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Comprehensive factorization and I-central extensions

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ABSTRACT

We show that, for a regular reflection functor I between efficiently regular categories, the reflection of an extension to an I-central extension is reduced to the comprehensive factorization of an explicit internal functor. We then analyse the Mal'tsev context where similar results are obtained under weaker conditions on I.

0. Introduction

Two well-known and, apparently, independent subjects are brought together in this work: central extensions and factorization of functors. The aim is to make explicit the unexpected relationship between them in order to provide the reflection of extensions to central extensions.

The categorical Galois theory developed by Janelidze in [22,23] gave a generalized interpretation of the classical Galois theory, completing the work of other authors (see references in [22,23]). Several examples were investigated and exhibited, eventually, a strong analogy between the notion of "covering" and the notion of a "central" morphism. This led to the introduction of the definition of a *central extension* with respect to a reflection $I : \mathbb{D} \to \mathbb{C}$ associated with the inclusion of an admissible subcategory of an exact category [1], $j : \mathbb{C} \rightarrowtail \mathbb{D}$; see [24] and also [20]. These I-central extensions form, in turn, a full subcategory of all extensions in $\mathbb{D}$ and simultaneously the question of its reflectiveness was raised.

Positive answers to this question have been given by several authors. For a quite general setting, G. Janelidze and G.M. Kelly give in [25] sufficient conditions for the existence of such a reflector. In fact, the problem is successively reduced to giving a reflection for more appropriate categories. Although the context that we work in does not fulfil these conditions, it is still possible to adapt the process developed in [25], mainly because I-central extensions are I-normal (Proposition 2.10). The resulting reflection can be shown to be the same as the one that we describe in Section 2. However, our construction is fairly direct and simple, besides exploring the connection with a special factorization of functors. Some other known results are given: Theorem 2.8.11 of [2] with an explicit description of the reflector for a finitely cocomplete Mal'tsev category $\mathbb{D}$ and its subcategory of abelian objects $\mathbb{C} = \mathbb{D}_{Ab}$; Remark 4.4 of [26] for a description in the Mal'tsev varietal case; Lemma 4.3 of [18] where the reflector is expressed for a semi-abelian category $\mathbb{D}$ and any Birkhoff (and thus admissible) subcategory $\mathbb{C}$ (also for higher-dimensional extensions).

The *comprehensive factorization* of a functor in the set-theoretical context gives a factorization into a final functor and a discrete fibration [33]. The generalization of such a factorization into an exact context was first done in [4]. Such a factorization represented an important tool in the development of the general non-abelian cohomology theory where internal n-groupoids played the role of chain complexes [5]. It turns out that the factorization of specific internal functors still holds in the more general context of *efficiently regular* categories [8], a notion intermediate between those of regular [1] and exact categories. So, the range of examples can be widened to include many topological situations such as the categories

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of topological or Hausdorff (abelian) groups, topological or Hausdorff Lie algebras and, more generally, any category of topological or Hausdorff algebras in the sense of [3].

The answer to our initial question (Theorem 2.17) is given for an efficiently regular category $\mathbb{C}$ such that the functor I satisfies suitable left exact conditions: given any extension $f : X \twoheadrightarrow Y$, the reflection to an I-central extension $\bar{f} : \bar{X} \twoheadrightarrow Y$ is reduced to the comprehensive factorization of the upper internal functor $\underline{\eta}_1 f : R[f] \to IR[f]$, of the following diagram where $R[f]$ denotes the kernel equivalence relation of f, and $\bar{f}$ the quotient of the domain R of the discrete fibration $\underline{\overline{\eta}}_1$ involved in this factorization:

$$\begin{tikzcd}
R[f] \arrow[r, "R(\hat{\eta})"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R \arrow[r, two heads, "\overline{\eta}_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\
X \arrow[r, "\hat{\eta}"'] \arrow[d, "f"'] & \bar{X} \arrow[r, two heads, "\overline{\eta}_0"'] \arrow[d, "\bar{f}"] & IX \arrow[d, "If"] \\
Y \arrow[r, equal] & Y \arrow[r, two heads, "\eta_Y"'] & IY
\end{tikzcd}$$

The suitable but relatively strong left exact conditions needed in the general situation become much simpler in the Mal'tsev context [15,16] on which we focus our attention and which deals with many subtle variations about the left exactness of the functor I. These conditions hold for any inclusion $j : \mathbb{C} \rightarrowtail \mathbb{D}$ where $\mathbb{D}$ is a Mal'tsev variety and $\mathbb{C}$ is any subvariety. In particular, they hold for the varieties $\mathbb{D}$ of (abelian) groups, quasi-groups, R-modules (for a fixed ring R), (commutative) rings, associative algebras, Lie algebras and Heyting algebras. Given any finitely cocomplete efficiently regular Mal'tsev category $\mathbb{D}$, they also hold when $\mathbb{C}$ is the subcategory $\mathbb{D}_{Ab}$ of *abelian* objects of $\mathbb{D}$. As shown in [10], an application of our results to this last inclusion produces some cohomology isomorphisms.

The article is organized along the following lines: the first section refreshes the results of [4] concerning the comprehensive factorization; Section 2 deals with the above specified result in the efficiently regular context; Section 3 modulates the assumptions to the regular Mal'tsev context, and finally the last section is devoted to the correlated, but more specific question of whether the functor I preserves products.

1. The comprehensive factorization

In this section we recall the basic notions and properties concerning internal groupoids and internal functors needed to obtain the comprehensive factorization of internal functors [4] for an efficiently regular context [8].

We shall suppose that all our categories $\mathbb{E}$ are finitely complete. Given the following right hand side commutative square, we denote the kernel equivalence relation of f by $R[f]$ and the induced map between the kernel equivalences by $R(x)$:

$$\begin{tikzcd}
R[f] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] \arrow[d, "R(x)"'] & X \arrow[l, "s_0"' description] \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\
R[f'] \arrow[r, shift left=2] \arrow[r, shift right=2] & X' \arrow[l] \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

1.1. The shifting functor Dec

An internal groupoid $\underline{X}_1$ in $\mathbb{E}$ will be presented (see [4]) as a reflexive graph $(d_0, d_1) : X_1 \rightrightarrows X_0$ endowed with an operation $p_2 : R[d_0] \to X_1$:

$$\begin{tikzcd}
R^2[d_0] \arrow[r, shift left=2, "p_2"] \arrow[r, "p_1" description] \arrow[r, shift right=2, "p_0"'] \arrow[r, bend left=60, "R(p_2)"] & R[d_0] \arrow[r, shift left=1, "p_1"] \arrow[r, shift right=1, "p_0"'] \arrow[r, bend left=60, "p_2"] & X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X_0 \arrow[l, "s_0" description]
\end{tikzcd}$$

making the previous diagram satisfy all the simplicial identities, including the degeneracies, which express the usual axioms for a groupoid. In a set-theoretical context, this operation p_2 associates the composite $\psi.\phi^{-1}$ with any pair (ϕ, ψ) of arrows of $\underline{X}_1$ with the same domain. In particular, any equivalence relation $(p_0, p_1) : R \rightrightarrows X$ on an object X in $\mathbb{E}$ provides an internal groupoid.

Let $Grd\,\mathbb{E}$ denote the category of internal groupoids and internal functors in $\mathbb{E}$, and $()_0 : Grd\,\mathbb{E} \to \mathbb{E}$ the forgetful functor associating with an internal groupoid $\underline{X}_1$ its "object of objects" X_0. This functor is a left exact fibration. Any fibre above an object X has a terminal object ∇X, the indiscrete equivalence relation on the object X given by the product projections $(p_0, p_1) : X \times X \rightrightarrows X$, and an initial object ΔX, the discrete equivalence relation on X given by $(1_X, 1_X) : X \rightrightarrows X$. They produce respectively a right adjoint and a left adjoint of the forgetful functor $()_0$. An internal functor $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ is

$()_0$-cartesian if and only if the following square is a pullback in $\mathbb{E}$, or in other words if and only if it is internally fully faithful:

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "{(d_0,d_1)}"'] & Y_1 \arrow[d, "{(d_0,d_1)}"] \\ X_0 \times X_0 \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0 \end{tikzcd}$$

We shall need the following definition:

Definition 1.1. An internal functor $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ is said to be a *discrete fibration* when the following solid square is a pullback:

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, shift right=2, "d_1"'] \arrow[d, dotted, shift left=2, "d_0"] & Y_1 \arrow[d, shift right=2, "d_1"'] \arrow[d, dotted, shift left=2, "d_0"] \\ X_0 \arrow[u, dotted] \arrow[r, "f_0"'] & Y_0 \arrow[u, dotted] \end{tikzcd}$$

Since the codomain $\underline{Y}_1$ is an internal groupoid, then the square with d_0 is a pullback as well. It is easy to check that, when $\underline{f}_1$ is a discrete fibration and its codomain $\underline{Y}_1$ is an equivalence relation, then the same holds for its domain $\underline{X}_1$. The class *Disf* of discrete fibrations contains the isomorphisms, is stable under composition and has the property that when $\underline{g}_1.\underline{f}_1$ and $\underline{g}_1$ are in *Disf*, then $\underline{f}_1$ is in *Disf*. The discrete fibrations are also stable under pullbacks.

Given an internal groupoid $\underline{X}_1$, we define *Dec* $\underline{X}_1$ (the *décalage* of $\underline{X}_1$) as the following internal groupoid obtained by shifting the indexation:

$$\begin{tikzcd} R^3[d_0] \arrow[r, shift left=2, "p_2"] \arrow[r, "p_1" description] \arrow[r, shift right=2, "p_0"'] \arrow[r, bend left=60, "p_3"] & R^2[d_0] \arrow[r, shift left=1, "p_1"] \arrow[r, shift right=1, "p_0"'] \arrow[r, bend left=60, "p_2"] & R[d_0] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] & X_1 \arrow[l, "s_0" description] \end{tikzcd}$$

It is the kernel equivalence relation of the map $d_0 : X_1 \to X_0$. We denote by $\underline{\epsilon}_1\underline{X}_1 : Dec\,\underline{X}_1 \to \underline{X}_1$ the following internal functor:

$$\begin{tikzcd} R[d_0] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] \arrow[d, "p_2"'] & X_1 \arrow[l] \arrow[d, "d_1"] \\ X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l] \end{tikzcd}$$

which is a discrete fibration and a regular epimorphism in $Grd\,\mathbb{E}$, since it is levelwise split. It is clear that this shifting construction *Dec* is functorial and left exact and that the internal functors $\underline{\epsilon}_1$ determine a natural transformation $Dec \Rightarrow Id$ (which is actually underlying a comonad; see [4]). Moreover the following diagram, in the category $Grd\,\mathbb{E}$, is a kernel equivalence relation with its quotient:

$$\begin{tikzcd} Dec^2\underline{X}_1 \arrow[r, shift left=1, "\underline{\epsilon}_1 Dec\,\underline{X}_1"] \arrow[r, shift right=1, "Dec\,\underline{\epsilon}_1\underline{X}_1"'] & Dec\,\underline{X}_1 \arrow[r, two heads, "\underline{\epsilon}_1\underline{X}_1"] & \underline{X}_1 \end{tikzcd} \tag{1}$$

1.2. *The support of an internal groupoid in the regular context*

Recall that a category $\mathbb{E}$ is regular [1] when regular epimorphisms are stable under pullbacks and any effective equivalence relation (i.e. which is the kernel equivalence relation of some map) admits a quotient. Regular categories are also characterized by the existence of a pullback-stable (regular epimorphism, monomorphism) factorization system. Let us begin by recalling a well-known theorem:

Theorem 1.2 (*Barr–Kock*)**.** *Let $\mathbb{E}$ be a regular category. Given any commutative diagram:*

$$\begin{tikzcd} R[f] \arrow[r, shift left=1, "p_0"] \arrow[r, shift right=1, "p_1"'] \arrow[d, "R(x)"'] & X \arrow[r, two heads, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\ R[f'] \arrow[r, shift left=1] \arrow[r, shift right=1] & X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

where f is a regular epimorphism, then the right hand side square is a pullback if and only if the internal functor $R(x) : R[f] \to R[f']$ is a discrete fibration.

