear that these sets provide identical information as the combinations of decisions represented by the rows in the nearest neighbor interaction matrix given in Equation C.6. For constant K each of the interaction sets V_j where $j = (1, ..., N)$ will have 2^{K+1} values based the various bit combinations in each strategy, analogous to Table C.1.

The interaction set construct allows us to represent the NK landscape configuration in terms of a single matrix $\mathbf{F}$. Each interaction set, V_j, is considered a functional contribution to the overall strategic performance value. A row in the model matrix consists of all the 2^{K+1} possible bitwise values, referred to as the functional contribution for that interaction set, for each of the N interaction sets for a given strategy. Each row will therefore be of length $2^{K+1} \times N$. Expressed more formally, each row will consist of

$$\mathbf{f}(\mathbf{s_i}) = \mathbf{f_1}(\mathbf{s_i}) \mid \mathbf{f_2}(\mathbf{s_i}) \mid \ \ldots \ \mid \mathbf{f_N}(\mathbf{s_i}) \tag{C.8}$$

where each $\mathbf{f_j}$ represents one of the N interaction sets and $|$ represents column concatenation. This interaction model matrix, $\mathbf{F}$, consists of 2^N rows, one per strategy, and $2^{K+1} \times N$ columns, where every group of N columns represents an interaction set.

The $\mathbf{f}(\mathbf{s_i})$ functions are maps from the given strategy to the correct bit representation of the interaction sets. Each function $\mathbf{f_j}(\mathbf{s_i})$ is a bit string of length 2^{K+1} with a 1 in the position indicating which of the 2^{K+1} possible representations is appropriate to a particular strategy.[2]

Concretely, in our $N = 4,\ K = 1$ example each interaction set has one of four possible bitwise representations:

[2] In order to ensure the mapping is consistent across the landscape each interaction set must be ordered.

Table C.2: Possible Interaction Set Representations for $K = 1$

1 :	(0, 0)
2 :	(0, 1)
3 :	(1, 0)
4 :	(1, 1).

Compare with the configuration for the π constructs in Table C.1.

Taking for example again $\mathbf{s}_4 = (0, 0, 1, 1)$, the interaction sets from Equation C.7 translate to:

$$V_j(s_4) = \begin{cases} (\ 0 \quad 0\ ) & \text{for } V_1 = \{1, 2\}, j = 1 \\ (\ 0 \quad 1\ ) & \text{for } V_2 = \{2, 3\}, j = 2 \\ (\ 1 \quad 1\ ) & \text{for } V_3 = \{3, 4\}, j = 3 \\ (\ 0 \quad 1\ ) & \text{for } V_4 = \{1, 4\}, j = 4. \end{cases} \tag{C.9}$$

From Table C.2 we see that $(0, 0)$ is the first representation in Table C.2 so we place a 1 in the first position of the four-bit string for $\mathbf{f_1}(\mathbf{s_4})$. Likewise, $(0, 1)$ is the second representation so we place a 1 in the second position in the bit string for $\mathbf{f_1}(\mathbf{s_4})$, and so on for the remaining two interaction sets.

$$\mathbf{f_j}(\mathbf{s_4}) = \begin{cases} (\ 1 \quad 0 \quad 0 \quad 0\ ) & \text{for } j = 1 \\ (\ 0 \quad 1 \quad 0 \quad 0\ ) & \text{for } j = 2 \\ (\ 0 \quad 0 \quad 0 \quad 1\ ) & \text{for } j = 3 \\ (\ 0 \quad 1 \quad 0 \quad 0\ ) & \text{for } j = 4. \end{cases} \tag{C.10}$$

Therefore the fourth row of the model matrix $\mathbf{F}$ is

$$1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0. \tag{C.11}$$

The entire $\mathbf{F}$ interaction model matrix for our $N = 4, K = 1$ next neighbor scenario is

$$\mathbf{F} = \begin{pmatrix} 1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0 \\ 1&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0 \\ 1&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0 \\ 1&0&0&0&0&1&0&0&0&0&0&1&0&1&0&0 \\ 0&1&0&0&0&0&1&0&1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&1&0&0&1&0&0&0&1&0&0 \\ 0&1&0&0&0&0&0&1&0&0&1&0&1&0&0&0 \\ 0&1&0&0&0&0&0&1&0&0&0&1&0&1&0&0 \\ 0&0&1&0&1&0&0&0&1&0&0&0&0&0&1&0 \\ 0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&1 \\ 0&0&1&0&0&1&0&0&0&0&1&0&0&0&1&0 \\ 0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&1 \\ 0&0&0&1&0&0&1&0&1&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&1&0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1 \end{pmatrix} \tag{C.12}$$

This $\mathbf{F}$ matrix can then be multiplied by a $2^{K+1} \times N$ array of uniform random $[0, 1)$, the weights array, to generate the $2^N \times 1$ array of performance values $\mathbf{\Pi}(\mathbf{s_i})$. Note that the number of uniform random values in the weights array are the same as in the classic $2^{K+1} \times N$ payoff matrix in Equation C.4.

