Open Mathematics

Research Article

Anthony Sofo*

Evaluation of integrals with hypergeometric and logarithmic functions

Abstract: We provide an explicit analytical representation for a number of logarithmic integrals in terms of the Lerch transcendent function and other special functions. The integrals in question will be associated with both alternating harmonic numbers and harmonic numbers with positive terms. A few examples of integrals will be given an identity in terms of some special functions including the Riemann zeta function. In general none of these integrals can be solved by any currently available mathematical package.

Keywords: Polylogarithm function, Integral representation, Lerch transcendent function, Logarithmic integral, Riemann zeta function

MSC: 05A10, 05A19, 33C20, 11B65, 11B83, 11M06

1 Introduction and Preliminaries

In this paper we will develop explicit analytical representations, identities, new families of integral representations, of the form:

$$\int_0^1 x^{2k-1} \ln x \; {}_3F_2 \left[\begin{array}{c} \frac{1}{2}, 2, 2 \\ \frac{3+p}{2}, \frac{4+p}{2} \end{array} \middle| x^{2k} \right] dx \tag{1}$$

for (k, p) being the set of positive integers and where ${}_3F_2 \left[\begin{array}{c} \cdot, \cdot, \cdot \\ \cdot, \cdot \end{array} \middle| z \right]$ is the classical generalized hypergeometric function. We also provide analytical solutions for integrals of the form

$$\int_0^1 x^{2k-1} \ln(1-x) \; \Phi\left(x^{2k}, 1, \frac{1+r}{2}\right) dx,$$

where the Lerch transcendent function Φ is defined as the analytic continuation of the series

$$\Phi(z, t, a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^t},$$

which converges for any real number $a > 0$ if z and t are any complex numbers with either $|z| < 1$ or $|z| = 1$ and $\Re(t) > 1$. It is known that the Lerch transcendent extends by analytic continuation to a function $\Phi(z, t, a)$ which is defined for all complex $t, z \in \mathbb{C} - [1, \infty)$ and $a > 0$, which can be represented, [3], by the integral formula

$$\Phi(z, t, a) = \frac{1}{\Gamma(t)} \int_0^{\infty} \frac{x^{t-1} e^{-(t-1)x}}{e^x - z} dx = \frac{1}{\Gamma(t)} \int_0^1 \frac{x^{a-1} \ln\left(\frac{1}{x}\right)}{1 - xz} dx$$

*Corresponding Author: Anthony Sofo: Victoria University, P.O. Box 14428, Melbourne City, Victoria 8001, Australia,
E-mail: anthony.sofo@vu.edu.au

for $\Re(t) > 0$. For a fuller account of the Lerch function see the excellent papers, [6], [7] and [8]. The Lerch transcendent generalizes the Hurwitz zeta function at $z = 1$,

$$\zeta(t, a) = \Phi(1, t, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^t}$$

and the Polylogarithm, or de Jonquière's function, when $a = 1$,

$$\text{Li}_t(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^t} = \Phi(z, t, 1), \quad t \in \mathbb{C} \text{ when } |z| < 1; \; \Re(t) > 1.$$

Moreover,

$$\int_0^1 \frac{\text{Li}_t(px)}{x} dx = \begin{cases} \zeta(1+t), & \text{for } p = 1 \\ \\ (2^{-t} - 1)\zeta(1+t), & \text{for } p = -1 \end{cases}.$$

Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the sets of real and complex numbers and let $\mathbb{N} := \{1, 2, 3, \ldots\}$ be the set of positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A generalized binomial coefficient $\binom{\lambda}{\mu}$ $(\lambda, \mu \in \mathbb{C})$ is defined, in terms of the familiar gamma function, by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)}, \qquad (\lambda, \mu \in \mathbb{C}),$$

which, in the special case when $\mu = n$, $n \in \mathbb{N}_0$, yields

$$\binom{\lambda}{0} := 1 \quad \text{and} \quad \binom{\lambda}{n} := \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!} \quad (n \in \mathbb{N}),$$

where $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is the Pochhammer symbol. Let

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi(n+1), \qquad (H_0 := 0) \tag{2}$$

be the nth harmonic number. Here, as usual, γ denotes the Euler-Mascheroni constant and $\psi(z)$ is the Psi (or Digamma) function defined by

$$\psi(z) := \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log \Gamma(z) = \int_1^z \psi(t) \, dt.$$

A generalized harmonic number $H_n^{(m)}$ of order m is defined, for positive integers n and m, as follows:

$$H_n^{(m)} := \sum_{r=1}^{n} \frac{1}{r^m}, \quad (m, n \in \mathbb{N}) \text{ and } \qquad H_0^{(m)} := 0 \qquad (m \in \mathbb{N}).$$

