Hopf bifurcation in an age-structured SIR epidemic model

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Abstract

In this paper, we study the occurrence of a sustained periodic solution via the Hopf bifurcation in an age-structured SIR epidemic model. Under the assumption that the transmission rate depends on the age of infective individuals and the product of the transmission rate and the population age distribution is concentrated in a specific age, we reformulate the model into an integral equation of Fredholm type. We then define the basic reproduction number $\mathcal{R}_0$ and show that the unique positive endemic equilibrium of the integral equation exists if and only if $\mathcal{R}_0 > 1$. We derive a characteristic equation for the endemic equilibrium, and regarding the specific age as a bifurcation parameter, we obtain a sufficient condition for the occurrence of the Hopf bifurcation. Finally, we provide a numerical example that supports our theoretical result.

Keywords: SIR epidemic model, Age structure, Basic reproduction number, Stability, Hopf bifurcation

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1. Introduction

SIR epidemic models are known as one of the most basic epidemic models in which total population is divided into three classes called susceptible, infective and recovered. Since the first work [1] by Kermack and McKendrick in 1927, many authors have developed the theory of SIR epidemic models with various structures such as time-delay structure [2], multigroup structure [3] and age structure [4].

The basic reproduction number $\mathcal{R}_0$ is known as one of the key concepts in the theory of epidemic models, which represents the expected number of secondary cases produced by a typical infective individual during its entire period of infectiousness in a fully susceptible population [5]. For an SIR epidemic model in a simple form of ordinary differential equations, $\mathcal{R}_0$ determines the complete global dynamics of the solution, that is, if $\mathcal{R}_0 \leq 1$, then the disease-free equilibrium with no infective population is globally asymptotically stable, whereas if $\mathcal{R}_0 > 1$, then the endemic equilibrium with positive infective population is so [6, Section 5.5.2].

In contrast, this is not the case for a class of age-structured SIR epidemic models in forms of partial differential equations. For an age-structured SIR epidemic model, the global asymptotic stability of the disease-free equilibrium for $\mathcal{R}_0 < 1$ has been proved [4], however, the stability of the endemic equilibrium for $\mathcal{R}_0 > 1$ has been proved in some restricted special cases [4, 7, 8], and the instability of the endemic equilibrium for $\mathcal{R}_0 > 1$ has also been proved in some other cases [9–12]. One of the most important questions in the cases of unstable endemic equilibria for $\mathcal{R}_0 > 1$ is whether a sustained periodic solution can arise via the Hopf bifurcation since such a solution can be responsible for the biennial outbreaks of diseases such as measles [9]. The purpose of this study is to obtain a new sufficient condition for the Hopf bifurcation in an age-structured SIR epidemic model.

The organization of this paper is as follows. In Section 2, we formulate an age-structured SIR epidemic model. Based on [9], we assume that the transmission rate depends only on the age of infective individuals and the product of the transmission rate and the population age distribution is concentrated in a specific age a^* in the sense of the Dirac delta function. We then reformulate the model into an integral equation of infective population with age a^*. In Section 3, we define the basic reproduction number $\mathcal{R}_0$ and show that

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the unique endemic equilibrium exists if $\mathcal{R}_0 > 1$. We then derive a characteristic equation for the stability analysis of the endemic equilibrium. By regarding a^* as a bifurcation parameter and applying a method for delay differential systems in [13], we obtain a sufficient condition for the Hopf bifurcation. Finally, in Section 4, we provide a numerical example that supports our theoretical result.

2. Model

As in [9], we focus on the following normalized SIR epidemic model with age structure.

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) s(t,a) = -s(t,a) \int_0^{+\infty} \kappa(a) p^*(a) i(t,a) \mathrm{d}a, & t > 0, \quad a > 0, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) i(t,a) = s(t,a) \int_0^{+\infty} \kappa(a) p^*(a) i(t,a) \mathrm{d}a - \gamma i(t,a), & t > 0, \quad a > 0, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) r(t,a) = \gamma i(t,a), & t > 0, \quad a > 0, \\ s(t,0) = 1, \quad i(t,0) = r(t,0) = 0, & t > 0, \end{cases} \tag{2.1}$$

where $s(t,a)$, $i(t,a)$ and $r(t,a)$ denote the fractions of susceptible, infective and recovered individuals of age a at time t, respectively. $\kappa(\cdot)$ denotes the transmission rate, $p^*(\cdot)$ denotes the stable population age distribution and $\gamma > 0$ denotes the recovery rate. In [9], it was shown that the endemic equilibrium can loose its stability if $\kappa(a)p^*(a)$ is sufficiently concentrated in one particular age in the sense of the Dirac delta function. However, any specific parameter regions for the instability of the endemic equilibrium and the occurrence of the Hopf bifurcation were not obtained in [9]. In this study, motivated by [9], we make the following assumption.