C.3 Fitness Landscapes as Parametric Equation

Even though this vectorized formulation is more computationally efficient than the classic NK formulation, the $\mathbf{F}$ matrix in Equation C.12 is a large matrix for higher values of N and K. For $N = 20, K = 19$ the matrix contains nearly 420 million elements. Buzas and Dinitz (2013) describe how the $\mathbf{F}$ matrix is over-specified, and how the column space of this matrix is the same as the column space of a parametric equation specification.[3] This parametric specification consists of the power sets for each interaction term V_j. Decision values are adjusted in order for terms to not be zeroed out, 0 decision bit values are changed to -1. For our working example with $N = 4, K = 1$

$$\Pi_i = \beta_0 + \beta_1 d_1 + \beta_2 d_2 + \beta_3 d_3 + \beta_4 d_4 + \beta_{1,2} d_1 d_2 + \beta_{2,3} d_2 d_3 + \beta_{3,4} d_3 d_4 + \beta_{1,4} d_1 d_4. \quad \text{(C.13)}$$

The matrix $\tilde{\mathbf{F}}$ representing this parametric formulation is constructed by considering the strategies as rows, and the transformed binary values for the decisions in each of the terms the columns. The intercept coefficient β_0 is represented by a column of ones, the single decision bits by the transformed binary value in a particular strategy, and the interactions by columns containing the results of multiplying the relevant decision bits. Thus we have a matrix that is the $2^N \times$ the number of parametric terms, smaller than the $\mathbf{F}$ matrix. Equation C.13 is represented by the matrix

$$\tilde{\mathbf{F}} = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{(C.14)}$$

[3]This is a critical discovery, and can be understood in a linear algebra context as the different matrix formulations representing sets of equations and unknowns, with the classic NK representation specifying many extraneous equations.

which has 16 rows but nine columns instead of the 16 in Equation C.12, and Buzas and Dinitz provide a proof that the column spaces for $\mathbf{F}$ and $\tilde{\mathbf{F}}$ are equivalent. This $\tilde{\mathbf{F}}$ can then be multiplied with a number of term $\times 1$ weights array to again obtain a $2^N \times 1$ array of fitness values corresponding to all the strategies on the landscape.

The parametric formulation is known as the NM representation, and not only does it simplify the computation by using a smaller matrix construct, but it also gives us the flexibility to essentially vary K since we there no limitations on how we explicitly specify the interaction terms, and we can therefore specify terms of varying order. For example, with $N = 5$ we could specify a $d_1 d_2 d_3$ term as well as a $d_4 d_5$ term without regard for the fact that the former is third order ($K = 2$) and the latter is second order ($K = 1$). Varying K allows for heterogeneity in the degree of impact each decision would have on other decisions. The weights array would be random uniform $(0, 1)$ of length equal to the total number of terms.

APPENDIX D

LOCAL OPTIMA NETWORK COMPUTATIONAL ALGORITHM

The algorithm used to construct the Local Optima Network maps is described in Figure D.1.

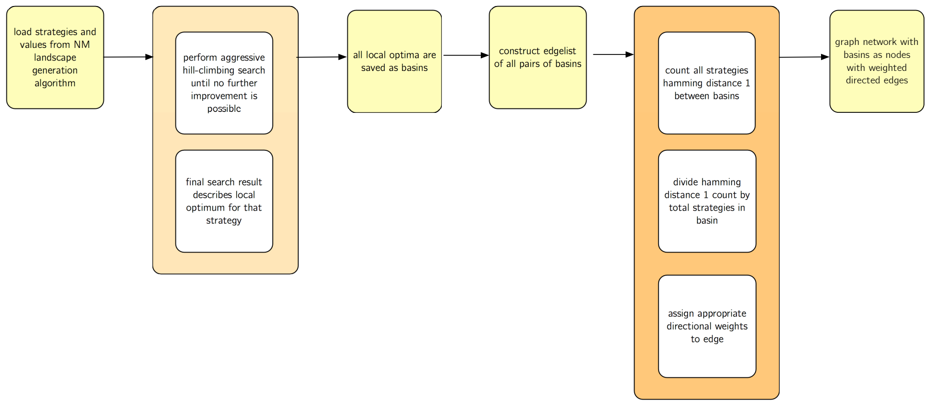

Figure D.1: LON Generation Algorithm

Algorithm to generate a LON map of the NM landscape with weighted edges based on probability of a bit flip moving an agent from one basin of attraction to another.

The NM landscape we selected as the basis for the Structural Inertia model is presented as an LON in Figure D.2.