In the case of non-integer values of n such as (for example) a value $\rho \in \mathbb{R}$, the generalized harmonic numbers $H_\rho^{(m+1)}$ may be defined, in terms of the Polygamma functions

$$\psi^{(n)}(z) := \frac{d^n}{dz^n}\{\psi(z)\} = \frac{d^{n+1}}{dz^{n+1}}\{\log \Gamma(z)\} \qquad (n \in \mathbb{N}_0),$$

by

$$H_\rho^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(\rho+1) \tag{3}$$

$$(\rho \in \mathbb{R} \setminus \{-1, -2, -3, \cdots\}; \; m \in \mathbb{N}),$$

where $\zeta(z)$ is the Riemann zeta function. Whenever we encounter harmonic numbers of the form $H_\rho^{(m)}$ at admissible real values of ρ, they may be evaluated by means of this known relation (3). In the exceptional case of (3) when $m = 0$, we may define $H_\rho^{(1)}$ by

$$H_\rho^{(1)} = H_\rho = \gamma + \psi(\rho+1) \qquad (\rho \in \mathbb{R} \setminus \{-1, -2, -3, \cdots\}).$$

We assume (as above) that

$$H_0^{(m)} = 0 \qquad (m \in \mathbb{N}).$$

In the case of non integer values of the argument $z = \frac{r}{q}$, we may write the generalized harmonic numbers, $H_z^{(\alpha+1)}$, in terms of polygamma functions

$$H_{\frac{r}{q}}^{(\alpha+1)} = \zeta(\alpha+1) + \frac{(-1)^\alpha}{\alpha!}\psi^{(\alpha)}\left(\frac{r}{q}+1\right), \ \frac{r}{q} \neq \{-1, -2, -3, \ldots\},$$

where $\zeta(z)$ is the zeta function. When we encounter harmonic numbers at possible rational values of the argument, of the form $H_{\frac{r}{q}}^{(\alpha)}$ they maybe evaluated by an available relation in terms of the polygamma function $\psi^{(\alpha)}(z)$ or, for rational arguments $z = \frac{r}{q}$, and we also define

$$H_{\frac{r}{q}}^{(1)} = \gamma + \psi\left(\frac{r}{q}+1\right), \text{ and } H_0^{(\alpha)} = 0.$$

The evaluation of the polygamma function $\psi^{(\alpha)}\left(\frac{r}{a}\right)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [4], or Choi and Cvijovic [1] in terms of the Polylogarithmic or other special functions. Polygamma functions at negative rational values of the argument can also be explicitly evaluated, for example

$$H_{-\frac{3}{4}} = -\frac{\pi}{2} - 3\ln 2, \ H_{-\frac{3}{4}}^{(2)} = -8G - 5\zeta(2), \ H_{-\frac{1}{6}}^{(3)} = -2\sqrt{3}\pi^3 - 90\zeta(3).$$

Some specific values are listed in the books [13] and [14]. Some results for sums of harmonic numbers may be seen in the works of [2], [15] and references therein.

The following lemma will be useful in the development of the main theorems.

Lemma 1.1. *Let k be a positive integer. Then:*

$$X(k, 0) = \sum_{n \geq 1} \frac{(-1)^{n+1} H_{kn}}{n} = -k \int_0^1 \frac{x^{k-1} \ln(1-x)}{1+x^k} dx \tag{4}$$

$$= \frac{1}{2k}\left(\zeta(2) - \ln^2 2\right) + \frac{1}{2}\sum_{j=1}^{k-1}\left(\frac{1}{k-j} \, {}_3F_2\left[\begin{array}{c}1,1,1\\2,2-\frac{j}{k}\end{array}\middle|\frac{1}{2}\right]\right) \tag{5}$$

$$= \frac{(1+k^2)}{4k}\zeta(2) - \frac{1}{2}\sum_{j=0}^{k-1}\ln^2\left(2\sin\left(\frac{(2j+1)\pi}{2k}\right)\right) \tag{6}$$

Proof. Consider, for $t \in [-1, 1)$ and $j \in \mathbb{R}^+ \cup \{0\}$

$$\sum_{n \geq 1} \frac{t^n}{n\binom{kn+j}{j}} = k\sum_{n \geq 1} t^n \ B(j+1, kn)$$

where the beta function

$$B(s, x) = B(x, s) = \frac{\Gamma(s)\Gamma(x)}{\Gamma(s+x)} = \int_0^1 t^{s-1}(1-t)^{x-1} dt$$

for $\Re(s) > 0$ and $\Re(x) > 0$. We have

$$k\sum_{n \geq 1} t^n \ B(j+1, kn) = k\int_0^1 \frac{(1-x)^j}{x}\sum_{n \geq 1}\left(tx^k\right)^n dx = kt\int_0^1 \frac{(1-x)^j x^{k-1}}{1-tx^k} dx,$$

now

$$\lim_{j\to 0}\left(\frac{d}{dj}\sum_{n\geq 1}\frac{t^n}{n\binom{kn+j}{j}}\right) = kt\lim_{j\to 0}\left(\frac{d}{dj}\int_0^1\frac{(1-x)^j\,x^{k-1}}{1-tx^k}dx\right)$$

and with $t = -1$ we obtain the result (4). To prove (5), we note, from the properties of the polygamma function with multiple argument, that

$$\psi^{(n)}(kz) = \delta_{n,0}\ln k + \frac{1}{k^{n+1}}\sum_{j=0}^{k-1}\psi^{(n)}\left(z+\frac{j}{k}\right),$$

where $\delta_{n,0}$ is the Kronecker delta. By the use of the digamma function in terms of harmonic numbers, we have

$$H_{kn} = \ln k + \frac{1}{k}H_n + \frac{1}{k}\sum_{j=1}^{k-1}H_{n-\frac{j}{k}},$$

where $H_{n-\frac{r}{p}}$ may be thought of as shifted harmonic numbers. Summing over the integers