Corollary 1.3 ([24]). *Let $\mathbb{E}$ be a regular category. Suppose that the following whole rectangle and the left hand side square are pullbacks:*

$$\begin{tikzcd} X' \arrow[r, two heads, "x"] \arrow[d, "f'"'] & X \arrow[r, "u"] \arrow[d, "f"'] & U \arrow[d, "\phi"] \\ Y' \arrow[r, two heads, "y"'] & Y \arrow[r, "v"'] & V \end{tikzcd}$$

If y is regular epimorphism, then the right hand square is a pullback.

Let $\mathbb{E}$ be a regular category. A regular epimorphism in a fibre, above an object X_0, of the fibration $()_0 : Grd\,\mathbb{E} \to E$ is an internal functor $\underline{f}_1 = (f_1, 1_{X_0})$, where f_1 is a regular epimorphism. Accordingly all of these fibres are regular, and any change of base functor is left exact and preserves the regular epimorphisms. The canonical (regular epimorphism, monomorphism) decomposition of the terminal map $\underline{X}_1 \to \nabla X_0$ in the fibre:

$$\underline{X}_1 \twoheadrightarrow \underline{\Sigma}_1\underline{X}_1 \rightarrowtail \nabla X_0$$

gives an equivalence relation $\underline{\Sigma}_1\underline{X}_1$ called the *support* of the internal groupoid. Clearly the construction of the support extends to a functor $\underline{\Sigma}_1 : Grd\,\mathbb{E} \to Req\,\mathbb{E}$, where $Req\,\mathbb{E}$ denotes the category of equivalence relations in $\mathbb{E}$; it is a reflection of the inclusion $Req\,\mathbb{E} \rightarrowtail Grd\,\mathbb{E}$ of the equivalence relations and, up to equivalence, a fibration, i.e. a fibred reflection in the sense of [4].

Definition 1.4. An internal functor $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ is called a $\underline{\Sigma}_1$-*discrete fibration* when $\underline{f}_1$ and $\underline{\Sigma}_1\underline{f}_1$ are both discrete fibrations.

Immediately we get:

Lemma 1.5. *A discrete fibration $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ is $\underline{\Sigma}_1$-discrete if and only if the following square in $Grd\,\mathbb{E}$ is a pullback:*

$$\begin{tikzcd} \underline{X}_1 \arrow[r, two heads] \arrow[d, "\underline{f}_1"'] & \underline{\Sigma}_1\underline{X}_1 \arrow[d, "\Sigma_1\underline{f}_1"] \\ \underline{Y}_1 \arrow[r, two heads] & \underline{\Sigma}_1\underline{Y}_1 \end{tikzcd}$$

Proof. This is a straightforward consequence of Corollary 1.3. □

Any discrete fibration between equivalence relations is $\underline{\Sigma}_1$-discrete. The class $\underline{\Sigma}_1$-*dis* of $\underline{\Sigma}_1$-discrete fibrations contains the isomorphisms, is stable under composition and has the property that when $\underline{g}_1.\underline{f}_1$ and $\underline{g}_1$ are in $\underline{\Sigma}_1$-*dis*, then $\underline{f}_1$ is in $\underline{\Sigma}_1$-*dis*. The $\underline{\Sigma}_1$-discrete fibrations are stable under pullbacks and $\underline{\Sigma}_1$ preserves these pullbacks. Given an internal groupoid $\underline{X}_1$, the discrete fibration $\underline{\epsilon}_1\underline{X}_1$ is $\underline{\Sigma}_1$-discrete if and only if $\underline{X}_1$ is actually an equivalence relation (by Lemma 1.5 and by the fact that $\underline{\epsilon}_1\underline{X}_1$ is levelwise split).

Definition 1.6. An internal groupoid $\underline{X}_1$ is said to have *effective support* when the equivalence relation $\underline{\Sigma}_1\underline{X}_1$ is effective. We shall denote by $Gref\ \mathbb{E}$ the full subcategory of $Grd\,\mathbb{E}$ whose objects are the internal groupoids with effective support.

The internal groupoids with effective support are stable under products. Note that, when $\mathbb{E}$ is exact [1] (i.e. when moreover any equivalence relation is effective), any internal groupoid has effective support and we have $Gref\ \mathbb{E} = Grd\,\mathbb{E}$. We denote by $\pi_0 : Gref\ \mathbb{E} \to \mathbb{E}$ the functor which associates with an internal groupoid $\underline{X}_1$ the quotient $q_{\underline{X}_1} : X_0 \twoheadrightarrow \pi_0\underline{X}_1$ of the effective equivalence relation $\underline{\Sigma}_1\underline{X}_1$; it is a left adjoint to the inclusion $\Delta : \mathbb{E} \to Gref\ \mathbb{E}$ and consequently is right exact. When an internal groupoid $\underline{X}_1$ has effective support, then the image under π_0 of the above kernel equivalence relation with quotient (1) in $Grd\,\mathbb{E}$ is the following coequalizer diagram in $\mathbb{E}$:

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \arrow[r, two heads, "q\underline{X}_1"] & \pi_0\underline{X}_1 \end{tikzcd}$$

Proposition 1.7. *Any discrete fibration $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ in $Gref\ \mathbb{E}$ is $\underline{\Sigma}_1$-discrete if and only if the following square is a pullback in $\mathbb{E}$:*

$$\begin{tikzcd} X_0 \arrow[r, two heads, "q\underline{X}_1"] \arrow[d, "f_0"'] & \pi_0\underline{X}_1 \arrow[d, "\pi_0\underline{f}_1"] \\ Y_0 \arrow[r, two heads, "q\underline{Y}_1"'] & \pi_0\underline{Y}_1 \end{tikzcd}$$

The functor π_0 preserves pullbacks, when they exist in $Gref\ \mathbb{E}$, of $\underline{\Sigma}_1$-discrete fibrations along any map.

Proof. It is a straightforward consequence of the Barr–Kock Theorem 1.2. □

1.3. The efficiently regular context

We recall here from [8] a notion intermediate between those of regular and exact categories which allows us to integrate many topological situations.

Definition 1.8. A regular category $\mathbb{E}$ is said to be *efficiently regular* when any equivalence relation T on an object X which is a subobject $j : T \rightarrowtail R[f]$ of an effective equivalence relation $R[f]$ on X by a regular monomorphism in $\mathbb{E}$ (i.e. j is the equalizer of some pair of maps in $\mathbb{E}$) is itself effective.

Example 1.9. Any exact category is always efficiently regular. The category of topological (abelian) groups is efficiently regular, but not exact. The same holds for Hausdorff (abelian) groups and topological or Hausdorff Lie algebras. More generally any category $Top^{\mathbb{T}}$ (resp. $Haus^{\mathbb{T}}$) of topological (resp. Hausdorff) protomodular algebras (where $\mathbb{T}$ is a protomodular theory) is efficiently regular: it is a regular category [3] and clearly an equivalence relation T on X is effective if and only if the object T is endowed with the topology induced by the topological product, which is the case when $j : T \rightarrowtail R[f]$ is a regular monomorphism. When $\mathbb{E}$ is efficiently regular, so is any slice category $\mathbb{E}/Y$, and any fibre of the fibration $()_0 : Grd\,\mathbb{E} \to \mathbb{E}$.

Let $\mathbb{E}$ represent an efficiently regular category. An important fact in such a context is that any discrete fibration above an effective equivalence relation $R[f]$:

$$\begin{tikzcd}
S \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "f_1"'] & U \arrow[r, dotted, two heads, "q"] \arrow[d, "f_0"] & Q \arrow[d, dotted, "\phi"] \\
R[f] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

makes its domain S an effective equivalence relation on U, i.e. regular epimorphisms are effective for descent [28]. Then we can complete the diagram with its quotient Q which makes the right hand side a pullback (by the Barr–Kock Theorem 1.2). So, any $\underline{\Sigma}_1$-discrete fibration $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ having its codomain $\underline{Y}_1$ with effective support has its domain $\underline{X}_1$ with effective support. Consequently, in this context, the category $Gref\,\mathbb{E}$ admits pullbacks of $\underline{\Sigma}_1$-discrete fibrations along any map, and the functor $\pi_0 : Gref\,\mathbb{E} \to \mathbb{E}$ preserves them (Proposition 1.7).

Now suppose that we have a discrete fibration $\underline{f}_1 : R[q_0] \to \underline{Y}_1$ with q_0 a regular epimorphism:

$$\begin{tikzcd}
R[f_1] \arrow[r, shift left, "R(p_0)"] \arrow[r, shift right, "R(p_1)"'] \arrow[d, shift left] \arrow[d, shift right] & R[f_0] \arrow[r, dotted, two heads, "q_1"] \arrow[d, shift left] \arrow[d, shift right] & T_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
R[q_0] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, "f_1"'] & X \arrow[r, two heads, "q_0"'] \arrow[d, "f_0"] & T_0 \\
Y_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & Y_0 &
\end{tikzcd} \tag{2}$$

Then completing the diagram by taking the vertical kernel equivalence relations makes the upper left hand side horizontal diagram an effective equivalence relation, which produces, by its quotient T_1, an internal groupoid $\underline{T}_1$ (see [30]) and a discrete fibration $\underline{q}_1 : R[f_0] \to \underline{T}_1$ (by the Barr–Kock Theorem 1.2). The following result enlarges a previous version only asserted in an efficiently regular Mal'tsev context [9]:

Theorem 1.10. *Let $\mathbb{E}$ be an efficiently regular category and consider the above construction (2). If $\underline{Y}_1$ is an equivalence relation (resp. has effective support), then $\underline{T}_1$ is an equivalence relation (resp. has effective support); the converse is true when f_0 is a regular epimorphism.*

Proof. According to Lemma 1.1 in [9], when $\underline{Y}_1$ is an equivalence relation, then we have $R[f_0] \cap R[q_0] = \Delta X$, and since q_0 is a regular epimorphism, $\underline{T}_1$ is an equivalence relation.