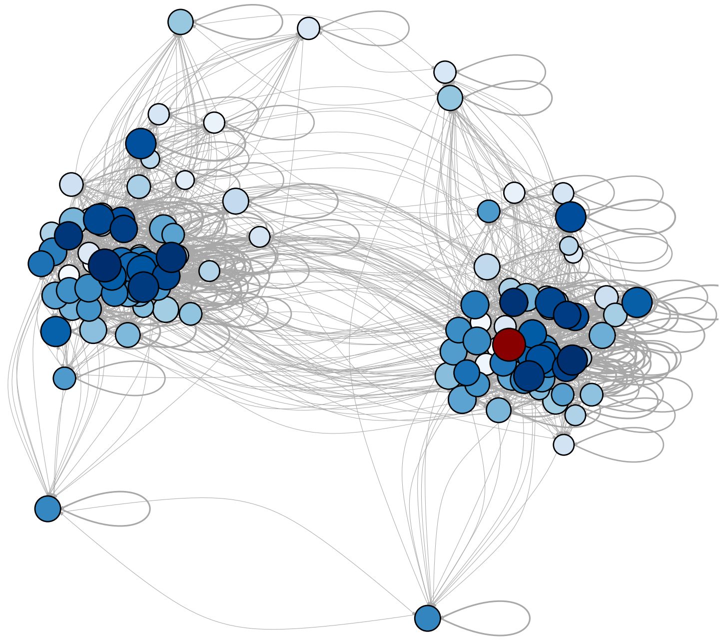

Figure D.2: LON of Sample Structured Landscape Used in the SIM

Local optima network graph representation of an NM landscape with $N = 20$ and M as fully connected divisions with partial second-order connection between divisions. Darker blue circles represent basins of attraction for higher-valued optima, and the size of the circle represents the size of the basin. The red circle is the global optimum. The lines connecting the basins represent the transition probabilities between basins for single bit-flips, with the weights of the lines representing the magnitude of the transition probabilities.

APPENDIX E

STRUCTURAL INERTIA ODD

The Structural Inertia Model describes the conflict between scope of control and the need for organizational agreement in an attempt to develop theory explaining structural inertia (Hannan and Freeman 1984). Firm strategies are described by 20-bit strings, and firms have control of from one to five divisions consisting of four bits each. The number of owned divisions is the scope, s. Under single bit-flip exploration, controlling a single division provides access to the search space consisting of 16 possible strategies, while controlling all five divisions provides access to a search space consisting of 1,048,576 possible strategies. Yet each division computes a division-specific value for each search string and have varying degrees of power to ensure that value doesn't decrease, so potential solutions that could improve the overall firm performance may be rejected.

Firms search across an NM landscape (Manukyan, Eppstein, and Buzas 2014) with $N = 20$ and designed such that each bit in a division is fully connected to all the other bits and groups of bits in that division, and partially connected by second-order and higher interactions to other divisions, yielding a search environment consisting of around basins of attraction on average.

Both the landscape generation and search algorithms are coded in Python3.

E.1 Entities, State Variables and Scale

The model entities are firms characterized by differing values of s which takes values from 1 to 5 and describes the number of 4-bit divisions that the firm controls. Each firm is randomly assigned an initial strategy with a value determined by the NM landscape.

The fitness landscape is constructed using $N = 20$ with fully connected bits within divisions and partial second-order or higher connections across divisions. The landscape is parameterized `terms, weights` and `values.` Each strategy, or location on the landscape, has a value derived from a linear combination of its terms and weights.

Each time step all firms search and evaluate trial strategies in search of fitness improvement.

E.2 Process Overview & Scheduling

E.2.1 NM Landscape Generation

To generate the customized NM landscape, all the intra-division interactions of four bits for each division are determined, and then all the second-order or higher interactions between divisions are identified. Some quantity of the possible inter-division interactions are randomly selected and the list of all terms for the landscape is now complete. A random coefficient value is found for each of the terms, and values for each of the possible strategies are obtained by summing over all the relevant terms for each bit-string.

The landscape generation functionality is summarized as a flowchart in Figure E.1.

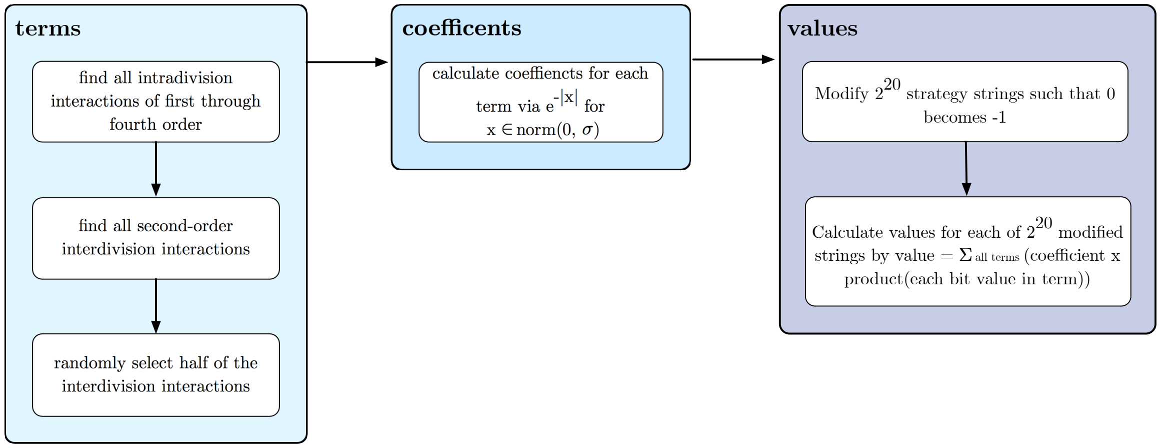

Figure E.1: Functional Flow for the Model Landscape Generation

E.2.2 Search Algorithms

The number of firms specified by `num_firms` is instantiated with a value for s and an initial strategy.