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_{kn}}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\ln k + \frac{1}{k}H_n + \frac{1}{k}\sum_{j=1}^{k-1}H_{n-\frac{j}{k}}\right)$$

$$= \ln 2\ln k + \frac{1}{k}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n} + \frac{1}{k}\sum_{j=1}^{k-1}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_{n-\frac{j}{k}}}{n}$$

$$= \ln 2\ln k + \frac{1}{2k}\left(\zeta(2) - \ln^2 2\right)$$

$$+\frac{1}{k}\sum_{j=1}^{k-1}\left(\ln 2 H_{-\frac{j}{k}} + \frac{k}{2(k-j)}\,_3F_2\left[\begin{array}{c}1,1,1\\2,2-\frac{j}{k}\end{array}\middle|\frac{1}{2}\right]\right),$$

the first sum is obtained from [11] and the second sum is deduced from [9]. Since $\sum_{j=1}^{k-1}H_{-\frac{j}{k}} = -k\ln k$ then (5) follows. The closed form representation (6) can be evaluated by contour integration, the details are in [5]. $\square$

Lemma 1.2. *Let k be a positive integer. Then:*

$$\sum_{n\geq 1}\frac{(-1)^{n+1}H_{kn}}{n+1} = X(k,1) \tag{7}$$

$$= k\int_0^1\frac{\ln(1-x)}{x}\left(\frac{1}{1+x^k} - \frac{\ln(1+x^k)}{x^k}\right)dx$$

$$= -X(k,0) + \frac{1}{2k}\zeta(2) - H_{k-1}\ln 2 + \frac{1}{2}\sum_{j=1}^{k-1}\frac{1}{j}\left(H_{-\frac{j}{2k}} - H_{-\frac{k+j}{2k}}\right), \tag{8}$$

where $X(k,0)$ is given in (4).

Proof. The proof of (7) is concluded in the same manner as used in Lemma 1.1. Consider

$$X(k,1) = \sum_{n\geq 1}\frac{(-1)^{n+1}H_{kn}}{n+1}$$

and by a change of summation index

$$X(k, 1) = \sum_{n \geq 1} \frac{(-1)^{n+1} H_{kn}}{n + 1} = \sum_{n \geq 1} \frac{(-1)^n H_{kn-k}}{n}$$

$$= -\sum_{n \geq 1} \frac{(-1)^{n+1} H_{kn}}{n} + \sum_{n \geq 1} \frac{(-1)^{n+1}}{kn^2} + \sum_{j=1}^{k-1} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n(kn - j)}$$

$$= -X(k, 0) + \frac{1}{2k} \zeta(2) - H_{k-1} \ln 2 + \frac{1}{2} \sum_{j=1}^{k-1} \frac{1}{j} \left(H_{-\frac{j}{2k}} - H_{-\frac{k+j}{2k}} \right).$$

The integral identity following (7) is obtained by the Beta method as described in Lemma 1.1 and therefore the details will not be outlined. It is of some interest to note that from (4) and (7)

$$k \int_0^1 \frac{\ln(1 - x)}{x} \left(\frac{1 - x^k}{1 + x^k} - \frac{\ln(1 + x^k)}{x^k} \right) dx = \sum_{n \geq 1} \frac{(-1)^{n+1} (2n + 1) H_{kn}}{n(n + 1)}$$

$$= \frac{1}{2k} \zeta(2) - H_{k-1} \ln 2 + \frac{1}{2} \sum_{j=1}^{k-1} \frac{1}{j} \left(H_{-\frac{j}{2k}} - H_{-\frac{k+j}{2k}} \right). \qquad \square$$

Lemma 1.3. *Let k and r be positive integers. Then:*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{kn}}{n + r} = X(k, r) = -\frac{k}{1 + r} \int_0^1 x^{k-1} \ln(1 - x) \, {}_2F_1 \left[\begin{array}{c} 2, 1 + r \\ 2 + r \end{array} \middle| -x^k \right] dx \qquad (9)$$

and

$$X(k, r) = (-1)^{r+1} X(k, 1) + \frac{(-1)^{r+1}}{k} \left(H_{\left[\frac{r-1}{2} \right]} - H_{r-1} \right) \ln 2$$

$$+ \frac{(-1)^{r+1}}{2k} \sum_{s=1}^{r-1} \frac{(-1)^s}{s} \left(H_{\frac{s-1}{2}} - H_{\frac{s}{2}} \right)$$

$$+ \frac{(-1)^{r+1}}{2} \sum_{s=1}^{r-1} (-1)^s \left(H_{\frac{s-1}{2}} - H_{\frac{s}{2}} \right) (H_{ks+k-1} - H_{ks})$$

$$+ \frac{(-1)^{r+1}}{2} \sum_{j=1}^{k-1} \sum_{s=1}^{r-1} \frac{(-1)^s}{ks + j} \left(H_{-\frac{j}{2k}} - H_{-\frac{k+j}{2k}} \right), \qquad (10)$$

with X(k, 1) given by (8).