Now, suppose that $\underline{Y}_1$ has effective support and let $\pi_0 \underline{Y}_1 = Q$. Then consider the following diagram:

$$\begin{tikzcd}
R[f_1] \arrow[r, shift left, "R(p_0)"] \arrow[r, shift right, "R(p_1)"'] \arrow[d, shift left] \arrow[d, shift right] & R[f_0] \arrow[r, two heads, "q_1"] \arrow[d, shift left] \arrow[d, shift right] \arrow[rr, bend left, "R(q_0)"] & T_1 \arrow[r, "\psi"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & R[q] \arrow[dl, shift right, "p_0"'] \arrow[dl, shift left, "p_1"] \\
R[q_0] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, "f_1"'] \arrow[dd, bend right=60, "R(f_0)"'] & X \arrow[r, two heads, "q_0"'] \arrow[d, "f_0"] & T_0 \arrow[d, dotted, "q"] & \\
Y_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, two heads, "\sigma_1"'] & Y_0 \arrow[r, two heads, "q_{\underline{Y}_1}"'] & Q & \\
\Sigma_1 Y_1 \arrow[ur, shift left, "d_0"] \arrow[ur, shift right, "d_1"'] & & &
\end{tikzcd}$$

We are going to show that $R(q_0)$ is a regular epimorphism, which will imply that the factorization ψ is a regular epimorphism and the effective equivalence relation $R[q]$ is the support of $\underline{T}_1$. This will come from the fact that σ_1 is a regular epimorphism. For that, consider the following diagram where $P_0 = Y_0 \times_Q T_0$, $q_0 = \bar{q} \cdot \hat{\pi}$, $P_1 = \Sigma_1 Y_1 \times_{Y_0} X$ and $p_0 = \delta_0 \cdot \hat{\sigma}_1$:

$$\begin{tikzcd}
R[q_0] \arrow[rr, "\hat{\sigma}_1"] \arrow[dr, "p_1"] \arrow[dd, "f_1"'] & & P_1 \arrow[rr, "\delta_0"] \arrow[dr, "\delta_1"] \arrow[dd, dotted] & & X \arrow[dr, two heads, "q_0"] \arrow[dd, dotted, "f_0"'] & \\
& X \arrow[rr, "\hat{\pi}"] \arrow[dd, "f_0"'] & & P_0 \arrow[rr, two heads, "\bar{q}"] \arrow[dd] & & T_0 \arrow[dd, "q"] \\
Y_1 \arrow[rr, dotted, two heads, "\sigma_1"] \arrow[dr, "d_1"'] & & \Sigma_1 Y_1 \arrow[rr, dotted, "d_0"] \arrow[dr, dotted, "d_1"'] & & Y_0 \arrow[dr, dotted, "q_{\underline{Y}_1}"'] & \\
& Y_0 \arrow[rr, equal] & & Y_0 \arrow[rr, two heads, "q_{\underline{Y}_1}"'] & & Q
\end{tikzcd}$$

Since $\underline{f}_1$ is a discrete fibration, the back left hand side square is a pullback, and the factorization $\hat{\sigma}_1$ is a regular epimorphism. For the right hand side cube, the back, bottom and front faces are pullbacks. Thus the top right face is a pullback and, consequently, the factorization δ_1 is a regular epimorphism. It is easy to check that the factorization $\hat{\pi}$ is such that $\hat{\pi}.p_1 = \delta_1.\hat{\sigma}_1$. This composite is a regular epimorphism since so are δ_1 and $\hat{\sigma}_1$. Accordingly the map $\hat{\pi}$ is a regular epimorphism. Finally, by taking the vertical kernel pairs of the front rectangle we get $R(q_0) = R(\bar{q}).R(\hat{\pi})$, which is a regular epimorphism as a composite of regular epimorphisms: the map $R(\bar{q})$ because the front right hand side square is a pullback and so $R(\bar{q})$ is a pullback of $\bar{q}$; the map $R(\hat{\pi})$ because it is the product of the regular epimorphism $\hat{\pi}$ by itself in the regular category $\mathbb{E}/Y_0$.

The converse is true in both cases when f_0 is a regular epimorphism, since, then, the roles of $\underline{Y}_1$ and $\underline{T}_1$ are totally symmetric. □

1.4. The comprehensive factorization

We shall suppose from now on that $\mathbb{E}$ is an efficiently regular category. Let $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ be an internal functor. Next, we give sufficient conditions for the comprehensive factorization $\underline{f}_1 = \bar{\underline{f}}_1.\hat{\underline{f}}_1$, into a final internal functor and a discrete fibration, to hold. For that we consider the following diagram in $Grd\,\mathbb{E}$ with the right hand side pullbacks:

$$\begin{tikzcd}
Dec^2 \underline{X}_1 \arrow[r, "\underline{\psi}_1^2"] \arrow[rr, bend left, "Dec^2 \underline{f}_1"] \arrow[d, shift right=2, "\epsilon_1 Dec\, \underline{X}_1"'] \arrow[d, shift left=2, "Dec\, \epsilon_1 \underline{X}_1"] & \underline{V}_1 \arrow[r, "\underline{\phi}_1^2"] \arrow[d, shift right=2] \arrow[d, shift left=2] & Dec^2 \underline{Y}_1 \arrow[d, shift right=2, "\epsilon_1 Dec\, \underline{Y}_1"'] \arrow[d, shift left=2, "Dec\, \epsilon_1 \underline{Y}_1"] \\
Dec\, \underline{X}_1 \arrow[r, "\underline{\psi}_1"'] \arrow[u] \arrow[d, two heads, "\epsilon_1 \underline{X}_1"'] & \underline{U}_1 \arrow[r, "\underline{\phi}_1"'] \arrow[u] \arrow[d, two heads, "\underline{e}_1"'] & Dec\, \underline{Y}_1 \arrow[u] \arrow[d, two heads, "\epsilon_1 \underline{Y}_1"] \\
\underline{X}_1 \arrow[r, equal] & \underline{X}_1 \arrow[r, "\underline{f}_1"'] & \underline{Y}_1
\end{tikzcd} \tag{3}$$

The middle vertical diagram is then a kernel equivalence relation with quotient, since so is the right hand side vertical one. Since the two upper right hand side vertical arrows are $\underline{\Sigma}_1$-discrete fibrations, so are the two unlabelled projections $\underline{V}_1 \rightrightarrows \underline{U}_1$. Consequently, if $\underline{U}_1$ has effective support, then so does $\underline{V}_1$; we denote their quotients by $\pi_0\underline{U}_1 = T_0$ and $\pi_0\underline{V}_1 = T_1$.

Proposition 1.11. *Suppose that $\mathbb{E}$ is efficiently regular, $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ is an internal functor and the internal groupoid $\underline{U}_1$ of the previous construction (3) has effective support. Then the image by π_0 of the upper part of (3):*

$$\begin{tikzcd}
X_1 \arrow[r, "\hat{f}_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & T_1 \arrow[r, "\bar{f}_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Y_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
X_0 \arrow[r, "\hat{f}_0"'] \arrow[u] \arrow[d, dotted, two heads, "q_{\underline{X}_1}"'] & T_0 \arrow[r, "\bar{f}_0"'] \arrow[u] \arrow[d, dotted, two heads, "q_{\underline{T}_1}"] & Y_0 \arrow[u] \\
Q \arrow[r, equal] & Q &
\end{tikzcd}$$

produces the universal decomposition of $\underline{f}_1 = \underline{\bar{f}}_1.\underline{\hat{f}}_1 : \underline{X}_1 \to \underline{T}_1 \to \underline{Y}_1$ through a discrete fibration $\underline{\bar{f}}_1$. If, moreover, the internal groupoid $\underline{X}_1$ has effective support, then the internal groupoid $\underline{T}_1$ has effective support and the same quotient Q as $\underline{X}_1$ (we can extend the diagram with the dotted coequalizers).

Proof. The upper right hand side pullbacks in (3) having $\underline{\Sigma}_1$-discrete fibrations as vertical edges are preserved by π_0 (Proposition 1.7) and produce the discrete fibration $\underline{\bar{f}}_1$. Suppose now that $\underline{f}_1 = \underline{g}_1.\underline{h}_1$ with $\underline{g}_1$ a discrete fibration. In the following diagram, the right hand side squares are pullbacks since $\underline{g}_1$ is a discrete fibration, whence the following factorizations with $\underline{\phi}_1 = Dec\, \underline{g}_1.\underline{\bar{h}}_1$:

$$\begin{tikzcd}
\underline{V}_1 \arrow[r, "\underline{\bar{h}}_1^2"] \arrow[rr, bend left, "\underline{\phi}_1^2"] \arrow[d, shift right=2] \arrow[d, shift left=2] & Dec^2\underline{Z}_1 \arrow[r, "Dec^2\underline{g}_1"] \arrow[d, shift right=2] \arrow[d, shift left=2] & Dec^2\underline{Y}_1 \arrow[d, shift right=2, "\underline{\epsilon}_1 Dec\, \underline{Y}_1"'] \arrow[d, shift left=2, "Dec\, \underline{\epsilon}_1\underline{Y}_1"] \\
\underline{U}_1 \arrow[r, "\underline{\bar{h}}_1"'] \arrow[u] \arrow[d, "\underline{e}_1"'] & Dec\, \underline{Z}_1 \arrow[r, "Dec\, \underline{g}_1"'] \arrow[u] \arrow[d, "\underline{\epsilon}_1\underline{Z}_1"'] & Dec\, \underline{Y}_1 \arrow[u] \arrow[d, "\underline{\epsilon}_1\underline{Y}_1"] \\
\underline{X}_1 \arrow[r, "\underline{h}_1"'] \arrow[rr, bend right, "\underline{f}_1"'] & \underline{Z}_1 \arrow[r, "\underline{g}_1"'] & \underline{Y}_1
\end{tikzcd}$$

which make the left hand side squares pullbacks. The image by π_0 of the upper pullbacks:

$$\begin{tikzcd}
T_1 \arrow[r, "\tau_1"] \arrow[rr, bend left, "\bar{f}_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Z_1 \arrow[r, "g_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Y_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
T_0 \arrow[r, "\tau_0"'] \arrow[rr, bend right, "\bar{f}_0"'] \arrow[u] & Z_0 \arrow[r, "g_0"'] \arrow[u] & Y_0 \arrow[u]
\end{tikzcd}$$

produces the factorization $\underline{\tau}_1 : \underline{T}_1 \to \underline{Z}_1$ such that $\underline{\bar{f}}_1 = \underline{g}_1.\underline{\tau}_1$ that we were looking for. It is easy to check that $\underline{\tau}_1.\underline{\hat{f}}_1 = \underline{h}_1$.

When $\underline{X}_1$ has effective support, we can apply π_0 to the left hand side of (3): both vertical coequalizers are preserved and coincide. To see that $\underline{T}_1$ has effective support, we just apply Theorem 1.10 to the following diagram:

$$\begin{tikzcd}
\Sigma_1 V_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, shift left] \arrow[d, shift right] & V_0 \arrow[r, two heads, "q_{\underline{V}_1}"] \arrow[d, shift left] \arrow[d, shift right] & T_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
\Sigma_1 U_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "\xi"'] \arrow[ddd, bend right, "\Sigma_1 e_1"'] & U_0 \arrow[r, two heads, "q_{\underline{U}_1}"'] \arrow[d, "e_0"] & T_0 \arrow[d, dotted, "q"] \\
\Xi_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[dd, "\bar{\xi}"'] & X_0 \arrow[r, two heads, "q_{\underline{X}_1}"'] & Q \\
& & \\
\Sigma_1 X_1 \arrow[uur, shift left, "d_0"] \arrow[uur, shift right, "d_1"'] & &
\end{tikzcd}$$

where Ξ_1 denotes the quotient of the effective equivalence relation $\Sigma_1 V_1 \rightrightarrows \Sigma_1 U_1$ (it is effective because it is a discrete fibration above the effective equivalence relation $V_0 \rightrightarrows U_0$ in an efficiently regular context). Note that, since $\Sigma_1 e_1 : \Sigma_1 U_1 \twoheadrightarrow \Sigma_1 X_1$ is a regular epimorphism, so is the factorization $\bar{\xi} : \Xi \twoheadrightarrow \Sigma_1 X_1$. Thus the internal groupoids $\underline{X}_1$ and $\underline{\Xi}_1$ have the same effective support $\underline{\Sigma}_1 \underline{X}_1$. $\square$

Recall that the discrete fibrations are stable under composition and pullbacks. Then necessarily the factorization $\hat{\underline{f}}_1$ given in Proposition 1.11 belongs to the class of morphisms which is *orthogonal* to the class *Disf* of discrete fibrations (see Theorem 1.8 in [17]) namely the class of those internal functors $\underline{\phi}_1 : \underline{M}_1 \to \underline{N}_1$ such that any commutative square with $\underline{k}_1$ a discrete fibration produces a unique diagonal factorization:

$$\begin{tikzcd}
\underline{M}_1 \arrow[r] \arrow[d, "\underline{\phi}_1"'] & \underline{A}_1 \arrow[d, "\underline{k}_1"] \\
\underline{N}_1 \arrow[r] \arrow[ur, dotted] & \underline{B}_1
\end{tikzcd}$$

So according to the terminology introduced in [33] when $\mathbb{E}$ is *Set*, and extended to any exact context in [4], we shall call this orthogonal class the class of *final* internal functors and the previous decomposition the *comprehensive factorization* of the internal functor $\underline{f}_1$. Clearly, when $\mathbb{E}$ is exact, any internal functor $\underline{f}_1$ admits a comprehensive factorization. This was first shown in [4]. Next, we describe two important situations where such a decomposition holds in an efficiently regular context; we use the notation of diagram (3) and Proposition 1.11.