(A1) There exist $a^* > 0$ and $\beta > 0$ such that $\kappa(a) b \mathrm{e}^{-\int_0^a \mu(\sigma)\mathrm{d}\sigma} = \beta \delta(a - a^*)$ for all $a \geq 0$, where $\delta(\cdot)$ denotes the Dirac delta function.

Under assumption (A1), model (2.1) can be rewritten as follows.

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) s(t,a) = -\beta s(t,a) i(t,a^*), & t > 0, \quad a > 0, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) i(t,a) = \beta s(t,a) i(t,a^*) - \gamma i(t,a), & t > 0, \quad a > 0, \\ s(t,0) = 1, \quad i(t,0) = 0, & t > 0. \end{cases} \tag{2.2}$$

By integration along the characteristic line, we obtain $s(t,a) = \mathrm{e}^{-\beta \int_0^a i(t-a+\rho, a^*)\mathrm{d}\rho}$, $t - a > 0$, and

$$\begin{aligned} i(t,a) =& \beta \int_0^a \mathrm{e}^{-\gamma(a-\sigma)} s(t-a+\sigma, \sigma) i(t-a+\sigma, a^*) \mathrm{d}\sigma = \beta \int_0^a \mathrm{e}^{-\gamma(a-\sigma)} \mathrm{e}^{-\beta \int_0^\sigma i(t-a+\rho, a^*)\mathrm{d}\rho} i(t-a+\sigma, a^*) \mathrm{d}\sigma \\ =& \beta \int_0^a \mathrm{e}^{-\gamma\tau} \mathrm{e}^{-\beta \int_0^{a-\tau} i(t-a+\rho, a^*)\mathrm{d}\rho} i(t-\tau, a^*) \mathrm{d}\tau = \beta \int_0^a \mathrm{e}^{-\gamma\tau} \mathrm{e}^{-\beta \int_\tau^a i(t-\eta, a^*)\mathrm{d}\eta} i(t-\tau, a^*) \mathrm{d}\tau, \quad t - a > 0. \end{aligned}$$

Let $Y(t) := i(t, a^*)$. We then obtain the following integral equation of Fredholm type.

$$Y(t) = \beta \int_0^{a^*} \mathrm{e}^{-\gamma\tau} \mathrm{e}^{-\beta \int_\tau^{a^*} Y(t-\eta)\mathrm{d}\eta} Y(t-\tau) \mathrm{d}\tau, \quad t > a^*. \tag{2.3}$$

The main interest in this paper is to study the occurrence of a sustained periodic solution via the Hopf bifurcation in equation (2.3).

3. Stability analysis

Let $Y^* > 0$ denote a positive equilibrium of equation (2.3). We see from (2.3) that Y^* should be a positive root of the following equation.

$$1 = \beta \int_0^{a^*} \mathrm{e}^{-\gamma\tau}\mathrm{e}^{-\beta Y^*(a^*-\tau)}\mathrm{d}\tau \tag{3.1}$$

We define the basic reproduction number by $\mathcal{R}_0 := \beta \int_0^{a^*} \mathrm{e}^{-\gamma\tau}\mathrm{d}\tau = (\beta/\gamma)\left(1 - \mathrm{e}^{-\gamma a^*}\right)$ and establish the following proposition.