During each time step, each firm unpacks its strategy values into contributions from each of its owned divisions. Then based on the exploration option a trial strategy will be derived through a bit-flip of a single bit within the owned divisions, or by a bit-flip within each owned division. Then based on the co-specialisation algorithm, the firms will explore either each single division flip separately, s trials, or any combination of those flips, 2^s trials.

If the evaluation algorithm is `decider`, the trial strategy with the highest improved value is chosed. If the evaluation algorithm is either `veto` or `vote` then if the trial improves the overall strategy value for the firm, each owned division compares its current value with the new division components of the trial value. In the case of `veto` any decrease in value of an owned division will lead to a rejection of that trial, while in the case of `vote` if a minority of divisions experience a loss in value the trial strategy may still be selected if it yields the highest improved overall value. In the case of ties, the trial is rejected or accepted randomly. Firm's strategy and value are updated if trial strategy is selected. Figure E.2 graphically demonstrates the possible search and decision routes.

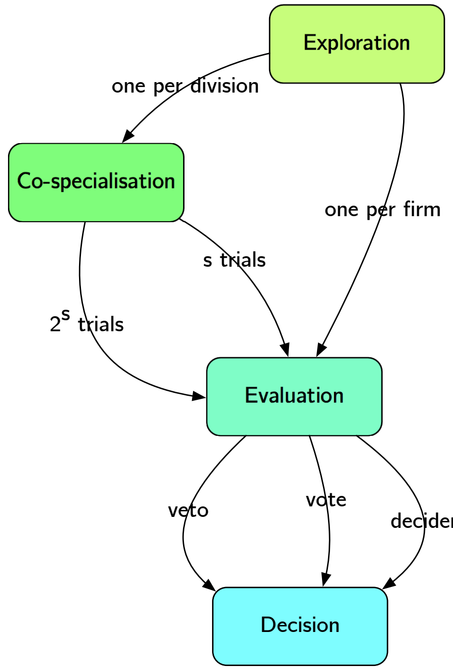

Figure E.2: Nine Decision Routes in the SIM

The model allows for the exploration of nine different decision routes: either firm-wide exploration resulting in a single trial strategy or per division exploration which could be result in either s or 2^s different trial strategies depending on whether or not there is co-specialisation. Trials are evaluated by either veto, vote or a decider.

The model continues to run for the specified number of `steps`. At the end of a step firms' strategies and values are recorded. The model search functionality is summarized as a flowchart in Figure E.3.