Proof. By a change of summation index

$$X(k, r) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{kn}}{n + r} = \sum_{n=1}^{\infty} \frac{(-1)^n H_{kn-k}}{n + r - 1}$$

$$= -\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{kn}}{n + r - 1} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{kn(n + r - 1)} + \sum_{j=1}^{k-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(kn - j)(n + r - 1)}$$

$$= -X(k, r - 1) + \frac{1}{k(r - 1)} \left(\ln 2 + \frac{1}{2} \left(H_{\frac{r-2}{2}} - H_{\frac{r-1}{2}} \right) \right)$$

$$+ \sum_{j=1}^{k-1} \frac{1}{j + k(r-1)} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{k}{kn-j} - \frac{1}{n+r-1} \right)$$

$$= -X(k, r-1) + \frac{1}{k(r-1)} \left(\ln 2 + \frac{1}{2} \left(H_{\frac{r-2}{2}} - H_{\frac{r-1}{2}} \right) \right)$$

$$+ \sum_{j=1}^{k-1} \frac{1}{j + k(r-1)} \left(\Phi \left(-1, 1, 1 - \frac{j}{k} \right) + \frac{1}{2} \left(H_{\frac{r-2}{2}} - H_{\frac{r-1}{2}} \right) \right).$$

Since the Lerch transcendent

$$\Phi \left(-1, 1, 1 - \frac{j}{k} \right) = \frac{1}{2} \left(\psi \left(\frac{2k-j}{2k} \right) - \psi \left(\frac{k-j}{2k} \right) \right) = \frac{1}{2} \left(H_{-\frac{j}{2k}} - H_{-\frac{j+k}{2k}} \right),$$

so that

$$X(k, r) = -X(k, r-1) + \frac{\ln 2}{k(r-1)} + \frac{1}{2k(r-1)} \left(H_{\frac{r-2}{2}} - H_{\frac{r-1}{2}} \right)$$

$$+ \frac{1}{2} \left(H_{\frac{r-2}{2}} - H_{\frac{r-1}{2}} \right) \left(H_{kr-1} - H_{kr-k} \right)$$

$$+ \frac{1}{2} \sum_{j=1}^{k-1} \frac{1}{k(r-1)+j} \left(H_{-\frac{j}{2k}} - H_{-\frac{k+j}{2k}} \right). \tag{11}$$

From (11) we have the recurrence relation

$$X(k, r) + X(k, r-1) = \frac{\ln 2}{k(r-1)} + \frac{1}{2k(r-1)} \left(H_{\frac{r-2}{2}} - H_{\frac{r-1}{2}} \right)$$

$$+ \frac{1}{2} \left(H_{\frac{r-2}{2}} - H_{\frac{r-1}{2}} \right) \left(H_{kr-1} - H_{kr-k} \right)$$

$$+ \frac{1}{2} \sum_{j=1}^{k-1} \frac{1}{k(r-1)+j} \left(H_{-\frac{j}{2k}} - H_{-\frac{k+j}{2k}} \right)$$

for $r \geq 2$, and with $X(k, 1)$ given by (8). The recurrence relation is solved by the subsequent reduction of the

$$X(k, r), \ X(k, r-1), \ X(k, r-2), \dots, X(k, 1)$$

terms, finally arriving at the relation (10). The integral identity (9) is obtained by the Beta method as described in Lemma 1.1 and details will not be outlined.   □

A slightly different re-arrangement of the terms in $X(k, r)$ leads to the following Lemma.

Lemma 1.4. *Let k and $r \geq 2$ be a positive integers. Then:*

$$Y(k, r) = \frac{k}{2} \int_0^1 x^{2k-1} \ln(1-x) \left(\begin{array}{c} (r-1) \Phi \left(x^{2k}, 1, \frac{r+1}{2} \right) \\ -\Phi \left(x^{2k}, 1, \frac{r+2}{2} \right) \end{array} \right) dx$$

$$\tag{12}$$

$$= X(k, r) + \frac{1}{2} H_{\frac{r-1}{2}} \left(H_{kr-1} - H_{kr-k-1} \right) - \frac{1}{2} \sum_{j=1}^{k-1} \frac{1}{j + (r-1)k} H_{-\frac{j}{2k}} \tag{13}$$

$$= \sum_{n=1}^{\infty} \frac{H_{2kn}}{(2n+r-1)(2n+r)}$$

with $X(k, r)$ given by (10).