Proposition 3.1. *(2.3) has the unique positive endemic equilibrium $Y^* > 0$ if and only if $\mathcal{R}_0 > 1$.*

Proof. Since the right-hand side of (3.1) is monotone decreasing with respect to Y^* and converges to zero as $Y^* \to +\infty$, $\mathcal{R}_0 = \beta \int_0^{a^*} \mathrm{e}^{-\gamma\tau}\mathrm{d}\tau > 1$ is equivalent to the existence of the unique positive root $Y^* > 0$ of (3.1). This completes the proof. □

In what follows, we assume that $\mathcal{R}_0 > 1$. We then obtain the following linearized equation of (2.3) around Y^* (note that $e^x \approx 1 + x$ if $|x| \ll 1$, $x \in \mathbb{R}$).

$$\begin{aligned} Z(t) =& \beta \int_0^{a^*} \mathrm{e}^{-\gamma\tau}\mathrm{e}^{-\beta Y^*(a^*-\tau)} Z(t-\tau)\mathrm{d}\tau - \beta Y^* \int_0^{a^*} \mathrm{e}^{-\gamma\tau}\beta \int_\tau^{a^*} Z(t-\eta)\mathrm{d}\eta \mathrm{e}^{-\beta Y^*(a^*-\tau)}\mathrm{d}\tau \\ =& \int_0^{a^*} p(\tau)Z(t-\tau)\mathrm{d}\tau - \beta Y^* \int_0^{a^*} p(\tau) \int_\tau^{a^*} Z(t-\eta)\mathrm{d}\eta\mathrm{d}\tau, \end{aligned} \tag{3.2}$$

where $p(\tau) := \beta \mathrm{e}^{-\gamma\tau}\mathrm{e}^{-\beta Y^*(a^*-\tau)}$, $0 \le \tau \le a^*$. We now prove the following lemma on $p(\cdot)$.

Lemma 3.1. (i) $p'(\tau) = (\beta Y^* - \gamma)\, p(\tau)$, $0 \le \tau \le a^*$. (ii) $\int_0^{a^*} p(\tau)\mathrm{d}\tau = 1$. (iii) $p(a^*) - p(0) = \beta Y^* - \gamma$.

Proof. (i) is obvious. (ii) follows from (3.1). From (i) and (ii), we have $\beta Y^* - \gamma = (\beta Y^* - \gamma) \int_0^{a^*} p(\tau)\mathrm{d}\tau = \int_0^{a^*} p'(\tau)\mathrm{d}\tau = p(a^*) - p(0)$ and hence, (iii) holds. This completes the proof. □

Substituting $Z(t) = Z\mathrm{e}^{\lambda t}$, $Z \neq 0$, $\lambda \in \mathbb{C}$ into (3.2) and dividing both sides by Z, we obtain the following characteristic equation.

$$1 = \int_0^{a^*} p(\tau)\mathrm{e}^{-\lambda\tau}\mathrm{d}\tau - \beta Y^* \int_0^{a^*} p(\tau) \int_\tau^{a^*} \mathrm{e}^{-\lambda\eta}\mathrm{d}\eta\mathrm{d}\tau. \tag{3.3}$$

Note that $\lambda \neq 0$ since $\int_0^{a^*} p(\tau)\mathrm{d}\tau - \beta Y^* \int_0^{a^*} p(\tau)\mathrm{d}\tau = 1 - \beta Y^* < 1$. We then have

$$\int_0^{a^*} p(\tau)\mathrm{e}^{-\lambda\tau}\mathrm{d}\tau = \frac{p(0) - p(a^*)\mathrm{e}^{-\lambda a^*}}{\lambda} + \frac{\beta Y^* - \gamma}{\lambda} \int_0^{a^*} p(\tau)\mathrm{e}^{-\lambda\tau}\mathrm{d}\tau \tag{3.4}$$

and

$$\int_0^{a^*} p(\tau) \int_\tau^{a^*} \mathrm{e}^{-\lambda\eta}\mathrm{d}\eta\mathrm{d}\tau = \int_0^{a^*} p(\tau) \left[\frac{\mathrm{e}^{-\lambda\tau} - \mathrm{e}^{-\lambda a^*}}{\lambda}\right] \mathrm{d}\tau = \frac{1}{\lambda} \int_0^{a^*} p(\tau)\mathrm{e}^{-\lambda\tau}\mathrm{d}\tau - \frac{\mathrm{e}^{-\lambda a^*}}{\lambda}. \tag{3.5}$$

We have from (3.4) that

$$[\lambda - (\beta Y^* - \gamma)] \int_0^{a^*} p(\tau)\mathrm{e}^{-\lambda\tau}\mathrm{d}\tau = p(0) - p(a^*)\mathrm{e}^{-\lambda a^*}. \tag{3.6}$$