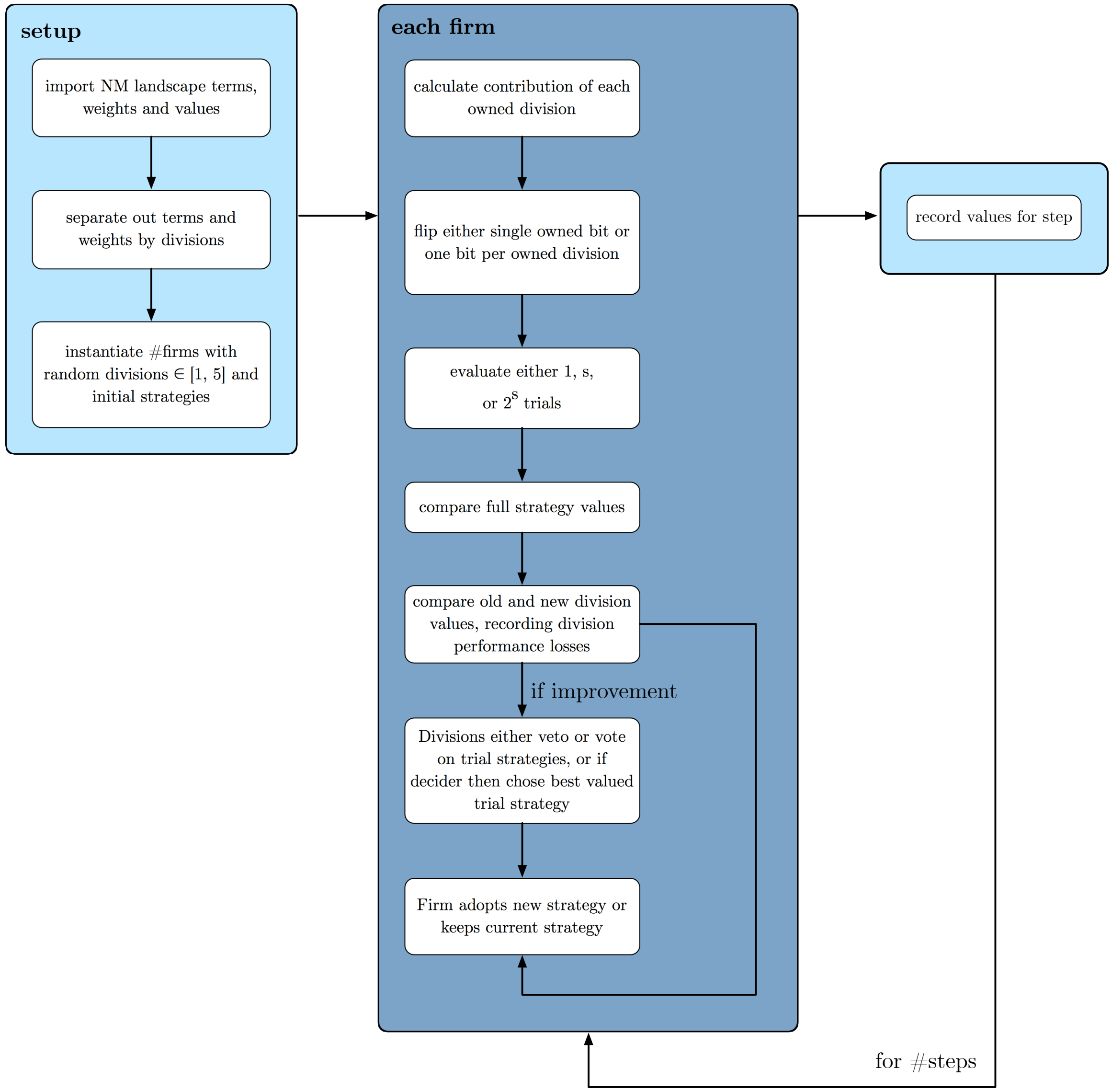

Figure E.3: Functional Flow for the Model Search Algorithm

E.3 Details

E.3.1 Initialisation & Inputs

The NM Landscape generation algorithm does not require any external input files. The Search Algorithms require the terms, coefficients and values from the landscape generation routines, which could be imported as files or run continuously as a single program.

Table E.1: Model Parameters with Initial Values

Parameter	Inputs
`N`, number of bits in each strategy	20
`d`, maximum number of possible firm divisions	5
σ, the mean of the normal distribution used to find coefficients	10
`num_firms`, number of firms exploring landscape each run	100
`steps`, the number of time steps, or search iterations	30
`exploration`, specifies exploration algorithm	`by_firm` or `by_division`
`co-specialization`, specifies whether to use s or 2^s trials	`yes` or `no`
`evaluation`, specifies evaluation algorithm	`decider`, `veto` or `vote`

E.3.2 Landscape Generation Submodels

Terms With `d` representing number of divisions, `division_bits` identifying the sets of bits in each division and `division_size` the size of each division,

```
intradivision_terms = [combinations(i, j) for i in division_bits
                        for k in range(0, division_size)]
```

and all second-order interdiversion terms are given by

```
for combinations for i, j in combinations(range(0, d), 2):
    a = product(divisions[i], divisions[j])
    interdivision_terms += a.
```

Half of the second-order interorder terms are selected randomly to produce the landscape.
Coefficients With `sumM` representing the total number of terms,

```
coefficients = [exp(-abs(i)) for i in normal(0, sigma, sumM)].
```

Values For each `strategy` convert number to N digit string of binary values and convert 0s to -1s. For each strategy,

```
value = sum over all terms of (coefficient x product of all modified bits in term).
```

E.3.3 Search Algorithm Submodels

Assign terms & weights to divisions Identify and collect all terms containing bits in each division.
Instantiate firms Instantiate `num_firms` firms with number of owned divisions between 1 and d and with an initial strategy.
Calculate contribution of each owned division For each owned division

```
division_value = sum over all division terms (division_coefficient x
                    product of all modified bits in term).
```

Explore If the exploration algorithm is `by_firm`, select single owned bit `n` and

```
trial_strategy = strategy ^ (1 << n).
```

If the exploration algorithm is `by_division` select bit n per owned division and generate list of s `trial_strategies` by flipping each value for n as above if `co-specialisation` is `no`. *If `co-specialisation` is `yes` then find all combinations of any number of n values and flip each combination of bits as above to generate list of 2^s `trial_strategies`.
Evaluate If evaluation algorithm is `decider` select best value among `trial_strategies` greater than `strategy`. If evaluation algorithm is `veto` calculate each owned `division_value` as above for each `trial` in `trial_strategies` where `trial` > `strategy`. If any `division_value` for a `trial` is less than `division_value` for `strategy`, that `trial` is rejected. Rejected trials are noted in a list construct called `losses`. Of the trials in `accepted_strategies`, the best value is selected. If there are no accepted trials the firm retains its current `strategy`. *If evaluation algorithm is `vote` then `losses` is evaluated to determine those trials with a minority of rejections, and those trials remain in `accepted_strategies`. Again, of the trials in `accepted_strategies`, the best value is selected.

E.4 Experiments

The SIM allows for multiple configurations of algorithms allowing for multiple experiments, examples of which are shown in Table E.2.

Table E.2: Possible Combinations of Search Paths

exporation	co-specialisation	evaluation
by_firm	no	decider
by_firm	no	veto
by_firm	no	vote
by_division	no	decider
by_division	no	veto
by_division	no	vote
by_division	yes	decider
by_division	yes	veto
by_division	no	vote

APPENDIX F

EMERGENT FIRMS ODD

F.1 Purpose

The Emergent Firm (EF) model is based on the premise that firms arise out of individuals choosing to work together to advantage themselves of the benefits of returns-to-scale and coordination. The Emergent Firm (EF) model is a new implementation and extension of Rob Axtell's Endogenous Dynamics of Multi-Agent Firms model (Axtell 2018). Like the Axtell model, the EF model describes how economies, composed of firms, form and evolve out of the utility maximizing activity on the part of individual agents. The EF model includes a cash-in-advance constraint on agents changing employment, as well as a universal credit-creating lender to explore how costs and access to capital affect the emergent economy and its macroeconomic characteristics such as firm size distributions, wealth, debt, wages and productivity.

