The Mean-field Behavior of Processor Sharing Systems with General Job Lengths Under the SQ(d) Policy

ABSTRACT

In this paper, we study the mean-field behavior of large-scale systems that consist of N (large) identical parallel processor sharing servers with Poisson arrival process having intensity $N\lambda$ and generally distributed job lengths under the randomized SQ(d) load balancing policy. Under this policy, an arrival is routed to the server with the least number of progressing jobs among d randomly chosen servers. The limit of the empirical distribution is then used to study the statistical properties of the system. In particular, this shows that in the limit as N grows, individual servers are statistically independent of others (propagation of chaos) and more importantly, the equilibrium point of the mean-field is insensitive to the job length distributions that has important engineering relevance for the robustness of such routing policies used in web server farms. We use a framework of measure-valued processes and martingale techniques to obtain our results. We also provide numerical results to support our analysis.

CCS CONCEPTS

• **Networks** → **Network performance analysis**;

KEYWORDS

Mean-field limit, Fixed-point, Insensitivity, Measure-valued processes

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1 INTRODUCTION

The emergence of the cloud computing paradigm and other centralized applications result in server farms that contain large numbers of servers to process the incoming job requests. The front end job dispatchers route an arriving job to one of the servers that provide minimal response times as the tasks in most cases are delay sensitive[20]. Therefore the key challenge in these systems is to design low complex load balancing algorithms that results in efficient use of resources thereby good system performance. In server farms, the resources are shared by processing requests in a round-robbin manner with small time granularity. This model can be well approximated by the processor sharing model [17, 36] where the processing speed of a server is equally shared by all the progressing jobs.

In practice, the server farms such as Cisco Local Director, IBM Network Dispatcher, Microsoft Sharepoint use the classical Join-the-shortest-queue (JSQ) policy to achieve the load balancing. It was shown in [14, 34] that the JSQ policy is nearly optimal and further, it is robust to the job length distributions since it is nearly insensitive. The notion of insensitivity implies that the stationary distribution of occupancy depends only on the mean job lengths but not on the type of job-length distributions. For large scale systems that contain hundreds of thousands of servers, the JSQ policy requires the information about the number of progressing jobs at all the servers. However it was shown in [6, 21–24, 32] that a randomized routing scheme SQ(d) based on sending jobs to the best amongst d randomly chosen servers can achieve almost the same gains at a much smaller sampling cost. This policy achieves near optimal performance even for $d = 2$ and hence, led to the popularization of the so-called *power-of-2* terminology.

In [32], under the assumption of exponential service time distributions, the SQ(d) policy with $d = 2$ was introduced for multi-server systems having N severs with FCFS service discipline and the job arrival process is a Poisson process with rate $N\lambda$, where the policy of routing the smaller of two randomly sampled servers was studied. Since under the SQ(d) or power-of-d routing policy, the servers are coupled in a finite N system, the exact analysis of the system is not tractable and it is extremely difficult. Their key contribution was that by first taking the limit as $N \to \infty$ the system decouples into independent queues where the limiting empirical distribution of a queue is described by a deterministic non-linear equation called the mean-field or hydrodynamic limit. The property of decoupling between servers as $N \to \infty$ is called propagation of chaos. Moreover they showed that the limit of the stationary distribution of the queues corresponds to the equilibrium or fixed point of the mean field equations (MFE) and the proof relies on the fact that the mean-field has quasi-monotonicity property. Furthermore, the fixed point of the MFE that represents the stationary distribution of a queue shows a dramatic reduction in the average response time due to the fact that the occupancy (Q) has tail distribution satisfying a super-exponential decay given by $\Pr(Q \geq k) = (\frac{\lambda}{\mu})^{2^k-1}$ instead of $(\frac{\lambda}{\mu})^k$ that would be the case if uniform routing had been used. Later the analysis is extended to the SQ(d) policy with $d > 2$ in [21] where it was shown that the fixed point of the MFE satisfies $\Pr(Q \geq k) = (\frac{\lambda}{\mu})^{\frac{d^k-1}{d-1}}$. These nice conclusions reflect that the mean-field techniques can help us to obtain deeper insights about a large-scale complex stochastic system when the exact analysis is not tractable.

The SQ(d) policy was introduced in [32] which treated the exponential service times case. The study of the SQ(d) policy for large-scale systems with general service time distributions has only

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been recently addressed with some partial progress and simulation observations that we recall later in this section. This is because for queuing systems with general service times, the job occupancy process (Queue-length process) is not Markov and this limits us in extending the analysis directly from exponential case to the general distributions case. In that case one needs to Markovianize the system by considering the occupancy state as well as the age or residual service times of the jobs in service which makes the mean-field analysis a challenging task. However, it is very important to make progress on understanding the large-scale multi-server system behavior when we employ the SQ(d) policy since more realistic models need us to dispense with the exponential job length hypothesis. For example, the service time distributions are Log-normal in call centers [8], and Gamma distributed in automatic teller machines (ATMs) [18] etc. Therefore it is necessary to study the impact of SQ(d) routing scheme when job lengths are generally distributed.

In this paper we consider a multi-server large scale processor sharing system with the generally distributed job lengths with finite mean $\frac{1}{\mu}$ under the SQ(d) policy. Earlier, the case with groups of heterogeneous servers under the assumption of exponential job lengths was studied in [22, 23] using mean-field techniques. In particular, they established the global asymptotic stability (GAS) of the mean-field limit that also establishes that the fixed point of the MFE corresponds to the limiting stationary distribution of a server occupancy.

The focus in this paper is to understand the probabilistic behavior of systems with a large number of servers by studying the limiting behavior when the number of servers goes to infinity. We show that in this context the limit of the empirical distribution satisfies a deterministic equation called the mean-field or hydrodynamic limit. One of the key insights we seek is how the distribution of the job lengths affects the mean-field behavior. We show that as $N \to \infty$ any finite set of servers become statistically independent and moreover the fixed point of the mean field only depends on the mean job lengths and not their distributions, the property termed as insensitivity. It is well known that Processor Sharing systems are insensitive to the service time distributions in equilibrium for Poisson inputs, and thus our result shows that this property is inherited under randomized SQ(d) routing. In prior works [6, 23] evidence of this property was presented via simulations but without complete proof.

We now discuss some of the results on the large-scale multi-server systems with general job length distributions under the SQ(d) policy. The SQ(d) policy for processor sharing, FIFO, and LIFO with preemptive resume models with general job length distributions in the homogeneous context were studied in [6] under the *ansatz* hypothesis that the underlying stationary distribution of the occupancy of the Markov process converges to a unique limiting distribution and any finite set of servers are asymptotically independent using the cavity method. As we show in Section 5.2, the *ansatz* carries the necessary and sufficient conditions needed to conclude insensitivity for processor sharing systems. This is because each individual server must have state dependent Poisson arrivals which is a consequence of the assumption of asymptotic independence of servers or propagation of chaos and furthermore, existence of unique stationary distribution is essential to conclude insensitivity. Bramson [7] succeeded in proving the *ansatz* only for the case of FIFO models with job length distributions having decreasing hazard rate functions.

Recently the FIFO case with generally distributed service times has been revisited in [1] via a mean-field approach where they show that the joint process that counts the jobs at a server and the age of the job in service has a mean-field that is described by a set of PDEs. However the analysis is only restricted to the finite time or transient case and no results are given for the stationary regime. They also established the propagation of chaos for the individual queues for any finite time t in the limiting system. However, in the processor sharing case, the analysis is now more complicated because in addition to the occupancy process one needs to keep track of the ages of all the jobs in the system. This results in a Markov process on $\bigcup_{n\in Z_+}(n, \mathfrak{R}_+^n)$ and the analysis and proofs are much more difficult because the ages do not increase linearly at the rate of processing speed of the server but the rate depends on the server occupancy or the first coordinate of the Markov process. This requires us to consider a measure-valued Markov process representation.

The use of measure-valued processes to deal with general service time distributions is natural. In [13] it was used to study the fluid limit of a single $GI/GI/1$ processor sharing system in critically loaded regime. The steady state analysis of the fluid limit established in [13] is considered in [25]. However, the mean-field limit that we study in this paper has not been addressed for processor sharing systems under SQ(d). The analysis is now more difficult because the mean-field limit represents the dynamics of a non-linear Markov process and the arrival rate to each server in the limiting system takes a specific form under the SQ(d) policy.

Recently there has been interest in PULL based policies such as Join-Idle-Queue (JIQ)[20]. In JIQ the dispatcher stores the identities of servers that are idle in a memory and an arrival is routed to a randomly selected idle server if there are any available in the list at the dispatcher, otherwise the job is routed to a server chosen uniformly at random. In [29], it was shown that under sub-critical system load, the steady state probability of waiting vanishes as $N \to \infty$. However, under high load, since random routing is used, there are often no idle servers in the list and it is observed in [20, 33] that the SQ(d) performs better than JIQ. To overcome this, extensions to JIQ have been considered. In one approach, the dispatcher stores the identities of all the servers with number of progressing jobs less than or equal to some threshold value such as one[20]. An arrival is routed to a randomly selected server from the list at the dispatcher if there are any available, otherwise the job is routed to a server chosen uniformly at random. In the second approach, the JIQ is combined with the SQ(d) where when there are no idle servers, the destination server is chosen according to the SQ(d) policy[33]. These extensions increase complexity significantly. Therefore understanding the impact of the SQ(d) policy when job lengths are general is an important problem.

Main contributions: The main contributions of this paper are listed below:

- *Mean-field limit:* We show that the measure-valued Markov process $(\overline{\nu}_t^N, t \geq 0)$ that tracks the fraction of servers lying in each possible server state converges in distribution to a unique deterministic measure-valued process called as the mean-field limit. As a consequence of this, we also establish the propagation of chaos for any finite time t.

- *Insensitivity of Equilibrium point of the MFE:* We then show that the partial differential equation (PDE) that describe the mean-field limit has a unique equilibrium point. Furthermore, the equilibrium point is insensitive to the job length distributions. We also provide simulation results that support the insensitivity and the GAS of the equilibrium point. This result has significant engineering implications in practice since the load balancing for large-scale processor sharing systems appear in server farms.

Organization of the paper

The rest of the paper is organized as follows: In Section 2, we introduce the system model, the SQ(d) policy, notation, and a Markovian description of the system by using a state descriptor. In Section 3, we give the main results of this paper. We then study the uniqueness of the fixed-point of the mean-field limit and its insensitivity in Section 4. After that, in Section 5, we provide a discussion on the relationship between propagation of chaos in the stationary regime and the mean-field where we also recollect existing works regarding this. In Section 5, we also provide numerical results that support insensitivity and GAS of the fixed-point of the mean-field limit. Finally, we conclude in Section 6 with a discussion on future work. The proofs are provided in Appendix (Section 7).

2 SYSTEM MODEL AND PRELIMINARIES

We first introduce the system model and the routing policy considered in this paper.

We consider a large-scale system that contains N identical processor sharing servers each having unit processing rate. Therefore if there are n jobs in progress at a server, then each job is processed at the rate of $\frac{1}{n}$.

Jobs arrive into the system according to a Poisson process with rate $N\lambda$ and there is a central job dispatcher that routes an incoming job to one of the servers according to the SQ(d) routing policy described below.

Definition 1. *The SQ(d) Routing:*
An incoming job is routed to the server with least number of ongoing jobs among d chosen servers uniformly at random with replacement[1]*. The ties if there are any, are broken by choosing a server uniformly at random. The d randomly chosen servers are called as potential destination servers and the server that is picked from the potential destination servers to route the arrived job is called as the destination server.*

We assume that the job lengths have general cumulative distribution function $G(\cdot)$ with density function $g(\cdot)$ such that the average job length is equal to $\frac{1}{\mu}$. The hazard rate function of the job length distribution is denoted by $\beta(\cdot) = \frac{g(\cdot)}{\bar{G}(\cdot)}$. For the system to be in stable region, we must have $\lambda < \mu$.

[1]Although we consider that servers are picked with replacement for simplification of the analysis, in both the cases with or without replacement, as $N \to \infty$, they would result in the same mean-field limit. For large N systems, simulation results also support this.

Notation: We now introduce the mathematical notations that we use in rest of the paper. Let $\mathcal{Z}, \mathcal{R}$ denote the space of integers and real numbers, respectively. The space of non-negative integers and non-negative real numbers are denoted by $\mathcal{Z}_+$, $\mathcal{R}_+$, respectively.

For any given metric space E, let $\mathcal{K}_b(E), C_b(E), C_k(E)$ be the space of real-valued bounded measurable, bounded continuous, and continuous functions with compact support, respectively. Further, let $C^1(E)$ be the space of once continuously differentiable functions defined on E. The subset of functions in $C^1(E)$ that are bounded functions whose first derivatives are also bounded is denoted by $C^1_b(E)$ and the space of functions with compact support in $C^1(E)$ is denoted by $C^1_k(E)$. For any function $f \in \mathcal{K}_b(E)$, we define

$$\|f\| = \sup_{x \in E} |f(x)| . \tag{1}$$

The space $C_b(E)$ is equipped with the uniform topology, *i.e.*, a sequence of functions $\{f_n\}_{n\geq 1}$ in $C_b(E)$ is said to converge to a function $f \in C_b(E)$ if $\|f_n - f\| \to 0$ as $n \to \infty$.

We next define for any function $f \in C^1_b(E)$,

$$\|f\|_1 = \|f\| + \|f'\|, \tag{2}$$

where f' is the first derivative of f. The space $C^1_b(E)$ is equipped with the topology induced by the norm $\|\cdot\|_1$. For a function $f \in C^1_b(\mathcal{R}^n_+)$, we define a function $f'_\Sigma : \mathcal{R}^n_+ \mapsto \mathcal{R}$ as follows

$$f'_\Sigma(x_1, \cdots, x_n) = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial f(x_1, \cdots, x_n)}{\partial x_i}. \tag{3}$$

For given metric space E, let the Borel σ-algebra be denoted by $\mathcal{B}(E)$. The space of finite non-negative measures defined on E is represented by $\mathcal{M}_F(E)$ and the space of probability measures is denoted by $\mathcal{M}_1(E)$. For a Borel set $B \in \mathcal{B}(E)$, the measure value of the set B with respect to the measure ν is given by $\nu(B)$ and the measure value at an element $y \in E$ is given by $\nu(\{y\})$. Further, the Dirac measure with unit mass at $x \in E$ is denoted by δ_x. We also define a set of probability measures $\mathcal{M}^N_1(E)$ as follows

$$\mathcal{M}^N_1(E) = \{\nu \in \mathcal{M}_1(E) : N\nu(B) \in \mathcal{Z}_+, \forall B \in \mathcal{B}(E)\}. \tag{4}$$

We next define for any $\phi \in C_b(E)$, $\nu \in \mathcal{M}_F(E)$,

$$\langle \nu, \phi \rangle = \int_{y \in E} \phi(y)\, d\nu(y). \tag{5}$$

The space $\mathcal{M}_F(E)$ is equipped with the weak topology, *i.e.*, a sequence $\{\nu_n\}$ in $\mathcal{M}_F(E)$ is said to converge weakly to $\nu \in \mathcal{M}_F(E)$ if and only if $\langle \nu_n, \phi \rangle \to \langle \nu, \phi \rangle$ as $n \to \infty$ for all $\phi \in C_b(E)$. We recall that if E is a Polish space, then $\mathcal{M}_F(E)$ equipped with the weak topology is also a Polish space.

We next introduce the notations that are required to model a processor sharing system evolution. To model the system evolution by a Markov process, we consider that the state of a server is given by $(n, a_1, \cdots, a_n)$ where a_i indicates the age of the i^{th} progressing job. We define the age of a progressing job as the amount of cumulative service it has received since its arrival. If $\gamma(t)$ denotes the number of jobs in service at time t at a server, the age a_i of the i^{th}

progressing job at time t that entered the server at time T_i ($t \geq T_i$) is given by

$$a_i = \int_{s=T_i}^{t} \frac{1}{\gamma(s)}\, ds. \tag{6}$$

Let U_n be the set of all possible states of a server when there are n progressing jobs, *i.e.*,

$$U_n = \{(n, a_1, \cdots, a_n) : a_i \in \mathcal{R}_+, 1 \leq i \leq n\}. \tag{7}$$

When there are no progressing jobs, then the server state lies in the set

$$U_0 = \{0\}. \tag{8}$$

Therefore at any given time t, the state of a server lies in the set

$$U = \cup_{n \in \mathcal{Z}_+} U_n. \tag{9}$$

Without loss of generality, we indicate an element $(n, u_1, \cdots, u_n) \in U$ for $n \geq 0$ by $\underline{u}$. For $\underline{u} = (n, u_1, \cdots, u_n)$ and $\underline{v} = (m, v_1, \cdots, v_m)$, we define the metric

$$d_U(\underline{u}, \underline{v}) = \begin{cases} \sum_{i=1}^{n} |u_i - v_i| & \text{if } n = m \\ \infty & \text{otherwise.} \end{cases} \tag{10}$$

We say that a sequence $\{\underline{u}_n, n \geq 1\}$ in U converges to $\underline{u} \in U$ if $\lim_{n\to\infty} d_U(\underline{u}_n, \underline{u}) = 0$.

For any Borel set $B \in \mathcal{B}(U)$, we define the indicator function of B as

$$I_{\{B\}}(\underline{u}) = \begin{cases} 1 & \text{if } \underline{u} \in B \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Further, we define a function $\mathbf{1} : U \mapsto \mathcal{R}$ as

$$\mathbf{1}(\underline{u}) = 1, \tag{12}$$

for $\underline{u} \in U$.

Any measure $\nu \in \mathcal{M}_F(U)$ restricted to the space U_0 is a Dirac measure with mass at (0). Further, for $n \geq 1$, we say that the measure $\nu \in \mathcal{M}_F(U)$ is absolutely continuous at $\underline{x} \in U_n$ if $\nu(\{\underline{x}\}) = 0$ and the measure ν is called absolutely continuous with respect to Lebesgue measure if $\nu(\{\underline{y}\}) = 0$ for all $\underline{y} \in U_n$, $n \geq 1$. We say that a function $f : U \mapsto \mathcal{R}$ is differentiable if for every $i \geq 1$, $\frac{\partial f(i, x_1, \cdots, x_i)}{\partial x_j}$ exists for every $1 \leq j \leq i$ at every $(x_1, \cdots, x_i) \in \mathcal{R}_+^i$. Hence a function of the type $f = I_{\{U_n\}}$, for $n \geq 1$, that we use frequently in our analysis is differentiable.

We next define the following two functions that are used in Assumption 2 in order to establish the mean-field limit. The first function is $\Xi : U \mapsto \mathcal{R}$ defined as

$$\Xi(n, x_1, \cdots, x_n) = n, \tag{13}$$

for $(n, x_1, \cdots, x_n) \in U$. The second function, $\Upsilon : U \mapsto \mathcal{R}$ is defined as

$$\Upsilon(n, x_1, \cdots, x_n) = \begin{cases} 0 & \text{for } n = 0, \\ x_1 + \cdots + x_n & \text{otherwise.} \end{cases} \tag{14}$$

We next define the transition operators on functions and measures that we use in describing the time evolution of the system. For any $\underline{u} \in U_n$, $n \geq 1$ and for $b > 0$, if a sever lies in state $\underline{u} \in U_n$ at time t, then if there are no arrivals or departures at this server in the interval $(t, t + b]$, then its updated state at time $t + b$ is denoted by

$$\tau_b^+(n, u_1, \ldots, u_n) = \left(n, u_1 + \frac{b}{n}, u_2 + \frac{b}{n}, \cdots, u_n + \frac{b}{n}\right) \tag{15}$$

and

$$\tau_b^+(0) = (0). \tag{16}$$

Further, for any $y > 0$, $f \in \mathcal{K}_b(U)$, we define a mapping

$$\tau_y : \mathcal{K}_b(U) \to \mathcal{K}_b(U) \tag{17}$$

satisfying

$$\tau_y f(\underline{u}) = f(\tau_y^+ \underline{u}). \tag{18}$$

For $y > 0$, now let us define a shifted measure $\tau_y \nu \in \mathcal{M}_F(U)$ such that for any Borel set $B \in \mathcal{B}(U)$, we have

$$\tau_y \nu(B) = \nu(\tau_y^+(B)). \tag{19}$$

For $\nu \in \mathcal{M}_F(U)$, the measure $\tau_y \nu \in \mathcal{M}_F(U)$ satisfies

$$\langle \tau_y \nu, f \rangle = \langle \nu, \tau_y f \rangle \tag{20}$$

for all $f \in \mathcal{K}_b(U)$. The Riesz-Markov-Kakutani theorem [27, 30] implies the existence of the unique measure $\tau_y \nu$ satisfying equation (20). For the measure-valued Markov process $(\overline{\nu}_t^N, t \geq 0)$ that describes the system evolution, equation (20) plays crucial rule in computing the expression of the generator of the process $(\overline{\nu}_t^N, t \geq 0)$. In particular, by using equation (20), the information about change in the process $(\overline{\nu}_t^N, t \geq 0)$ in a given time interval can be treated as a change in the function f. Based on this idea, by choosing the class of functions of the type $\overline{\nu}_t^N \mapsto \langle \overline{\nu}_t^N, \phi \rangle$ for $\phi \in C_b^1(U)$, one can compute the expression of the generator of the Markov process $(\overline{\nu}_t^N, t \geq 0)$.

We next define a norm on the measure $\nu \in \mathcal{M}_F(U)$ that we use in proving the uniqueness of a solution to the MFE given an initial point. For $\nu \in \mathcal{M}_F(U)$, $\langle \nu, \phi \rangle$ is a continuous linear operator on the space of functions $\phi \in C_b(U)$, we define

$$\|\nu\| = \sup_{\phi \in C_b(U)} \frac{|\langle \nu, \phi \rangle|}{\|\phi\|}. \tag{21}$$

When $\mathcal{H}$ is a Polish space, let $\mathcal{D}_{\mathcal{H}}([0,T])$, $\mathcal{D}_{\mathcal{H}}([0,\infty))$ denote the càdlàg[2] functions that take values in $\mathcal{H}$ defined on $[0,T]$, $[0,\infty)$, respectively. The space of the continuous functions that take values in $\mathcal{H}$ defined on $[0,T]$, $[0,\infty)$ are denoted by $C_{\mathcal{H}}([0,T])$, $C_{\mathcal{H}}([0,\infty))$, respectively. We assume that the spaces $\mathcal{D}_{\mathcal{H}}([0,T])$,$\mathcal{D}_{\mathcal{H}}([0,\infty))$ are equipped with the Skorokhod J_1-topology and in that case, they are Polish spaces. For two local martingales $(M_t^1, t \geq 0)$ and $(M_t^2, t \geq 0)$, we denote the covariation and quadratic variation in $\mathcal{D}_{\mathcal{R}}([0,T])$ by $(< M_\cdot^1, M_\cdot^2 >_t, t \geq 0)$ and $(< M_\cdot^1 >_t, t \geq 0) = (< M_\cdot^1, M_\cdot^1 >_t, t \geq 0)$, respectively.