Proof. By expansion,

$$X(k, r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{kn}}{n+r} = \sum_{n=1}^{\infty} \frac{H_{2kn}}{(2n+r-1)(2n+r)} - \sum_{j=0}^{k-1} \sum_{n=1}^{\infty} \frac{1}{(2n+r-1)(2nk-j)},$$

by re arrangement

$$Y(k, r) = \sum_{n=1}^{\infty} \frac{H_{2kn}}{(2n+r-1)(2n+r)} = X(k, r) + \sum_{j=0}^{k-1} \frac{H_{\frac{r-1}{2}} - H_{\frac{-j}{2k}}}{2(j+(r-1)k)}$$

$$= X(k, r) + \frac{1}{2} H_{\frac{r-1}{2}} \left(H_{kr-1} - H_{kr-k-1} \right) - \frac{1}{2} \sum_{j=1}^{k-1} \frac{1}{j+(r-1)k} H_{-\frac{j}{2k}}.$$

The integral (12) is obtained by considering for $t \in [-1, 1)$ and $j \in \mathbb{R}^+ \cup \{0\}$

$$\sum_{n \geq 1} \frac{t^n}{(2n+r-1)(2n+r) \binom{2kn+j}{j}} = 2k \sum_{n \geq 1} \frac{n t^n \ B(j+1, 2kn)}{(2n+r-1)(2n+r)}$$

$$2k \sum_{n \geq 1} \frac{n t^n \ B(j+1, 2kn)}{(2n+r-1)(2n+r)} = 2k \int_0^1 \frac{(1-x)^j}{x} \sum_{n \geq 1} \frac{n \left(t x^{2k} \right)^n}{(2n+r-1)(2n+r)} dx.$$

Now differentiating with respect to j and replacing the limit as j approaches zero, with $t = -1$, we obtain the result (12). Two special cases, furnish the following. For $r = 0$,

$$\frac{k}{2} \int_0^1 x^{k-1} \ln(1-x) \ln\left(\frac{1+x^k}{1-x^k} \right) dx = -X(k, 0) - H_k \ln 2 + \frac{1}{2} \sum_{j=1}^{k-1} \frac{1}{j-k} H_{-\frac{j}{2k}}. \tag{14}$$

For $r = 1$,

$$k \int_0^1 \frac{\ln(1-x)}{x} \left(1 - \frac{1}{2x^k} \ln\left(\frac{1+x^k}{1-x^k} \right) \right) dx = X(k, 1) + \frac{1}{4k} \zeta(2) - \frac{1}{2} \sum_{j=1}^{k-1} \frac{1}{j} H_{-\frac{j}{2k}},$$

from which we deduce the integral identity,

$$\frac{k}{2} \int_0^1 \frac{\ln(1-x)}{x^{k+1}} \ln\left(\frac{1+x^k}{1-x^k} \right) dx = \frac{1}{2} \sum_{j=1}^{k-1} \frac{1}{j} H_{-\frac{j}{2k}} - X(k, 1) - \frac{1+4k^2}{4k} \zeta(2),$$

and for $k = 4$,

$$\int_0^1 \frac{\ln(1-x)}{x^5} \ln\left(\frac{1+x^4}{1-x^4} \right) dx = \frac{\pi}{48} \left(1 - 8\sqrt{2} \right) - \frac{25}{16} \zeta(2) - \frac{1}{16} \ln^2 2$$

$$- \frac{19}{24} \ln 2 + \ln\left(\sqrt{2} + 1 \right) \left(\frac{\sqrt{2}}{6} + \frac{1}{4} \ln\left(\sqrt{2} - 1 \right) \right). \qquad \square$$

The next few theorems relate the main results of this investigation, namely the closed form representation of integrals of the type (1).

2 Integral and Closed form identities

In this section we investigate integral identities in terms of closed form representations of infinite series of harmonic numbers and inverse binomial coefficients. First we indicate the closed form representation of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{kn}}{n^q \binom{n+p}{p}} \tag{15}$$

for $q = 0, 1,$ and $k, \ p \geq 1$ are positive integers.

Theorem 2.1. *Let $k \geq 1$ be real positive integer, then from (15) with $q = 0$ and p be real positive integer:*

$$-\frac{k}{p+1} \int_0^1 x^{k-1} \ln(1-x) \ {}_2F_1 \left[\begin{array}{c} 2, 2 \\ p+2 \end{array} \middle| -x^k \right] dx \tag{16}$$

$$= \Lambda(k, p) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{kn}}{\binom{n+p}{p}}$$

$$= \sum_{r=1}^{p} (-1)^{1+r} \, r \binom{p}{r} X(k, r), \tag{17}$$

where $X(k, r)$ is given by (10).

Proof. Consider the expansion

$$\Lambda(k, p) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{kn}}{\binom{n+p}{p}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} p! \, H_{kn}}{(n+1)_p} = \sum_{n=1}^{\infty} (-1)^{n+1} p! \, H_{kn} \sum_{r=1}^{k} \frac{\lambda(r)}{n+r}$$

where

$$\lambda(r) = \lim_{n \to -r} \left\{ \frac{n+r}{\prod_{r=1}^{p}(n+r)} \right\} = \frac{(-1)^{1+r} \, r}{p!} \binom{p}{r}. \tag{18}$$

We can now express

$$\Lambda(k, p) = \sum_{n=1}^{\infty} (-1)^{n+1} p! \, H_{kn} \sum_{r=1}^{p} \frac{\lambda(r)}{n+r} = \sum_{r=1}^{p} (-1)^{1+r} \, r \binom{p}{r} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{H_{kn}}{n+r}. \tag{19}$$

From (10) we have $X(k, r)$, hence substituting into (19), (17) follows. The integral (16) is evaluated as in Lemma 1.4. □

The other case of $q = 1$ can be evaluated in a similar fashion. We list the result in the next Theorem.