F.2 Entities, State Variables and Scale

Agents are individuals with preferences for income, θ, production characteristics, `a`, `b` and β, a savings rate, `s` and a position in a social network. During the course of the simulation, agents will make utility calculations and chose to change employment, either by starting a new firm or joining another firm, in order to maximize their utility. The model captures at each time step whether or not an agent recalculated its utility values, whether those calculations resulted in a particular change of employment, if the agent was thwarted in making a change, the agent's current effort, wage, loans and savings and its firm affiliation.

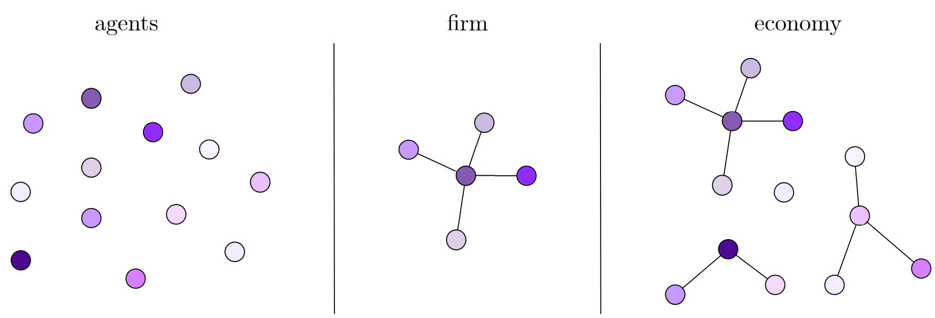

Figure F.1: Hierarchical Depiction of Model Entities: Individuals Compose Firms, Firms Compose the Economy

This agent-level information can be aggregated to describe firm level information such as size, wage and productivity, and firm level information in turn can be aggregated to describe macroeconomic characteristics of the economy as whole, such as the size distributions of firm populations, mean wage, per capita wealth and debt.

Firms are modeled as star graphs connecting all employees to the firm owner The economy is modeled as a collection of all the star subgraphs and singleton agents, illustrated in Figure F.2. Thus each agent has two types of neighbors: those in the social network which define an agent's alternative employment opportunities, and those in the star graph networks which represent an agent's employees or employer.

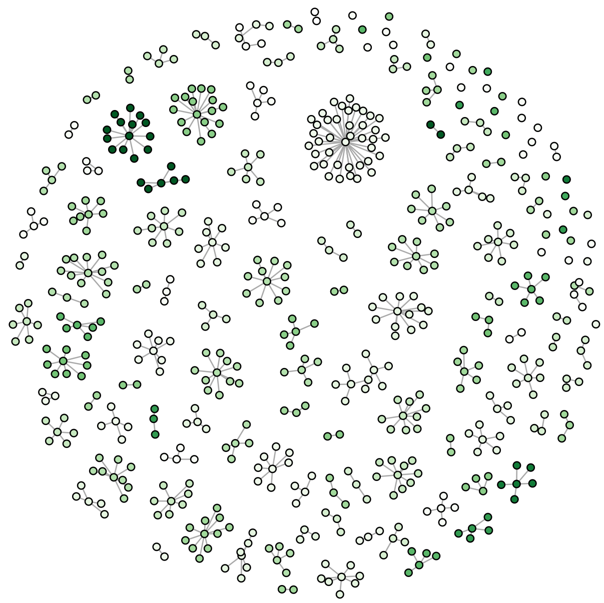

Figure F.2: Network Graph of an Emergent Economy

Network graph of an emergent economy consisting of a collection of firms modeled as star sub-graphs connecting individuals. Nodes are colored on employee savings; the more intense the color the higher the savings. Neighbors in this network are either employees or employers.

F.3 Process Overview & Scheduling

Each time step agents explore options for utility improvement. Agents decide whether to stay in their current situation, join another firm in their social network, or start their own firms based on which of the options maximizes their utility. The ability to make a desired change requires either sufficient savings or the ability to obtain a loan. After all agents have made changes, firm outputs are calculated and distributed, and loans are repaid. The EF model process is summarized as a flowchart in Figure F.3.