In this paper, we work with $\mathcal{H}-$valued stochastic processes where $\mathcal{H} = \mathcal{M}_F(U)$. We assume that the considered stochastic processes are random elements defined on $(\Omega, \mathbb{F}, \mathcal{P})$ with sample paths lying in $\mathcal{D}_{\mathcal{H}}([0,\infty))$. The space $\mathcal{D}_{\mathcal{H}}([0,\infty))$ is equipped with the Borel $\sigma-$algebra generated by the open sets under the Skorokhod J_1- topology [4]. A sequence $\{X_n\}$ of $\mathcal{H}$-valued càdlàg processes defined on $(\Omega_n, \mathbb{F}_n, \mathcal{P}_n)$ is said to converge in distribution to a $\mathcal{H}$-valued càdlàg process X defined on $(\Omega, \mathbb{F}, \mathcal{P})$ if, for every bounded, continuous, real valued functional $F : \mathcal{D}_{\mathcal{H}} : [0,\infty) \to \mathcal{R}$, we have

$$\lim_{n\to\infty} \mathbb{E}_n(F(X_n)) = \mathbb{E}(F(X)) \tag{22}$$

[2] Also referred to as RCLL (right continuous with left limits).

where the expectation operators $\mathbb{E}_n, \mathbb{E}$ are defined with respect to $\mathcal{P}_n, \mathcal{P}$, respectively. The convergence of $\{X_n\}$ in distribution to X is denoted by $X_n \Rightarrow X$. Law of a random variable X is denoted by $\mathcal{L}(X)$.

Markovian description: We first introduce the system state descriptor that represents the system state at any given time t. We then describe the system evolution.

In our analysis, we consider age of each progressing job in modeling the system by a Markov process. At any time t, a server's state is represented by $(n, a_1, \cdots, a_n)$ where n denotes the number of progressing jobs at the server at time t and a_i denotes the age of the i^{th} progressing job at time t. If a server has no job at time t, then its state is considered to be (0). We now define the system state descriptor as follows.

Definition 2. *System State Descriptor*
If the state of server "i" at time t is denoted by $s_t^i = (n_i, a_{1,i}, a_{2,i}, \cdots, a_{n_i,i})$, *then the system state at time t is defined by the measure valued process*

$$\nu_t^N = \sum_{i=1}^{N} \delta_{(s_t^i)} \tag{23}$$

where $\delta_{(n,x_1,\cdots,x_n)}$ *denotes the Dirac measure with unit mass at* $(n, x_1, \cdots, x_n)$. *Note that* $\nu_t^N(\{(m, z_1, \cdots, z_m)\})$ *denotes the number of servers with state* $(m, z_1, \cdots, z_m)$ *at time t.*

Once we define the state descriptor, the time evolution of the system is modeled by tracking the time evolution of the the measure-valued process $(\nu_t^N, t \geq 0)$. For given $h > 0$ and ν_t^N, to know the value of ν_{t+h}^N, we need to exactly track how each server state changes in the time interval $(t, t+h]$. In this interval, there can be no event (arrival or departure) or some events(arrivals and departures) can occur in the system. The departure events can be modeled by using hazard rate function $\beta(x) = \frac{g(x)}{\overline{G}(x)}$ that defines the instantaneous rate of departure of a job conditioned on its age value equal to x. Precisely, if a job achieves age x at time t, then it departs in the interval $(t, t+y]$ with probability given by $\frac{G(x+y)-G(x)}{\overline{G}(x)} = \beta(x)y + o(y)$. When the number of progressing jobs does not change at a server with state say $(n, a_1, \cdots, a_n)$ at time t in the interval $(t, t+h]$, then its state becomes $\tau_h^+(n, a_1, \cdots, a_n)$ at time $t+h$. By assuming h is a small value, we consider that multiple events do not occur in the interval $(t, t+h]$.

3 MAIN RESULTS

We now present the main results of this paper.

The goal of this paper is to study the limit of the following normalized process $(\overline{\nu}_t^N, t \geq 0)$ defined as

$$\overline{\nu}_t^N = \frac{\nu_t^N}{N} \tag{24}$$

when $N \to \infty$. Furthermore, we would like to study the fixed-point of the limit to obtain some Engineering insights.

We next state the following simple result on probability of choosing a server with state $(n, l_1, \cdots, l_n)$ as the destination server upon an arrival.

Lemma 1. *Suppose* $\nu_t^N = \eta$, *then according to the SQ(d) routing policy, if a job arrives at time t, the probability that it is routed to a server with state* $(n, l_1, \cdots, l_n)$ *is equal to*

$$p_r(n, l_1, \cdots, l_n; \eta) = \frac{\eta\{(n, l_1, \cdots, l_n)\}}{N} \Phi_n\left(\frac{\eta}{N}\right), \tag{25}$$

where

$$\Phi_n\left(\frac{\eta}{N}\right) = \frac{(\overline{R}_n(\frac{\eta}{N})^d - \overline{R}_{n+1}(\frac{\eta}{N})^d)}{(\overline{R}_n(\frac{\eta}{N}) - \overline{R}_{n+1}(\frac{\eta}{N}))} \tag{26}$$

and $\overline{R}_n(\frac{\eta}{N}) = \frac{\sum_{j=n}^{\infty} \eta(\{U_j\})}{N}$ *represents the fraction of servers with at least n jobs at time t.*

The proof of Lemma 1 is given in Appendix 7.1.

Our analysis shows that under the SQ(d) policy, the probability of choosing a server as destination server given by (25) has the following implications in the limiting system as $N \to \infty$. Once we establish the mean-field $(\overline{\nu}_t, t \geq 0)$, in the limiting system, due to propagation of chaos, each individual server has Poisson arrival process having rate $\lambda \Phi_n(\overline{\nu}_t)$ when there are n progressing jobs at time t.

Before stating our results precisely, we briefly summarize our analysis and findings below.

Sketch of the analysis: *In this paper, we show the weak convergence of the measure-valued Markov process* $(\overline{\nu}_t^N, t \geq 0)$ *to the mean-field limit via the following arguments. We first construct the Dynkin martingale[11] in Theorem 1 associated with the process* $(\nu_t^N, t \geq 0)$. *We then state the result on establishing the mean-field limit in Theorem 2. In the first part of Theorem 2, we show that there exists unique solution to the MFE for given initial point. In the second part of Theorem 2, we show that the process* $(\overline{\nu}_t^N, t \geq 0)$ *converges to the unique solution of the MFE referred as the mean-field solution. For this, we use the constructed martingale Theorem 1 to establish the tightness of the process* $(\overline{\nu}_t^N, t \geq 0)$ *by using Jakubowski's criteria[10]. After that, by showing that the sequence of martingales converge to the null process as* $N \to \infty$ *and the tightness of* $(\overline{\nu}_t^N, t \geq 0)$, *we get that every limit point satisfies the MFE. Further, since there exists unique mean-field solution for given initial point, under the assumption of* $\overline{\nu}_0^N \Rightarrow \overline{\nu}_0$ *where* $\overline{\nu}_0$ *is a deterministic measure in* $\mathcal{M}_1(U)$, *we get that all the limit points have identical distribution coinciding with that of the unique deterministic mean-field solution. This establishes the mean-field limit and as a consequence, we also establish the propagation of chaos for any finite time t in Theorem 4. In order to study the fixed-point of the mean-field, we translate the integral form of the MFE to the PDEs satisfied by the mean-field. Finally, we state our main result on the insensitivity of the fixed-point of the mean-field in Theorem 3. We also provide some insights about the stationary behavior of the limiting system in Section 5 where we also discuss drawbacks of [6] in concluding the insensitivity.*

The starting point of our analysis depends on finding a suitable martingale process that can be used to establish the mean-field limit. For this, the starting step is to characterize the Markov process $(\nu_t^N, t \geq 0)$ and finding its generator $A^N(\cdot)$. We consider the filtration defined by

$$\mathcal{F}_t^N = \sigma(\nu_s^N(B) : s \leq t, B \in \mathcal{B}(U)). \tag{27}$$

Theorem 1. *The process* $(\nu_t^N, t \geq 0)$ *is a Feller-Dynkin process[10, 11] of* $\mathcal{D}_{\mathcal{M}_F(U)}([0, \infty))$. *Let* $\phi \in C_b^1(U)$, *then the process* $(M_t^N(\phi), t \geq$

0) *defined as*

$$M_t^N(\phi) = \langle v_t^N, \phi\rangle - \langle v_0^N, \phi\rangle - \int_{s=0}^t A^N \langle v_s^N, \phi\rangle ds \quad (28)$$

is a square integrable $\mathcal{F}_t^N$-martingale and it is right continuous with left limits (RCLL) process. Further, for $\phi, \psi \in C_b^1(U)$, the mutual variation of $(M_t^N(\phi), t \geq 0)$ with $(M_t^N(\psi), t \geq 0)$ is given by

$$\begin{aligned}< M^N(\phi)_., M^N_.(\psi) >_t = \int_{s=0}^t \Bigg(\sum_{n=1}^\infty \sum_{j=1}^n \int_{x_1} \cdots \int_{x_n} \left(\frac{\beta(x_j)}{n}\right) \\ \times (\phi(n-1, x_1, \cdots, x_{j-1}, x_{j-1}, x_{j+1}, \cdots, x_n) - \phi(n, x_1, \cdots, x_n)) \\ \times (\psi(n-1, x_1, \cdots, x_{j-1}, x_{j-1}, x_{j+1}, \cdots, x_n) - \psi(n, x_1, \cdots, x_n)) \\ \times dv_s^N(n, x_1, \cdots, x_n) \\ + N\lambda \Bigg[\frac{v_s^N(\{0\})}{N} \Phi_0 \left(\frac{v_s^N}{N}\right) (\phi(1,0) - \phi(0))(\psi(1,0) - \psi(0)) \\ + \sum_{n=1}^\infty \sum_{j=1}^{n+1} \int_{x_1} \cdots \int_{x_n} \frac{1}{N(n+1)} \Phi_n \left(\frac{v_s^N}{N}\right) \\ \times (\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \phi(n, x_1, \cdots, x_n)) \\ \times (\psi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \psi(n, x_1, \cdots, x_n)) \\ \times dv_s^N(n, x_1, \cdots, x_n)\Bigg]\Bigg) ds. \quad (29)\end{aligned}$$

.

The proof of Theorem 1 is given in Appendix 7.2.

We now look at establishing the mean-field limit. For this, we require the following assumptions:

Assumption 1. *The hazard rate function $\beta(\cdot)$ satisfies*

$$\beta \in C_b(\mathcal{R}_+) \text{ and } \|\beta\| < \infty. \quad (30)$$

Assumption 2. *The sequence of initial measures of the normalized measure-valued processes $(\overline{v}_t^N, t \geq 0)$ satisfy*

$$(\overline{v}_0^N, \langle \overline{v}_0^N, \Xi\rangle, \langle \overline{v}_0^N, \Upsilon\rangle) \Rightarrow (\overline{v}_0, \langle \overline{v}_0, \Xi\rangle, \langle \overline{v}_0, \Upsilon\rangle) \quad (31)$$

where $\overline{v}_0 \in \mathcal{M}_1(U)$ is absolutely continuous measure satisfying $\langle \overline{v}_0, \Xi\rangle < \infty$ and $\langle \overline{v}_0, \Upsilon\rangle < \infty$.

Definition 3. *Mean-field equations (MFE): The following evolution equations are referred to as the MFE with initial point $\overline{\eta}_0 \in \mathcal{M}_1(U)$ satisfied by a process $(\overline{\eta}_t \in \mathcal{M}_1(U), t \geq 0) \in C_{\mathcal{M}_1(U)}([0, \infty))$, for all $\phi \in C_b(U)$,*

$$\begin{aligned}\langle \overline{\eta}_t, \phi\rangle = \langle \overline{\eta}_0, \tau_t \phi\rangle + \int_{r=0}^t \Bigg(\sum_{n=1}^\infty \sum_{j=1}^n \int_{x_1} \cdots \int_{x_n} \frac{\beta(x_j)}{n} \\ \times (\tau_{t-r}\phi(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) - \tau_{t-r}\phi(n, x_1, \cdots, x_n)) \\ \times d\overline{\eta}_r(n, x_1, \cdots, x_n) \\ + \lambda \Bigg[(\overline{\eta}_r(\{0\}) \Phi_0 (\overline{\eta}_r) (\tau_{t-r}\phi(1,0) - \tau_{t-r}\phi(0))) \\ + \sum_{n=1}^\infty \sum_{j=1}^{n+1} \int_{x_1} \cdots \int_{x_n} \frac{1}{(n+1)} \Phi_n (\overline{\eta}_r) \\ \times (\tau_{t-r}\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \tau_{t-r}\phi(n, x_1, \cdots, x_n)), \\ \times d\overline{\eta}_r(n, x_1, \cdots, x_n)\Bigg]\Bigg) dr, \quad (32)\end{aligned}$$

where $\Phi_n (\overline{\eta}_r) = \frac{(\overline{R}_n(\overline{\eta}_r)^d - \overline{R}_{n+1}(\overline{\eta}_r)^d)}{(\overline{R}_n(\overline{\eta}_r) - \overline{R}_{n+1}(\overline{\eta}_r))}$ *and* $\overline{R}_j(\overline{\eta}_r) = \sum_{n=j}^\infty \overline{\eta}_r(U_n)$.

Any solution of the above equation is referred to as the mean-field solution. Note that since for $\phi \in C_b(U)$, $t \mapsto \langle \overline{\eta}_t, \phi\rangle$ is a continuous mapping and $C_b(U)$ is a separating class of $\mathcal{M}_1(U)$, $t \mapsto \overline{\eta}_t$ is continuous and hence any mean-field solution belongs to the set $C_{\mathcal{M}_1(U)}([0, \infty))$.

We next state our main result of the paper.

Theorem 2. *We establish the following two results*

- *There exists unique solution in $C_{\mathcal{M}_1(U)}([0, \infty))$ satisfying the MFE for given initial point $\overline{v}_0 \in \mathcal{M}_1(U)$. Furthermore, if $(\overline{v}_t^1, t \geq 0)$ and $(\overline{v}_t^2, t \geq 0)$ are two mean-field solutions that start at the initial measures $\overline{v}_0^1 \in \mathcal{M}_1(U), \overline{v}_0^2 \in \mathcal{M}_1(U)$, respectively, then*

$$\|\overline{v}_t^1 - \overline{v}_t^2\| \leq \|\overline{v}_0^1 - \overline{v}_0^2\| \, e^{(2\|\beta\| + 8d^2\lambda)t}. \quad (33)$$

- *If the sequence $\{\overline{v}_0^N\}$ satisfies the assumption 2, we then have for every $T > 0$, $(\overline{v}_t^N, 0 \leq t \leq T) \Rightarrow (\overline{v}_t, 0 \leq t \leq T)$ where the process $(\overline{v}_t, 0 \leq t \leq T)$ is a deterministic process referred to as the mean-field limit is the unique solution of the MFE* (32) *with initial point $\overline{v}_0$.*

The proof of above Theorem 2 is given in Appendix 7.3.

We now look at the probabilistic interpretation to the MFE. By using the propagation of chaos result that we state later in this section, $\overline{v}_t$ represents the distribution of a server state in the limiting system. We next obtain the partial differential equations satisfied by the mean-field limit by using MFE (32). Since $\overline{v}_0$ is absolutely continuous $w.r.t.$ Lebesgue measure, then at every $t \geq 0$, $\overline{v}_t$ is also absolutely continuous resulted from the fact that $\overline{v}_0$ is absolutely continuous and the mapping $t \mapsto \overline{v}_t$ is continuous. Let $p_t(0)$ denotes $\overline{v}_t(\{0\})$ and $p_t(n, x_1, \cdots, x_n)$ denotes the Radon-Nikodym derivative of the measure $\overline{v}_t$ $w.r.t.$ Lebesgue measure at $(n, x_1, \cdots, x_n)$. Let us define the process $P_t = (P_t(\underline{u}), \underline{u} \in U)$ as

$$P_t(n, y_1, \cdots, y_n) = \int_{x_1=0}^{y_1} \cdots \int_{x_n=0}^{y_n} p_t(n, x_1, \cdots, x_n) \, dx_1 \cdots dx_n. \quad (34)$$

Hence $P_t(n, y_1, \cdots, y_n)$ denotes the probability that a server has n jobs and i^{th} job, $1 \leq i \leq n$, has age atmost y_i in the limiting system at time t.

Corollary 1. *The process $P_t = (P_t(\underline{u}), \underline{u} \in U)$ satisfies the PDEs*

$$\frac{dP_t(0)}{dt} = \int_{y=0}^\infty \beta(y) \left(\frac{\partial P_t(1, y)}{\partial y}\right) dy - \lambda \Phi_0(P_t) P_t(0), \quad (35)$$

for $n \geq 1$,

$$\begin{aligned}\frac{dP_t(n, y_1, \cdots, y_n)}{dt} = -\sum_{i=1}^n \frac{1}{n} \frac{\partial P_t(n, y_1, \cdots, y_n)}{\partial y_i} \\ + \sum_{j=1}^{n+1} \int_{x_j=0}^\infty \frac{\beta(x_j)}{n+1} \left(\frac{\partial P_t(n+1, y_1, \cdots, y_{j-1}, x_j, y_j, \cdots, y_n)}{\partial x_j}\right) dx_j\end{aligned}$$

$$- \sum_{j=1}^{n} \int_{x_j=0}^{y_j} \frac{\beta(x_j)}{n} \left(\frac{\partial P_t(n, y_1, \cdots, y_{j-1}, x_j, y_{j+1}, \cdots, y_n)}{\partial x_j} \right) dx_j$$
$$+ \sum_{j=1}^{n} \frac{\lambda \Phi_{n-1}(P_t)}{n} P_t(n-1, y_1, \cdots, y_{j-1}, y_{j+1}, \cdots, y_n)$$
$$- \lambda \Phi_n(P_t) P_t(n, y_1, \cdots, y_n), \quad (36)$$

where $\Phi_n(P_t) = \frac{(\overline{R}_n(P_t)^d - \overline{R}_{n+1}^d(P_t))}{(\overline{R}_n(P_t) - \overline{R}_{n+1}(P_t))}$ and $\overline{R}_n(P_t) = \sum_{j=n}^{\infty} P_t(j, \infty, \cdots, \infty)$ denotes the probability that there are at least n jobs in progress at a server in the limiting system at time t. The proof of above Corollary 1 is given in Appendix 7.6.

Notice that the PDEs (35)-(36) represent the evolution equations of $(P_t(\underline{u}), \underline{u} \in U)$ where $P_t(n, u_1, \cdots, u_n)$ denotes the probability that there are n jobs in progress and i^{th} job has age atmost y_i in a single server processor sharing system in which jobs arrive according to a Poisson process with state-dependent time varying arrival rates $\lambda \frac{(\overline{R}_n(P_t)^d - \overline{R}_{n+1}^d(P_t))}{(\overline{R}_n(P_t) - \overline{R}_{n+1}(P_t))}$ when there are n progressing jobs at the server and the job length distributions are same as given in Section 2. Hence the mean-field PDEs represents the evolution of a non-linear Markov process. We now look at what each term on the right side of equation (36) represent. The first term represents the change in $P_t(n, y_1, \cdots, y_n)$ due to change of the age of each progressing job at the rate of $\frac{1}{n}$ when there are no arrivals or departures. The second term corresponds to the departure event when there are $n+1$ jobs resulting in server state $(n, a_1, \cdots, a_n)$ such that $a_i \leq y_i$, $1 \leq i \leq n$. The departure event when the server state is $(n, a_1, \cdots, a_n)$ such that $a_i \leq y_i$, $1 \leq i \leq n$ is represented by the third term. The fourth term corresponds to the arrival event when the server has $n-1$ jobs. Finally, the fifth term corresponds to the case of arrival event when server state is $(n, a_1, \cdots, a_n)$ such that $a_i \leq y_i$, $1 \leq i \leq n$.

We now state our main result on the insensitivity of the fixed point of the mean-field limit. We first define a class of fixed-points $\mathcal{Y}$ as

$$\mathcal{Y} = \{\boldsymbol{\theta} : \text{ If } \overline{R}_n(\boldsymbol{\theta}) = \sum_{j=n}^{\infty} \theta(n, \infty, \ldots), \text{ we have } \lim_{n \to \infty} \overline{R}_n(\boldsymbol{\theta}) = 0\}. \quad (37)$$

Note that the class $\mathcal{Y}$ contains the fixed-points under which the average queue length is finite

Theorem 3. *There exists a unique fixed-point for the process* $(P_t, t \geq 0) = (P_t(\underline{u}), \underline{u} \in U, t \geq 0)$ *denoted by* $\boldsymbol{\pi}$ *among class of fixed-points* $\mathcal{Y}$ *given by,*

$$\pi(n, y_1, \cdots, y_n) = \pi^{(exp)}(n) \mu^n \prod_{i=1}^{n} \int_{x_i=0}^{y_i} \overline{G}(x_i)\, dx_i. \quad (38)$$

where $\boldsymbol{\pi}^{(exp)} = (\pi^{(exp)}(n), n \geq 0)$ *denotes the unique fixed-point of the mean-field limit when job lengths are exponentially distributed with mean* $\frac{1}{\mu}$ *and* $\pi^{(exp)}(n)$ *is the stationary probability that there are* n *jobs in the limiting system at a server. Furthermore, as* $\int_{x=0}^{\infty} \overline{G}(x)\, dx = \frac{1}{\mu}$, *the fixed-point is insensitive since*

$$\pi(n, \infty, \cdots, \infty) = \pi^{(exp)}(n). \quad (39)$$

The proof of Theorem 3 is given in section 4.

Remark 1. *For any closed or open subset* $B \in U$, *by having* $\overline{\nu}_t^N \Rightarrow \overline{\nu}_t$, *since* $\overline{\nu}_t$ *is absolutely continuous w.r.t. Lebesgue measure for every* $t \geq 0$, *we then have by using continuous mapping theorem that* $\langle \overline{\nu}_t^N, I_{\{B\}} \rangle \Rightarrow \langle \overline{\nu}_t, I_{\{B\}} \rangle$. *This means that for large N, we can approximate* $\langle \overline{\nu}_t^N, I_{\{B\}} \rangle$ *by* $\langle \overline{\nu}_t, I_{\{B\}} \rangle$. *In particular the tail distributions are obtained by taking* $B = \bigcup_{j \geq n} U_j$, $n \geq 1$. *These results are reported in [1] for the case of FCFS queueing models.*

Propagation of chaos: The existence of the mean-field limit allows us to show that any finite subset of servers become independent of each other in the limiting system. We first define the needed notation below.