Corollary 1.12. *Suppose that $\mathbb{E}$ is an efficiently regular category. Then the comprehensive factorization of an internal functor of the form $\underline{f}_1 : R[\hat{q}] \to \underline{Y}_1$ exists. Moreover the internal groupoid $\underline{T}_1$ is itself an effective equivalence relation which has the same quotient as $R[\hat{q}]$.*

Proof. The internal functor $\underline{e}_1 : \underline{U}_1 \to R[\hat{q}]$ being a discrete fibration and $\mathbb{E}$ efficiently regular, the internal groupoid $\underline{U}_1$ is an effective equivalence relation; the comprehensive factorization holds. By adapting the last diagram in the proof of Proposition 1.11 to this context, the fact that $\underline{X}_1 = R[\hat{q}]$ is an effective equivalence relation implies that $\underline{T}_1$ is the effective equivalence relation $R[q]$ (again Theorem 1.10). Not only does it have the quotient Q as $R[\hat{q}]$, but according to Proposition 1.11, the internal functor $\hat{\underline{f}}_1 : R[\hat{q}] \to R[q]$ induces the identity on Q. $\square$

Corollary 1.13. *Let $\mathbb{E}$ be an efficiently regular category. Then the comprehensive factorization of an internal functor of the form $\underline{f}_1 : \nabla X \to \underline{Y}_1$ exists and is given by $\nabla X \to \nabla T \to \underline{Y}_1$.*

Proof. The indiscrete equivalence relation ∇X is effective and its quotient is a subobject Q of the terminal object 1. By the previous proposition, the internal groupoid $\underline{T}_1$ is then an effective equivalence relation with quotient Q, namely an indiscrete equivalence relation ∇T. $\square$

2. Reflections and *I*-centralization functors

Given an extension $f : X \twoheadrightarrow Y$, the comprehensive factorization of a specific internal functor provides the main tool which allows the construction of the reflection of an extension to a central extension (with respect to a reflection functor I). In this section we establish sufficient conditions for which such a construction holds (see also [25]).

Let $j : \mathbb{C} \rightarrowtail \mathbb{D}$ be a full replete reflective inclusion.

2.1. The reg-epi reflections

Recall the following:

Definition 2.1. A reflection $I : \mathbb{D} \to \mathbb{C}$ of the inclusion j is said to be a *reg-epi reflection* when any reflection morphism $\eta_X : X \twoheadrightarrow IX$ is a regular epimorphism.

When $\mathbb{D}$ is a regular category and $I : \mathbb{D} \to \mathbb{C}$ is a reg-epi reflection, a map in $\mathbb{C}$ is a regular epimorphism in $\mathbb{C}$ if and only if it is a regular epimorphism in $\mathbb{D}$ and $\mathbb{C}$ is also a regular category. The fact that the reflection morphisms are regular epimorphisms is equivalent to saying that $\mathbb{C}$ is closed under subobjects. If $\mathbb{C}$ is also closed under regular epimorphisms, then it is called a *Birkhoff* subcategory of $\mathbb{D}$ [24]. For instance, a Birkhoff subcategory of a variety is just a subvariety. From now on $\mathbb{D}$ represents a regular category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection.

Following [24], we shall now be interested in certain classes of maps with respect to the reflection I:

Definition 2.2. Given a regular category $\mathbb{D}$ and a reg-epi reflection $I : \mathbb{D} \to \mathbb{C}$, a map $f : X \to Y$ in $\mathbb{D}$ is said to be *I-trivial* when the following square is a pullback:

$$\begin{tikzcd} X \arrow[r, two heads, "\eta_X"] \arrow[d, "f"'] & IX \arrow[d, "If"] \\ Y \arrow[r, two heads, "\eta_Y"'] & IY \end{tikzcd}$$

Clearly the isomorphisms are I-trivial, and the I-trivial maps are stable under composition and have the property that, when $g.f$ and g are I-trivial, then f is I-trivial. Also I-trivial maps are stable under the pullbacks which are preserved by the reflection I. This last point emphasizes the importance of those pullbacks in $\mathbb{D}$ which are preserved by I. In fact, analysing the left exact properties of I plays a main role in this work; in particular, the preservation of products investigated in Section 4. Finally, an I-trivial map f is certainly I-cartesian, namely universal among the maps above If.

Example 2.3. Suppose that $\mathbb{D}$ is a regular category.
1. The category $Grd\,\mathbb{D}$ is not necessarily regular, but the reflection $\underline{\Sigma}_1 : Grd\,\mathbb{D} \to Req\,\mathbb{D}$ is a reg-epi reflection. In this context, any $()_0$-cartesian functor $\underline{f}_1$ is $\underline{\Sigma}_1$-trivial. By Lemma 1.5, a discrete fibration is $\underline{\Sigma}_1$-discrete if and only if it is $\underline{\Sigma}_1$-trivial.
2. The category $Gref\,\mathbb{D}$ is not necessarily regular, but the reflection $\pi_0 : Gref\,\mathbb{D} \to \mathbb{D}$ is a reg-epi reflection. An internal functor is π_0-trivial if and only if it is $\underline{\Sigma}_1$-discrete by Proposition 1.7.

Definition 2.4. Given a regular category $\mathbb{D}$ and a reg-epi reflection $I : \mathbb{D} \to \mathbb{C}$, a map $f : X \to Y$ is called *I-central* when it is I-trivial up to a regular epimorphism; that is, there exists a regular epimorphism along which f is pulled back onto an I-trivial map. A map f is called *I-normal*, when its projection $p_0 : R[f] \to X$ (or p_1) is I-trivial.

The class of I-central morphisms contains the I-trivial morphisms and thus the isomorphisms. It is not stable under composition, nor under pullbacks, in general. Any I-normal extension f is I-central (pullback f along itself). Notice that, although any I-trivial extension is I-central, it is not necessarily I-normal. However it is clear that when I-central extensions and I-normal extensions coincide, any I-trivial extension is I-normal.

2.2. I-central and I-normal extensions

We now analyse under which conditions are we able to show that I-trivial extensions are I-normal, and that I-central and I-normal extensions coincide. Eventually, thanks to the comprehensive factorization, we shall associate with any regular epimorphism an I-central ($=I$-normal) extension.

Proposition 2.5. *Let $\mathbb{D}$ be a regular category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. Consider the following conditions for a regular epimorphism $f : X \twoheadrightarrow Y$:*

1. *f is an I-trivial extension;*
2. *f is an I-normal extension;*
3. $IR[f] \simeq R[If]$.

If any two conditions are satisfied, then the third one holds. More generally, an I-trivial map f is I-normal if and only if $IR[f] \simeq R[If]$.

Proof. It is a straightforward consequence of Corollary 1.3. □

Example 2.6. Let $\mathbb{C}$ be a Birkhoff subcategory of an exact Goursat [14] (and a fortiori Mal'tsev [15,16]) category $\mathbb{D}$ (with reflection I). Then the notions of I-central and I-normal extensions coincide (Theorem 4.8 of [24]).

Definition 2.7. Given a regular category $\mathbb{D}$, a reg-epi reflection $I : \mathbb{D} \to \mathbb{C}$ is called a *regular reflection* when it preserves pullbacks of split epimorphisms along regular epimorphisms.

Example 2.8. Regular reflections.
1. Let $\mathbb{D}$ be a semi-abelian category [27]. Then the abelianization functor $I : \mathbb{D} \to \mathbb{D}_{Ab}$ is a regular reflection (Lemma 8.2 of [21]; see also Section 4.2 and Theorem 3.16).
2. If $\mathbb{D}$ is a regular Mal'tsev category, then $\underline{\Sigma}_1 : Grd\mathbb{D} \to Req\mathbb{D}$ is a regular reflection (Proposition 3.2).
3. Any Birkhoff subcategory of an exact Mal'tsev category determines a regular reflection (Theorem 3.16).

Regular reflections preserve, in particular, kernel equivalence relations of split epimorphisms, and thus preserve internal groupoids. This produces a functor still denoted by $I : Grd\,\mathbb{D} \to Grd\,\mathbb{C}$ for the sake of simplicity.

Lemma 2.9. *Let $\mathbb{D}$ be a regular category and $I : \mathbb{D} \to \mathbb{C}$ a regular reflection. Then any I-trivial map is I-normal.*

Proof. When $f : X \to Y$ is I-trivial, the pullback defining it produces the following whole rectangle of pullbacks of split epimorphisms along the regular epimorphism η_X:

$$\begin{tikzcd}
R[f] \arrow[r, two heads, "\eta_{R[f]}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \arrow[rr, bend left=40, "R(\eta_X)"] & IR[f] \arrow[r, "\chi"] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] & R[If] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\
X \arrow[r, two heads, "\eta_X"'] \arrow[rr, bend right=40, "\eta_X"'] & IX \arrow[r, equal] & IX
\end{tikzcd}$$

It is preserved by I. This makes the right hand side squares pullbacks, and the factorization χ an isomorphism. Then $p_0 : R[f] \to X$ is I-trivial and f is I-normal. □

Proposition 2.10. *Given a regular category $\mathbb{D}$ and a regular reflection $I : \mathbb{D} \to \mathbb{C}$, any I-central map is I-normal. Accordingly the I-central extensions and I-normal extensions coincide.*

Proof. Suppose that f is an I-central map and $h : Y' \twoheadrightarrow Y$ is the regular epimorphism along which f is pulled back onto an I-trivial map $f' : X' \to Y'$. Then consider the following diagram:

$$\begin{tikzcd}
R[f'] \arrow[rr, two heads, "\eta_{R[f']}"] \arrow[dr, "R(g)"] \arrow[dd, shift right, "p_0"'] \arrow[dd, shift left, "p_1"] & & IR[f'] \arrow[dr, "IR(g)"] \arrow[dd, dotted, shift right, "Ip_0"'] \arrow[dd, dotted, shift left, "Ip_1"] & \\
& R[f] \arrow[rr, two heads, "\eta_{R[f]}"] \arrow[dd, shift right, "p_0"'] \arrow[dd, shift left, "p_1"] & & IR[f] \arrow[dd, shift right, "Ip_0"'] \arrow[dd, shift left, "Ip_1"] \\
X' \arrow[rr, dotted, two heads, "\eta_{X'}"] \arrow[dr, "g"'] & & IX' \arrow[dr, dotted, "Ig"] & \\
& X \arrow[rr, two heads, "\eta_X"'] & & IX
\end{tikzcd}$$

where g denotes the pullback of h along f. The left hand side and back faces (by Lemma 2.9 f' is I-normal) are pullbacks. Since I preserves the left pullbacks, then the right faces are pullbacks. By Corollary 1.3, the front faces are pullbacks. So, the projections $p_i : R[f] \to X$ are I-trivial, and the map f is I-normal. We conclude that any I-central extension is I-normal. □

It is known from Proposition 4.7 of [24] that, for an exact category $\mathbb{D}$ and an admissible reflection $I : \mathbb{D} \to \mathbb{C}$ (Definition 2.12 below), the notions of I-centrality and I-normality coincide if and only if every split I-central extension is I-trivial. In our context, we get the following:

Corollary 2.11. *Given a regular category $\mathbb{D}$ and a regular reflection $I : \mathbb{D} \to \mathbb{C}$, any split epimorphism f which is I-central is I-trivial.*

Proof. Being split and I-central, f is an I-central extension, and thus an I-normal extension. On the other hand, being split, its kernel equivalence relation is preserved by I. According to Proposition 2.5, it is I-trivial. □

2.3. The I-centralization functor

Given a regular category $\mathbb{D}$ and a reg-epi reflection $I : \mathbb{D} \to \mathbb{C}$, we denote by $Ext(Y)$ the category of extensions over Y and by $CExt_I(Y)$ its full subcategory of I-central extensions. When $\mathbb{C}$ is also efficiently regular and I is a regular reflection, then the inclusion functor $j_1 : CExt_I(Y) \hookrightarrow Ext(Y)$ admits a reflection $I_1 : Ext(Y) \to CExt_I(Y)$, called the ($I$-)*centralization* functor (see [25]). We are going to show now that I_1 can be defined through the comprehensive factorization given in Section 1.4, combining two, apparently, independent subjects.