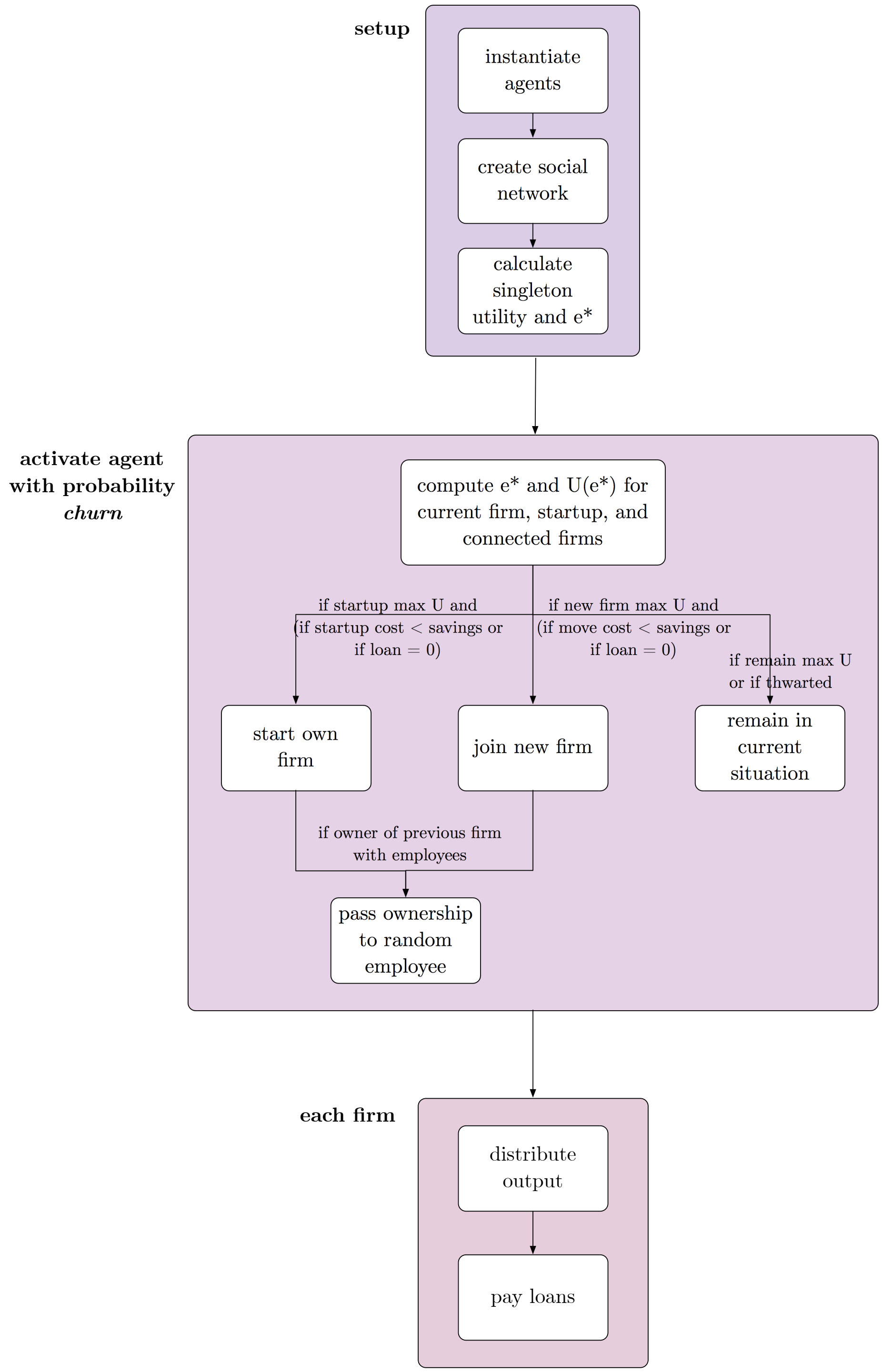

Figure F.3: Functional Flow for the Emergent Firm Model

To start, the number of agents specified by `N` are instantiated with individual values for parameters θ, `a`, `b`, β and `s`. A random graph is generated to organize the agents as nodes in a social network with a specified range of edges per node. The model then calculates the utility and effort values for each agent as a singleton firm and stores these values, and model setup is complete.

The model activates individual agents with a probability given by `churn`. For each activated agent, the model calculates a Cobb-Douglas utility value for that agent's current position, as well as the expected utility if it were employed by each of the firms employing its neighbors in the social network. These values are compared with the agent's singleton utility calculated during setup, which is the utility for forming a startup. The scenario yielding the highest utility value is the agent's choice: either remain, form a startup or change firms. If the choice requires a change, the model computes the cost of the change, and if the agent has enough savings to cover the costs of the move, or if it can take out a loan to cover any residual costs, the agent's firm affiliation and firm links are changed. An agent cannot take out a loan if he already has an outstanding loan balance. If the agent making a change is the owner of a firm with employees, that firm's ownership is reassigned to another random employee. Wage, effort, savings and loan values are updated accordingly. An agent is considered to be 'thwarted' if it cannot make a desired move.

After all agents have had the opportunity to be activated, the model calculates the output for each firm and distributes a share of that output to all employees and owners, and wage and savings values are updated accordingly. Finally, any agent with a loan will apply their savings toward repaying the loan, loan values are compounded, and savings and loan values are updated.

F.3.1 Initialization & Inputs

The initial values for the model parameters are described in Table F.1.

Table F.1: Agent and Global Parameters with Starting Values

Attribute	Description	Value
Agent Variables		
`a`	effort multiplier in output formula	$\mathcal{U}(0, .5)$
`b`	exponential effort multiplier	$\mathcal{U}(.75, 1.25)$
β	effort exponent	$\mathcal{U}(1.5, 2)$
θ	preference for income	$\mathcal{U}(0, 1)$
ω	time endowment	1
`rate`	savings rate, multiplies wage each time step	$\mathcal{N}(.03, .01)$, truncated at 0
Global Variables		
`N`	number of agents	600
ν	number of social network links	$\mathcal{U}(2, 6)$
`churn`	agent activation rate	.1
`tmax`	number of steps	500
`move`	job change cost, multiplies last wage	1
`startup`	startup cost, multiplies last wage	2
`lendingrate`	cost of loan each time step	.03
	compensation rule	equal shares
	initial condition	all singleton firms

F.3.2 Input & Outputs

The EF model does not require any external input files.