Theorem 2.2. *Under the assumptions of Theorem 2.1, with $q = 1$, we have,*

$$-\frac{k}{p+1} \int_0^1 x^{k-1} \ln(1-x) \ {}_2F_1 \left[\begin{array}{c} 1, 2 \\ p+2 \end{array} \middle| -x^k \right] dx = M(k, p) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{kn}}{n \binom{n+p}{p}} = \sum_{r=0}^{p} (-1)^r \binom{p}{r} X(k, r).$$

$$\tag{20}$$

and where $X(k, r)$ is given by (10).

Proof. The proof of (20) follows using the same technique as used in Theorem 2.1 and also using (18). □

It is possible to gain some further integral identities from Theorems 2.1 and 2.2 regarding the representation of a sequence of alternating shifted harmonic numbers as follows.

Theorem 2.3. *For $p \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$:*

$$\frac{1}{k(p+1)} \sum_{r=1}^{k-1} \int_0^1 \frac{\ln(1-x)}{x^{\frac{r}{k}}} \left(k\,{}_2F_1 \left[\begin{array}{c} 1,2 \\ p+2 \end{array} \middle| -x \right] - r\,{}_2F_1 \left[\begin{array}{c} 1,1 \\ p+2 \end{array} \middle| -x \right] \right) dx$$

(21)

$$= kM(k,p) - \frac{k\ln k}{p+1}\,{}_2F_1 \left[\begin{array}{c} 1,1 \\ p+2 \end{array} \middle| -1 \right] - S(p)$$

$$= \sum_{r=1}^{k-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{r}{k}}}{n \left(\begin{array}{c} n+p \\ p \end{array} \right)},$$

(22)

where $M(k,p)$ is given by (20) and

$$S(p) = \frac{1}{2}\zeta(2) + \left(2^{p-1} - 1\right)\ln^2 2$$

(23)

$$+ \sum_{m=1}^{p} m \left(\begin{array}{c} p \\ m \end{array} \right) \left(\begin{array}{c} \left(2H_{m-1} - H_{\left[\frac{m-1}{2}\right]}\right)\ln 2 + H_{m-1}\left(H_{\left[\frac{m}{2}\right]} - H_m\right) \\ -\sum_{j=1}^{m-1} \frac{(-1)^j}{j}\left(H_{\left[\frac{m-j}{2}\right]} - H_{m-j} + \frac{jH_j}{j+1}\right) \end{array} \right),$$

where $[x]$ is the integer part of x.

Proof. From the properties of harmonic numbers,

$$H_{kn} = \ln k + \frac{1}{k}\sum_{r=0}^{k-1} H_{n-\frac{r}{k}},$$

$$\sum_{r=1}^{k-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{r}{k}}}{n \left(\begin{array}{c} n+p \\ p \end{array} \right)} = k \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{kn}}{n \left(\begin{array}{c} n+p \\ p \end{array} \right)} - k\ln k \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \left(\begin{array}{c} n+p \\ p \end{array} \right)} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n \left(\begin{array}{c} n+p \\ p \end{array} \right)}$$

$$= kM(k,p) - \frac{k\ln k}{p+1}\,{}_2F_1 \left[\begin{array}{c} 1,1 \\ p+2 \end{array} \middle| -1 \right] - S(p),$$

the details for the calculation of (23) may be seen in [11]. The integral representation (21) is obtained in the same manner as in Lemma 1.4.

For the simple case of $p = 0$, we have

$$\sum_{r=1}^{k-1} \int_0^1 \frac{\ln(1-x)}{x^{1+\frac{r}{k}}} \left(\frac{r}{k}\ln(1+x) - \frac{x}{1+x} \right) dx = kX(k,0) - k\ln 2\ln k - \frac{1}{2}\zeta(2) + \frac{1}{2}\ln^2 2,$$

and when $k = 6$,

$$\sum_{r=1}^{5} \int_0^1 \frac{\ln(1-x)}{x^{1+\frac{r}{6}}} \left(\frac{r}{6}\ln(1+x) - \frac{x}{1+x} \right) dx = \frac{35}{4}\zeta(2) - 10\ln^2 2 - 6\ln 2\ln 3 + 12\ln\left(\sqrt{3}+1\right)\ln\left(\sqrt{3}-1\right).$$

It is also possible to represent, individually, some results of shifted harmonic numbers of (22), see for example, [9] and [10]. □

The following integral identities can be exactly evaluated by using the alternating harmonic number sums in Theorems 2.1 and 2.2.