By Corollary 1.12, the internal functor $\underline{\eta}_1 f = (\eta_{R[f]}, \eta_X) : R[f] \to IR[f]$ factors through an effective equivalence relation with the same quotient as $R[f]$; say $R[f] \to R[\bar{f}] \to IR[f]$. So $\bar{f} : \bar{X} \twoheadrightarrow Y$ is an extension with the same codomain as f and gives a candidate for the (I-)centralization of f. To be I-central (=I-normal), the projections $p_i : R[\bar{f}] \to \bar{X}$ must be I-trivial. Since $R[\bar{f}] \to IR[f]$ is a discrete fibration, these projections are pullbacks of the extensions $Ip_i : IR[f] \to IX$ (see the first diagram of the proof of Theorem 2.17). Therefore, it is convenient for I to have the following property:

Definition 2.12. Given a regular category $\mathbb{D}$, a reg-epi reflection $I : \mathbb{D} \to \mathbb{C}$ is said to be *admissible* when any pullback of a regular epimorphism ϕ in $\mathbb{C}$ of the form

$$\begin{tikzcd} X \arrow[r, two heads, "g"] \arrow[d, two heads, "f"'] & A \arrow[d, two heads, "\phi"] \\ Y \arrow[r, two heads, "\eta_Y"'] & IY \end{tikzcd}$$

is I-trivial.

Lemma 2.13. *Let $\mathbb{D}$ be a regular category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. Then I is admissible if and only if any pullback of a regular epimorphism in $\mathbb{C}$ is I-trivial. Then I preserves such pullbacks. Hence a reg-epi reflection I is admissible if and only if the I-trivial extensions are stable under pullbacks.*

Proof. The first point is Proposition 3.3 of [24] and the second one a straightforward consequence of the first one. $\square$

Remark 2.14. From the previous lemma, we conclude that when I is admissible, the I-trivial extensions, being also stable under composition, are then stable under products and that I-normal maps and I-central extensions are stable under pullbacks.

Example 2.15. For every Birkhoff subcategory $\mathbb{C}$ of an exact Goursat (and a fortiori Mal'tsev) category $\mathbb{D}$, the reflection $I : \mathbb{D} \to \mathbb{C}$ is admissible (Theorem 3.4 of [24]).

We shall suppose now that $\mathbb{C}$ is an efficiently regular category.

Proposition 2.16. *Let $\mathbb{D}$ be a regular category, $I : \mathbb{D} \to \mathbb{C}$ a regular reflection and $\mathbb{C}$ an efficiently regular category. Then I is admissible.*

Proof. Consider the following diagram where the regular epimorphism ϕ is in $\mathbb{C}$ and the lower rectangle is a pullback:

$$\begin{tikzcd} R[f] \arrow[r, two heads, "\eta_{R[f]}"] \arrow[rr, bend left, "R(g)"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f] \arrow[r, "IR(g)"] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] & R[\phi] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\ X \arrow[r, two heads, "\eta_X"'] \arrow[rr, two heads, bend left=15, "g"] \arrow[d, "f"'] & IX \arrow[r, "Ig"'] \arrow[d, "If"] & A \arrow[d, "\phi"] \\ Y \arrow[r, two heads, "\eta_Y"'] & IY \arrow[r, equal] & IY \end{tikzcd}$$

Then, so are the corresponding two upper ones. The maps g and $R(g)$ being regular epimorphisms, these pullbacks are preserved by I. This makes the internal functor $IR(g) : IR[f] \to R[\phi]$ a discrete fibration. Since its codomain is an effective equivalence relation and $\mathbb{C}$ is efficiently regular, its domain $IR[f]$ is an effective equivalence relation whose quotient is If; we have $IR[f] \simeq R[If]$. Moreover, the top left hand squares are also pullbacks, i.e. f is I-normal. Thus f is I-trivial by Proposition 2.5. $\square$

Theorem 2.17. *Let $\mathbb{C}$ be and $\mathbb{D}$ be efficiently regular categories and $I : \mathbb{D} \to \mathbb{C}$ a regular reflection. Then the centralization functor $I_1 : Ext(Y) \to CExt_I(Y)$ assigns to each extension $f : X \twoheadrightarrow Y$ the I-central extension $\bar{f} : \bar{X} \twoheadrightarrow Y$ given by the comprehensive factorization of the internal functor $\underline{\eta}_1 f : R[f] \to R[\bar{f}] \to IR[f]$. Moreover, we have $I\bar{X} \simeq IX$, $I\bar{f} \simeq If$ and $IR[\bar{f}] \simeq IR[f]$.*

Proof. Let $\underline{\eta}_1 f = (\eta_{R[f]}, \eta_X) : R[f] \to IR[f]$ represent the internal functor with codomain the internal groupoid $IR[f]$. By Corollary 1.12, its comprehensive factorization exists and has the form

$$\begin{tikzcd}
R[f] \arrow[r, "R(\hat\eta)"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R[\bar f] \arrow[r, two heads, "\bar\eta_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\
X \arrow[r, "\hat\eta"'] \arrow[d, "f"'] & \bar X \arrow[r, two heads, "\bar\eta_0"'] \arrow[d, "\bar f"] & IX \arrow[d, "If"] \\
Y \arrow[r, equal] & Y \arrow[r, two heads, "\eta_Y"'] & IY
\end{tikzcd}$$

The upper right hand side squares are pullbacks, so the projections $p_i : R[\bar f] \to \bar X$ are I-trivial (Proposition 2.16) and the extension $\bar f$ is I-normal and, therefore, I-central (Proposition 2.10). As pullbacks of split epimorphisms along regular epimorphisms, they are preserved by I, and consequently the image $I\underline{\bar\eta}_1$ of the discrete fibration $\underline{\bar\eta}_1$ is still a discrete fibration; and so is $IR(\hat\eta)$ since $I\underline{\bar\eta}_1 \cdot IR(\hat\eta) = 1_{IR[f]}$. This way we get the following diagram in $Grd\,\mathbb{D}$:

$$\begin{tikzcd}
R[f] \arrow[r, two heads, "\underline{\eta}_1 f"] \arrow[d, "R(\hat\eta)"'] & IR[f] \arrow[d, "IR(\hat\eta)"] \\
R[\bar f] \arrow[r, two heads, "\underline{\eta}_1 \bar f"'] \arrow[ur, dotted, two heads, "\underline{\bar\eta}_1"] & IR[\bar f]
\end{tikzcd}$$

which commutes by naturality. The internal functor $R(\hat\eta)$ being final and the internal functor $IR(\hat\eta)$ being a discrete fibration, we have, thanks to the diagonality property associated with the comprehensive factorization system, a unique factorization, necessarily $\underline{\bar\eta}_1 (= I\underline{\bar\eta}_1 \cdot \underline{\eta}_1 \bar f)$, such that both triangles commute. The equality $IR(\hat\eta) \cdot \underline{\bar\eta}_1 = \underline{\eta}_1 \bar f$ comes from the uniqueness of the diagonal induced by $I\underline{\bar\eta}_1 \cdot IR(\hat\eta) \cdot \underline{\eta}_1 f = I\underline{\bar\eta}_1 \cdot \underline{\eta}_1 \bar f \cdot R(\hat\eta)$, where the internal functor $R(\hat\eta)$ is final and $I\underline{\bar\eta}_1$ is a discrete fibration. Thus $IR(\hat\eta) \cdot I\underline{\bar\eta}_1 = 1_{IR[\bar f]}$ and, consequently, $IR(\hat\eta)$ is an isomorphism of internal groupoids $IR[f] \simeq IR[\bar f]$, which determines the isomorphism of their quotient maps $I\bar f \simeq If$.

The definition of I_1 for morphisms is obvious. The unit of the adjunction $I_1 \dashv j_1$, for an extension $f : X \twoheadrightarrow Y$, is given by $\hat\eta$. As for the universal property, suppose that we have a factorization $f = f'.h : X \to X' \to Y$ with f' an I-normal extension. We have then the following right hand side commutative square in $Grd\,\mathbb{D}$:

$$\begin{tikzcd}
 & R[f] \arrow[r, "R(h)"] \arrow[dl, "R(\hat\eta)"'] \arrow[dd, "\underline{\eta}_1 f"] & R[f'] \arrow[dd, "\underline{\eta}_1 f'"] \\
R[\bar f] \arrow[urr, dotted, "\underline{\tau}_1"'] \arrow[dr, "\underline{\bar\eta}_1"'] & & \\
 & IR[f] \arrow[r, "IR(h)"'] & IR[f']
\end{tikzcd}$$

The internal functor $R(\hat\eta)$ being final and the internal functor $\underline{\eta}_1 f'$ being a discrete fibration (f' is I-normal), we have, thanks to the diagonality property associated with the comprehensive factorization system, a factorization $\underline{\tau}_1$ such that $\underline{\tau}_1.R(\hat\eta) = R(h)$ and $\underline{\eta}_1 f'.\underline{\tau}_1 = IR(h).\underline{\bar\eta}_1$. The first equality implies, at the level of objects, that $\tau_0 : \bar X \to X'$ is such that $\tau_0.\hat\eta = h$. Moreover the image by π_0 of this same equality gives $\pi_0(\underline{\tau}_1) = \pi_0(R(h)) = 1_Y$, which implies that $f'.\tau_0 = \bar f$. If we suppose for the sake of simplicity that $IR(\hat\eta)$ is an identity (and thus $\underline{\bar\eta}_1 = \underline{\eta}_1 \bar f$), we have necessarily $I\underline{\tau}_1 = IR(h)$. Suppose that we have another factorization $\tau : \bar X \to X'$ such that $\tau.\hat\eta = h$ and $f'.\tau = \bar f$. Then the internal functor $R(\tau) : R[\bar f] \to R[f']$ is such that $R(\tau).R(\hat\eta) = R(h) = \underline{\tau}_1.R(\hat\eta)$ and consequently $IR(\tau) = IR(h) = I\underline{\tau}_1$. Thanks to the unicity of the diagonality condition, we get $R(\tau) = \underline{\tau}_1$ and $\tau = \tau_0$. $\square$

3. The regular Mal'tsev context

In this section we adapt the theory developed in Section 2 to the Mal'tsev context. The outcome provides similar results with simpler initial conditions.

3.1. The definition and main properties

Recall that a finitely complete category $\mathbb{D}$ is called Mal'tsev when any reflexive relation is an equivalence relation [15,16].

Example 3.1. As examples of Mal'tsev categories there are the categories of groups (or any category of π-algebras, where π is an algebraic theory which contains a group operation), Heyting algebras, topological or Hausdorff (abelian) groups and the dual category of an elementary topos. Also, if $\mathbb{D}$ is a Mal'tsev category, then this is also the case for $Grd\,\mathbb{D}$ or any (co)slice of $\mathbb{D}$.

If $\mathbb{D}$ is a regular and Mal'tsev category, then the regular epimorphisms in $Grd\,\mathbb{D}$ are the internal functors which are levelwise regular epimorphisms, and then $Grd\,\mathbb{D}$ is still a regular Mal'tsev category [19]. This context will allow us to give many examples of regular reflections.