- Let the state of the k^{th} server at finite time $t \geq 0$ be denoted by the random variable $q_k^{(N)}(t) \in U$.
- Assuming the assumptions 1-2 are true, from Theorem 2, we denote the mean-field limit by $(\overline{\nu}_t, t \geq 0)$ defined on U.

Definition 4. *: Let* $\{S_k^{(N)}, 1 \leq k \leq N\}$ *denote a collection of N random variables. Then the collection is called exchangeable if the joint law of collection is invariant under any permutation of indices,* $1 \leq k \leq N$, *of random variables.*

Theorem 4. *If* $\{q_k^{(N)}(0), 1 \leq k \leq N\}$ *are exchangeable and if the assumptions 1-2 are true, then the following holds*

- *For each fixed k and* $t \in [0, \infty)$, $\mathcal{L}(q_k^{(N)}(t)) \Rightarrow \overline{\nu}_t$ *as* $N \to \infty$.
- *For any fixed positive integer l and for each* $t \in [0, \infty)$, *we have* $\{q_k^{(N)}(t), 1 \leq k \leq l\} \Rightarrow \{V_k(t), 1 \leq k \leq l\}$ *as* $N \to \infty$, *where* $V_k(t)$, $1 \leq k \leq l$ *are independent random variables with* $\mathcal{L}(V_k(t))$ *is equal to* $\overline{\nu}_t$ *for all* $1 \leq k \leq l$.

The proof of Theorem 4 is given in Appendix 7.8.

4 INSENSITIVITY: PROOF OF THEOREM 3

Proof. Now let us look at the uniqueness of the fixed-point $\boldsymbol{\pi} = (\pi(\underline{u}), \underline{u} \in U)$ of the mean-field limit and its insensitivity. Let $\boldsymbol{\theta} = (\theta(\underline{u}), \underline{u} \in U)$ be a fixed-point for the process $(P_t, t \geq 0)$. We first show that any fixed-point $\boldsymbol{\theta}$ must satisfy

$$\theta(n, y_1, \ldots, y_n) = \frac{\left(\prod_{i=1}^{n} \frac{\lambda_{i-1}^{(GEN)}(\boldsymbol{\theta})}{\mu}\right)}{1 + \sum_{m=1}^{\infty} \left(\prod_{i=1}^{m} \frac{\lambda_{i-1}^{(GEN)}(\boldsymbol{\theta})}{\mu}\right)} \mu^n \prod_{i=1}^{n} \int_{x_i=0}^{y_i} \overline{G}(x_i)\, dx_i \quad (40)$$

and

$$\theta(0) = \frac{1}{1 + \sum_{m=1}^{\infty} \left(\prod_{i=1}^{m} \frac{\lambda_{i-1}^{(GEN)}(\boldsymbol{\theta})}{\mu}\right)} \quad (41)$$

where $\lambda_n^{(GEN)}(\boldsymbol{\theta}) = \lambda \frac{\overline{R}_n(\boldsymbol{\theta})^d - \overline{R}_{n+1}(\boldsymbol{\theta})^d}{\overline{R}_n(\boldsymbol{\theta}) - \overline{R}_{n+1}(\boldsymbol{\theta})}$. Since $0 \leq \overline{R}_n(\theta) \leq 1$ for $n \geq 0$, $\overline{R}_n(\boldsymbol{\theta}) \geq \overline{R}_{n+1}(\boldsymbol{\theta})$ and $\lim_{n \to \infty} \overline{R}_n(\theta) = 0$, it is verified that we have $\sum_{m=1}^{\infty} \prod_{i=1}^{m} \frac{\lambda_{i-1}^{(GEN)}(\boldsymbol{\theta})}{\mu} < \infty$.

We now draw an analogy between the single server system with pre-specified state dependent arrival rates and the mean-field limit. This analogy is used in the proof. We first recall in Appendix 7.7, the dynamics of the probabilities of server state of a single server processor sharing system in which the job arrival process is a Poisson process with pre-specified state-dependent arrival rates.

On comparing the equation (157) satisfied by the mean-field limit with the single server Kolomogorov equation (186), it is clear that both the dynamics are identical except that α_i in equation (186) is replaced by $\lambda \frac{(\overline{R}_n(\overline{\nu}_s)^d - \overline{R}_{n+1}(\overline{\nu}_s)^d)}{(\overline{R}_n(\overline{\nu}_s) - \overline{R}_{n+1}(\overline{\nu}_s))}$ when the probability measure for server occupancies is $\overline{\nu}_s$ at time s. Hence we have that the equation (186) represents the evolution of a linear Markov process whereas equation (157) represents the evolution of a non-linear Markov process.

We use contradiction arguments to establish equations (40)-(41). Let $\boldsymbol{\gamma}$ be a fixed-point that does not satisfy equations (40)-(41). Using this fixed-point $\boldsymbol{\gamma}$, we first compute the set of arrival rates $(\lambda_i^{(GEN)}(\boldsymbol{\gamma}), i \geq 0)$. Now let us consider a single server processor sharing system where pre-specified state-dependent arrival rate is equal to $\lambda_i^{(GEN)}(\boldsymbol{\gamma})$ when there are i jobs in progress and the job length distributions are same as given in the system model. The expression for the unique stationary distribution is given by equation (190) where we replace α_i by $\lambda_i^{(GEN)}(\boldsymbol{\gamma})$. On comparing the stationary evolution equations corresponding to single server dynamics given in equations (188)-(189) and the mean-field dynamics given in equations (35)-(36), we have that $\boldsymbol{\gamma}$ is also another stationary distribution for single server system with pre-specified arrival rates $(\lambda_i^{(GEN)}(\boldsymbol{\gamma}), i \geq 0)$. This contradicts the result established in [9] that there exists unique stationary distribution for single server system with pre-specified state-dependent arrival rates. Hence the equations (40)-(41) must be true.

Now let $\Gamma = (\Gamma_n, n \geq 0)$ is defined such that $\Gamma_n = \theta(n, \infty, \cdots, \infty))$ and $\Gamma_0 = \theta(0)$. We then have from equations (40)-(41),

$$\Gamma_n = \frac{\left(\prod_{i=1}^{n} \frac{\lambda_{i-1}^{(exp)}(\Gamma)}{\mu}\right)}{1 + \sum_{m=1}^{\infty} \left(\prod_{i=1}^{m} \frac{\lambda_{i-1}^{(exp)}(\Gamma)}{\mu}\right)} \tag{42}$$

and

$$\Gamma_0 = \frac{1}{1 + \sum_{m=1}^{\infty} \left(\prod_{i=1}^{m} \frac{\lambda_{i-1}^{(exp)}(\Gamma)}{\mu}\right)} \tag{43}$$

where

$$\lambda_n^{(exp)}(\Gamma) = \lambda \frac{(\sum_{j=n}^{\infty} \Gamma_j)^d - (\sum_{j=n+1}^{\infty} \Gamma_j)^d}{(\sum_{j=n}^{\infty} \Gamma_j) - (\sum_{j=n+1}^{\infty} \Gamma_j)}. \tag{44}$$

We also have

$$\lambda_n^{(exp)}(\Gamma)\Gamma_n = \mu \Gamma_{n+1}. \tag{45}$$

From [23], the only probability measure satisfying equations (44)-(45) is the unique fixed-point $\boldsymbol{\pi}^{(exp)}$ of the mean-field limit when job lengths are exponentially distributed with mean $\frac{1}{\mu}$. Hence from equations (40) and (41), every fixed point $\boldsymbol{\theta}$ satisfies,

$$\Gamma_n = \theta(n, \infty, \cdots, \infty) = \pi^{(exp)}(n). \tag{46}$$

This concludes the insensitivity of the fixed-point of the mean-field limit. By using equation (40), every fixed point $\boldsymbol{\theta}$ satisfies

$$\theta(n, y_1, \ldots, y_n) = \pi^{(exp)}(n)\mu^n \prod_{i=1}^{n} \int_{x_i=0}^{y_i} \overline{G}(x_i)\, dx_i \tag{47}$$

This concludes that the fixed-point is unique since $\boldsymbol{\pi}^{(exp)}$ is unique. □

5 ON THE STATIONARY REGIME

In this section, we provide some numerical results to support insensitivity and the global asymptotic stability (GAS) of the fixed-point of the mean-field by numerically evaluating the MFE when job lengths have mixed-Erlang distributions. Further, we also discuss later on the propagation of chaos in the stationary regime and recollect some existing relevant works. If one can prove the GAS of the mean-field, then we can exploit Prokhorov's theorem to conclude the convergence of the stationary distribution for a server occupancy of a finite N system to the fixed-point of the mean-field[23]. Proving the GAS of the mean-field is extremely difficult since the mean-field does not possess any monotonicity properties when job lengths are generally distributed unlike the exponential case[22, 23]. Recently, the case of loss models has been considered in [31] under the assumption of mixed-Erlang distributions where the existence, uniqueness and insensitivity of the fixed-point of the mean-field was shown but the GAS of the mean-field is not shown and was only studied numerically. Here also we provide simulation results that support the convergence of the stationary distribution for a server occupancy of a finite N system as $N \to \infty$ to the fixed-point of the mean-field. As a result, since our analysis proves the insensitivity of the fixed-point of the mean-field, it supports the insensitivity of the stationary distribution of the limiting system as $N \to \infty$.

From a computational point we consider the numerical evaluation of the MFE when job length distributions are mixed-Erlang using Euler's method with step size of 2×10^{-3}. From the case of exponential distributions we know that the stationary probability that there are atleast k jobs at a server under the SQ(d) policy in the limiting system is given by $(\frac{\lambda}{\mu})^{\frac{d^k - 1}{d-1}}$ [32] for given values of $\frac{\lambda}{\mu}(< 1)$. We assume that the servers have a finite buffer size of C chosen such that $(\frac{\lambda}{\mu})^{\frac{d^C - 1}{d-1}}$ is negligible. We consider the system parameters as follows. The job length distributions have Mixed-Erlang distributions under which a job length is sampled with probability p_i ($i \in \{1, 2, \ldots, M\}$) from an Erlang distribution having i exponential phases with rate μ_p. As a consequence, from the average job length, we have

$$\frac{1}{\mu} = \frac{\sum_{i=1}^{M} i p_i}{\mu_p}. \tag{48}$$

Let us define the state of a server with n progressing jobs having l_j phases remaining for j^{th} progressing job by $\underline{l} = (n, l_1, \ldots, l_n)$ with $1 \leq l_j \leq M$, $1 \leq j \leq n$. For $n \geq 1$, let $S_n = \{(n, l_1, \ldots, l_n) : 1 \leq l_i \leq M, 1 \leq i \leq n\}$ be the set of all possible states of a server when there are n progressing jobs and $S_0 = \{(0)\}$ denotes the state of a server when there are no progressing jobs. We then define S to be the set of all possible server states given by

$$S = \cup_{n=0}^{C} S_n. \tag{49}$$

We can model the system evolution by using a Markov process $\mathbf{x}^N(t) = (x_{\underline{l}}^N(t), \underline{l} \in S)$ where $x_{\underline{l}}^N(t)$ denotes the fraction of servers lying in state $\underline{l}$ at time t. Since the underlying space S is countable, the mean-field limit can be established by the same procedure as that of the exponential case[23]. Hence we claim the following result and the proof is omitted.

Claim 1. *If $\mathbf{x}^N(0)$ converges in distribution to a state $\mathbf{u}$, then the process $\mathbf{x}^N(\cdot)$ converges in distribution to a deterministic process*

$\mathbf{x}(\cdot,\mathbf{u})$ *as* $N \to \infty$ *referred to as the mean-field. The process* $\mathbf{x}(\cdot,\mathbf{u})$ *is the unique solution of the following system of ordinary differential equations.*

$$\mathbf{x}(0,\mathbf{u}) = \mathbf{u}, \tag{50}$$

$$\dot{x}_{\underline{l}}(t,\mathbf{u}) = h_{\underline{l}}(\mathbf{x}(t,\mathbf{u})), \tag{51}$$

and $\mathbf{h} = (h_{\underline{l}}, \underline{l} \in S)$ *with the mapping* $h_{\underline{l}}$ *given by*

$$\begin{aligned} h_{(n,l_1,\cdots,l_n)}(\mathbf{x}) = &\sum_{b=1}^{n} \left(\frac{p_{l_b}}{n}\right) x_{(n-1,l_1,l_2,\ldots,l_{b-1},l_{b+1},\ldots,l_n)}(t) \\ &\times \lambda_{n-1}^{(ME)}(\mathbf{x}) - x_{\underline{l}}(t)\lambda_n^{(ME)}(\mathbf{x}) I_{\{n<C\}} \\ &+ \sum_{b=1}^{n+1} \frac{\mu_p}{n+1} I_{\{n<C\}} x_{(n+1,l_1,\ldots,l_{b-1},1,l_b,\ldots,l_n)}(t) \\ + \sum_{b=1}^{n} \frac{\mu_p}{n} & x_{(n,l_1,\ldots,l_{b-1},l_b+1,l_{b+1},\ldots,l_n)}(t) - \mu_p x_{(n,l_1,\cdots,l_n)}(t), \end{aligned} \tag{52}$$

where

$$\lambda_n^{(ME)}(\mathbf{v}) = \frac{\lambda}{(\sum_{\underline{l}\in S_n} v_{\underline{l}})} \left[\left(\sum_{i=n}^{C} \sum_{\underline{l}\in S_i} v_{\underline{l}} \right)^d - \left(\sum_{j=n+1}^{C} \sum_{\underline{l}\in S_j} v_{\underline{l}} \right)^d \right]. \tag{53}$$

We now numerically evaluate the MFE by choosing the following parameters: $d = 2$, $\mu = 1$, $C = 7$, $M = 2$, $p_1 = 0.4$ and $p_2 = 0.6$.

The unique fixed-point $\pi = (\pi_{\underline{l}}, \underline{l} \in S)$ is given by

$$\pi_{(n,l_1,\cdots,l_n)} = \pi^{(exp)}(n) \prod_{i=1}^{n} \left(\frac{\sum_{j=l_i}^{M} p_j}{\sum_{r=1}^{M} r p_r} \right), \tag{54}$$

where $\pi^{(exp)} = (\pi^{(exp)}(n), 0 \le n \le C)$ is the unique fixed-point of the mean-field in exponential case.

In Figure 1, we plot $d_{tv}(\mathbf{x}(t,\mathbf{u}),\pi)$ as a function of t where d_{tv} is the total variation distance defined by

$$d_{tv}(\mathbf{a},\mathbf{b}) = \sum_{\underline{l}\in S} \left| a_{\underline{l}} - b_{\underline{l}} \right|. \tag{55}$$

In Figure 2, by defining a process $y(t,\mathbf{v}) = (y_n(t,\mathbf{v}), 0 \le n \le C)$ referred to as the tail mean-field that satisfies $y(0,\mathbf{v}) = \mathbf{v}$ and $y_j(t,\mathbf{v}) = \sum_{i=j}^{C} \sum_{\underline{l}\in S_i} x_{\underline{l}}(t,\mathbf{u})$, we plot $\vartheta_{tv}(\mathbf{y}(t,\mathbf{v}),\pi^*)$ as a function of t where π^* is the fixed-point of $y(t,\mathbf{v})$ and ϑ_{tv} is the total variation distance defined by

$$\vartheta_{tv}(\mathbf{w},\mathbf{z}) = \sum_{0\le n\le C} |w_n - z_n|. \tag{56}$$

It is clear from Figure 1 and Figure 2 that the mean-field $x(t,\mathbf{u})$ and its tail mean-field $y(t,\mathbf{v})$ converge to their fixed-points for three different initial points when λ takes 0.7 and 0.9. This supports that the mean-field $x(t,\mathbf{u})$ and $y(t,\mathbf{v})$ are globally stable.

From Figure 2, it is clear that $\vartheta_{tv}(y(t,\mathbf{v}),\pi^*)$ the total variation distance between the tail mean-field $y(t,\mathbf{v})$ and its fixed-point π^* is not monotonically decreasing. Further, let $\vartheta_E(y(t,\mathbf{v}),\pi^*)$ be the euclidean distance between $y(t,\mathbf{v})$ and π^* defined by

$$\vartheta_E(\mathbf{w},\mathbf{z}) = \sqrt{\sum_{0\le n\le C} |w_n - z_n|^2}. \tag{57}$$

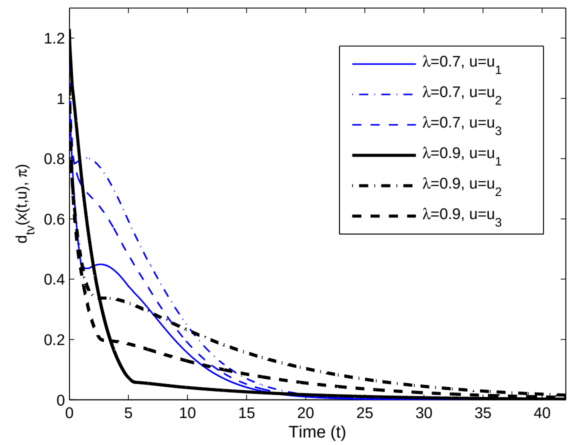

Figure 1: Convergence of the mean-field to its fixed-point

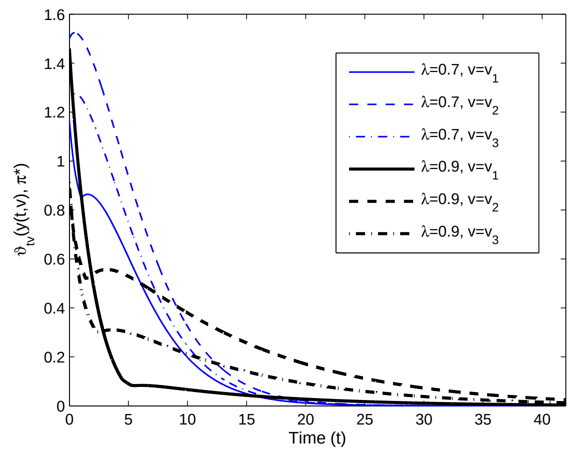

Figure 2: Convergence of the tail mean-field to its fixed-point

Then from Figure 3, $\vartheta_E(y(t,\mathbf{v}),\pi^*)$ is also not decreasing monotonically. The case with $\lambda = 0.9$ and $v = v_2$ for the region where $\vartheta_E(y(t,\mathbf{v}),\pi^*)$ is increasing is shown in Figure 4. Therefore from Figure 2 and Figure 3, both the total variation distance and the euclidean distance cannot be used for constructing a Lyapunov function to show the GAS of the tail mean-field.

We now present the simulation results that support insensitivity of the fixed-point of the mean-field limit and the convergence of the stationary distribution of a finite N system to the fixed-point of the mean-field as $N \to \infty$ for various class of job length distributions. Let $\theta^N = (\theta_i^N, i \ge 1)$ such that θ_i^N denotes the stationary probability that there are atleast i jobs in progress at a server in the system with N servers in which arriving jobs are routed according to the SQ(d) policy. In our simulation results, when we implement the SQ(d) policy, d servers are sampled without replacement whereas

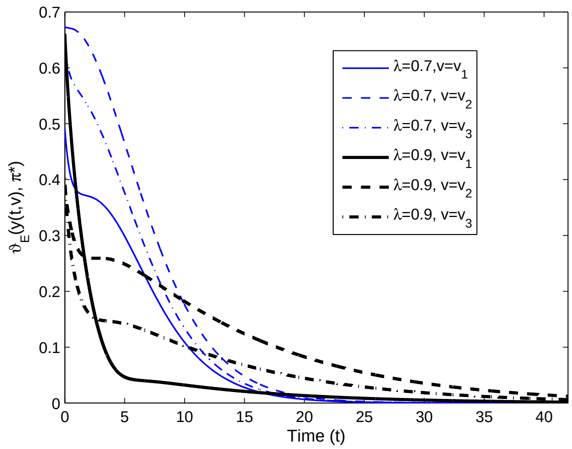

Figure 3: Convergence of the tail mean-field to its fixed-point

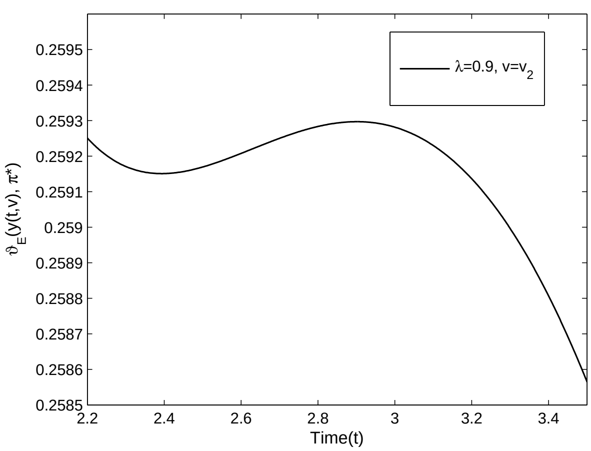

Figure 4: Convergence of the tail mean-field to its fixed-point

in our analysis we assume that servers are sampled with replacement. By using PASTA property, we compute θ^N for various type of job length distributions by simulating the system up to 2000000 job arrivals. Let $\theta^{(exp)} = (\theta_i^{(exp)}, i \geq 0)$ be the fixed point of the mean-field limit of the tail queue length process, then as shown in [32],

$$\theta_i^{(exp)} = \left(\frac{\lambda}{\mu}\right)^{\frac{d^i-1}{d-1}}. \tag{58}$$

We now compute the total variation distance between θ^N and $\theta^{(exp)}$ defined as

$$\vartheta_{tv}(\theta^N, \theta^{(exp)}) = \sum_i \left|\theta_i^N - \theta_i^{(exp)}\right|. \tag{59}$$

We assume that the parameters λ, μ, d are fixed at 0.7, 1, and 2, respectively. The different types of job length distributions that we consider are exponential (Exp), constant (Const), power-law (PL), and mixed-Erlang (ME) distributions. The power-law distribution has CDF $G(y) = 1 - \frac{1}{3y^{\frac{3}{2}}}$ for $y \geq \frac{1}{3}$ and zero otherwise. In the mixed-Erlang case, the distribution has i ($1 \leq i \leq 2$) exponential phases with probability p_i and each exponential phase has rate μ_p. We choose $p_1 = .4, p_2 = 0.6$ and μ_p is chosen by looking at the formula of average job length given by

$$\sum_{i=1}^{2} \frac{ip_i}{\mu_p} = \frac{1}{\mu}. \tag{60}$$

It is clear from Table 1 that for large N system, θ^N for different job length distributions having same average job length can be approximated by the fixed-point of the mean-field limit under exponential job length distribution having the same average job length. This supports insensitivity and the global asymptotic stability of the mean-field limit.