Theorem 2.4. *Let k and p be real positive integers, then:*

$$\frac{-4k}{(p+1)(p+2)} \int_0^1 x^{2k-1} \ln(1-x) \, {}_3F_2 \left[\begin{array}{c} 1, \frac{3}{2}, 2 \\ \frac{p+3}{2}, \frac{p+4}{2} \end{array} \middle| x^{2k} \right] dx \tag{24}$$

$$= \Omega(k,p) = \sum_{n=1}^{\infty} \frac{H_{2kn}}{n \binom{2n+p}{p}} = \frac{2}{p} \sum_{r=1}^{p} (-1)^{1+r} \, r \binom{p}{r} X(k,r)$$

$$+ \frac{2}{p(p+1)} \sum_{j=0}^{k-1} \frac{1}{2k-j} \, {}_4F_3 \left[\begin{array}{c} 1,1,\frac{3}{2},\frac{2k-j}{2k} \\ \frac{p+2}{2}, \frac{p+3}{2}, \frac{4k-j}{2k} \end{array} \middle| 1 \right], \tag{25}$$

where X (k, r) is given by (10).

Proof. From

$$\Lambda(k,p) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{kn}}{\binom{n+p}{p}} = \sum_{n=1}^{\infty} \frac{H_{(2n-1)k}}{\binom{2n-1+p}{p}} - \sum_{n=1}^{\infty} \frac{H_{2kn}}{\binom{2n+p}{p}}$$

$$= \sum_{n=1}^{\infty} \frac{p \, H_{2kn}}{2n \binom{2n+p}{p}} - \sum_{j=1}^{k-1} \sum_{n=1}^{\infty} \frac{1}{(2nk-j)\binom{2n-1+p}{p}} - \sum_{n=1}^{\infty} \frac{1}{2nk\binom{2n-1+p}{p}}.$$

Re-arranging

$$\Omega(k,p) = \sum_{n=1}^{\infty} \frac{H_{2kn}}{n \binom{2n+p}{p}} = \frac{2}{p} \Lambda(k,p) + \frac{2}{p(p+1)} \sum_{j=0}^{k-1} \frac{1}{2k-j} \, {}_4F_3 \left[\begin{array}{c} 1,1,\frac{3}{2},\frac{2k-j}{2k} \\ \frac{p+2}{2}, \frac{p+3}{2}, \frac{4k-j}{2k} \end{array} \middle| 1 \right]$$

and (25) follows. The integral (24) is evaluated as in Lemma 1.4.   □

Theorem 2.5. *Let k and p be real positive integers, then:*

$$\frac{-4k}{(p+1)(p+2)} \int_0^1 x^{2k-1} \ln(1-x) \, {}_3F_2 \left[\begin{array}{c} \frac{1}{2}, 1, 2 \\ \frac{p+3}{2}, \frac{p+4}{2} \end{array} \middle| x^{2k} \right] dx$$

$$= \Xi(k,p) = \sum_{n=1}^{\infty} \frac{H_{2kn}}{n(2n-1)\binom{2n+p}{p}} = \frac{2}{p+1} \sum_{r=0}^{p} (-1)^{r} \binom{p}{r} X(k,r)$$

$$+ \frac{2}{(p+1)^2} \sum_{j=0}^{k-1} \frac{1}{2k-j} \, {}_4F_3 \left[\begin{array}{c} \frac{1}{2},1,1,\frac{2k-j}{2k} \\ \frac{p+2}{2}, \frac{p+3}{2}, \frac{4k-j}{2k} \end{array} \middle| 1 \right],$$

where X (k, r) is given by (10).

Proof. Follows the same pattern as used in Theorem 2.4.   □

Theorem 2.6. *Let k and p be real positive integers, then:*

$$\frac{4k}{(p+1)(p+2)} \int_0^1 x^{2k-1} \ln(1-x) \, {}_3F_2 \left[\begin{array}{c} \frac{1}{2}, 2, 2 \\ \frac{p+3}{2}, \frac{p+4}{2} \end{array} \middle| x^{2k} \right] dx$$

$$= \Upsilon(k,p) = \sum_{n=1}^{\infty} \frac{H_{2kn}}{(2n-1)\binom{2n+p}{p}} = \frac{2}{p}\Lambda(k,p) + \frac{2}{p+1}M(k,p)$$

$$+ \frac{2}{(p+1)^2} \sum_{j=0}^{k-1} \frac{1}{2k-j} \, {}_4F_3 \left[\begin{array}{c} \frac{1}{2},1,1,\frac{2k-j}{2k} \\ \frac{p+2}{2},\frac{p+3}{2},\frac{4k-j}{2k} \end{array} \bigg| 1 \right]$$

$$+ \frac{2}{p(p+1)} \sum_{j=0}^{k-1} \frac{1}{2k-j} \, {}_4F_3 \left[\begin{array}{c} 1,1,\frac{3}{2},\frac{2k-j}{2k} \\ \frac{p+2}{2},\frac{p+3}{2},\frac{4k-j}{2k} \end{array} \bigg| 1 \right],$$

where $\Lambda(k,p)$ is given by (17) and $M(k,p)$ by (20).

Proof. Follows the same pattern as used in Theorem 2.4.   □

A number of special cases follow in the next Corollary.