From (3.5) and (3.6), multiplying $\lambda\,[\lambda - (\beta Y^* - \gamma)]$ by both sides of (3.3), we obtain

$$\lambda\,[\lambda - (\beta Y^* - \gamma)] = (\lambda - \beta Y^*)\,[p(0) - p(a^*)\mathrm{e}^{-\lambda a^*}] + \beta Y^* \mathrm{e}^{-\lambda a^*}\,[\lambda - (\beta Y^* - \gamma)]\,. \tag{3.7}$$

Rearranging (3.7), we obtain

$$\lambda^2 - [p(0) + (\beta Y^* - \gamma)]\,\lambda + \beta Y^* p(0) + [(p(a^*) - \beta Y^*)\lambda + \beta Y^*(\beta Y^* - \gamma - p(a^*))]\,\mathrm{e}^{-\lambda a^*} = 0 \tag{3.8}$$

By Lemma 3.1 (iii), we can rewrite (3.8) as

$$P(\lambda) + Q(\lambda)\mathrm{e}^{-\lambda a^*} = 0. \tag{3.9}$$

where $P(\lambda) := \lambda^2 - p(a^*)\lambda + \beta Y^* p(0)$ and $Q(\lambda) := (p(a^*) - \beta Y^*)\lambda - \beta Y^* p(0)$. Characteristic equations with forms similar to (3.9) have appeared in the stability analysis of many delay differential systems (see, for instance, [13–20]). Substituting $\lambda = i\omega$, $\omega > 0$ into (3.9), we obtain as in [13] that

$$\begin{pmatrix} \cos(\omega a^*) \\ \sin(\omega a^*) \end{pmatrix} = \frac{1}{|Q(i\omega)|^2} \begin{pmatrix} -\operatorname{Re} Q(i\omega) & -\operatorname{Im} Q(i\omega) \\ -\operatorname{Im} Q(i\omega) & \operatorname{Re} Q(i\omega) \end{pmatrix} \begin{pmatrix} \operatorname{Re} P(i\omega) \\ \operatorname{Im} P(i\omega) \end{pmatrix} = \begin{pmatrix} -\operatorname{Re}\left(\dfrac{P(i\omega)}{Q(i\omega)}\right) \\ \operatorname{Im}\left(\dfrac{P(i\omega)}{Q(i\omega)}\right) \end{pmatrix}, \tag{3.10}$$

and thus, $1 = |P(i\omega)|^2/|Q(i\omega)|^2$. To apply the method in [13], we define $F(\omega) := |P(i\omega)|^2 - |Q(i\omega)|^2$ and find a positive real root of $F(\omega) = 0$.

Proposition 3.2. *If $2\gamma - \beta Y^* > 0$, then $F(\omega) = 0$ has a positive real root $\omega^* = \sqrt{\beta Y^*(2\gamma - \beta Y^*)} > 0$.*

Proof. We have $P(i\omega) = -\omega^2 + \beta Y^* p(0) - ip(a^*)\omega$ and $Q(i\omega) = -\beta Y^* p(0) + i\,(p(a^*) - \beta Y^*)\,\omega$, and thus,

$$\begin{aligned} F(\omega) &= \left[-\omega^2 + \beta Y^* p(0)\right]^2 + [p(a^*)]^2\omega^2 - [\beta Y^* p(0)]^2 - [p(a^*) - \beta Y^*]^2\,\omega^2 \\ &= \omega^2 \left\{\omega^2 - 2\beta Y^* p(0) + [p(a^*)]^2 - [p(a^*) - \beta Y^*]^2\right\} = \omega^2\left\{\omega^2 - \beta Y^*\,[2p(0) - 2p(a^*) + \beta Y^*]\right\} \\ &= \omega^2\left[\omega^2 - \beta Y^*\,(2\gamma - \beta Y^*)\right]. \end{aligned} \tag{3.11}$$

Hence, $F(\omega) = 0$ has a positive real root $\omega^* = \sqrt{\beta Y^*\,(2\gamma - \beta Y^*)}$ if $2\gamma - \beta Y^* > 0$. This completes the proof. □