Table 1: $\vartheta_{tv}(\theta^N, \theta^{(exp)})$ for different job length distributions

N	Exp	$Const$	PL	ME
10	0.0424	0.0409	0.0419	0.0421
50	0.0078	0.0068	0.0072	0.0077
100	0.0060	0.0037	0.0071	0.0038
300	0.0012	0.0016	0.0016	0.0017

The SQ(d) policy even for small value of d improves the system performance significantly. We plot the average sojourn time ($E(T_s)$) of a job under the SQ(d) or Power-of-d policy and the random routing scheme ($d = 1$) in Figure 5. The expression for $E(T_s)$ under the SQ(d) or Power-of-d policy is given by

$$E(T_s) = \sum_{i \geq 1} \theta_i^{exp} \tag{61}$$

and for the random routing, we have

$$E(T_s) = \sum_{i \geq 1} \left(\frac{\lambda}{\mu}\right)^i. \tag{62}$$

We also plot the simulation results in Figure 5 by considering a system with $N = 100$ and exponential job length distributions. It is clear that the SQ(d) policy reduces the average sojourn time significantly over the random routing policy ($d = 1$).

5.1 On the propagation of chaos in the stationary regime:

We now discuss the relationship between the propagation of chaos in the stationary regime, the tightness of $(\boldsymbol{\pi}^N)_N$, and the GAS of the equilibrium of the mean-field. For simplicity, we assume that the job length distributions are mixed-Erlang and each server has finite buffer C. When $\frac{\lambda}{\mu} < 1$, a finite N system is stable[5], and hence there exists a unique invariant distribution $\boldsymbol{\pi}^N$ for the Markov process $\mathbf{x}^N(t) = (x_{\underline{l}}^N(t), \underline{l} \in S)$. In this case, the mean-field equations are given by equations (50)-(52). However the system does not exhibit

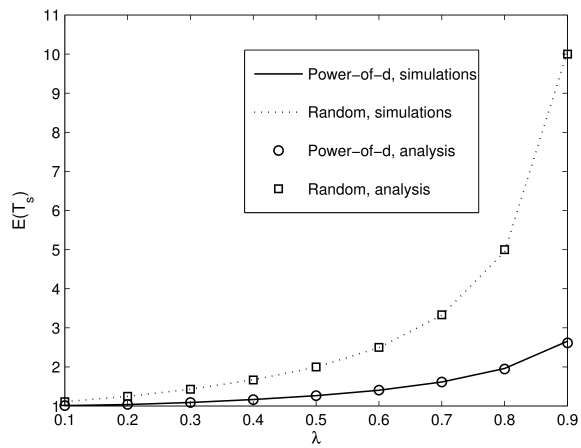

Figure 5: The average sojourn time ($E(T_s)$) versus λ

monotonicity properties unlike the simple exponential case and thus establishing propagation of chaos is a challenging problem as noted in [7]. When the GAS of the mean-field is true, by invoking Prokhorov's theorem we can establish that $\boldsymbol{\pi}^N \Rightarrow \delta_{\boldsymbol{\pi}}$ where $\boldsymbol{\pi}$ is the fixed-point of the mean-field[23]. This implies the validity of the interchange of limits

$$\lim_{N\to\infty}\lim_{t\to\infty} x_t^N = \lim_{t\to\infty}\lim_{N\to\infty} x_t^N. \tag{63}$$

Further, this would then imply that propagation of chaos holds in the stationary regime[2, 23]. Thus it appears that GAS, propagation of chaos, and the coincidence of the stationary distribution and the fixed point of the mean field are all inter-related. We now discuss what happens when we cannot show GAS of the mean-field.

Since the space of probability measures on S denoted by $\mathcal{M}_1(S)$ with metric induced by total variation distance is compact, from Prokhorov's theorem[4], the sequence $(\boldsymbol{\pi}^N)_N$ is tight in $\mathcal{M}_1(\mathcal{M}_1(S))$ under the topology induced by the weak convergence. Let $(\boldsymbol{\pi}^{N_k})_k$ be a converging subsequence with limiting point $Z \in \mathcal{M}_1(\mathcal{M}_1(S))$. We say that for the sequence of systems with index $(N_k)_k$, the limiting system is said to have stationary distribution Z for the stationary empirical random variable. Then form Theorem 1 of [2], Z is an invariant distribution of the mean-field $\mathbf{x}(t,\cdot)$ that means

$$\int_{\mathcal{M}_1(S)} f(\mathbf{x}(t,\mathbf{u}))dZ(\mathbf{u}) = \int_{\mathcal{M}_1(S)} f(\mathbf{u})dZ(\mathbf{u}). \tag{64}$$

Furthermore, from Theorem 3 of [3], the support of Z is a compact set included in the Birkhoff center of the mean-field where the Birkhoff center is the closure of the set of recurrent points. Hence the Birkhoff center includes the existing limit cycles, fixed-points of the mean-field.

Let $q_i^{N_k}(\infty)$ be the random variable that denotes the state of server i in the stationary regime in a finite N_k system. Let $V^{N_k}(\infty)$, $V(\infty)$ be random variables with distribution $\boldsymbol{\pi}^{N_k}$ and Z, respectively. Note that since the system behavior is symmetric to servers as servers' labels do not play any role, the set $(q_i^{N_k}(\infty), 1 \le i \le N_k)$ is exchangeable irrespective of the initial conditions on $(q_i^{N_k}(t), 1 \le i \le N_k)$ in the transient regime. Let us consider continuous bounded mappings $\phi_i : S \to \mathcal{R}_+$, $1 \le i \le l$.

Theorem 5. *If $\boldsymbol{\pi}^{N_k} \Rightarrow Z$, then*

$$\mathbb{E}\left[\prod_{i=1}^{l}\phi_i(q_i^{N_k}(\infty))\right] \to \mathbb{E}\left[\prod_{i=1}^{l}\langle V(\infty),\phi_i\rangle\right] \tag{65}$$

as $k \to \infty$. Any finite set of servers $(n_i)_{1\le i\le l}$ in the limiting system of the sequence $(\boldsymbol{\pi}^{N_k})_k$ are mutually independent iff Z is a Dirac measure. Furthermore, if $Z = \delta_a$ for some $a \in \mathcal{M}_1(S)$, then each server state is a random variable with distribution a.

Proof. We can write

$$\begin{aligned}
&\left|\mathbb{E}\left[\prod_{i=1}^{l}\phi_i\left(q_i^{N_k}(\infty)\right)\right] - \mathbb{E}\left[\prod_{i=1}^{l}\langle V(\infty),\phi_i\rangle\right]\right| \\
&\quad\le \left|\mathbb{E}\left[\prod_{i=1}^{l}\phi_i\left(q_i^{N_k}(\infty)\right)\right] - \mathbb{E}\left[\prod_{i=1}^{l}\langle V^{N_k}(\infty),\phi_i\rangle\right]\right| \\
&\quad\quad + \left|\mathbb{E}\left[\prod_{i=1}^{l}\langle V^{N_k}(\infty),\phi_i\rangle\right] - \mathbb{E}\left[\prod_{i=1}^{l}\langle V(\infty),\phi_i\rangle\right]\right|.
\end{aligned} \tag{66}$$

Note that since $V^{N_k}(\infty) \Rightarrow V(\infty)$, the second term on the right hand side of the above inequality vanishes as $N_k \to \infty$. Now, due to exchangeability, the permutation of states between servers does not affect the joint distribution. Hence, we have

$$\mathbb{E}\left[\prod_{i=1}^{l}\phi_i\left(q_i^{N_k}(\infty)\right)\right] = \frac{1}{(N_k)_l}\mathbb{E}\left[\sum_{\sigma\in Q(l,N_k)}\prod_{i=1}^{l}\phi_i\left(q_{\sigma(i)}^{N_k}(\infty)\right)\right] \tag{67}$$

where $(N)_j = N(N-1)\dots(N-j+1)$, and $Q(r,n)$ denotes the set of all permutations of the numbers $\{1,2,\dots,n\}$ taken r at a time. Also, by definition of $V^{N_k}(\infty)$ we have

$$\mathbb{E}\left[\prod_{i=1}^{l}\langle V^{N_k}(\infty),\phi_i\rangle\right] = \mathbb{E}\left[\left(\prod_{i=1}^{l}\frac{1}{N_k}\sum_{j=1}^{N_k}\phi_i\left(q_j^{(N_k)}(\infty)\right)\right)\right] \tag{68}$$

Hence, the first term on the right hand side of (193) can be bounded as follows

$$\begin{aligned}
&\left|\mathbb{E}\left[\prod_{i=1}^{l}\phi_i\left(q_i^{N_k}(\infty)\right)\right] - \mathbb{E}\left[\prod_{i=1}^{l}\langle V^{N_k}(\infty),\phi_i\rangle\right]\right| \\
&\le 2B^l\left(1-\frac{(N_k)_l}{(N_k)^l}\right) \qquad \to 0 \text{ as } N_k \to \infty
\end{aligned}$$

where $\max_i \|\phi_i\| = B$.

Finally, from equation (65), any finite set of servers are independent of each other iff Z is a Dirac measure. Otherwise they are coupled through the sample value of the random variable $V(\infty)$. If $Z = \delta_a$, then it implies that in the limiting system the stationary empirical random variable $V(\infty)$ is a deterministic value coinciding with a. Then the following equation concludes that each server has distribution a

$$\mathbb{E}\left[\prod_{i=1}^{l}\phi_i(q_i^{N_k}(\infty))\right] \to \prod_{i=1}^{l}\langle a, \phi_i\rangle \tag{69}$$

as $N_k \to \infty$. This completes the proof. □

Since Z is an invariant distribution of the mean-field $\mathbf{x}(t, \cdot)$, from equation (64)

$$\mathbb{E}\left[\prod_{i=1}^{l}\phi_i(q_i^{N_k}(\infty))\right] \to \int_{\mathbf{u}\in\mathcal{M}_1(S)}\left(\prod_{i=1}^{l}\langle\mathbf{x}(t,\mathbf{u}), \phi_i\rangle\right) dZ(\mathbf{u}) \tag{70}$$

as $N_k \to \infty$. The equation (70) implies that in the stationary regime, at any time t, servers are coupled through the position of the mean-field $\mathbf{x}(t, \cdot)$ which is a random element since its initial point is random with distribution Z. Furthermore, in the limiting system, at any instant t in the stationary regime, each servers' state is a random variable with distribution coinciding with the position of the mean-field. For example, if support of Z contains limit cycles or multiple fixed-points of the mean-field, then at any instant in the stationary, the position of the mean-field is random as a result, any finite set of servers are coupled.

5.2 Discussion on prior work in [6]:

In the literature, the only work that claims to prove the insensitivity of the stationary distribution of the limiting system is [6] based on an *ansatz* that we recall below. Using Theorem 5, we demonstrate that the *ansatz* in [6] carries the necessary and sufficient conditions required to establish insensitivity of the limiting system. As a result, the insensitivity of the stationary distribution of a server state in the limiting system is an immediate consequence. Infact, the *ansatz* is the result that we aim to establish in studying the large-scale systems under randomized load balancing in order to understand the impact of the load balancing policy on system performance by using the stationary distribution in the limiting system.

Let $Q^N(t) = (r^{1,N}(t), r^{1,N}(t), \cdots, r^{N,N}(t))$ is the joint queue-size process at time t where $r^{i,N}(t)$ (notation $q^{i,n}(t)$ is used in [6]) denotes the number of jobs at server i at time t. Let Γ^N (Γ is replaced with $\boldsymbol{\pi}$ in [6]) be the stationary distribution of $Q^N(t)$. We now recall exactly the *ansatz* stated in [6].

Ansatz in [6]:
Demonstrate $(\Gamma^N) \Rightarrow (\Gamma)$ *as* $N \to \infty$, *where* Γ *is a stationary and ergodic measure on* $\mathcal{Z}_+^\infty$. *Show that the limit* Γ *is unique, depending only on the service distribution, service discipline and load balancing rule. Let* $\Gamma^{(k)}$ *be the restriction of* Γ *to its first* k *coordinates, with* $\gamma = \Gamma^{(1)}$ *being the one-dimensional marginal of* Γ. *Show that, for every* k,

$$\Gamma^{(k)} = \otimes_{i=1}^{k}\gamma. \tag{71}$$

Let $\overline{Q}^N(\infty) = (\overline{Q}_i^N(\infty), 0 \le i \le C)$ where $\overline{Q}_i^N(\infty)$ denotes the random variable in the stationary regime indicating the faction of servers with i jobs. Then from Theorem 5 (also Proposition 2.1 of [12]), $\overline{Q}^N(\infty) \Rightarrow \gamma$ where γ is a deterministic measure in $\mathcal{M}_1(S)$. Since the $SQ(d)$ policy uses the queue-size information of a finite set of d randomly sampled servers that have independent and identical distributions coinciding with γ, the arrival process is a Poisson process to any particular server which is a necessary condition to have insensitivity in processor sharing systems. Furthermore, the arrival process to each server is a state-dependent Poisson arrival process with rate $\lambda_k = \lambda\frac{(\sum_{j=k}^{C}\gamma_j)^d - (\sum_{j=k+1}^{C}\gamma_j)^d}{\gamma_k}$ when there are k jobs at the server. Therefore the set of arrival rates $\Lambda = (\lambda_k, 0 \le k \le C)$ can be written as a function of γ as

$$\Lambda = F_1(\gamma). \tag{72}$$

Further, for given set of arrival rates Λ, the stationary distribution for a server occupancy in the limiting system can be written through a mapping F_2 as

$$\gamma = F_2(\Lambda). \tag{73}$$

Therefore γ must be an unique fixed-point of the mapping $F_2(F_1)$ for the case of general job length distributions which is not shown in [6] except for the case of FIFO queues with service time distributions having decreasing hazard rate functions. To have insensitivity, γ must be same for all the general job length distributions having same average job length. In [6], from uniqueness of γ in *ansatz*, insensitivity is concluded from reversibility since arrival process to each server is a state-dependent Poisson arrival process. Note that since the mappings F_2, F_1 are same for both exponential and general distributions, the uniqueness of the fixed-point of $F_2(F_1)$ follows from the GAS of the mean-field in exponential case. Therefore the fixed-point of the mapping $F_2(F_1)$ is same for both exponential and general job-length distributions when they have same average job lengths. Since *ansatz* implies the Poisson arrival process to servers and uniqueness of the stationary distribution in the limiting system, the insensitivity follows immediately. However, the proof of *ansatz* remains an open problem and has been shown only for the case of FIFO queuing models with service time distributions having decreasing hazard rate functions in [7].

6 CONCLUSIONS AND FUTURE WORK

In this paper we have obtained the mean-field limit for PS systems under SQ(d) routing and have shown that its equilibrium point is *unique and insensitive*. In [6], the analysis was restricted to studying the limit $\lim_{N\to\infty}\lim_{t\to\infty}\overline{\nu}_t^N$ under the assumption of the *ansatz*. Later [7], proved the *ansatz* for the case of FIFO queuing models with service time distributions having decreasing hazard rate functions by exploiting the monotonic behavior of the system. However, as stated in [7, page 252] that the proof techniques cannot be extended to processor sharing models since the preordering of states is not possible unlike FIFO systems which is the key idea in showing monotonicity. On the other hand, the mean-field limit obtained by studying $\lim_{N\to\infty}\overline{\nu}_t^N$ is a deterministic process. It is enough to show the GAS of the mean-field since the Prokhorov's theorem would then imply the interchange of limits $\lim_{N\to\infty}\lim_{t\to\infty}\overline{\nu}_t^N = \lim_{t\to\infty}\lim_{N\to\infty}\overline{\nu}_t^N$. This will be addressed in future work.

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7 APPENDIX

7.1 Proof of Lemma 1

Proof. When a potential destination server is randomly chosen from N servers, the probability that the chosen server lies in state $(m, a_1, \cdots, a_m)$ is given by $\frac{\eta(\{(m,a_1,\cdots,a_m)\})}{N}$. Suppose out of d potential destination servers, j servers have n jobs and the remaining $d-j$ potential destination servers have atleast $n+1$ jobs. Further, out of j servers that have n jobs, let i servers lie in state $(n, l_1, \cdots, l_n)$. Then according to the SQ(d) policy, the probability that the destination server lies in state $(n, l_1, \cdots, l_n)$ is equal to

$$\binom{d}{j}\binom{j}{i}\frac{i}{j}\left(\frac{\eta(\{(n,l_1,\cdots,l_n)\})}{N}\right)^i\left(\frac{\eta(U_n)-\eta(\{(n,l_1,\cdots,l_n)\})}{N}\right)^{j-i} \times\left(\frac{\sum_{k=n+1}^{\infty}\eta(U_k)}{N}\right)^{d-j}. \quad (74)$$

Then by summing over all possible values of i and j, we get equation (25). □

7.2 Proof of Theorem 1

Proof. The proof of Theorem 1 involves establishing three results. The first result is to obtain the expression for the semigroup operator of the Markov process $(\nu_t^N, t \ge 0)$. Using this, in the second result, we then show that the Markov process $(\nu_t^N, t \ge 0)$ is a Feller-Dynkin process[10, 11] of $\mathcal{D}_{\mathcal{M}_F(U)}([0,\infty))$. Finally, in the third result, by using the generator $A^N(\cdot)$ of the Markov process $(\nu_t^N, t \ge 0)$, we study the martingale process defined in equation (102) by using Dynkin's lemma[11].

We next derive expression for the semigroup operator of the Markov process $(\nu_t^N, t \ge 0)$. Given that the initial state is $\nu_0^N = \eta$, let A_h and D_h be the number of arrivals and departures, respectively, in the interval $(0, h]$. Further, let the measure η contains the mass at m points denoted by $\underline{u}^{(l)} = (n_l, u_{1l}, \cdots, u_{n_l l})$, $1 \le l \le m$, and the number of servers with state $\underline{u}^{(l)}$ is given by $\eta(\{\underline{u}^{(l)}\})$. Let us denote the probability that there is no departure at a server with state $\underline{b} = (n, b_1, \cdots, b_n)$ at time t in the interval $(t, t+h]$ by $p_{ND}(\underline{b}; h)$. Then

$$p_{ND}(\underline{b}; h) = \prod_{i=1}^{n}\left(\frac{\overline{G}(b_i + \frac{h}{n})}{\overline{G}(b_i)}\right). \quad (75)$$

Further, we can write

$$p_{ND}(\underline{b}; h) = \prod_{i=1}^{n}\left((1-\beta(b_i)\frac{h}{n})\right) + o(h). \quad (76)$$

We next obtain expression for the semigroup operator of the Markov process $(\nu_t^N, t \ge 0)$ defined by

$$T_h^N f(\eta) = \mathbb{E}\left[f(\nu_h^N)|\nu_0^N = \eta\right], \quad (77)$$

where the mapping $f : \mathcal{M}_F(U) \mapsto \mathcal{R}$ is a continuous and bounded mapping.

Lemma 2. *If f is a bounded continuous function on $\mathcal{M}_F(U)$, then the semigroup operator $T_h^N f(\eta)$ of the Markov process $(\nu_t^N, t \ge 0)$ is given by*

$$\begin{aligned}T_h^N f(\eta) &= (1-N\lambda h)\prod_{l=1,\, n_l>0}^{m}(p_{ND}(\underline{u}^{(l)}; h))^{\eta(\{\underline{u}^{(l)}\})} f(\tau_h\eta)\\ &+ (1-N\lambda h)\sum_{i=1,\, n_i>0}^{m}\eta(\{\underline{u}^{(i)}\})\\ &\times\sum_{j=1}^{n_i}\int_{\tilde{h}=0}^{\frac{h}{n_i}}\frac{g(u_{ji}+\tilde{h})}{\overline{G}(u_{ji})}\prod_{k=1,\,k\neq j}^{n_i}\frac{\overline{G}(u_{ki}+\tilde{h})}{\overline{G}(u_{ki})}\left((p_{ND}(\underline{u}^{(i)}; h))^{(\eta(\{\underline{u}^{(i)}\})-1)}\right)\\ &\times\left(\prod_{r=1,\,r\neq i}^{m}(p_{ND}(\underline{u}^{(r)}; h))^{\eta(\{\underline{u}^{(r)}\})}\right)\\ &\times f\left(\tau_h\eta + \delta_{(B(\underline{u}^{(i)},j,\tilde{h},h))} - \delta_{(n_i, u_{1i}+\frac{h}{n_i},\cdots,u_{n_i i}+\frac{h}{n_i})}\right)d\tilde{h}\\ &+ \mathcal{P}(\{D_h=0\})N\lambda h\\ &\times\mathbb{E}\left[\left(f\left(\tau_h(\eta+\delta_{(M+1,Z_1,\cdots,Z_{L-1},0,Z_L,\cdots,Z_M)}-\delta_{(M,Z_1,\cdots,Z_M)})\right)\right)|\nu_0^N=\eta\right]\\ &+\epsilon(\eta,h), \quad (78)\end{aligned}$$

where

$$\begin{aligned}&B(\underline{u}^{(i)}, j, \tilde{h}, h)\\ &= \Big(n_i - 1, u_{1i}+\tilde{h}+\frac{(h-n_i\tilde{h})}{n_i-1}, \cdots, u_{(j-1)i}+\tilde{h}+\frac{(h-n_i\tilde{h})}{n_i-1},\\ &\quad u_{(j+1)i}+\tilde{h}+\frac{(h-n_i\tilde{h})}{n_i-1}, \cdots, u_{n_i i}+\tilde{h}+\frac{(h-n_i\tilde{h})}{n_i-1}\Big) \quad (79)\end{aligned}$$

and $\epsilon(\eta, h)$ is a $o(h)$ term. Further, $(M, Z_1, \cdots, Z_M)$ denotes the random variable that denotes the state of the destination server at time $t = 0$ when the arrived job is routed according to the power-of-d policy with system state as $\nu_0^N = \eta$ (job is considered to be arrived at $T_1 = 0$). Further, L is the random variable that denotes the position of the routed job that is picked up uniformly at random from $M + 1$ possible positions at the destination server.