Corollary 2.7. *Some examples of integrals are given below. For $p = 0$, Theorem 2.5 reduces to (14).*

For $p = 1$, from Theorem 2.6 we have

$$-k \int_0^1 \left(\frac{1+x^{2k}}{x^{k+1}} \right) \ln(1-x)\,\ln\left(\frac{1+x^k}{1-x^k} \right) dx = \frac{3+4k^2}{2k}\zeta(2) + \frac{2}{k}\ln 2 + 4H_{k-1} + \sum_{j=1}^{k-1} \frac{1}{k-j}\left(H_{-\frac{j}{2k}} - H_{\frac{j}{2k}} \right),$$

for $k = 3$, we have

$$-3 \int_0^1 \left(\frac{1+x^6}{x^4} \right) \ln(1-x)\,\ln\left(\frac{1+x^3}{1-x^3} \right) dx = \frac{13}{2}\zeta(2) + \frac{5\sqrt{3}}{6}\pi + \frac{2}{3}\ln 2,$$

this integral is highly oscillatory near the origin of x. From Theorem 2.5, with $k = 6$.

$$\int_0^1 \left(\frac{x^{12}-1}{x^7} \right) \ln(1-x)\,\ln\left(\frac{1+x^6}{1-x^6} \right) dx = \frac{73}{72}\zeta(2) + \frac{1}{2}\ln^2 2 + \frac{39}{40}\ln 3$$

$$-\frac{4}{3}\ln\left(1+\sqrt{3}\right)\left(\frac{2\sqrt{3}}{5} + \ln\left(\sqrt{3}-1\right) \right) + \frac{1}{15}\left(4\sqrt{3} + \frac{11}{6} \right) \ln 2,$$

this integral is highly oscillatory near the origin of x.

Conclusion 2.8. *We have established a number of integral identities in closed form in terms of special functions. A number of oscillatory integrals are also given in closed form. The integral identities established in this paper complement and extend the results in the paper [12]. Some particular identities obtained are*

$$\frac{1}{8} \int_0^1 \left(\frac{x+1}{\sqrt{x}} \right) \ln^2 x\,\ln\left(\frac{1+x}{1-x} \right) dx = -\frac{10}{3}\zeta(2) - \frac{14}{3}\zeta(3) + \frac{26}{27}\pi + \frac{32}{9}G - \frac{56}{27}\ln 2 + \frac{104}{27} + \frac{\pi^3}{12}.$$

$$\frac{1}{2} \int_0^1 \frac{1-x^4}{(1-x)\,x^{\frac{5}{2}}} \ln^2 x\,\ln\left(\frac{1+x^2}{1-x^2} \right) dx = \frac{16}{27}\left(14\sqrt{2}-13 \right)\pi + \frac{16}{3}\left(2\sqrt{2}-5 \right)\zeta(2) + \frac{1}{3}\left(3\sqrt{2}-2 \right)\pi^3.$$

References

[1] Choi J., Cvijović D., Values of the polygamma functions at rational arguments, *J. Phys. A: Math. Theor.* **40** (2007), 15019–15028, Corrigendum, ibidem, **43** (2010), 239801 (1 p).

[2] Choi J., Srivastava H.M., Some summation formulas involving harmonic numbers and generalized harmonic numbers. *Math. Comput. Modelling.* **54** (2011), 2220-2234.

[3] Guillera J., Sondow J., Double integrals and infinite constants via analytic continuations of Lerch's transcendent. *Ramanujan J.* **16** (2008), 247-270.

[4] Kölbig K., The polygamma function $\psi(x)$ for $x = 1/4$ and $x = 3/4$. *J. Comput. Appl. Math.* **75** (1996), 43-46.

[5] Kouba O., The sum of certain series related to harmonic numbers. *Octogon Math. Mag.* **19(1)** (2011), 3-18.

[6] Lagarias J.C., Li W.-C., The Lerch zeta function I. *Forum Math.* **24** (2012), 1-48.

[7] Lagarias J.C., Li W.-C., The Lerch zeta function II. *Forum Math.* **24** (2012), 49-84.

[8] Lagarias J.C., Li W.-C., The Lerch function III. Polylogarithms and special values. *Res. Math. Sci.* **3** (2016). Art 2, 54 pp.

[9] Sofo A., Harmonic numbers at half integer values. *Integral Transforms Spec. Funct.* **27** (2016), no. 6, 430–442.

[10] Sofo A., Srivastava H.M., A family of shifted harmonic sums. *Ramanujan J.* **37** (2015), no. 1, 89–108.

[11] Sofo A., New families of alternating harmonic number sums. *Tbilisi Math. J.* **8** (2015), no. 2, 195–209.

[12] Sofo A., A master integral in four parameters. *J. Math. Anal. Appl.* **448** (2017), no. 1, 81–92.

[13] Srivastava H.M., Choi J., Series Associated with the Zeta and Related Functions. Kluwer Academic Publishers, London, 2001.

[14] Srivastava H.M., Choi J., *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[15] Xu C., Yan Y., Shi Z., Euler sums and integrals of polylogarithmic functions. *J. Number Theory.* **165** (2016), 84-108.