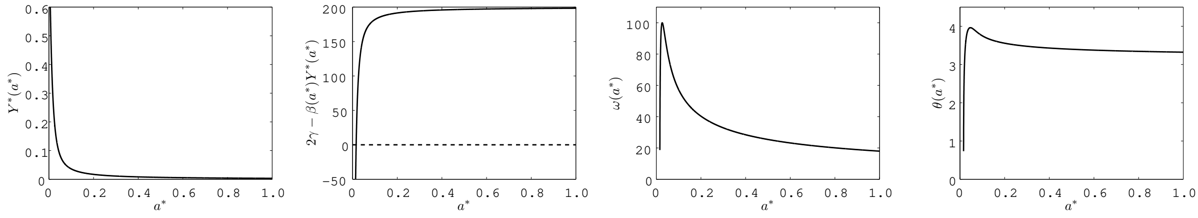
In what follows, we regard $a^* > 0$ as the bifurcation parameter under fixed $\mathcal{R}_0$ and γ. Note that $\beta = \mathcal{R}_0\gamma/(1 - \mathrm{e}^{-\gamma a^*})$ and Y^* are functions of a^* if we fix $\mathcal{R}_0$ and γ. Thus, we see that $p(0) = \beta\mathrm{e}^{-\beta Y^* a^*}$ and $p(a^*) = \beta\mathrm{e}^{-\gamma a^*}$ are also functions of a^*. Let us define a set $\mathcal{A} \subset \mathbb{R}_+$ by $\mathcal{A} := \{a^* \in \mathbb{R}_+ : 2\gamma - \beta(a^*)Y^*(a^*) > 0\}$. By Proposition 3.2, for any $a^* \in \mathcal{A}$, $\omega^*(a^*) = \sqrt{\beta(a^*)Y^*(a^*)[2\gamma - \beta(a^*)Y^*(a^*)]}$ satisfies $F(\omega^*(a^*)) = 0$. By (3.10), we define $\theta(a^*) \in [0, 2\pi)$ as the solution of $\cos\theta(a^*) = -\operatorname{Re}\left(P(i\omega^*(a^*))/Q(i\omega^*(a^*))\right)$ and $\sin\theta(a^*) = \operatorname{Im}\left(P(i\omega^*(a^*))/Q(i\omega^*(a^*))\right)$. To apply the method in [13], we define the following function.

$$\mathcal{S}_n(a^*) := a^* - \frac{\theta(a^*) + 2n\pi}{\omega^*(a^*)}, \quad a^* \in \mathcal{A}, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}.$$

By [13, Theorem 2.2], we obtain the following theorem.

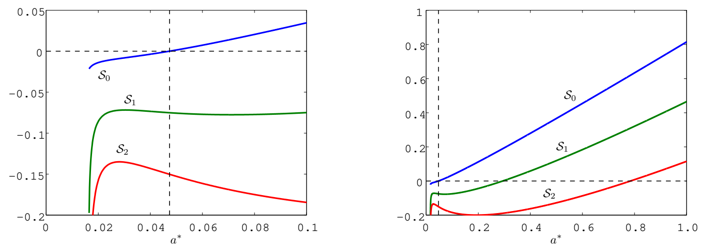
Theorem 3.1. *If $\mathcal{S}_n(a^*) = 0$ holds for some $a^* = \tau^* \in \mathcal{A}$ and $n = n^* \in \mathbb{N}_0$, then the characteristic equation (3.9) has a pair of simple conjugate pure imaginary roots $\pm\omega^*(\tau^*)$ and it crosses the imaginary axis from left to right if $\delta(\tau^*) > 0$, and from right to left if $\delta(\tau^*) < 0$, where $\delta(\tau^*) = \operatorname{sign}\{\mathrm{d}\mathcal{S}_{n^*}(\tau^*)/\mathrm{d}a^*\}$.*

Proof. By [13, Theorem 2.2], we have $\delta(\tau^*) = \operatorname{sign}\{\mathrm{d}F(\omega^*(\tau^*))/\mathrm{d}\omega\}\operatorname{sign}\{\mathrm{d}\mathcal{S}_{n^*}(\tau^*)/\mathrm{d}a^*\}$. Hence, it suffices to show that $\mathrm{d}F(\omega^*(\tau^*))/\mathrm{d}\omega > 0$. In fact, by (3.11), we have $\mathrm{d}F(\omega)/\mathrm{d}\omega = 2\omega\left[\omega^2 - \beta Y^*\,(2\gamma - \beta Y^*)\right] + 2\omega^3$, and hence, $\mathrm{d}F(\omega^*(\tau^*))/\mathrm{d}\omega = 2\left[\omega^*(\tau^*)\right]^3 > 0$. This completes the proof. □