Proof. From the definition of $T_h^N f(\eta)$, we can write

$$T_h^N f(\eta) = \sum_{i\ge0}\sum_{j\ge0}\mathbb{E}\left[f(\nu_h^N)I_{\{A_h=i\}}I_{\{D_h=j\}}|\nu_0^N=\eta\right]. \quad (80)$$

Hence we can write

$$\begin{aligned}T_h^N f(\eta) &= \mathbb{E}\left[f(\nu_h^N)I_{\{A_h=0\}}I_{\{D_h=0\}}|\nu_0^N=\eta\right]\\ &+ \mathbb{E}\left[f(\nu_h^N)I_{\{A_h=0\}}I_{\{D_h=1\}}|\nu_0^N=\eta\right]\\ &+ \mathbb{E}\left[f(\nu_h^N)I_{\{A_h=1\}}I_{\{D_h=0\}}|\nu_0^N=\eta\right]\\ &+ \sum_{i\ge1}\sum_{j\ge1}\mathbb{E}\left[f(\nu_h^N)I_{\{A_h=i\}}I_{\{D_h=j\}}|\nu_0^N=\eta\right]. \quad (81)\end{aligned}$$

We first look at the expression for $\mathbb{E}\left[f(\nu_h^N)I_{\{A_h=0\}}I_{\{D_h=0\}}|\nu_0^N=\eta\right]$ that corresponds to the event that there are no arrivals and no departures in the interval $(0, h]$. In this case, we have $\nu_h^N = \tau_h\eta$. Therefore, we have

$$\mathbb{E}\left[f(\nu_h^N)I_{\{A_h=0\}}I_{\{D_h=0\}}|\nu_0^N=\eta\right] = \mathcal{P}(\{A_h=0\})\mathcal{P}(\{D_h=0\})f(\tau_h\eta). \quad (82)$$

We can write

$$\mathbb{E}\left[f(\nu_h^N)I_{\{A_h=0\}}I_{\{D_h=0\}}|\nu_0^N=\eta\right]$$
$$=(\mathcal{P}(\{A_h=0\})+(1-N\lambda h)-(1-N\lambda h))\mathcal{P}(\{D_h=0\})f(\tau_h\eta)$$
$$=(1-N\lambda h)\mathcal{P}(\{D_h=0\})f(\tau_h\eta)+\epsilon_1(\eta,h)$$
$$=(1-N\lambda h)\prod_{l=1,\,n_l>0}^{m}(p_{ND}(\underline{u}^{(l)};h))^{\eta(\{\underline{u}^{(l)}\})}f(\tau_h\eta)+\epsilon_1(\eta,h),\quad(83)$$

where

$$\epsilon_1(\eta,h)=(\mathcal{P}(\{A_h=0\})-(1-N\lambda h))\mathcal{P}(\{D_h=0\})f(\tau_h\eta)\quad(84)$$

is a $o(h)$ term.

The expression for the second term $\mathbb{E}\left[f(\nu_h^N)I_{\{A_h=0\}}I_{\{D_h=1\}}|\nu_0^N=\eta\right]$ that corresponds to the event that there is no arrival and one job departs in the interval $(0,h]$ is obtained as follows. We can write

$$\mathbb{E}\left[f(\nu_h^N)I_{\{A_h=0\}}I_{\{D_h=1\}}|\nu_0^N=\eta\right]$$
$$=\mathcal{P}(\{A_h=0\})\mathbb{E}\left[f(\nu_h^N)I_{\{D_h=1\}}|\nu_0^N=\eta\right]$$
$$=(1-N\lambda h)\mathbb{E}\left[f(\nu_h^N)I_{\{D_h=1\}}|\nu_0^N=\eta\right]+\epsilon_2(\eta,h)\quad(85)$$

where $\epsilon_2(\eta,h)=(\mathcal{P}(\{A_h=0\})-(1-N\lambda h))\mathbb{E}\left[f(\nu_h^N)I_{\{D_h=1\}}|\nu_0^N=\eta\right]$ is a $o(h)$ term. Let a job departs from a server that had state $(n,a_1,\cdots,a_n)$ at time $t=0$ and assume that j^{th} job departs at time $n\tilde{h}$, *i.e.*, when j^{th} job age reaches $a_j+\tilde{h}$, then the server state would be equal to $\left(n-1,a_1+\tilde{h}+\frac{(h-n\tilde{h})}{n-1},\cdots,a_{j-1}+\tilde{h}+\frac{(h-n\tilde{h})}{n-1},a_{j+1}+\tilde{h}+\frac{(h-n\tilde{h})}{n-1},\cdots,a_n+\tilde{h}+\frac{(h-n\tilde{h})}{n-1}\right)$ at time h. By using the index i to represent that j^{th} job departs at time $n\tilde{h}$ from a server that had state $\underline{u}^{(i)}=(n_i,u_{i1},\cdots,u_{n_ii})$ at time $t=0$, we have

$$\mathbb{E}\left[f(\nu_h^N)I_{\{A_h=0\}}I_{\{D_h=1\}}|\nu_0^N=\eta\right]=(1-N\lambda h)\sum_{i=1,\,n_i>0}^{m}\eta(\{\underline{u}^{(i)}\})$$
$$\times\sum_{j=1}^{n_i}\int_{\tilde{h}=0}^{\frac{h}{n_i}}\frac{g(u_{ji}+\tilde{h})}{\overline{G}(u_{ji})}\prod_{k=1,k\neq j}^{n_i}\frac{\overline{G}(u_{ki}+\tilde{h})}{\overline{G}(u_{ki})}\left((p_{ND}(\underline{u}^{(i)};h))^{(\eta(\{\underline{u}^{(i)}\})-1)}\right)$$
$$\times\left(\prod_{r=1,r\neq i,n_r>0}^{m}(p_{ND}(\underline{u}^{(r)};h))^{\eta(\{\underline{u}^{(r)}\})}\right)$$
$$f\left(\tau_h\eta+\delta_{\left(B(\underline{u}^{(i)},j,\tilde{h},h)\right)}-\delta_{(n_i,u_{1i}+\frac{h}{n_i},\cdots,u_{n_ii}+\frac{h}{n_i})}\right)d\tilde{h}+\epsilon_2(\eta,h).\quad(86)$$

We now consider expression for $\mathbb{E}\left[f(\nu_h^N)I_{\{A_h=1\}}I_{\{D_h=0\}}|\nu_0^N=\eta\right]$ that corresponds to the the event that there is an arrival in the interval $(0,h]$ and none of the ongoing jobs depart the system. Note that if a job arrives at time $T_1(T_1\leq h)$, we use the system state to be $\nu_{T_1}^N$ in implementing the SQ(d) policy since the ages of progressing jobs increase with time. Suppose the arrived job achieves age of $\tilde{h}$ by the time of h and further, suppose it has joined at j^{th} position of a server that had state $(n,a_1,\cdots,a_n)$ at time $t=0$. This happens only if job arrives at time $h-(n+1)\tilde{h}$. In this case, server will have state $(n+1,a_1+\frac{h-(n+1)\tilde{h}}{n}+\tilde{h},\cdots,a_{j-1}+\frac{h-(n+1)\tilde{h}}{n}+\tilde{h},\tilde{h},a_{j+1}+\frac{h-(n+1)\tilde{h}}{n}+\tilde{h},\cdots,a_n+\frac{h-(n+1)\tilde{h}}{n}+\tilde{h})$. Suppose the job arrives at time T_1 which is sampled according to exponential distribution with rate $N\lambda$, then we have

$$\mathbb{E}\left[f(\nu_h^N)I_{\{A_h=1\}}I_{\{D_h=0\}}|\nu_0^N=\eta\right]=\mathcal{P}(\{D_h=0\})$$
$$\times\mathbb{E}\left[\left(f(\tau_h\eta+\delta_{(C((M,Z_1,\cdots,Z_M),L,T_1,h))}-\delta_{(M,Z_1+\frac{h}{M},\cdots,Z_M+\frac{h}{M})}\right)\times I_{\{A_h=1\}}\right)|\nu_0^N=\eta\right]\quad(87)$$

where

$$C((M,Z_1,\cdots,Z_M),L,T_1,h)$$
$$=\left(M+1,Z_1+\frac{T_1}{M}+\frac{h-T_1}{M+1},\cdots,Z_{L-1}+\frac{T_1}{M}+\frac{h-T_1}{M+1},\frac{h-T_1}{M+1},Z_L+\frac{T_1}{M}+\frac{h-T_1}{M+1},\cdots,Z_M+\frac{T_1}{M}+\frac{h-T_1}{M+1}\right)\quad(88)$$

and $(M,Z_1,\cdots,Z_M)$ denotes the random variable that represents the state of the destination server (chosen according to power-of-d policy at arrival instant T_1) at time $t=0$ and L is the random variable that indicates the position that is picked up uniformly among $M+1$ positions at the destination server for the job. Note that while choosing the destination server, the system state is considered as $\nu_{T_1}^N=\tau_{T_1}\eta$ when we implement the SQ(d) policy.

Now we can write

$$\mathbb{E}\left[f(\nu_h^N)I_{\{A_h=1\}}I_{\{D_h=0\}}|\nu_0^N=\eta\right]=\mathcal{P}(\{D_h=0\})N\lambda h$$
$$\times\mathbb{E}\left[\left(f\left(\tau_h(\eta+\delta_{(M+1,Z_1,\cdots,Z_{L-1},0,Z_L,\cdots,Z_M)}-\delta_{(M,Z_1,\cdots,Z_M)})\right)\right)|\nu_0^N=\eta\right]$$
$$+\mathcal{P}(\{D_h=0\})(\mathcal{P}(\{A_h=1\})-N\lambda h)$$
$$\times\mathbb{E}\left[\left(f\left(\tau_h(\eta+\delta_{(M+1,Z_1,\cdots,Z_{L-1},0,Z_L,\cdots,Z_M)}-\delta_{(M,Z_1,\cdots,Z_M)})\right)\right)|\nu_0^N=\eta\right]$$
$$+\mathcal{P}(\{D_h=0\})$$
$$\times\left(\mathbb{E}^{(1)}\left[\left(f(\tau_h\eta+\delta_{(S((M,Z_1,\cdots,Z_M),L,T_1,h))}-\delta_{(M,Z_1+\frac{h}{M},\cdots,Z_M+\frac{h}{M})}\right)\times I_{\{A_h=1\}}\right)|\nu_0^N=\eta\right]$$
$$-\mathbb{E}^{(2)}\left[\left(f\left(\tau_h(\eta+\delta_{(M+1,Z_1,\cdots,Z_{L-1},0,Z_L,\cdots,Z_M)}-\delta_{(M,Z_1,\cdots,Z_M)})\right)\times I_{\{A_h=1\}}\right)|\nu_0^N=\eta\right]\right),\quad(89)$$

where

$$S((M,Z_1,\cdots,Z_M),L,T_1,h)$$
$$=\left(M+1,Z_1+\frac{T_1}{M}+\frac{h-T_1}{M+1},\cdots,Z_{L-1}+\frac{T_1}{M}+\frac{h-T_1}{M+1},\frac{h-T_1}{M+1},Z_L+\frac{T_1}{M}+\frac{h-T_1}{M+1},\cdots,Z_M+\frac{T_1}{M}+\frac{h-T_1}{M+1}\right).\quad(90)$$

Further, on the right side of equation (89), in the first and second terms, we have that $T_1 = 0$ and hence we use η as the system state in choosing the destination server while in the third term, $\mathbb{E}^{(1)}[\cdot]$ is obtained by assuming that T_1 takes any arbitrary value sampled according to exponential distribution and $\mathbb{E}^{(2)}[\cdot]$ is computed by taking $T_1 = 0$. The sum of second and third terms on the right side of equation (89) is denoted by $\epsilon_3(\eta, h)$ and it is checked that $\epsilon_3(\eta, h)$ is a $o(h)$ term.

Finally, by the fact that f is a bounded function, then the fourth term is a $o(h)$ term denoted by $\epsilon_4(\eta, h)$. By defining

$$\epsilon(\eta, h) = \epsilon_1(\eta, h) + \epsilon_2(\eta, h) + \epsilon_3(\eta, h) + \epsilon_4(\eta, h), \tag{91}$$

we obtain equation (78). □

Proposition 1. *The process $(\nu_t^N, t \geq 0)$ is a Feller-Dynkin process[10, 11] of $\mathcal{D}_{\mathcal{M}_F(U)}([0, \infty))$.*

Proof. The process $(\nu_t^N, t \geq 0)$ has Feller-Dynkin property if we have the following properties (Using Lemma 3.5.1 and Corollary 3.5.2 of [10]):
For $f \in C_k^1(U), \eta \in \mathcal{M}_F(U)$, let $Q_f : \mathcal{M}_F(U) \longmapsto \mathcal{R}$ be defined by $Q_f(\eta) = e^{-\langle \eta, f \rangle}$, then we must have

(1) The mapping $\eta \longmapsto \mathbb{E}\left[Q_f(\nu_h^N)|\nu_0^N = \eta\right]$ is continuous for all $f \in C_k^1(U)$ and $h > 0$.
(2) For all $h > 0$, we have
$$\mathbb{E}\left[Q_1(\nu_h^N)|\nu_0^N = \eta\right] \to 0 \tag{92}$$
as $\eta(U) \to \infty$.
(3) For all $\eta \in \mathcal{M}_F(U)$ and $f \in C_k^1(U)$, we have
$$\mathbb{E}\left[Q_f(\nu_h^N)|\nu_0^N = \eta\right] \to Q_f(\eta) \tag{93}$$
as $h \to 0$.

By using equation (78), we have

$$\begin{aligned}\mathbb{E}\left[Q_f(\nu_h^N)|\nu_0^N = \eta\right] &= e^{-\langle \tau_h \eta, f\rangle}\Bigg\{(1 - N\lambda h)\\ &\times\left(\prod_{j=1, n_j>0}^{m} (p_{ND}(\underline{u}^{(j)}; h))^{\eta(\{\underline{u}^{(j)}\})}\right)\\ &+ (1 - N\lambda h)\sum_{j=1, n_j>0}^{m}\sum_{r=1}^{n_j}\eta(\{\underline{u}^{(j)}\})\left(\frac{G(u_{rj}+h) - G(u_{rj})}{\overline{G}(u_{rj})}\right)\\ &\times\left(\prod_{w=1;w\neq r}^{n_j}\left(\frac{\overline{G}(u_{wj}+h)}{\overline{G}(u_{wj})}\right)\right)(p_{ND}(\underline{u}^{(j)}; h))^{(\eta(\{\underline{u}^{(j)}\})-1)}\\ &\times\left(\prod_{i=1, n_i>0, i\neq j}^{m}(p_{ND}(\underline{u}^{(i)}; h))^{\eta(\{\underline{u}^{(i)}\})}\right)\\ &\times Q_f\left(\tau_h(\delta_{(n_j-1, u_{1j},\ldots,u_{r-1j},u_{r+1j},\ldots,u_{n_jj})} - \delta_{(n_j,u_{1j},\ldots,u_{n_jj})})\right)\\ &+ \mathcal{P}(\{D_h = 0\})(N\lambda h)\\ \times\mathbb{E}&\left[Q_f\left(\tau_h(\delta_{(M+1,Z_1,\ldots,Z_{(L_1-1)},0,Z_L,\ldots,Z_M)} - \delta_{(M,Z_1,\ldots,Z_M)})\right)|\nu_0^N = \eta\right]\\ &+ \epsilon_f(\eta, h)\Bigg\}\end{aligned} \tag{94}$$

where $\epsilon_f(\eta, h)$ is given by

$$\epsilon_f(\eta, h) = \epsilon_{1f}(\eta, h) + \epsilon_{2f}(\eta, h) + \epsilon_{3f}(\eta, h) + \epsilon_{4f}(\eta, h) \tag{95}$$

such that

$$\epsilon_{1f}(\eta, h) = (\mathcal{P}(\{A_h = 0\}) - (1 - N\lambda h))\mathcal{P}(\{D_h = 0\}), \tag{96}$$

$$\begin{aligned}\epsilon_{2f}(\eta, h) &= (\mathcal{P}(\{A_h = 0\}) - (1 - N\lambda h))\sum_{i=1, n_i>0}^{m}\eta(\{\underline{u}^{(i)}\})\\ \times\sum_{j=1}^{n_i}\int_{\tilde{h}=0}^{\frac{h}{n_i}}&\frac{g(u_{ji}+\tilde{h})}{\overline{G}(u_{ji})}\prod_{k=1, k\neq j}^{n_i}\frac{\overline{G}(u_{ki}+\tilde{h})}{\overline{G}(u_{ki})}\left((p_{ND}(\underline{u}^{(i)}; h))^{(\eta(\{\underline{u}^{(i)}\})-1)}\right)\\ &\times\left(\prod_{r=1, r\neq i, n_r>0}^{m}(p_{ND}(\underline{u}^{(r)}; h))^{\eta(\{\underline{u}^{(r)}\})}\right)\\ &Q_f\left(\delta_{\left(B(\underline{u}^{(i)}, j, \tilde{h}, h)\right)} - \delta_{(n_i, u_{1i}+\frac{h}{n_i},\cdots,u_{n_ii}+\frac{h}{n_i})}\right)d\tilde{h},\end{aligned} \tag{97}$$

$$\begin{aligned}\epsilon_{3f}(\eta, h) &= \mathcal{P}(\{D_h = 0\})[\mathcal{P}(\{A_h = 1\}) - N\lambda h]\\ \times\mathbb{E}&\left[Q_f\left(\tau_h(\delta_{(M+1,Z_1,\ldots,Z_{(L_1-1)},0,Z_{L_1},\ldots,Z_M)} - \delta_{(M,Z_1,\ldots,Z_M)})\right)|\nu_0^N = \eta\right]\\ &+ \mathcal{P}(\{D_h = 0\})\\ \times&\left(\mathbb{E}^{(1)}\left[Q_f\left((\delta_{(S((M,Z_1,\cdots,Z_M),L,T_1,h))} - \delta_{(M,Z_1+\frac{h}{M},\ldots,Z_M+\frac{h}{M})}\right)\right.\right.\\ &\left.\times I_{\{A_h=1\}}|\nu_0^N = \eta\right]\\ -\mathbb{E}^{(2)}&\left[Q_f\left(\tau_h(\delta_{(M+1,Z_1,\ldots,Z_{(L_1-1)},0,Z_{L_1},\ldots,Z_M)} - \delta_{(M,Z_1,\ldots,Z_M)})\right)\right.\\ &\left.\left.\times I_{\{A_h=1\}}|\nu_0^N = \eta\right]\right)\end{aligned} \tag{98}$$

and

$$\begin{aligned}&\epsilon_{4f}(\eta, h) =\\ &\sum_{i\geq1, j\geq1}\mathbb{E}\Bigg[Q_f\Bigg(\sum_{r=1}^{i}(\delta_{(S((M_r,Z_{1r},\cdots,Z_{M_rr}),L_r,0,h-T_r))}\\ &\quad-\delta_{(M_r,Z_{1r}+\frac{h-T_r}{M_r},\ldots,Z_{M_rr}+\frac{h-T_r}{M_r})})\\ &\quad+\sum_{l=1}^{j}(\delta_{(B((n_l,X_{1l},\ldots,X_{n_ll}),J_l,0,h-W_l))}\\ &\quad-\delta_{(n_l,X_{1l}+\frac{(h-W_l)}{n_l},\ldots,X_{n_ll}+\frac{(h-W_l)}{n_l})}))\Bigg)\\ &\quad\times I_{\{A_h=i,D_h=j\}}|\nu_0^N = \eta\Bigg].\end{aligned} \tag{99}$$

In equation (99), T_r denotes the arrival time of r^{th} job which is routed to a server with state $(M_r, Z_1^{(r)}, \ldots, Z_{M_r}^{(r)})$ at time T_r and L_r

is the position of r^{th} arriving job at its destination server. Corresponding to departures, suppose l^{th} departure occurs at time W_l at a server with state $(n_l, X_1^{(l)}, \ldots, X_{n_l}^{(l)})$ at time W_l and the position of the departing job is J_l. By using the same arguments as for $\epsilon(\nu, h)$ in equation (78), $\epsilon_f(\nu, h)$ is also a $o(h)$ term.

Now let us look at the proof of the first condition required for Feller property, we write equation (94) as

$$\mathbb{E}\left[Q_f(\nu_h^N)|\nu_0^N = \eta\right] = (e^{-\langle \tau_h \eta, f\rangle})V(\eta, h). \tag{100}$$

We have that $e^{-\langle \tau_h \eta, f\rangle}$ is a continuous mapping of η. Now if we show $V(\eta, h)$ is a continuous mapping of η, then $\mathbb{E}\left[Q_f(\nu_h^N)|\nu_0^N = \eta\right]$ is a continuous mapping of η. By the fact that η is a point measure at finite N, the continuity of $V(\eta, h)$ $w.r.t.$ η follows from the fact that the routing probabilities under the SQ(d) policy as shown in equation (25) and the departure probabilities are continuous mappings of η. Since $\tau_h\eta(U) = \eta(U) = N$, the second condition is satisfied. Finally, since $\langle \tau_h \eta, f\rangle = \langle \eta, \tau_h f\rangle$, by applying the dominated convergence theorem we have $\langle \tau_h \eta, f\rangle \to \langle \eta, f\rangle$ as $h \to 0$. This establishes the third condition. Hence the process $(\nu_t^N, t \geq 0)$ is a Feller process. □

Before looking at the third result, we now recall the definition of the generator $A^N(\cdot)$ of the Markov process $(\nu_t^N, t \geq 0)$ by using the semigroup operator $T_h^N(\cdot)$ that satisfies equation (78). For any $F \in C(\mathcal{M}_F(U))$, the generator $A^N(\cdot)$ is defined as

$$A^N F(\eta) = \lim_{h\to 0} \frac{\mathbb{E}\left[F(\nu_h^N)|\nu_0^N = \eta\right] - F(\eta)}{h}, \tag{101}$$

where $F \in C(\mathcal{M}_F(U))$ such that the limit exists. We now define a process $(M_t^N(\phi), t \geq 0)$ for $\phi \in C_b^1(U)$ using the generator $A^N(\cdot)$ and the Dynkin's formula.