Proposition 3.2. *Let $\mathbb{D}$ be a regular Mal'tsev category. The functor $\underline{\Sigma}_1 : Grd\,\mathbb{D} \to Req\,\mathbb{D}$ preserves pullbacks of split epimorphisms along any map. Hence the functor $\underline{\Sigma}_1$ is a regular reflection and the pullbacks of split epimorphisms along any map exist in $Gref\ \mathbb{D}$.*

Proof. This a straightforward consequence of Lemma 2.5.7 in [2]. $\square$

Recall Lemma 2.5.6 in [2]:

Lemma 3.3. *Let $\mathbb{D}$ be a regular Mal'tsev category. Given a commutative diagram of vertical split epimorphisms:*

$$\begin{tikzcd} X' \arrow[r, two heads, "x"] \arrow[d, shift right, "f'"'] & X \arrow[d, shift right, "f"'] \\ Y' \arrow[r, two heads, "y"'] \arrow[u, shift right, "s'"'] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

where x (and thus y) is a regular epimorphism, then the factorization $(f', x) : X' \to Y' \times_Y X$ is a regular epimorphism.

From that we get the following very powerful property:

Proposition 3.4. *Let $\mathbb{D}$ be a regular Mal'tsev category. Suppose that the following whole rectangle is a pullback and the left hand side is a commutative square of vertical split epimorphisms:*

$$\begin{tikzcd} X' \arrow[r, two heads, "x"] \arrow[d, shift right, "f'"'] & X \arrow[r, "u"] \arrow[d, shift right, "f"'] & U \arrow[d, "\phi"] \\ Y' \arrow[r, two heads, "y"'] \arrow[u, shift right, "s'"'] & Y \arrow[r, "v"'] \arrow[u, shift right, "s"'] & V \end{tikzcd}$$

If x (and thus y) is a regular epimorphism, then the two squares are pullbacks.

Proof. By Lemma 3.3, the factorization $(f', x) : X' \twoheadrightarrow Y' \times_Y X$ is a regular epimorphism. But it is also a monomorphism, since $(f', u \cdot x)$ is a monomorphism, and thus it is an isomorphism. This proves that the left hand side square is a pullback. Since y is a regular epimorphism, then Corollary 1.3 allows us to conclude that the right hand side square is also a pullback. $\square$

This result was stated in [20] (see Lemma 1.1) in the stricter context of exact Mal'tsev categories.

3.2. Left exact properties of reg-epi reflections

Thanks to this last property, we are going to be able to show that, in the Mal'tsev context, the reg-epi reflections I satisfy some significant left exact properties which allow us to recover some aspects of regular reflections (Propositions 3.6, 3.7 and 3.19). This will eventually lead, by gradually increasing the assumptions (Definition 3.8), to the preservation of pullbacks of split epimorphisms along regular epimorphisms, that is to the property of being a regular reflection (Theorem 3.16). Let us begin with the following:

Proposition 3.5. *Let $\mathbb{D}$ be a regular Mal'tsev category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. Consider any pullback with vertical split epimorphisms:*

$$\begin{tikzcd} X' \arrow[r, "x"] \arrow[d, shift right, "f'"'] & X \arrow[d, shift right, "f"'] \\ Y' \arrow[r, "y"'] \arrow[u, shift right, "s'"'] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

Then the factorization γ towards the following pullback P is a regular epimorphism:

$$\begin{tikzcd} IX' \arrow[rr, "Ix"] \arrow[dd, shift right, "If'"'] \arrow[dr, two heads, "\gamma"] & & IX \arrow[dd, shift right, "If"'] \\ & P \arrow[ur] \arrow[dl] & \\ IY' \arrow[rr, "Iy"'] \arrow[uu, shift right] & & IY \arrow[uu, shift right, "Is"'] \end{tikzcd}$$

If moreover Ix is a monomorphism, then I preserves this pullback.

Proof. This is a consequence of the fact that the factorization $\eta : X' \to P$ between the two pullbacks, induced by the three regular epimorphisms η_X, η_Y and $\eta_{Y'}$, is a regular epimorphism in a Mal'tsev category by Lemma 2.5.7 in [2]. When moreover Ix is a monomorphism, then γ is also a monomorphism. We conclude that it is an isomorphism, and I preserves the pullback in question. $\square$

Then we get to the next important point:

Proposition 3.6. *Let $\mathbb{D}$ be a regular Mal'tsev category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. The functor I preserves pullbacks of pairs of split epimorphisms. Accordingly it preserves kernel equivalence relations of split epimorphisms, and the image $I(\underline{X}_1)$ of any internal groupoid $\underline{X}_1$ is an internal groupoid.*

Proof. Using the same notation as in the last proposition, suppose that the map y is split by ξ. It suffices to prove that the factorization γ is a monomorphism, which we shall obtain by proving that the two kernel equivalence relations $R[\eta_{X'}]$ and $R[\eta]$ are the same. Consider now the following diagram:

$$\begin{tikzcd}
R[\eta_{X'}] \arrow[rr, shift left, "R(x)"] \arrow[dr, "\bar{\gamma}"'] \arrow[dd, shift right, "R(f')"'] & & R[\eta_X] \arrow[ll, shift left] \arrow[dl, shift left] \arrow[dd, shift right, "R(f)"'] \\
& R[\eta] \arrow[ur, shift left] \arrow[dl, shift left] & \\
R[\eta_{Y'}] \arrow[rr, shift left, "R(y)"] \arrow[uu, shift right] \arrow[ur, shift left] & & R[\eta_Y] \arrow[ll, shift left] \arrow[uu, shift right, "R(s)"']
\end{tikzcd}$$

The lower quadrangle is a pullback of split epimorphisms since it is constructed from the pullbacks defining P and X'. The factorization $\bar{\gamma}$ comes from the factorization γ. It is a monomorphism since it compares the two kernel equivalence relations $R[\eta_{X'}]$ and $R[\eta]$. But also it is a regular epimorphism by Lemma 3.3. We have $R[\eta_{X'}] \simeq R[\eta]$, and, as expected, γ is an isomorphism. $\square$

In Section 2, we saw that, in a regular context, regular reflections have the properties stated in Proposition 3.6. Now, in a regular Mal'tsev context, these properties are still fulfilled just by reg-epi reflections. This is also the case for the following property (cf. Lemma 2.9):

Proposition 3.7. *Let $\mathbb{D}$ be a regular Mal'tsev category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. If a morphism f is I-trivial, then we have $IR[f] \simeq R[If]$. So, any I-trivial map (and a fortiori any I-trivial extension) is I-normal.*

Proof. Let f be an I-trivial map. We want to show that the factorization γ:

$$\begin{tikzcd}
IR[f] \arrow[rr, shift left, "Ip_1"] \arrow[dr, "\gamma"'] \arrow[dd, shift right, "Ip_0"'] & & IX \arrow[ll, shift left] \arrow[dl, shift left] \arrow[dd, "If"] \\
& R[If] \arrow[ur, shift left, "p_1"] \arrow[dl, shift left, "p_0"] & \\
IX \arrow[rr, "If"'] \arrow[uu, shift right] \arrow[ur, shift left] & & IY
\end{tikzcd}$$

is an isomorphism. To do so, we use arguments similar to those used in the last two proofs. In the first place, here $\eta = R(\eta_X) : R[f] \twoheadrightarrow R[If]$ is a regular epimorphism since f is I-trivial ($R(\eta_X)$ is the pullback of the regular epimorphism η_X). So, γ is a regular epimorphism as well. Secondly, $\bar{\gamma} : R[\eta_{R[f]}] \xrightarrow{\sim} R[R(\eta_X)]$ is an isomorphism and implies that γ is a monomorphism. According to Proposition 2.5, the I-trivial map is I-normal. $\square$

Unlike what happens in a regular context for regular reflections (Proposition 2.10), I-central extensions and I-normal extensions do not coincide in general. However, from Propositions 3.6 and 2.5, it is straightforward to see that, in this context, any I-normal split epimorphism is I-trivial (cf. Corollary 2.11).

3.3. The Birkhoff reflections

We just saw that in a regular Mal'tsev context, reg-epi reflections $I : \mathbb{D} \to \mathbb{C}$ recover some aspects of the regular reflections. However, the construction of the centralization functor given in Section 2.3 was based on the property of I being a regular reflection (and thus admissible), so I-central extensions are I-normal. Having this in mind, we shall require a slightly stronger assumption for the reg-epi reflection I.

Definition 3.8. Given a regular category $\mathbb{D}$, a reg-epi reflection $I : \mathbb{D} \to \mathbb{C}$ is said to be a *Birkhoff* reflection when any regular epimorphism $f : X \twoheadrightarrow Y$ induces a regular epimorphic factorization $R(f)$:

$$\begin{tikzcd}
R[\eta_X] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, "R(f)"'] & X \arrow[r, two heads, "\eta_X"] \arrow[d, "f"'] & IX \arrow[d, "If"] \\
R[\eta_Y] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & Y \arrow[r, two heads, "\eta_Y"'] & IY
\end{tikzcd}$$

When I is a Birkhoff reflection, it is easy to check that the right hand square above is a pushout for any regular epimorphism f. Then $\mathbb{C}$ is closed under regular epimorphism (Proposition 3.1 in [24]). Since $\mathbb{C}$ is also closed under monomorphism, we conclude that $\mathbb{C}$ is a Birkhoff subcategory of $\mathbb{D}$. When $\mathbb{D}$ is an exact Mal'tsev category, we have the converse: if $\mathbb{C}$ is closed under regular epimorphism, then the right hand square above is a pushout for any regular epimorphism f and, consequently, any reg-epi reflection is a Birkhoff reflection (see Theorem 5.7 in [14]). So any Birkhoff subcategory $\mathbb{C}$ of an exact Mal'tsev category $\mathbb{D}$ determines a Birkhoff reflection, therefore justifying this choice of designation.

Counterexample 3.9. The reg-epi reflection $\underline{\Sigma}_1 : Grd\,\mathbb{D} \to Req\,\mathbb{D}$ is not a Birkhoff reflection, since $Req\,\mathbb{D}$ is not closed under regular epimorphisms in $Grd\,\mathbb{D}$ (see the projections $\underline{\epsilon}_1\underline{X}_1$).

Example 3.10. Let $\mathbb{D}$ be a finitely cocomplete exact Mal'tsev category. Then the abelianization functor $I : \mathbb{D} \to \mathbb{D}_{Ab}$ is a Birkhoff reflection (Section 4.2).

Remark 3.11. Let $\mathbb{D}$ be a regular Mal'tsev category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. Given a regular epimorphism $f : X \twoheadrightarrow Y$, the factorization $R(f)$ is a regular epimorphism if and only if the factorization $R(\eta_X)$ is a regular epimorphism by the denormalized 3×3 lemma [7]. This provides an alternative characterization for Birkhoff reflections in this context.

Lemma 3.12. *Let $\mathbb{D}$ be a regular Mal'tsev category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. Then I is a Birkhoff reflection if and only if, given any regular epimorphism $f : X \twoheadrightarrow Y$, the internal groupoid $IR[f]$ has effective support. Accordingly I is a Birkhoff reflection if and only if I preserves the internal groupoids with effective support.*

Proof. In this proof we use Remark 3.11. Suppose that I is a Birkhoff reflection. Consider the following diagram:

$$\begin{tikzcd}
R[f] \arrow[rr, shift left, "p_0"] \arrow[rr, shift right, "p_1"'] \arrow[d, "\eta_{R[f]}"'] \arrow[dr, "R(\eta_X)"] & & X \arrow[r, two heads, "f"] \arrow[d, "\eta_X"] & Y \arrow[d, "\eta_Y"] \\
IR[f] \arrow[r, dotted, "\gamma"'] & R[If] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & IX \arrow[r, two heads, "If"'] & IY
\end{tikzcd}$$

As the map $R(\eta_X)$ is a regular epimorphism, so is the dotted factorization γ which makes $R[If]$ the effective support of the internal groupoid $IR[f]$. As a consequence I preserves any internal groupoid with effective support.