The model produces two files, a `csv` and a network graph, `gml`. Each row in the `csv` file contains an agent's parameters for a given time step. For `tmax`= 500 and $N = 600$ the file will contain 300,000 rows. The agent parameters compose the columns and are described in Table F.2.

Table F.2: Agent Parameters Captured in the EF Model CSV Output File

Parameter	Description
`id`	id number
ω	time endowment
θ	preference for income
`links`	number of neighbors
`component`	component membership in the social network graph
`a`	effort multiplier
`b`	exponential effort multiplier
β	effort exponent
`rate`	savings rate
`U_self`	singleton utility
`e_self`	singleton effort
`e_star`	current profit maximizing effort
`firm`	firm affiliation
`wage`	current wage
`savings`	current savings
`loan`	current loan balance
`borrow`	binary flag indicating agent borrowed current time step
`startup`	binary flag indicating agent formed startup current time step
`move`	binary flag indicating agent changed firms current time step
`thwart`	binary flag indicating agent was thwarted current time step
`go`	binary flag indicating agent was activated current step

The `gml` graph file contains the network graph describing all the star subgraph firms and singleton firms at the end of the simulation.

F.3.3 Submodels

The EF model is coded in Python 3 with agents implemented as dictionary objects and firms implemented as a network graph. The nine submodels pseudo-coded and described below provide further implementation and functional details.

1. `instantiate agents`

```
for i to 1 to N:
     id = i,
     omega = 1.0,
     theta = random value in uniform(0, 1),
     a = random value in uniform(0, .5),
     b = random value in uniform(.75, 1.25),
```

```
        beta = random value in uniform(1.5, 2),
        rate = savingrate if sigma = 0,
                else random value in truncated normal(savingrate, sigma),
        firm = i
    F = network with N unconnected nodes
```

2. `create social network`

```
    degree_list = N random values in uniform(mindegree, maxdegree)
    G = graph with N nodes and degree_list
    find components of G
    for i in agents:
        links = degree for node i
        component = component membership for node i
```

3. `calculate singleton utility and e*`

```
    optimize e for maximum utility for singleton firm
    for i in agents:
        wage = output
        e_star = e_single
```

4. `compute` $e*$ `and U(e*) for opportunities`

```
    go = 1
```

An agent's utility is calculated via a Cobb-Douglas function with his individual preference set for income and leisure

$$U = \left(\frac{O}{n}\right)^{\theta} (\omega - e^*)^{1-\theta}, \tag{F.1}$$

where O is total firm output, n the number of persons in the firm, such that $\frac{O}{n}$ is the individual's wage in the current firm configuration. The individual's preference for income is given by θ, therefore preference for leisure is $1 - \theta$. The individual's total time endowment is ω and e^* is the individual's utility-maximizing work effort, thus the individual's leisure is $\omega - e^*$.

Each firm has an owner with unique parameters a, b and β, returns to scale, that characterize the firm's production function

$$O = \frac{aE + bE^{\beta}}{n}, \tag{F.2}$$

where E is the sum of all the firm members' efforts, e.

To find utility for current position:

```
    E = sum off e for all agents in firm
    optimize e for maximized utility function
```

To find utility for changing firms:

```
for each neighbor identify firm
find all unique connected firms
for each connected firm:
     E = sum of e for all agents in firm
     optimize e for maximized utility function
choose maximum utility and associated firm
```

Startup utility was already calculated during setup and is stored in the agent dictionary.

5. `start own firm` If starting a new firm provides the maximum utility then the firm affiliation for the agent needs to change its own *id*, and that node needs to remove its links in the F graph.

```
cost = startup * wage
if savings >= cost:
        change firm affiliation and remove firm network links
          savings = savings - cost
          update wage and e* values
          startup = 1
else if loans < 0:
          loans = costs - savings
          update wage and e* values
          startup, borrow = 1
else thwart = 1
```

6. `join new firm` If joining a new firm provides the maximum utility then the firm affiliation for the agent needs to change, as well as the linkages in the F graph.

```
cost = move * wage
if savings >= cost:
        change firm affiliation and firm network links
          savings = savings - cost
          update wage and e* values
          move = 1
else if loans < 0:
          loans = costs - savings
          update wage and e* values
          move, borrow = 1
else thwart = 1
```

If the agent is a firm owner, then the change ownership routine will run before changing affiliation.

7. `pass ownership`

```
new_owner = random neighbor
firm affiliation for new owner changes to id
firm affiliation for remaining employees changes to new owner
remaining employees link to new owner
```

8. `distribute output`

```
E = sum of the effort for all members of a star subgraph
O_total = (a * E + b * E **beta)
share = O_total / n
for all agents in star subgraph:
     wage = share
     savings = savings + share * rate
```

9. pay loans

```
for all agents with loans:
   loan = loan * (1 + lendingrate) - savings
   if loan < 0:
        savings = abs(loan)
        loan = 0
   else:
        savings = 0
        loan = loan
```