Lemma 3. *Let $\phi \in C_b^1(U)$, then the process $(M_t^N(\phi), t \geq 0)$ defined as*

$$M_t^N(\phi) = \langle \nu_t^N, \phi\rangle - \langle \nu_0^N, \phi\rangle - \int_{s=0}^t A^N \langle \nu_s^N, \phi\rangle ds \tag{102}$$

is a square integrable $\mathcal{F}_t^N$-martingale and it is right continuous with left limits (RCLL) process. Further, for $\phi, \psi \in C_b^1(U)$, the mutual variation of $(M_t^N(\phi), t \geq 0)$ with $(M_t^N(\psi), t \geq 0)$ is given by

$$\begin{aligned}
< M^N(\phi)_., M^N_.(\psi) >_t = &\int_{s=0}^t \left(\sum_{n=1}^{\infty}\sum_{j=1}^{n} \int_{x_1}\cdots\int_{x_n} \left(\frac{\beta(x_j)}{n}\right)\right.\\
&\times (\phi(n-1, x_1, \cdots, x_{j-1}, x_{j-1}, x_{j+1}, \cdots, x_n) - \phi(n, x_1, \cdots, x_n))\\
&\times (\psi(n-1, x_1, \cdots, x_{j-1}, x_{j-1}, x_{j+1}, \cdots, x_n) - \psi(n, x_1, \cdots, x_n))\\
&\times d\nu_s^N(n, x_1, \cdots, x_n)\\
&+ N\lambda\Bigg[\frac{\nu_s^N(\{0\})}{N}\Phi_0\left(\frac{\nu_s^N}{N}\right)(\phi(1,0) - \phi(0))(\psi(1,0) - \psi(0))\\
&+ \sum_{n=1}^{\infty}\sum_{j=1}^{n+1}\int_{x_1}\cdots\int_{x_n} \frac{1}{N(n+1)}\Phi_n\left(\frac{\nu_s^N}{N}\right)\\
&\times (\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \phi(n, x_1, \cdots, x_n))\\
&\times (\psi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \psi(n, x_1, \cdots, x_n))\\
&\left.\left.\times d\nu_s^N(n, x_1, \cdots, x_n)\right]\right) ds.
\end{aligned} \tag{103}$$

.

Proof. We first look at the expression for the generator $A^N(\cdot)$. By using equation (78) and since the set of linear combinations of Q_f for $f \in C_k^1(U)$ defined by $Q_f(\eta) = e^{-\langle \eta, f\rangle}$ is dense in the set $C(\mathcal{M}_F(U))$[26, proposition 7.10], by using expression for $A^N Q_f(\eta)$, we get

$$\begin{aligned}
A^N F(\eta) = &\lim_{h\to 0}\left(\frac{F(\tau_h\eta) - F(\eta)}{h}\right) - N\lambda F(\eta)\\
&- \sum_{n=1}^{\infty}\sum_{j=1}^{n}\int_{x_1}\cdots\int_{x_n}\left(\frac{\beta(x_j)}{n}\right)F(\eta)\, d\eta(n, x_1, \cdots, x_n)\\
&+ \sum_{n=1}^{\infty}\sum_{j=1}^{n}\int_{x_1}\cdots\int_{x_n}\left(\frac{\beta(x_j)}{n}\right)\\
&\times\Big(F(\eta + \delta_{(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)} - \delta_{(n, x_1, \cdots, x_n)})\Big)\, d\eta(n, x_1, \cdots, x_n)\\
&+ N\lambda\Bigg[\frac{\eta(\{0\})}{N}\Phi_0\left(\frac{\eta}{N}\right)F(\eta + \delta_{(1,0)} - \delta_{(0)})\\
&+ \sum_{n=1}^{\infty}\sum_{j=1}^{n+1}\int_{x_1}\cdots\int_{x_n}\frac{1}{N(n+1)}\Phi_n\left(\frac{\eta}{N}\right)\\
&\times F(\eta + \delta_{(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n)} - \delta_{(n, x_1, \cdots, x_n)})\\
&\times\, d\eta(n, x_1, \cdots, x_n)\Bigg].
\end{aligned} \tag{104}$$

We make it clear that when $\phi \in C_b^1(U)$, $\eta \in \mathcal{M}_F(U)$, then $A^N\langle \eta, \phi\rangle$ is well defined. By using the Dynkin's formula [11], the process $(M_t^N(\phi), t \geq 0)$ defined as

$$M_t^N(\phi) = \langle \nu_t^N, \phi\rangle - \langle \nu_0^N, \phi\rangle - \int_{s=0}^t A^N\langle \nu_s^N, \phi\rangle ds \tag{105}$$

is a RCLL $\mathcal{F}_t^N$−local martingale. Upon simplification, we have

$$\begin{aligned}
M_t^N(\phi) = &\langle \nu_t^N, \phi\rangle - \langle \nu_0^N, \phi\rangle - \int_{s=0}^t \langle \nu_s^N, \phi'_{\Sigma}\rangle\, ds\\
&- \int_{s=0}^t\left(\sum_{n=1}^{\infty}\sum_{j=1}^{n}\int_{x_1}\cdots\int_{x_n}\frac{\beta(x_j)}{n}\right.\\
&\times(\phi(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \ldots, x_n) - \phi(n, x_1, \cdots, x_n))\\
&\times d\nu_s^N(n, x_1, \cdots, x_n)\\
&+ N\lambda\Bigg[\left(\frac{\nu_s^N(\{0\})}{N}\Phi_0\left(\frac{\nu_s^N}{N}\right)(\phi(1,0) - \phi(0))\right)\\
&+ \sum_{n=1}^{\infty}\sum_{j=1}^{n+1}\int_{x_1}\cdots\int_{x_n}\frac{1}{N(n+1)}\Phi_n\left(\frac{\nu_s^N}{N}\right)\\
&\times(\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \phi(n, x_1, \cdots, x_n))\\
&\left.\left.\times\, d\nu_s^N(n, x_1, \cdots, x_n)\right]\right)ds,
\end{aligned} \tag{106}$$

where

$$\langle \nu_s^N, \phi'_{\sum}\rangle = \sum_{n=1}^{\infty}\sum_{i=1}^{n}\int_{x_1}\cdots\int_{x_n}\frac{1}{n}\frac{\partial\phi(n,x_1,\ldots,x_n)}{\partial x_i}\,d\nu_s^N(n,x_1,\cdots,x_n). \tag{107}$$

Further, let $\psi \in C_b^1(U)$, then the mapping $\eta \mapsto \langle\eta,\phi\rangle\langle\eta,\psi\rangle$ also belongs to the domain of A^N. We now define a process $(\tilde{M}_t^N(\phi,\psi), t \geq 0)$ as

$$\tilde{M}_t^N(\phi,\psi) = \langle \nu_t^N,\phi\rangle\langle\nu_t^N,\psi\rangle - \langle\nu_0^N,\phi\rangle\langle\nu_0^N,\psi\rangle - \int_{s=0}^{t} A^N\langle\nu_s^N,\phi\rangle\langle\nu_s^N,\psi\rangle ds \tag{108}$$

is a RCLL $\mathcal{F}_t^N-$local martingale. It is verified that, we have

$$\begin{aligned}
A^N\langle\eta,\phi\rangle\langle\eta,\psi\rangle &= \langle\eta,\phi\rangle A^N\langle\eta,\psi\rangle + \langle\eta,\psi\rangle A^N\langle\eta,\phi\rangle \\
&+ \sum_{n=1}^{\infty}\sum_{j=1}^{n}\int_{x_1}\cdots\int_{x_n}\frac{\beta(x_j)}{n} \\
&\times\left(\phi(n-1,x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_n)-\phi(n,x_1,\cdots,x_n)\right)\\
&\times\left(\psi(n-1,x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_n)-\psi(n,x_1,\cdots,x_n)\right)\\
&\times d\eta(n,x_1,\cdots,x_n)\\
&+ N\lambda\Bigg[\left(\frac{\eta(\{0\})}{N}\Phi_0\left(\frac{\eta}{N}\right)(\phi(1,0)-\phi(0))\right) \\
&+ \sum_{n=1}^{\infty}\sum_{j=1}^{n+1}\int_{x_1}\cdots\int_{x_n}\frac{1}{N(n+1)}\Phi_n\left(\frac{\eta}{N}\right)\\
&\times(\phi(n+1,x_1,\cdots,x_{j-1},0,x_j,\cdots,x_n)-\phi(n,x_1,\cdots,x_n))\\
&\times(\psi(n+1,x_1,\cdots,x_{j-1},0,x_j,\cdots,x_n)-\psi(n,x_1,\cdots,x_n))\\
&\times d\eta(n,x_1,\cdots,x_n)\Bigg].
\end{aligned} \tag{109}$$

By using Itô's formula, we have

$$\begin{aligned}
\langle\nu_t^N,\phi\rangle\langle\nu_t^N,\psi\rangle &= \langle\nu_0^N,\phi\rangle\langle\nu_0^N,\psi\rangle + \int_{s=0}^{t}\langle\nu_s^N,\phi\rangle\,dM_s^N(\psi)\\
&+ \int_{s=0}^{t}\langle\nu_s^N,\psi\rangle\,dM_s^N(\phi) + \int_{s=0}^{t}\langle\nu_s^N,\phi\rangle A^N\langle\nu_s^N,\psi\rangle\,ds\\
&+ \int_{s=0}^{t}\langle\nu_s^N,\psi\rangle\,A^N\langle\nu_s^N,\phi\rangle\,ds + <\langle\nu_\cdot^N,\phi\rangle,\langle\nu_\cdot^N,\psi\rangle>_t .
\end{aligned} \tag{110}$$

Further, by using equations (108)-(109), we have

$$\begin{aligned}
&\int_{s=0}^{t}\langle\nu_s^N,\phi\rangle\,dM_s^N(\psi) + \int_{s=0}^{t}\langle\nu_s^N,\psi\rangle\,dM_s^N(\phi)\\
&+ \int_{s=0}^{t}\langle\nu_s^N,\psi\rangle\,A^N\langle\nu_s^N,\phi\rangle\,ds + <\langle\nu_\cdot^N,\phi\rangle,\langle\nu_\cdot^N,\psi\rangle>_t =\\
&\tilde{M}_t^N(\phi,\psi) + \int_{s=0}^{t}\Bigg(\sum_{n=1}^{\infty}\sum_{j=1}^{n}\int_{x_1}\cdots\int_{x_n}\frac{\beta(x_j)}{n}\\
&\times\left(\phi(n-1,x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_n)-\phi(n,x_1,\cdots,x_n)\right)\\
&\times\left(\psi(n-1,x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_n)-\psi(n,x_1,\cdots,x_n)\right)\\
&\times d\nu_s^N(n,x_1,\cdots,x_n)\\
&+ N\lambda\Bigg[\left(\frac{\nu_s^N(\{0\})}{N}\Phi_0\left(\frac{\nu_s^N}{N}\right)(\phi(1,0)-\phi(0))\,(\psi(1,0)-\psi(0))\right)\\
&+ \sum_{n=1}^{\infty}\sum_{j=1}^{n+1}\int_{x_1}\cdots\int_{x_n}\frac{1}{N(n+1)}\Phi_n\left(\frac{\nu_s^N}{N}\right)\\
&\times(\phi(n+1,x_1,\cdots,x_{j-1},0,x_j,\cdots,x_n)-\phi(n,x_1,\cdots,x_n))\\
&\times(\psi(n+1,x_1,\cdots,x_{j-1},0,x_j,\cdots,x_n)-\psi(n,x_1,\cdots,x_n))\\
&\times d\nu_s^N(n,x_1,\cdots,x_n)\Bigg]\Bigg)ds.
\end{aligned} \tag{111}$$

By identifying the finite variation process, $\mathcal{P}-$a.s. we have

$$\begin{aligned}
&<\langle\nu_\cdot^N,\phi\rangle,\langle\nu_\cdot^N,\psi\rangle>_t =\\
&\int_{s=0}^{t}\Bigg(\sum_{n=1}^{\infty}\sum_{j=1}^{n}\int_{x_1}\cdots\int_{x_n}\frac{\beta(x_j)}{n}\\
&\times\left(\phi(n-1,x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_n)-\phi(n,x_1,\cdots,x_n)\right)\\
&\times\left(\psi(n-1,x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_n)-\psi(n,x_1,\cdots,x_n)\right)\\
&\times d\nu_s^N(n,x_1,\cdots,x_n)\\
&+ N\lambda\Bigg[\left(\frac{\nu_s^N(\{0\})}{N}\Phi_0\left(\frac{\nu_s^N}{N}\right)(\phi(1,0)-\phi(0))\,(\psi(1,0)-\psi(0))\right)\\
&+ \sum_{n=1}^{\infty}\sum_{j=1}^{n+1}\int_{x_1}\cdots\int_{x_n}\frac{1}{N(n+1)}\Phi_n\left(\frac{\nu_s^N}{N}\right)\\
&\times(\phi(n+1,x_1,\cdots,x_{j-1},0,x_j,\cdots,x_n)-\phi(n,x_1,\cdots,x_n))\\
&\times(\psi(n+1,x_1,\cdots,x_{j-1},0,x_j,\cdots,x_n)-\psi(n,x_1,\cdots,x_n))\\
&\times d\nu_s^N(n,x_1,\cdots,x_n)\Bigg]\Bigg)ds.
\end{aligned} \tag{112}$$

By using equation (105), we have

$$<\langle\nu_\cdot^N,\phi\rangle,\langle\nu_\cdot^N,\psi\rangle>_t = <M_\cdot^N(\phi),M_\cdot^N(\psi)>_t . \tag{113}$$

Hence as $\phi,\psi \in C_b^1(U)$ and $\beta \in C_b(\mathcal{R}_+)$, we have

$$\mathbb{E}\left[<M_\cdot^N(\phi),M_\cdot^N(\psi)>_t\right] < \infty \tag{114}$$

and hence $(M_t^N(\phi))_{t\geq 0}$ is a square integrable martingale. □

□

7.3 The mean-field Limit: Proof of Theorem 2

Proof. We first give the first part of the proof that corresponds to the existence and uniqueness of the mean-field solution. This is an essential requirement in proving the convergence of $(\frac{\nu_t^N}{N}, t \geq 0)$ as $N \to \infty$. We then give the proof of the second part that corresponds to the convergence of $(\frac{\nu_t^N}{N}, t \geq 0)$ as $N \to \infty$.

Existence and Uniqueness of Mean-field Solution: From equation (32), for $\phi \in C_b(U)$, the operator $\phi \mapsto \langle\overline{\nu}_t,\phi\rangle$ is a linear operator and $\overline{\nu}_t(U) = 1$. Therefore from Riesz-Markov-Kakutani theorem [27, 30], for $\nu_t \in \mathcal{M}_1(U)$, showing the existence of unique probability measure $\overline{\nu}_t$ is equivalent to showing the existence of unique operator $\phi \mapsto \langle\overline{\nu}_t,\phi\rangle$.

We next prove that given an initial measure $\overline{\nu}_0$, there exists atmost one mean-field solution by establishing that there exists atmost one real valued process $\langle\overline{\nu}_t,\phi\rangle$ satisfying the MFE. Now let

$(\overline{\nu}_t^1, t \geq 0), (\overline{\nu}_t^2, t \geq 0)$ be two solutions that satisfy MFE with initial points $\overline{\nu}_0^1, \overline{\nu}_0^2$, respectively. For $\phi \in C_b(U)$, we have

$$\langle \overline{\nu}_t^1 - \overline{\nu}_t^2, \phi \rangle = \langle \overline{\nu}_0^1 - \overline{\nu}_0^2, \tau_t \phi \rangle + \int_{s=0}^t \Bigg(\sum_{n=1}^\infty \sum_{j=1}^n \int_{x_1} \cdots \int_{x_n} \frac{\beta(x_j)}{n} \times \big(\tau_{t-s}\phi(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) - \tau_{t-s}\phi(n, x_1, \cdots, x_n)\big) \times d(\overline{\nu}_s^1 - \overline{\nu}_s^2)(n, x_1, \cdots, x_n) \Bigg) ds$$
$$+ \int_{s=0}^t \Bigg(\lambda \Bigg[\left(\overline{\nu}_s^1(\{0\}) \Phi_0(\overline{\nu}_s^1) \left(\tau_{t-s}\phi(1,0) - \tau_{t-s}\phi(0)\right) \right) + \sum_{n=1}^\infty \sum_{j=1}^{n+1} \int_{x_1} \cdots \int_{x_n} \frac{1}{(n+1)} \Phi_n(\overline{\nu}_s^1) \times \left(\tau_{t-s}\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \tau_{t-s}\phi(n, x_1, \cdots, x_n)\right) \times d\overline{\nu}_s^1(n, x_1, \cdots, x_n) \Bigg]$$
$$- \lambda \Bigg[\left(\overline{\nu}_s^2(\{0\}) \Phi_0(\overline{\nu}_s^2) \left(\tau_{t-s}\phi(1,0) - \tau_{t-s}\phi(0)\right) \right) - \sum_{n=1}^\infty \sum_{j=1}^{n+1} \int_{x_1} \cdots \int_{x_n} \frac{1}{(n+1)} \Phi_n(\overline{\nu}_s^2) \times \left(\tau_{t-s}\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \tau_{t-s}\phi(n, x_1, \cdots, x_n)\right) \times d\eta_s^2(n, x_1, \cdots, x_n) \Bigg] \Bigg) ds. \quad (115)$$

We would like to achieve a result of the form

$$\|\overline{\nu}_t^1 - \overline{\nu}_t^2\| \leq b + c \int_{s=0}^t \|\overline{\nu}_s^1 - \overline{\nu}_s^2\| ds \quad (116)$$

for some $b, c > 0, t \in [0, T]$. This implies from Gronwall's inequality[11] that

$$\|\overline{\nu}_t^1 - \overline{\nu}_t^2\| \leq b\, e^{ct} \quad (117)$$

for $t \in [0, T]$. By using the first term on the right side of equation (115), we can write

$$\left|\langle \overline{\nu}_0^1 - \overline{\nu}_0^2, \tau_t \phi \rangle\right| \leq \|\overline{\nu}_0^1 - \overline{\nu}_0^2\| \|\phi\|. \quad (118)$$

To simplify the second term, we define a function $w_{t,s}$ as follows:

$$w_{t,s}(n, x_1, \cdots, x_n) = \sum_{k=1}^n \frac{\beta(x_k)}{n} (\tau_{t-s}\phi(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) - \tau_{t-s}\phi(n, x_1, \cdots, x_n)) \quad (119)$$

and $w_{t,s}(0) = 0$. Since $\phi \in C_b(U)$ and $\beta \in C_b(\mathcal{R}_+)$, we have $w_{t,s} \in C_b(U)$. Further, we have

$$\|w_{t,s}\| \leq 2\|\beta\| \|\phi\|. \quad (120)$$

Using the definition of $w_{t,s}$, we have

$$\int_{s=0}^t \Bigg(\sum_{n=1}^\infty \sum_{j=1}^n \int_{x_1} \cdots \int_{x_n} \frac{\beta(x_j)}{n} \times \Big(\tau_{t-s}\phi(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) - \tau_{t-s}\phi(n, x_1, \cdots, x_n) \Big) d(\overline{\nu}_s^1 - \overline{\nu}_s^2)(n, x_1, \cdots, x_n)\, ds = \int_{s=0}^t \langle \overline{\nu}_s^1 - \overline{\nu}_s^2, w_{t,s} \rangle ds. \quad (121)$$

Now consider the third term, let us define a function $h_{t,s,\eta}$ as follows,

$$h_{t,s,\eta}(n, x_1, \cdots, x_n) = \sum_{j=1}^{n+1} \frac{1}{(n+1)} \frac{(\overline{R}_n(\eta)^d - \overline{R}_{n+1}(\eta)^d)}{(\overline{R}_n(\eta) - \overline{R}_{n+1}(\eta))} \times (\tau_{t-s}\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \tau_{t-s}\phi(n, x_1, \cdots, x_n)) \quad (122)$$

for $x_i \geq 0$ for all i. Then the third term is equal to $\int_{s=0}^t \lambda \left(\langle \overline{\nu}_s^1, h_{t,s,\overline{\nu}_s^1} \rangle - \langle \overline{\nu}_s^2, h_{t,s,\overline{\nu}_s^2} \rangle \right) ds$. Further, we can write

$$\left| \langle \overline{\nu}_s^1, h_{t,s,\overline{\nu}_s^1} \rangle - \langle \overline{\nu}_s^2, h_{t,s,\overline{\nu}_s^2} \rangle \right| \leq \left| \langle \overline{\nu}_s^1 - \overline{\nu}_s^2, h_{t,s,\overline{\nu}_s^1} \rangle \right| + \left| \langle \overline{\nu}_s^2, h_{t,s,\overline{\nu}_s^1} - h_{t,s,\overline{\nu}_s^2} \rangle \right|. \quad (123)$$

Hence we have,

$$\left| \langle \overline{\nu}_s^1, h_{t,s,\overline{\nu}_s^1} \rangle - \langle \overline{\nu}_s^2, h_{t,s,\overline{\nu}_s^2} \rangle \right| \leq \|\overline{\nu}_s^1 - \overline{\nu}_s^2\| \|h_{t,s,\overline{\nu}_s^1}\| + \|\overline{\nu}_s^2\| \|h_{t,s,\overline{\nu}_s^1} - h_{t,s,\overline{\nu}_s^2}\|. \quad (124)$$

As $\overline{\nu}_s^2$ is a probability measure, we have $\|\overline{\nu}_s^2\| = 1$. Further, we also have that $\|h_{t,s,\overline{\nu}_s^1}\| \leq 2d\|\phi\|$. We also have

$$\left| h_{t,s,\overline{\nu}_s^1}(n, x_1, \cdots, x_n) - h_{t,s,\overline{\nu}_s^2}(n, x_1, \cdots, x_n) \right| \leq 2d^2 \|\phi\| \left(\left| \overline{R}_n(\overline{\nu}_s^1) - \overline{R}_n(\overline{\nu}_s^2) \right| + \left| \overline{R}_{n+1}(\overline{\nu}_s^1) - \overline{R}_{n+1}(\overline{\nu}_s^2) \right| \right). \quad (125)$$

Further, by defining a function h^* such that for $m \geq n$ and for all x_i, $1 \leq i \leq m$, we have

$$h^*(m, x_1, \ldots, x_m) = 1 \quad (126)$$

and for $m < n$ and for all x_i, $1 \leq i \leq m$, we have $h^*(m, x_1, \cdots, x_m) = 0$, then we can write

$$\overline{R}_n(\overline{\nu}_s^1) = \langle \overline{\nu}_s^1, h^* \rangle. \quad (127)$$

We then have

$$\left| \overline{R}_n(\overline{\nu}_s^1) - \overline{R}_n(\overline{\nu}_s^2) \right| \leq \|\overline{\nu}_s^1 - \overline{\nu}_s^2\| \|h^*\| = \|\overline{\nu}_s^1 - \overline{\nu}_s^2\|. \quad (128)$$

By using bounds for all the terms, we have

$$\left| \langle \overline{\nu}_t^1 - \overline{\nu}_t^2, \phi \rangle \right| \leq \left(\|\overline{\nu}_0^1 - \overline{\nu}_0^2\| + \int_{s=0}^t 2\|\beta\| \|\overline{\nu}_s^1 - \overline{\nu}_s^2\| ds + \int_{s=0}^t 8d^2 \lambda \|\overline{\nu}_s^1 - \overline{\nu}_s^2\| ds \right) \|\phi\|. \quad (129)$$

Therefore we get

$$\|\overline{\nu}_t^1 - \overline{\nu}_t^2\| \leq \|\overline{\nu}_0^1 - \overline{\nu}_0^2\| + (2\|\beta\| + 8d^2\lambda) \int_{s=0}^t \|\overline{\nu}_s^1 - \overline{\nu}_s^2\| ds. \quad (130)$$

Finally, from equation (116), we have

$$\|\overline{\nu}_t^1 - \overline{\nu}_t^2\| \leq \|\overline{\nu}_0^1 - \overline{\nu}_0^2\| e^{(2\|\beta\| + 8d^2\lambda)t}. \quad (131)$$

Therefore for the given initial measure $\overline{\nu}_0$, there exists atmost one solution for the MFE.