Conversely suppose that the internal groupoid $IR[f]$ has effective support R. Since the map If is the coequalizer of $IR[f]$, it is the effective quotient of R, and we get $R = R[If]$. So $R[If]$ is the support of the internal groupoid $IR[f]$ and the factorization $R(\eta_X)$

$$\begin{tikzcd}
R[f] \arrow[r, two heads, "\eta_{R[f]}"] \arrow[rr, bend left, "R(\eta_X)"] & IR[f] \arrow[r, two heads] & R = R[If]
\end{tikzcd}$$

is a regular epimorphism. $\square$

Corollary 3.13. *Let $\mathbb{D}$ be a regular Mal'tsev category. Any exact reg-epi reflection $I : \mathbb{D} \to \mathbb{C}$ is a Birkhoff reflection (a functor is called exact when it preserves the kernel equivalence relations of any regular epimorphism).*

The following result is also stated in Theorem 3.5 of [32] for any Birkhoff subcategory of a regular Gumm category [12] such that regular epimorphisms are effective for descent. The proof follows the same structure as is given below, although we get some extra simplifications from working in a Mal'tsev context. See also Theorem 3.4 of [24] for the proof in an exact Mal'tsev context.

Proposition 3.14. *Let $\mathbb{D}$ be a regular Mal'tsev category and $I : \mathbb{D} \to \mathbb{C}$ a Birkhoff reflection. Then I is admissible.*

Proof. Consider the diagram in the proof of Proposition 2.16. The map $\eta_{R[f]}$ (and η_X) is a regular epimorphism; then any of the upper squares are pullbacks by Proposition 3.4. This makes f an I-normal extension and the internal functor $IR(g) : IR[f] \to R[\phi]$ a discrete fibration. Since its codomain is an equivalence relation, its domain $IR[f]$ is an equivalence relation, which is effective by Lemma 3.12. We have $IR[f] \simeq R[If]$ and f is I-trivial by Proposition 2.5. $\square$

Corollary 3.15. *Under the conditions of the previous proposition, a regular epimorphism f is I-cartesian if and only if it is I-trivial.*

Proof. We already noticed that an I-trivial map is certainly I-cartesian. Conversely, consider a regular epimorphism $f : X \twoheadrightarrow Y$. By the previous proposition, the pullback $\bar{f}$ of If along η_Y is I-trivial and thus I-cartesian above $I\bar{f} = If$. So, the maps f and $\bar{f}$ are I-cartesian above the same map $If = I\bar{f}$; they are the same up to isomorphism and f is I-trivial. □

Eventually, we get a large class of examples of regular reflections:

Theorem 3.16. *Let $\mathbb{D}$ be a regular Mal'tsev category and $I : \mathbb{D} \to \mathbb{C}$ a Birkhoff reflection. Then I is a regular reflection.*

Proof. We can use the proof of Proposition 3.6, where x and y are now just regular epimorphisms. Knowing that $R(x)$ and $R(y)$ are regular epimorphisms since x and y are such, the monomorphic factorization $\bar{\gamma}$ is still a regular epimorphism by Lemma 3.3. □

This result was already noticed in Lemma 8.2 of [21] in the special case of the abelianization functor for a semi-abelian context.

3.4. The reg-epi reflection I_1

Recall from Section 2.3 that, when $\mathbb{D}$ is an efficiently regular Mal'tsev category and $I : \mathbb{D} \to \mathbb{C}$ is a Birkhoff reflection (and thus a regular reflection), we can define the centralization functor $I_1 : Ext(Y) \to CExt_I(Y)$. The Mal'tsev context yields an important extra property:

Lemma 3.17. *Let $\mathbb{D}$ be an efficiently regular Mal'tsev category, $I : \mathbb{D} \to \mathbb{C}$ a Birkhoff reflection and $\underline{f}_1 : \underline{X}_1 \twoheadrightarrow \underline{Y}_1$ a regular epimorphic internal functor. When it exists, the comprehensive factorization of $\underline{f}_1 = \underline{\bar{f}}_1 \cdot \underline{\hat{f}}_1$ is such that the internal functor $\underline{\hat{f}}_1$ is a regular epimorphism.*

Proof. Let us go back to the structural diagram (3) and the diagram of Proposition 1.11. When $\mathbb{D}$ is a regular Mal'tsev category, then the internal functor $\underline{\psi}_1 : Dec\, \underline{X}_1 \to \underline{U}_1$ is a regular epimorphism, since it is a levelwise regular epimorphism by Lemma 3.3. So $\hat{f}_0 = \pi_0(\underline{\psi}_1)$ is a regular epimorphism (and similarly for $\hat{f}_1$); thus $\underline{\hat{f}}_1$ is a regular epimorphism in $Grd\,\mathbb{D}$. □

Corollary 3.18. *Let $\mathbb{D}$ be an efficiently regular Mal'tsev category and $I : \mathbb{D} \to \mathbb{C}$ a Birkhoff reflection. Then the centralization functor I_1 is a reg-epi reflection.*

As mentioned before, the construction of the centralization functor was based on the property of I being a regular reflection (and thus admissible), so I-central extensions are I-normal. However, when considering efficiently regular Mal'tsev categories, it is possible to proceed just with reg-epi reflections.

Proposition 3.19. *Let $\mathbb{D}$ be a regular Mal'tsev category, $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection and $\mathbb{C}$ an efficiently regular category. Then I is admissible.*

Proof. We can follow exactly the proof of Proposition 3.14. The only difference is the reason why the equivalence relation $IR[f]$ is effective. This time, it comes from the fact that the discrete fibration $IR(g) : IR[f] \to R[\phi]$ lies in an efficiently regular category. □

Remark 3.20. Under the conditions of Proposition 3.19, we can define a reg-epi (I-)*normalization* functor just like the centralization functor given in Theorem 2.17 and with the same properties. In fact, in the proof of Theorem 2.17, the assumption *"I is a regular reflection"* was mainly used since it implied the coincidence of I-central and I-normal extensions, but the construction itself dealt with the associated I-normal extension. We can mimic here, step by step, the proof of this theorem. The main point was that the regular reflection I preserved the following pullbacks of split epimorphisms along regular epimorphisms:

$$\begin{tikzcd}
R[\bar{f}] \arrow[r, two heads, "\bar{\eta}_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\
\bar{X} \arrow[r, two heads, "\bar{\eta}_0"'] & IX
\end{tikzcd}$$

But, in the Mal'tsev context, thanks to Proposition 3.4, this is true for this particular pullback even when I is only a reg-epi reflection.

4. Preservation of products

Given a regular category $\mathbb{D}$ and a full replete reflective inclusion $j : \mathbb{C} \rightarrowtail \mathbb{D}$, we investigated the preservation by a reflection $I : \mathbb{D} \to \mathbb{C}$ of certain kinds of pullbacks in the previous two sections. Now, we shall have a look at the particular case of the preservation of (binary) products. To do this we analyse the special product of an object with itself. We give a necessary and sufficient condition for this product to be preserved by I, through the behaviour of I-normal objects, i.e. objects whose terminal map is I-normal. Then we see how this applies to the case where I is the abelianization functor. In this setting we also explore the link between (algebraically) central maps and I-normality and I-centrality.

When $\mathbb{D}$ is pointed (i.e. when it has a zero object), any terminal map is split and any product is a special case of a pullback of a split epimorphism along a split epimorphism. Then, products are preserved by regular reflections. If $\mathbb{D}$ is also Mal'tsev, then products are preserved by reg-epi reflections (Proposition 3.6). So we shall be interested in the non-pointed case.

Given an object X, the (regular epimorphism, monomorphism) factorization of the terminal map, $X \twoheadrightarrow Q \rightarrowtail 1$, is made through an object Q, the *support* of X, that belongs to $\mathbb{C}$ since it is a subobject of 1. We say that X has *global support* when the terminal map $X \twoheadrightarrow 1$ is a regular epimorphism, i.e. the support of X is 1.

4.1. Special products and I-normal objects

Proposition 4.1. *Let $\mathbb{D}$ be a regular category and $I : \mathbb{D} \to \mathbb{C}$ an admissible reflection. If the projection $p_X : X \times C \twoheadrightarrow X$ is a regular epimorphism, with $C \in \mathbb{C}$, then it is I-trivial and we have $I(X \times C) \simeq IX \times C$.*

Proof. Since the following rectangle is a pullback, the regular epimorphism p_X is I-trivial (Lemma 2.13). So the following left square is also a pullback:

$$\begin{tikzcd}
X \times C \arrow[r, two heads, "\eta_{X \times C}"] \arrow[d, two heads, "p_X"'] \arrow[rr, bend left, "\eta_X \times 1_C"] & I(X \times C) \arrow[r, two heads, "(Ip_X, Ip_C)"] \arrow[d, two heads, "Ip_X"] & IX \times C \arrow[d, two heads, "p_{IX}"] \\
X \arrow[r, two heads, "\eta_X"'] & IX \arrow[r, equal] & IX
\end{tikzcd}$$

Accordingly the right hand square is a pullback (by Corollary 1.3) and, consequently, $I(X \times C) \simeq IX \times C$. $\square$

Note that such projections are regular epimorphisms when C has global support, or when there is a map $x : X \to C$. In particular, we have $I(X \times IX) \simeq IX \times IX$. Also, for the support Q of X we have $I(X \times Q) \simeq IX \times Q$.

Proposition 4.2. *Let $\mathbb{D}$ be a regular category, $I : \mathbb{D} \to \mathbb{C}$ a regular reflection and $\mathbb{C}$ an efficiently regular category. If X is an object such that $I(X \times X) \simeq IX \times IX$ and if X and Y have global supports, then I preserves their product.*

Proof. Consider the following diagram where Y has global support. Its left hand part is made of pullbacks of split epimorphisms along a regular epimorphism:

$$\begin{tikzcd}
X \times X \times Y \arrow[r, shift left, "p_0 \times 1_Y"] \arrow[r, shift right, "p_1 \times 1_Y"'] \arrow[d, two heads, "p_{X \times X}"'] & X \times Y \arrow[r, two heads, "p_Y"] \arrow[d, two heads, "p_X"] & Y \arrow[d, two heads] \\
X \times X \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[r, two heads] & 1
\end{tikzcd}$$

Then the left pullbacks are preserved by I and produce a vertical discrete fibration on the left:

$$\begin{tikzcd}
I(X \times X \times Y) \arrow[r, shift left, "I(p_0 \times 1_Y)"] \arrow[r, shift right, "I(p_1 \times 1_Y)"'] \arrow[d, "Ip_{X \times X}"'] & I(X \times Y) \arrow[r, two heads, "Ip_Y"] \arrow[d, "Ip_X"] & IY \arrow[d, two heads] \\
I(X \times X) \arrow[r, shift left, "Ip_0"] \arrow[r, shift right, "Ip_1"'] & IX \arrow[r, two heads] & 1
\end{tikzcd}$$

Since $I\nabla X = \nabla IX$ is an effective equivalence relation and $\mathbb{C}$ is efficiently regular, the upper horizontal internal groupoid is an effective equivalence relation. If, moreover X has global support, then $p_Y : X \times Y \twoheadrightarrow Y$ is a regular epimorphism and is the quotient of the upper equivalence relation in our first diagram. Accordingly Ip_Y is the quotient of the upper effective equivalence relation in the second one. Then the Barr–Kock Theorem 1.2 makes the right hand side square a pullback which shows that I preserves the product in question. $\square$

Concerning objects with global support, the problem of preserving products is reduced to that of the preservation of products of an object with itself. So we are now looking for those objects X such that $I(X \times X) \simeq IX \times IX$. For that let us call *$I$-normal* an object X which has an I-normal terminal map $X \to 1$, i.e. such that the projections $p_0, p_1 : X \times X \to X$ are I-trivial.