Let us now look at the existence of a solution for MFE. From the proof of the second part that we state next, we have the relative compactness of the sequence $\{\overline{\nu}_t^N, t \geq 0\}$ in $\mathcal{D}_{\mathcal{M}_1(U)}([0,\infty))$. Every limit point of the sequence $\{\overline{\nu}_t^N, t \geq 0\}$ has sample paths *a.s.* satisfying the equation (32). This establishes that there exists a solution to the MFE.

Convergence of $(\overline{\nu}_t^N, t \geq 0)$: We now look at the convergence of $(\overline{\nu}_t^N, t \geq 0)$ in $\mathcal{D}_{\mathcal{M}_1(U)}([0,\infty)$ as $N \to \infty$. We have that $\overline{\nu}_t^N(\{\underline{u}\})$ is equal to the fraction of servers that lie in $\underline{u}$ at time t. Further, suppose $(\overline{\mathcal{F}}_t^N, t \geq 0)$ is the natural filtration associated with the process $(\overline{\nu}_t^N, t \geq 0)$.

By using assumption 2, we first show that the sequence of processes $(\overline{\nu}_t^N, t \geq 0)$ is relatively compact and we then prove that every limit point $(\chi_t, t \geq 0)$ has sample paths evolving almost surely according to the MFE. Since the deterministic measure $\overline{\nu}_0$ is the initial point for all the limiting points, from the uniqueness of the mean-field solution for given initial measure, we have that all limiting points have almost surely identical sample paths coinciding with the unique mean-field solution. Hence we call the unique mean-field solution as the the mean-field limit denoted by $(\overline{\nu}_t, t \geq 0)$.

Using Theorem 3, for $\phi \in C_b^1(U)$, the process $(\overline{M}_t^N(\phi), t \geq 0)$ defined as follows is an RCLL square integrable $\overline{\mathcal{F}}_t^N$ −martingale

$$\overline{M}_t^N(\phi) = \langle \overline{\nu}_t^N, \phi\rangle - \langle \overline{\nu}_0^N, \phi\rangle - \int_{s=0}^t \overline{A}^N \langle \overline{\nu}_s^N, \phi\rangle \, ds, \tag{132}$$

where $\overline{A}^N(\cdot)$ is the generator of the Markov process $(\overline{\nu}_t^N, t \geq 0)$. Upon simplification, we have

$$\begin{aligned}\overline{M}_t^N(\phi) = \langle \overline{\nu}_t^N, \phi\rangle - \langle \overline{\nu}_0^N, \phi\rangle - \int_{s=0}^t \langle \overline{\nu}_s^N, \phi'_{\Sigma}\rangle \, ds \\ - \int_{s=0}^t \Bigg(\sum_{n=1}^{\infty}\sum_{j=1}^{n} \int_{x_1}\cdots\int_{x_n} \frac{\beta(x_j)}{n} \\ \times \big(\phi(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) - \phi(n, x_1, \cdots, x_n)\big) \\ \times\, d\overline{\nu}_s^N(n, x_1, \ldots, x_n) \\ + \lambda\Bigg[\Big(\overline{\nu}_s^N(\{0\})\Phi_0(\overline{\nu}_s^N)\,(\phi(1,0)-\phi(0))\Big) \\ + \sum_{n=1}^{\infty}\sum_{j=1}^{n+1} \int_{x_1}\cdots\int_{x_n} \frac{1}{(n+1)}\Phi_n(\overline{\nu}_s^N) \\ \times (\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \phi(n, x_1, \cdots, x_n)) \\ \times\, d\overline{\nu}_s^N(n, x_1, \cdots, x_n)\Bigg]\Bigg) ds.\end{aligned} \tag{133}$$

Further, for $\phi, \psi \in C_b^1(U)$, we have

$$\begin{aligned}< \overline{M}_{\cdot}^N(\phi), \overline{M}_{\cdot}^N(\psi) >_t = \frac{1}{N}\Bigg[\int_{s=0}^t \Bigg(\sum_{n=1}^{\infty}\sum_{j=1}^{n} \int_{x_1}\cdots\int_{x_n} \frac{\beta(x_j)}{n} \\ \times \big(\phi(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) - \phi(n, x_1, \cdots, x_n)\big) \\ \times \big(\psi(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) - \psi(n, x_1, \cdots, x_n)\big) \\ \times\, d\overline{\nu}_s^N(n, x_1, \cdots, x_n) \\ + \lambda\Bigg[\Big(\overline{\nu}_s^N(\{0\})\Phi_0(\overline{\nu}_s^N)\,(\phi(1,0)-\phi(0))\,(\psi(1,0)-\psi(0))\Big) \\ + \sum_{n=1}^{\infty}\sum_{j=1}^{n+1} \int_{x_1}\cdots\int_{x_n} \frac{1}{(n+1)}\Phi_n(\overline{\nu}_s^N) \\ \times (\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \phi(n, x_1, \cdots, x_n)) \\ \times (\psi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \psi(n, x_1, \cdots, x_n)) \\ \times\, d\overline{\nu}_s^N(n, x_1, \cdots, x_n)\Bigg]\Bigg) ds\Bigg].\end{aligned} \tag{134}$$

For establishing the convergence of $(\overline{\nu}_t^N, t \geq 0)$, we first show that the sequence of the processes $\{(\overline{\nu}_t^N)_{t\geq 0}\}$ is relatively compact in $\mathcal{D}_{\mathcal{M}_1(U)}([0,\infty))$. By Prohorov's theorem [4], as the space $\mathcal{M}_1(U)$ endowed with the weak topology is complete and separable, establishing the relative compactness of the sequence of the processes $\{(\overline{\nu}_t^N, t \geq 0)\}$ is equivalent to proving the tightness of the processes $\{(\overline{\nu}_t^N, t \geq 0)\}$.

We next recall the Jakubowski's criteria (From Theorem 4.6 of [15]) which gives the necessary and sufficient condition to have the relative compactness of the sequence of the processes $\{(\overline{\nu}_t^N, t \geq 0)\}$ in Section 7.5. By using conditions J1, J2, C1, and C2 given in Section 7.5, we next give proof of the second part of Theorem 2.

We first focus on establishing the relative compactness of the sequence $(\overline{\nu}_t^N, t \geq 0)$. In this direction, we establish the relative compactness of $(\langle \overline{\nu}_t^N, \phi\rangle, t \geq 0)$ for $\phi \in C_b^1(U)$ in $D_{\mathcal{R}}([0,\infty))$ by establishing condition J2. For this, we need to establish conditions C1 and C2. For any $T > 0$, $t \in [0,T]$, we have

$$\langle \overline{\nu}_t^N, \phi\rangle \leq \|\phi\|_1 \langle \overline{\nu}_t^N, \mathbf{1}\rangle \tag{135}$$

and since $\langle \overline{\nu}_t^N, \mathbf{1}\rangle = 1$, with $b = \|\phi\|_1$, the condition C1 is satisfied.

Now let us look at the proof of condition C2. By using equation (134) and Doob's inequality, for $\epsilon > 0$, we have

$$\mathcal{P}\left(\sup_{t\leq T}\left|\overline{M}_t^N(\phi)\right| \geq \epsilon\right) \leq \frac{4}{\epsilon^2}\mathbb{E}\left[< \overline{M}_{\cdot}^N(\phi) >_T\right] \tag{136}$$

$$\leq \frac{4T}{\epsilon^2}\|\phi\|^2 \frac{1}{N}(\|\beta\| + d\lambda) \to 0 \tag{137}$$

as $N \to \infty$. Therefore from standard convergence criterion in $\mathcal{D}_{\mathcal{R}}([0,T])$, the sequence of processes $(\overline{M}_t^N(\phi), t \geq 0)$ converges in distribution to the null process. Further, we have that the sequence of processes $(\overline{M}_t^N(\phi), t \geq 0)$ is tight in $\mathcal{D}_{\mathcal{R}}([0,T])$ and hence, there exists $\rho_1 > 0$ and $N_1 > 0$ such that for all $N \geq N_1$, we have

$$\mathcal{P}\left(\sup_{u,v\leq T,\,|u-v|\leq\rho_1}\left|\overline{M}_v^N(\phi) - \overline{M}_u^N(\phi)\right| \geq \frac{\gamma}{2}\right) \leq \frac{\epsilon}{2} \tag{138}$$

For any $u < v \leq T$, from equation (133), we have

$$\begin{aligned}\left|\langle \overline{\nu}_v^N, \phi\rangle - \langle \overline{\nu}_u^N, \phi\rangle\right| \leq \int_{s=u}^{v}\left|\langle \overline{\nu}_s^N, \phi'_{\Sigma}\rangle\right| ds + 2\|\beta\|\|\phi\|\,|u-v| \\ + 2\|\phi\|\lambda\,|u-v| \\ + \left|\overline{M}_v^N(\phi) - \overline{M}_u^N(\phi)\right|.\end{aligned} \tag{139}$$

We therefore have

$$\left|\langle \overline{\nu}_v^N, \phi\rangle - \langle \overline{\nu}_u^N, \phi\rangle\right| \leq |v-u|\,\|\phi\|_1(1+2\|\beta\|+2d\lambda) + \left|\overline{M}_v^N(\phi) - \overline{M}_u^N(\phi)\right|. \tag{140}$$

Therefore from equations (138) and (140), there exists some $\rho_2 > 0$ and $N_2 > 0$ such that for $N \geq N_2$, we have

$$\mathcal{P}\left(\sup_{u,v\leq T, |u-v|\leq\rho_2} \left|\langle\overline{v}_v^N, \phi\rangle - \langle\overline{v}_u^N, \phi\rangle\right| \geq \gamma\right) \leq \epsilon. \tag{141}$$

This completes the proof of condition J2.

Now let us look at compact containment condition J1. Suppose at time t, $(n_i(t), x_{i1}(t)\ldots, x_{in_i(t)}(t))$ denotes the state of the i^{th} server where $x_{ij}(t)$ denotes the age of the j^{th} progressing job. We then have

$$\langle\overline{v}_t^N, \Upsilon\rangle = \frac{1}{N}\sum_{i=1, n_i(t)>0}^{N}(x_{i1}(t) + \cdots + x_{in_i(t)}(t)). \tag{142}$$

Let Y_t be the random variable representing the age of a progressing job at time t, and X is a random variable sampled with job length distribution G, then for any $b \geq 0$, we have

$$\mathcal{P}(Y_t \geq b) \leq \mathcal{P}(X \geq b). \tag{143}$$

Further, we have

$$\langle\overline{v}_t^N, \Xi\rangle = \sum_{n=0}^{\infty}\int_{x_1}\cdots\int_{x_n} n\, d\overline{v}_t^N(n, x_1, \cdots, x_n). \tag{144}$$

Let $E^N(t)$ be the number of jobs that arrive into the system in the interval $(0, t]$. At any t, a progressing job in the system could be the one which stays in the system at time $t = 0$ or it could be the one which arrived into the system in the interval $(0, t]$. If a job that is present initially in the system at time $t = 0$ has age a, then its age is upper bounded by $a + t'$ at time $t = t'$. If $Z(0)$ are the number of jobs in the system at time $t = 0$, then at time t, the number of jobs that are in progress from time $t = 0$ is bounded by its initial value $Z(0)$. Further, at time t, the number of jobs that are in progress at time t which had arrived in the interval $(0, t]$ is upper bounded by the total number of arrived jobs $E^N(t)$. Therefore we can write

$$\mathcal{P}(\langle\overline{v}_t^N, \Upsilon\rangle \geq b) \leq \mathcal{P}\left(\left[\langle\overline{v}_0^N, \Upsilon\rangle + t\langle\overline{v}_0^N, \Xi\rangle + \frac{\sum_{j=1}^{E^N(t)} Y_j}{N}\right] \geq b\right) \tag{145}$$

where $(Y_j, 1 \leq j \leq E^N(t))$ are i.i.d random variables sampled according to job length distribution G. Now let us look at convergence of $\frac{\sum_{j=1}^{E^N(t)} Y_j}{N}$ in distribution sense. Here $E^N(t)$ denotes the number of jobs arrived according to a Poisson process with intensity $N\lambda$. However, a Poisson process with intensity $N\lambda$ is equal to the sum of N independent Poisson processes with intensity λ. Therefore, we can write

$$E^N(t) = \sum_{i=1}^{N} E_{(i)}(t) \tag{146}$$

where $E_{(i)}(t)$ denotes the number of arrivals in the time $[0, t]$ from i^{th} Poisson process with intensity λ. We can write

$$\frac{\sum_{j=1}^{E^N(t)} Y_j}{N} = \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{k=1}^{E_{(i)}(t)} Y_{ik}\right) \tag{147}$$

where $\{Y_{ik}\}$ are i.i.d. random variables with job length distribution G. Then by law of large numbers, we have

$$\frac{\sum_{j=1}^{E^N(t)} Y_j}{N} \Rightarrow \frac{\lambda t}{\mu}. \tag{148}$$

By using assumption 2, we have

$$\langle\overline{v}_0^N, \Upsilon\rangle + t\langle\overline{v}_0^N, \Xi\rangle + \frac{\sum_{j=1}^{E^N(t)} Y_j}{N} \Rightarrow \langle\overline{v}_0, \Upsilon\rangle + t\langle\overline{v}_0, \Xi\rangle + \frac{\lambda t}{\mu}. \tag{149}$$

Further, from assumption 2, there exists some M_0 such that

$$\liminf_{N\to\infty} \mathcal{P}(\max(\langle\overline{v}_0^N, \Xi\rangle, \langle\overline{v}_0^N, \Upsilon\rangle) < M_0) > 1 - \gamma. \tag{150}$$

By choosing $M_T = M_0(1 + T) + \frac{2\lambda T}{\mu}$, we have

$$\liminf_{N\to\infty} \mathcal{P}(\sup_{t\in[0,T]} \langle\overline{v}_t^N, \Upsilon\rangle < M_T) > 1 - \gamma. \tag{151}$$

For all $0 < \gamma < 1$, let

$$\mathcal{W}_{T,\gamma} \triangleq \{\zeta \in \mathcal{M}_1(U) : \langle\zeta, \Upsilon\rangle < M_T\}. \tag{152}$$

For $a > 0$ and as $\langle\zeta, \Upsilon\rangle \leq M_T$ for $\zeta \in \mathcal{W}_{T,\gamma}$, for any Borel set of the form $B_n = ([0, a], \cdots, [0, a]) \in \mathcal{B}(U_n)$ with $n \geq 1$ and $B = \{(0)\} \cup (\cup_n B_n)$ and if $\overline{B}$ denotes the complement of B, then

$$\zeta(\overline{B}) \leq \frac{M_T}{a} \tag{153}$$

and hence

$$\lim_{a\to\infty} \sup_{\zeta\in\mathcal{W}_{T,\gamma}} \zeta(\overline{B}) = 0. \tag{154}$$

Now using Lemma A7.5 of [16], $\mathcal{W}_{T,\gamma}$ is relatively compact in $\mathcal{M}_1(U)$. Further, from equation (151), we have

$$\liminf_{N\to\infty} \mathcal{P}(\overline{v}_t^N \in \mathcal{W}_{T,\gamma} \forall t \in [0, T]) > 1 - \gamma. \tag{155}$$

Let $\mathbb{K}_{T,\gamma}$ is the closure of $\mathcal{W}_{T,\gamma}$, we then found a compact set $\mathbb{K}_{T,\gamma}$ such that

$$\liminf_{N\to\infty} \mathcal{P}(\overline{v}_t^N \in \mathbb{K}_{T,\gamma} \forall t \in [0, T]) \geq 1 - \gamma. \tag{156}$$

This proves the condition J1 and therefore the tightness of the sequence of processes $(\overline{v}_t^N, t \geq 0)$ is true.

Suppose $(\boldsymbol{\chi}_t, t \geq 0)$ be a limiting point of a converging subsequence of $(\overline{v}_t^N, t \geq 0)$ then $\boldsymbol{\chi}_0$ almost surely coincides with $\overline{v}_0$ from assumption 2. By using the continuous mapping theorem, we have

$$\begin{aligned}
\langle\chi_t, \phi\rangle = \langle\chi_0, \phi\rangle &+ \int_{s=0}^{t}\langle\chi_s, \phi'_{\Sigma}\rangle\, ds \\
&- \int_{s=0}^{t}\Bigg(\sum_{n=1}^{\infty}\sum_{j=1}^{n}\int_{x_1}\cdots\int_{x_n}\frac{\beta(x_j)}{n} \\
&\times \left(\phi(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) - \phi(n, x_1, \cdots, x_n)\right) \\
&\times\, d\chi_s(n, x_1, \cdots, x_n) \\
&+ \lambda\Bigg[(\chi_s(\{0\})\Phi_0(\chi_s)(\phi(1, 0) - \phi(0))) \\
&+ \sum_{n=1}^{\infty}\sum_{j=1}^{n+1}\int_{x_1}\cdots\int_{x_n}\frac{1}{(n+1)}\Phi_n(\chi_s) \\
&\times (\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \phi(n, x_1, \cdots, x_n))
\end{aligned}$$

$$\times\, d\chi_s(n, x_1, \cdots, x_n)\Bigg]\Bigg)ds. \quad (157)$$

We now show that the sample paths of $(\boldsymbol{\chi}_t, t \geq 0)$ coincide almost surely with the unique mean-field solution. For this, we first show that by observing that the sample paths $(\chi_t, t \geq 0) \in C_{\mathcal{M}_1(U)}([0,\infty))$ since $C_b^1(U)$ is a separating class of $\mathcal{M}_1(U)$, for given initial point η_0, we establish any process $(\eta_t, t \geq 0) \in C_{\mathcal{M}_1(U)}([0,\infty))$ is a solution to equation (157) iff it is a solution to the mean-field equation (32). We give proof of this in Section 7.4. Finally, since there exists unique solution to the mean-field equation for given initial point, from assumption 2, we have that all the limiting points have almost surely identical sample paths coinciding with the mean-field solution. Therefore the sequence of processes $(\overline{v}_t^N, t \geq 0)$ converges in distribution to the unique mean-field solution denoted by $(\overline{v}_t, t \geq 0)$. □

7.4 Evolution of $(\langle \eta_t, \psi\rangle, t \geq 0)$ for $\psi \in C_b(U)$

Proof. We first show that any process $(\eta_t, t \geq 0)$ that satisfies equation (157) also satisfies equation (32). For this, for $\phi \in C_b^1(U)$, if the integrand in equation (157) is a continuous function of s, a real valued process $(\langle \eta_t, \phi\rangle, t \geq 0)$ satisfying the equation (157) is a solution to the following differential equation

$$\frac{d\langle \eta_t, \phi\rangle}{dt} = \langle \eta_t, \phi'_{\Sigma}\rangle + \Bigg(\sum_{n=1}^{\infty}\sum_{j=1}^{n}\int_{x_1}\cdots\int_{x_n}\frac{\beta(x_j)}{n}$$
$$\times\left(\phi(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) - \phi(n, x_1, \cdots, x_n)\right)$$
$$\times\, d\eta_t(n, x_1, \cdots, x_n)$$
$$+\lambda\Bigg[(\eta_t(\{0\})\Phi_0(\eta_t)(\phi(1,0) - \phi(0)))$$
$$+\sum_{n=1}^{\infty}\sum_{j=1}^{n+1}\int_{x_1}\cdots\int_{x_n}\frac{1}{(n+1)}\Phi_n(\eta_t)$$
$$\times(\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \phi(n, x_1, \cdots, x_n))$$
$$\times\, d\eta_t(n, x_1, \cdots, x_n)\Bigg]\Bigg). \quad (158)$$

Therefore we need to show that the two terms on the right side of equation (158) are continuous functions of t. The first term $\langle \eta_t, \phi'_{\Sigma}\rangle$ is a continuous function of t since $\phi \in C_b^1(U)$ and the mapping $t \mapsto \eta_t$ is continuous. The second term that corresponds to the case of departures can be written as

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n}\int_{x_1}\cdots\int_{x_n}\frac{\beta(x_j)}{n}$$
$$\times\left(\phi(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) - \phi(n, x_1, \cdots, x_n)\right)$$
$$\times\, d\eta_t(n, x_1, \cdots, x_n) = \langle \eta_t, \tilde{\psi}\rangle, \quad (159)$$

where the function $\tilde{\psi}$ is defined such that

$$\tilde{\psi}(0) = 0 \quad (160)$$

and for $n \geq 1$

$$\tilde{\psi}(n, x_1, \cdots, x_n) =$$
$$\sum_{j=1}^{n}\frac{\beta(x_j)}{n}((\phi(n-1, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) - \phi(n, x_1, \ldots, x_n)). \quad (161)$$

Since $\phi \in C_b^1(U)$ and $\beta \in C_b(\mathcal{R}_+)$, we have that $\tilde{\psi} \in C_b(U)$. Therefore $\langle \eta_t, \tilde{\psi}\rangle$ is a continuous function of t. Now let us look at the expression that corresponds to the case of arrivals, we can write

$$\lambda\Bigg[(\eta_t(\{0\})\Phi_0(\eta_t)(\phi(1,0) - \phi(0)))$$
$$+\sum_{n=1}^{\infty}\sum_{j=1}^{n+1}\int_{x_1}\cdots\int_{x_n}\frac{1}{(n+1)}\Phi_n(\eta_t)$$
$$\times(\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \phi(n, x_1, \cdots, x_n))$$
$$\times\, d\eta_t(n, x_1, \cdots, x_n)$$
$$= \langle \eta_t, \psi_{(\eta_t)}\rangle, \quad (162)$$

where $\psi_{(\eta_t)}$ is defined as

$$\psi_{(\eta_t)}(n, x_1, \cdots, x_n) = \frac{\lambda}{(n+1)}\Phi_n(\eta_t)$$
$$\times(\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \phi(n, x_1, \cdots, x_n)). \quad (163)$$

We have that $\psi_{\eta_t} \in C_b(U)$ since $\phi \in C_b(U)$. Hence for some fixed $b \geq 0$, the mapping $t \mapsto \langle \eta_t, \psi_{(\eta_b)}\rangle$ is continuous. To establish continuity of the mapping $t \mapsto \langle v_t, \psi_{(\eta_t)}\rangle$, we need to show $\langle \eta_{t+b}, \psi_{(\eta_{t+b})}\rangle \to \langle \eta_t, \psi_{(\eta_t)}\rangle$ as $b \to 0$. We can write

$$\left|\langle \eta_{t+b}, \psi_{(\eta_{t+b})}\rangle - \langle \eta_t, \psi_{(\eta_t)}\rangle\right| \leq \left|\langle \eta_{t+b}, \psi_{(\eta_{t+b})}\rangle - \langle \eta_{t+b}, \psi_{(\eta_t)}\rangle\right|$$
$$+\left|\langle \eta_{t+b}, \psi_{(\eta_t)}\rangle - \langle \eta_t, \psi_{(\eta_t)}\rangle\right|. \quad (164)$$

As $\psi_{(\eta_t)} \in C_b(U)$, we have

$$\lim_{b\to 0}\left|\langle \eta_{t+b}, \psi_{(\eta_t)}\rangle - \langle \eta_t, \psi_{(\eta_t)}\rangle\right| = 0. \quad (165)$$

We next show

$$\lim_{b\to 0}\left|\langle \eta_{t+b}, \psi_{(\eta_{t+b})} - \psi_{(\eta_t)}\rangle\right| = 0. \quad (166)$$

Consider a $L > 0$ and let

$$V^{(L)} = \{(n, x_1, \ldots, x_n) \in U_n : n \geq 1, x_i > L \text{ for all } 1 \leq i \leq n\}. \quad (167)$$

For given $\epsilon > 0$, we can choose $L > 0$ such that we have

$$\langle \eta_t, I_{\{V^{(L)}\}}\rangle < \epsilon. \quad (168)$$

From continuity of $t \mapsto \eta_t$, there exists some $r_1 > 0$ such that for all $b \in [-\min(t, r_1), r_1]$,

$$\langle \eta_{t+b}, I_{\{V^{(L)}\}}\rangle < \epsilon. \quad (169)$$

Further, $\psi_{(\eta_t)}$ is a continuous function of t as $\overline{R}_n(\eta_t) = \langle \eta_t, I_{\{\cup_{j=n}^{\infty} U_j\}}\rangle$ is a continuous function of t. Hence, $\psi_{(\eta_{t+b})}$ is uniformly continuous on the interval $b \in [-\min(t, r_1), r_1]$ and $\underline{u} \in \overline{V}^{(L)}$ (the complement of $V^{(L)}$). Therefore there exists some $r_2 \in (0, r_1)$ such that for $b \in [-\min(t, r_2), r_2]$, $\underline{u} \in \overline{V}^{(L)}$, we have

$$\left|\psi_{(\eta_{t+b})}(\underline{u}) - \psi_{(\eta_t)}(\underline{u})\right| < \epsilon. \quad (170)$$

Using equations (169)-(170), for $b \in [-\min(t, r_2), r_2]$, we have,

$$\left|\langle \eta_{t+b}, \psi_{(\eta_{t+b})} - \psi_{(\eta_t)} \rangle\right| \leq \epsilon \langle \eta_{t+b}, I_{\left\{\overline{V}^{(L)}\right\}} \rangle + 2d\lambda \|\phi\| \epsilon$$
$$\leq \epsilon + 2d\lambda \|\phi\| \epsilon. \quad (171)$$

Now by letting $b \to 0$ and then $\epsilon \to 0$ in equation (164), we have continuity of the mapping $t \mapsto \langle \eta_t, \psi_{(\eta_t)} \rangle$.

By using the change of variables, we now obtain an alternative form of the equations that are satisfied by any solution to the equation (158). For this, let us define a function $\tilde{\phi}$ from $\phi \in C_b^1(U)$ as follows: For $r \leq t$, let

$$\tilde{\phi}(n, x_1, \ldots, x_n) = \phi\left(n, x_1 + \frac{t-r}{n}, \cdots, x_n + \frac{t-r}{n}\right) \quad (172)$$
$$= \phi(\tau_{t-r}^{+}(n, x_1, \cdots, x_n)) \quad (173)$$
$$= \tau_{t-r}\phi(n, x_1, \cdots, x_n) \quad (174)$$

and $\tilde{\phi}(0) = \phi(0)$. Now consider the change of $\langle \eta_r, \tilde{\phi} \rangle$ $w.r.t.$ the variable 'r'. We have

$$\frac{d\langle \eta_r, \tilde{\phi} \rangle}{dr} = \frac{d\langle \eta_r, \tilde{\phi} \rangle}{dr}|(\text{fixed } \tilde{\phi}) + \frac{d\langle \eta_r, \tilde{\phi} \rangle}{dr}|(\text{fixed } \eta_r) \quad (175)$$

where the first term on the right side represents the change in $\langle \eta_r, \tilde{\phi} \rangle$ for fixed $\tilde{\phi}$ due to change in η_r as a function of r and the second term represents the change in $\langle \eta_r, \tilde{\phi} \rangle$ for fixed η_r due to change in $\tilde{\phi}$ as a function of r. Hence the first term can be computed by using equation (158) and the second term is equal to $-\langle \eta_t, \tilde{\phi}'_{\sum} \rangle$. Therefore, we have

$$\frac{d\langle \eta_r, \tilde{\phi} \rangle}{dr} = \Bigg(\sum_{n=1}^{\infty} \sum_{j=1}^{n} \int_{x_1} \cdots \int_{x_n} \frac{\beta(x_j)}{n}$$
$$\times \left(\tilde{\phi}(n-1, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) - \tilde{\phi}(n, x_1, \ldots, x_n) \right)$$
$$\times\, d\eta_r(n, x_1, \cdots, x_n)$$
$$+ \lambda \Bigg[\left(\eta_r(\{0\}) \Phi_0(\eta_r) \left(\tilde{\phi}(1, 0) - \tilde{\phi}(0) \right) \right)$$
$$+ \sum_{n=1}^{\infty} \sum_{j=1}^{n+1} \int_{x_1} \cdots \int_{x_n} \frac{1}{(n+1)} \Phi_n(\eta_r)$$
$$\times (\tilde{\phi}(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \tilde{\phi}(n, x_1, \cdots, x_n))$$
$$\times\, d\eta_r(n, x_1, \cdots, x_n) \Bigg] \Bigg). \quad (176)$$

By integrating $\frac{d\langle \eta_r, \tilde{\phi} \rangle}{dr}$ with respect to r from 0 to t, we get equation (32).

We next show that for $\phi \in C_b^1(U)$, any solution $(\langle \overline{\nu}_t, \phi \rangle, t \geq 0)$ of the equation (32) also satisfies the equation (157). For this, we need to show that the differentiation of $\langle \overline{\nu}_t, \phi \rangle$ with respect to t exists. The existence of $\frac{d\langle \overline{\nu}_0, \tau_t \phi \rangle}{dt}$ follows from bounded convergence theorem since $\phi \in C_b^1(U)$. The existence of the differentiation of the second term on the right side of equation (32) with respect to t follows from Leibniz integral rule. According to this rule, we first need to show that the integrand is continuous with respect to both the variables r and t. This follows from the same arguments as that of the continuity of the integrand in equation (157). After that we have to show that the differentiation of the integrand with respect to t exists and further, the differential should be continuous with respect to both the variables r and t. Since $\phi \in C_b^1(U)$, by the bounded convergence theorem, the differentiation of the integrand exists and further, it is continuous with respect to r and t from the similar arguments as that of the continuity of the integrand in equation (157). Hence, any process $\overline{\nu}_t \in C_{\mathcal{M}_1(U)}([0, \infty))$ is a solution to the equation (157) if and only if it is a solution to the equation (32). Note that ϕ in equation (32) need not be differentiable. □

7.5 Conditions J1, J2, C1, C2

Jakubowski's criteria:

A sequence of $\{X^N\}$ of $\mathcal{D}_{\mathcal{M}_1(U)}([0, \infty))$– valued random elements defined on $(\Omega, \mathbb{F}, \mathbb{P})$ is tight if and only if the following two conditions are satisfied:

J1: For each $T > 0$ and $\gamma > 0$, there exists a compact set $\mathbb{K}_{T,\gamma} \subset \mathcal{M}_1(U)$ such that

$$\liminf_{N \to \infty} \mathbb{P}(X_t^N \in \mathbb{K}_{T,\gamma} \forall t \in [0, T]) > 1 - \gamma. \quad (177)$$

This condition is also referred as the compact-containment condition.

J2: There exists a family Q of real valued continuous functions F defined on $\mathcal{M}_1(U)$ that separates points in $\mathcal{M}_1(U)$ and is closed under addition such that for every $F \in Q$, the sequence $\{(F(X_t^N), t \geq 0)\}$ is tight in $\mathcal{D}_{\mathcal{R}}([0, \infty))$.

To prove condition J2, we consider a class of functions Q as follows.

$$Q \triangleq \{F : \exists f \in C_b^1(U) \text{ such that } F(\eta) = \langle \eta, f \rangle,\ \forall \eta \in \mathcal{M}_1(U)\} \quad (178)$$

The class of functions Q defined in equation (178) can be considered to prove condition J2 as every function $F \in Q$ is continuous w.r.t. the weak topology on $\mathcal{M}_1(U)$ and further, the class of functions Q separates points in $\mathcal{M}_1(U)$ and closed under addition.

We next state the following sufficient condition (From Theorem $C.9$,[26]) to prove condition J2.

Tightness in $\mathcal{D}_{\mathcal{R}}([0, T])$: If $S = \mathcal{D}_{\mathcal{R}}([0, T])$ and $(\mathbb{P}_n)$ is a sequence of probability distributions on S, then $(\mathbb{P}_n)$ is tight if for any $\epsilon > 0$,

C1: There exists b such that

$$\mathbb{P}_n(|X(0)| > b) \leq \epsilon \quad (179)$$

for all $n \in \mathcal{Z}_+$

C2: For any $\gamma > 0$, there exists $\rho > 0$ such that

$$\mathbb{P}_n(w_X(\rho) > \gamma) \leq \epsilon \quad (180)$$

for n sufficiently large, where

$$w_X(\rho) = \sup\{|X(t) - X(s)| : s, t \leq T, |s - t| \leq \rho\} \quad (181)$$

and any limiting point $\mathbb{P}$ satisfies $\mathbb{P}(C_{\mathcal{R}}([0, T])) = 1$.

7.6 Proof of Corollary 1:

Proof. By assuming that $\overline{\nu}_0$ is absolutely continuous $w.r.t.$ Lebesgue measure at all $\underline{u} \in U_n$ for $n \geq 1$, we have absolutely continuity of $\overline{\nu}_t$ at all $\underline{u} \in U_n$ for $n \geq 1$ and $t \geq 0$. Let $p_t(0)$ denotes $\overline{\eta}_t(\{0\})$ and the Radon-Nikodym derivative of $\overline{\nu}_t$ $w.r.t.$ Lebesgue measure at $(n, x_1, \ldots, x_n)$ be denoted by $p_t(n, x_1, \ldots, x_n)$. Now let us construct a process $P_t = (P_t(\underline{u}), \underline{u} \in U)$ as follows,

$$P_t(n, y_1, \ldots, y_n) = \int_{x_1=0}^{y_1} \cdots \int_{x_n=0}^{y_n} p_t(n, x_1, \cdots, x_n)\, dx_1 \cdots dx_n. \tag{182}$$

We now look at $\langle \overline{\nu}_t, \hat{\phi} \rangle$ where $\hat{\phi} = I_{\{\underline{l} \in U_n : 0 \le l_i \le y_i, \forall i\}}$. For given absolutely continuous measure η that has no atoms,

$$\langle \eta, \hat{\phi} \rangle = \langle \eta, \psi \rangle, \tag{183}$$

where $\psi = I_{\{\underline{u} \in U_n : 0 < l_i < y_i, \forall i\}}$. We first recall the property that for any open set O in U_n, $n \ge 1$, there exists a sequence of functions $\{f_n\} \in C_b(U)$ that increase point wise to $I_{\{O\}}$. Now by using monotone convergence theorem and equation (32), we get that the equation (32) is also true for the function ψ (Indicators on open sets). Due to absolutely continuity of $\overline{\nu}_s$ for all $s \ge 0$, we have that equation (32) is true with the function $\hat{\phi}$ (Indicators on closed sets). Using equation (32), we now obtain the evolution equations for the process $(P_t, t \ge 0)$ that satisfy $P_t(n, y_1, \ldots, y_n) = \langle \overline{\nu}_t, \hat{\phi} \rangle$.

We then get final expression for the process $(P_t(\underline{u}), \underline{u} \in U, t \ge 0)$ using equation (32) and using the following observation

$$\langle \overline{\nu}_s, \tau_b I_{\{\underline{l} \in U_n : 0 \le l_i \le y_i, \forall i\}} \rangle = \langle \overline{\nu}_s, I_{\{\underline{l} \in U_n : 0 \le l_i + \frac{b}{n} \le y_i, \forall i\}} \rangle \tag{184}$$

$$= \langle \overline{\nu}_s, I_{\{\underline{l} \in U_n : 0 \le l_i \le y_i - \frac{b}{n}, \forall i\}} \rangle. \tag{185}$$

By simplifications, we obtain the set of partial differential equations for the process $P_t(n, y_1, \ldots, y_n)$ as in equations (35)-(36). ◻

7.7 Single server system with pre-specified arrival rates

Consider a single server system in which jobs arrive according to a Poisson process with intensity α_n when there are n jobs in progress at the server. The job lengths are sampled according to general distribution $G(\cdot)$ as in the system model. If $\nu_t^{(single)}$ denotes the probability measure for server occupancies at time t, then it is verified that the Kolmogorov equations are given by, for $\phi \in C_b^1(U)$,

$$\begin{aligned}
\langle \nu_t^{(single)}, \phi \rangle &= \langle \nu_0^{(single)}, \phi \rangle + \int_{s=0}^{t} \langle \nu_s^{(single)}, \phi'_{\sum} \rangle\, ds \\
&- \int_{s=0}^{t} \Bigg(\sum_{n=1}^{\infty} \sum_{j=1}^{n} \int_{x_1} \cdots \int_{x_n} \frac{\beta(x_j)}{n} \\
&\times \left(\phi(n-1, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) - \phi(n, x_1, \cdots, x_n)\right) \\
&\times d\nu_s^{(single)}(n, x_1, \cdots, x_n) \\
&+ \Bigg[\left(\alpha_0 \nu_s^{(single)}(\{0\}) \left(\phi(1,0) - \phi(0)\right)\right) + \sum_{n=1}^{\infty} \sum_{j=1}^{n+1} \int_{x_1} \cdots \int_{x_n} \frac{1}{(n+1)} \\
&\times \alpha_n (\phi(n+1, x_1, \cdots, x_{j-1}, 0, x_j, \cdots, x_n) - \phi(n, x_1, \cdots, x_n)) \\
&\times d\nu_s^{(single)}(n, x_1, \cdots, x_n) \Bigg] \Bigg) ds.
\end{aligned} \tag{186}$$

Now let the Radon-Nikodym derivative of the measure $\nu_t^{(single)}$ at $\underline{u} \in U$ be denoted by $p_t^{(single)}(\underline{u})$. We can derive the differential equations satisfied by the density function $p_t^{(single)} = (p_t^{(single)}(\underline{u}), \underline{u} \in U)$ by using the similar procedure as in [19, 28, 35, 36]. We then obtain the differential equations for the process $P_t^{(single)} = (P_t^{(single)}(\underline{u}), \underline{u} \in U)$ where

$$P_t^{(single)}(n, y_1, \cdots, y_n) = \int_{x_1=0}^{y_1} \cdots \int_{x_n=0}^{y_n} p_t^{(single)}(n, x_1, \cdots, x_n)\, dx_1 \cdots dx_n, \tag{187}$$

are given by

$$\frac{dP_t^{(single)}(0)}{dt} = \int_{y=0}^{\infty} \beta(y) \left(\frac{\partial P_t^{(single)}(1, y)}{\partial y} \right) dy - \alpha_0 P_t^{(single)}(0), \tag{188}$$

for $n \ge 1$,

$$\begin{aligned}
\frac{dP_t^{(single)}(n, y_1, \cdots, y_n)}{dt} &= -\sum_{i=1}^{n} \frac{1}{n} \frac{\partial P_t^{(single)}(n, y_1, \ldots, y_n)}{\partial y_i} \\
+ \sum_{j=1}^{n+1} \int_{x_j=0}^{\infty} \frac{\beta(x_j)}{n+1} &\left(\frac{\partial P_t^{(single)}(n+1, y_1, \cdots, y_{j-1}, x_j, y_j, \cdots, y_n)}{\partial x_j} \right) dx_j \\
- \sum_{j=1}^{n} \int_{x_j=0}^{y_j} \frac{\beta(x_j)}{n} &\left(\frac{\partial P_t^{(single)}(n, y_1, \cdots, y_{j-1}, x_j, y_{j+1}, \cdots, y_n)}{\partial x_j} \right) dx_j \\
+ \sum_{j=1}^{n} &\left(\frac{\alpha_{n-1}}{n} \right) P_t^{(single)}(n-1, y_1, \cdots, y_{j-1}, y_{j+1}, \cdots, y_n) \\
&- \alpha_n P_t^{(single)}(n, y_1, \cdots, y_n).
\end{aligned} \tag{189}$$

From [9], for single server processor sharing system with pre-specified state-dependent arrival rate α_i when there are i jobs in progress and job lengths are generally distributed with finite mean $\frac{1}{\mu}$, the unique stationary distribution $\boldsymbol{\pi}^{(single)} = (\pi^{(single)}(\underline{u}), \underline{u} \in U)$ is given by,

$$\pi^{(single)}(n, y_1, \cdots, y_n) = \frac{\left(\prod_{i=1}^{n} \frac{\alpha_{i-1}}{\mu} \right)}{1 + \sum_{m=1}^{\infty} \left(\prod_{i=1}^{m} \frac{\alpha_{i-1}}{\mu} \right)} \times \mu^n \prod_{i=1}^{n} \int_{x_i=0}^{y_i} \overline{G}(x_i)\, dx_i \tag{190}$$

and

$$\pi^{(single)}(0) = \frac{1}{1 + \sum_{m=1}^{\infty} \left(\prod_{i=1}^{m} \frac{\alpha_{i-1}}{\mu} \right)}. \tag{191}$$

7.8 Proof of Theorem 4

Proof. Note that the first part of Theorem 4 is a special case of the second part. Hence, it is sufficient to prove the second part.

For $t \ge 0$, we define $\overline{\nu}_t^N$, the probability measure on U such that $\overline{\nu}_t^N(\{\underline{u}\})$ for $\underline{u} \in U$ denotes the fraction of servers lying in state $\underline{u} \in U$ at time t.

From the dynamics of the system under SQ(d) scheme and the exchangeability of $\{q_k^{(N)}(0), 1 \le k \le N\}$ implies that the collection $\{q_k^{(N)}(t), 1 \le k \le N\}$ is also exchangeable for all $t \in [0, \infty)$. Further, from Theorem 2, we have $\overline{\nu}_t^N \Rightarrow \overline{\nu}_t$ for $t \in [0, \infty)$ as $N \to \infty$.

To prove the result, it is sufficient to show that the following holds:

$$\mathbb{E}\left[\prod_{k=1}^{l}\phi_k(q_k^{(N)}(t))\right] \to \prod_{k=1}^{l}\langle\overline{\nu}_t,\phi_k\rangle, \tag{192}$$

for all continuous bounded mappings $\phi_k : U \to \mathcal{R}_+$ as $N \to \infty$.

We can write

$$\begin{aligned}&\left|\mathbb{E}\left[\prod_{k=1}^{l}\phi_k\left(q_k^{(N)}(t)\right)\right] - \prod_{k=1}^{l}\langle\overline{\nu}_t,\phi_k\rangle\right| \\ &\quad\le \left|\mathbb{E}\left[\prod_{k=1}^{l}\phi_k\left(q_k^{(N)}(t)\right)\right] - \mathbb{E}\left[\prod_{k=1}^{l}\langle\overline{\nu}_t^N,\phi_k\rangle\right]\right| \\ &\qquad + \left|\mathbb{E}\left[\prod_{k=1}^{l}\langle\overline{\nu}_t^N,\phi_k\rangle\right] - \prod_{k=1}^{l}\langle\overline{\nu}_t,\phi_k\rangle\right|. \end{aligned} \tag{193}$$

Note that from Theorem 2, the second term on the right hand side of the above inequality vanishes as $N \to \infty$. Now, due to exchangeability, the permutation of states between servers does not affect the joint distribution. Hence, we have

$$\mathbb{E}\left[\prod_{k=1}^{l}\phi_k\left(q_k^{(N)}(t)\right)\right] = \frac{1}{(N)_l}\mathbb{E}\left[\sum_{\sigma\in Q(l,N)}\prod_{k=1}^{l}\phi_k\left(q_{\sigma(k)}^{(N)}(t)\right)\right] \tag{194}$$

where $(N)_k = N(N-1)\dots(N-k+1)$, and $Q(r,n)$ denotes the set of all permutations of the numbers $\{1,2,\dots,n\}$ taken r at a time. Also, by definition of $\overline{\nu}_t^N$ we have

$$\mathbb{E}\left[\prod_{k=1}^{l}\langle\overline{\nu}_t^N,\phi_k\rangle\right] = \mathbb{E}\left[\left(\prod_{k=1}^{l}\frac{1}{N}\sum_{j=1}^{N}\phi_k\left(q_j^{(N)}(t)\right)\right)\right] \tag{195}$$

Hence, the first term on the right hand side of (193) can be bounded as follows

$$\begin{aligned}&\left|\mathbb{E}\left[\prod_{k=1}^{l}\phi_k\left(q_k^{(N)}(t)\right)\right] - \mathbb{E}\left[\prod_{k=1}^{l}\langle\overline{\nu}_t^N,\phi_k\rangle\right]\right| \\ &\le 2B^l\left(1-\frac{(N)_l}{(N)^l}\right) \qquad \to 0 \text{ as } N\to\infty\end{aligned}$$

where $\max_k \|\phi_k\| = B$. This completes the proof. □