Lemma 4.3. *Suppose that $\mathbb{D}$ is a regular category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. An I-normal object X is such that $I(X \times X) \simeq IX \times IX$ if and only if X is in $\mathbb{C}$.*

Proof. It is clear that if X is in $\mathbb{C}$, we have $I(X \times X) \simeq IX \times IX$. Conversely let X be an I-normal object such that $I(X \times X) \simeq IX \times IX$. Consider the following diagram:

$$\begin{tikzcd} X \times X \arrow[r, two heads, "\eta_{X\times X}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & I(X \times X) \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\ X \arrow[r, two heads, "\eta_X"'] \arrow[d, "q"'] & IX \arrow[d, "Iq"] \\ Q \arrow[r, two heads, "\sim", "\eta_Q"'] & IQ \end{tikzcd}$$

where Q is the support of X. Then $I\nabla X \simeq \nabla IX$ is an effective equivalence relation, and the upper squares are pullbacks since X is I-normal. Hence the lower square is a pullback (by Barr–Kock Theorem 1.2), η_X is an isomorphism and X is in $\mathbb{C}$. □

Proposition 4.4. *Let $\mathbb{C}$ and $\mathbb{D}$ be efficiently regular Mal'tsev categories and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. Then $I(X \times X) \simeq IX \times IX$ for all objects X if and only if all I-normal objects are in $\mathbb{C}$.*

Proof. Suppose that all I-normal objects are in $\mathbb{C}$. Given any object X, we take the comprehensive factorization of the internal functor $\nabla X \to I\nabla X$:

$$\begin{tikzcd} X \times X \arrow[r, two heads] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & \bar{X} \times \bar{X} \arrow[r, two heads] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & I(X \times X) \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\ X \arrow[r, two heads] \arrow[d, "q"'] & \bar{X} \arrow[r, two heads] \arrow[d, "\bar{q}"] & IX \arrow[d, "Iq"] \\ Q \arrow[r, equal] & Q \arrow[r, two heads, "\eta_Q"'] & IQ \end{tikzcd}$$

It is the same construction as was mentioned in Remark 3.20 (see also Corollary 1.13). Then $\bar{X}$ is I-normal, and thus in $\mathbb{C}$. We have moreover $IX \simeq I\bar{X} = \bar{X}$, which implies that $I(X \times X) \simeq \bar{X} \times \bar{X} \simeq IX \times IX$. Conversely suppose that we have $I(X \times X) \simeq IX \times IX$ for any X. In particular, this holds for any I-normal object X which must be in $\mathbb{C}$ by the previous lemma. □

4.2. The reflection to abelian objects

We are now going to study the particular case of the abelianization functor $I : \mathbb{D} \to \mathbb{D}_{Ab}$ for a finitely cocomplete efficiently regular Mal'tsev category $\mathbb{D}$. We will prove that all I-normal objects are abelian to conclude that I preserves products of the type $X \times X$ (Proposition 4.7) and all products of objects with global support in an exact context (Proposition 4.8).

It appears that the context of Mal'tsev categories $\mathbb{D}$ particularly fits with the notion of the commutator of equivalence relations [11]; see also [31]. It is then possible to define an object X in $\mathbb{D}$ as being *abelian* when we have $[\nabla X, \nabla X] = 0$, i.e. when the commutator $[\nabla X, \nabla X]$ is trivial, or, equivalently, when the object X is equipped with a (necessarily uniquely determined) Mal'tsev operation $p : X \times X \times X \to X$ (see also Remark 2.6.2 of [2]). We shall denote by $\mathbb{D}_{Ab}$ the subcategory of the abelian objects in $\mathbb{D}$. It is easy to see that the subcategory $\mathbb{D}_{Ab}$ is stable under finite limits and under subobjects. It also clear that $\mathbb{D}_{Ab}$ is a naturally Mal'tsev category in the sense of [29]: any object is endowed with a natural Mal'tsev operation, or, equivalently, any reflexive graph is an internal groupoid. Another equivalent property is given by the fact that any pair of equivalence relations R and S on the same object is such that $[R, S] = 0$, i.e. admits a centralizing double (equivalence) relation (see also Corollary 2.7.6 of [2]). When moreover $\mathbb{D}$ is regular, $\mathbb{D}_{Ab}$ is stable under regular epimorphisms (by Proposition 4.1 of [11]). If $\mathbb{D}$ is efficiently regular, then so is $\mathbb{D}_{Ab}$ [8]. When $\mathbb{D}$ is also finitely cocomplete, the inclusion $j : \mathbb{D}_{Ab} \rightarrowtail \mathbb{D}$ admits a (reg-epi) reflection (see [2,21]) $I : \mathbb{D} \to \mathbb{D}_{Ab}$. Then $\mathbb{D}_{Ab}$ is a Birkhoff subcategory of $\mathbb{D}$ and, when $\mathbb{D}$ is exact, I is a Birkhoff reflection (Section 3.3).

In this section we shall suppose that $\mathbb{D}$ is a finitely cocomplete efficiently regular Mal'tsev category. Let us begin by recalling the following two results of [13].

Lemma 4.5 ([13]). *Let $\mathbb{A}$ be an efficiently regular naturally Mal'tsev category and $f : X \to Y$ a morphism in $\mathbb{A}$. Then there is an object $N[f]$ in $\mathbb{A}$ such that the following right hand side square is a pullback:*

$$\begin{tikzcd}
\Theta \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, shift right=2, "p_0"'] \arrow[d, dotted, shift left=2, "p_1"] & R[f] \arrow[r, two heads, "\nu(f)"] \arrow[d, shift right=2, "p_0"'] \arrow[d, dotted, shift left=2, "p_1"] & N[f] \arrow[d] \\
X \times X \arrow[u] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[u] \arrow[r] \arrow[d, "f"'] & 1 \\
& Y &
\end{tikzcd}$$

Proof. Since $\mathbb{A}$ is naturally Mal'tsev, the pair of equivalence relations ∇X and $R[f]$ on X admit a centralizing double relation represented by the left hand side pullbacks. So the vertical maps p_0 produce a discrete fibration between the two horizontal equivalence relations. Now, the lower one, ∇X, is effective and, since $\mathbb{A}$ is efficiently regular, the same holds for the upper one which, consequently, admits a quotient $\nu(f) : R[f] \twoheadrightarrow N[f]$. The Barr–Kock Theorem 1.2 implies that the right hand side square is a pullback. □

The object $N[f]$ (actually the pair $(N[f], \nu(f))$) is called the *metakernel* of the map f. The terminology is motivated by the fact that, when $\mathbb{A}$ is pointed and, thus, additive, this metakernel $N[f]$ coincides with the kernel $K[f]$ of the map f. When f is a terminal map $X \to 1$, we call its metakernel the *direction* of the object X. The terminology is adopted from affine geometry (see [6] for further reading).

Proposition 4.6 ([13]). *Let $\mathbb{A}$ be an efficiently regular naturally Mal'tsev category, and $\underline{X}_1$ an internal groupoid in $\mathbb{A}$. Then the following square is a pullback:*

$$\begin{tikzcd}
X_1 \arrow[r, "\nu(d_0).s_1"] \arrow[d, "d_0"'] & N[d_0] \arrow[d] \\
X_0 \arrow[r] & 1
\end{tikzcd}$$

Proof. Consider the following two pullbacks:

$$\begin{tikzcd}
X_1 \arrow[r, "s_1"] \arrow[d, "d_0"'] & R[d_0] \arrow[r, "\nu(d_0)"] \arrow[d, "p_0"'] & N[d_0] \arrow[d] \\
X_0 \arrow[r, "s_0"'] & X_1 \arrow[r] & 1
\end{tikzcd}$$

where $s_1 = (s_0 \cdot d_0, 1_{X_1})$. We shall set $\lambda_{\underline{X}_1} = \nu(d_0).s_1$. □

Proposition 4.7. *Let $\mathbb{D}$ be a finitely cocomplete efficiently regular Mal'tsev category. Then the reg-epi reflection $I : \mathbb{D} \to \mathbb{D}_{Ab}$ satisfies $I(X \times X) \simeq IX \times IX$ for all objects X.*

Proof. Using Proposition 4.4, we have to show that any I-normal object is abelian. For that consider the following diagram, with X an I-normal object:

$$\begin{tikzcd}
\Theta \arrow[r, dotted, shift left, "p_0"] \arrow[r, dotted, shift right, "p_1"'] \arrow[d, dotted, shift right, "p_0"'] \arrow[d, dotted, shift left, "p_1"] & X \times X \arrow[r, "\eta_{X \times X}"] \arrow[d, shift right, "p_0"'] \arrow[d, dotted, shift left, "p_1"] & I(X \times X) \arrow[r, "\lambda_{I\nabla X}"] \arrow[d, shift right, "Ip_0"'] \arrow[d, dotted, shift left, "Ip_1"] & N[Ip_0] \arrow[d] \\
X \times X \arrow[r, dotted, shift left, "p_0"] \arrow[r, dotted, shift right, "p_1"'] & X \arrow[r, "\eta_X"'] & IX \arrow[r] & 1
\end{tikzcd}$$

The middle square is a pullback since X is I-normal, and the right hand side one too according to the previous proposition, so the right hand side rectangle is a pullback. Its completion by the horizontal kernel equivalence relations produces the centralizing double relation on the left which shows that we have $[\nabla X, \nabla X] = 0$, and that X is abelian. □

Proposition 4.8. *Let $\mathbb{D}$ be a finitely cocomplete exact Mal'tsev category. Suppose that X and Y are two objects with global support. Then the Birkhoff reflection $I : \mathbb{D} \to \mathbb{D}_{Ab}$ preserves their product.*

Proof. The category $\mathbb{D}$ being an exact Mal'tsev category, the reg-epi reflection I is a Birkhoff reflection, and thus a regular reflection (Theorem 3.16). So we can apply Proposition 4.2. □

We conclude this work with a remark on centrality. In the Mal'tsev context, a morphism $f : X \to Y$ is classically said to be (algebraically) *central* when we have $[R[f], \nabla X] = 0$. The following proposition gathers parts of Theorem 4.6 of [32] and Theorem 6.1 of [21]:

Proposition 4.9. *Let $\mathbb{D}$ be a finitely cocomplete efficiently regular Mal'tsev category. A map f is central if and only if it is I-normal. When $\mathbb{D}$ is exact, f is a central extension if and only if f is an I-central extension.*

Proof. Suppose that f is I-normal. Consider the following rectangle:

$$\begin{tikzcd} R[f] \arrow[r, "\eta_{R[f]}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, dotted, "p_1"] & IR[f] \arrow[r, "\lambda_{IR[f]}"] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, dotted, "Ip_1"] & N[Ip_0] \arrow[d] \\ X \arrow[r, "\eta_X"'] & IX \arrow[r] & 1 \end{tikzcd}$$

It is a pullback made of two pullbacks. The completion by the horizontal kernel equivalences produces the centralizing double relation which gives us $[\nabla X, R[f]] = 0$. Conversely suppose that f is central. Then we can adapt Proposition 5.3 of [11] to an efficiently regular context to get $R[f] \simeq X \times C$, with C an abelian object. So the following rectangle is a pullback:

$$\begin{tikzcd} R[f] \arrow[r, two heads, "\eta_{R[f]}"] \arrow[d, shift right, "p_0"'] \arrow[rr, bend left, "q"] & IR[f] \arrow[r, two heads, "Iq"] \arrow[d, shift right, "Ip_0"'] & C \arrow[d] \\ X \arrow[r, two heads, "\eta_X"'] \arrow[u, shift right] & IX \arrow[r] \arrow[u, shift right] & 1 \end{tikzcd}$$

and, by applying Proposition 3.4, we conclude that f is I-normal. When $\mathbb{D}$ is exact, I is a Birkhoff reflection, and therefore a regular reflection (Theorem 3.16), and the I-normal and I-central extensions coincide (Proposition 2.10). $\square$

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