(a) $Y^*(a^*)$ (b) $2\gamma - \beta(a^*)Y^*(a^*)$ (c) $\omega(a^*)$ (d) $\theta(a^*)$

Figure 1: Variation of each function for $a^* \in [0,1]$ ((a) and (b)), and for $a^* \in \mathcal{A} = \{a^* \in \mathbb{R}_+ : 0.016 \lesssim a^* \leq 1\}$ ((c) and (d)).



(a) (b)

Figure 2: Variation of $\mathcal{S}_0(a^*)$, $\mathcal{S}_1(a^*)$ and $\mathcal{S}_2(a^*)$ for $a^* \in \mathcal{A} = \{a^* \in \mathbb{R}_+ : 0.016 \lesssim a^* \leq 1\}$.

4. Numerical simulation

We fix $\mathcal{R}_0 = 5$ and $\gamma = 100$ and manipulate the bifurcation parameter a^* in $[0,1]$. By applying the Newton method to nonlinear equation $\beta Y^* - \gamma - p(a^*) + p(0) = \beta Y^* - \gamma - \beta e^{-\gamma a^*} + \beta e^{-\beta Y^* a^*} = 0$, we obtain $Y^* = Y^*(a^*)$ for $a^* \in [0,1]$ as shown in Figure 1 (a). Using this $Y^*(\cdot)$, we obtain $2\gamma - \beta(a^*)Y^*(a^*)$ for $a^* \in [0,1]$ as shown in Figure 1 (b). In this case, we see that $2\gamma - \beta(a^*)Y^*(a^*) > 0$ for $0.016 \lesssim a^* \leq 1$. Thus, $\mathcal{A} = \{a^* \in \mathbb{R}_+ : 0.016 \lesssim a^* \leq 1\}$. For $a^* \in \mathcal{A}$, we obtain $\omega(a^*)$ and $\theta(a^*)$ as shown in Figure 1 (b) and (c), respectively. Using these $\omega(\cdot)$ and $\theta(\cdot)$, we obtain $\mathcal{S}_0(a^*)$, $\mathcal{S}_1(a^*)$ and $\mathcal{S}_2(a^*)$ for $a^* \in \mathcal{A}$ as shown in Figure 2. From Figure 2, we see that $\mathcal{S}_0(a^*) = 0$ for $a^* = \tau^* \approx 0.047$, and $\mathrm{d}\mathcal{S}_0(\tau^*)/\mathrm{d}a^* > 0$. Hence, by Theorem 3.1, we see that the Hopf bifurcation occurs at $a^* = \tau^* \approx 0.047$. In fact, Figure 3 illustrates that the endemic equilibrium is stable for $a^* = 0.04 < \tau^*$, whereas it is unstable and a sustained periodic solution exists for $a^* = 0.05, 0.06 > \tau^*$. Finally, Figure 4 illustrates that the solution $i(t,a)$ of the original model (2.2) can also oscillate when a^* varies in a similar way as in Figure 3.

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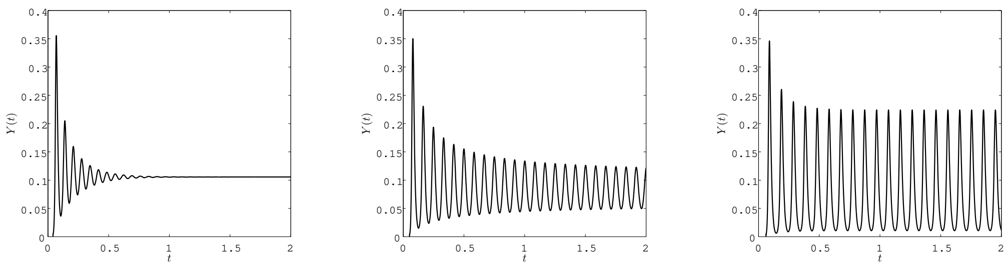
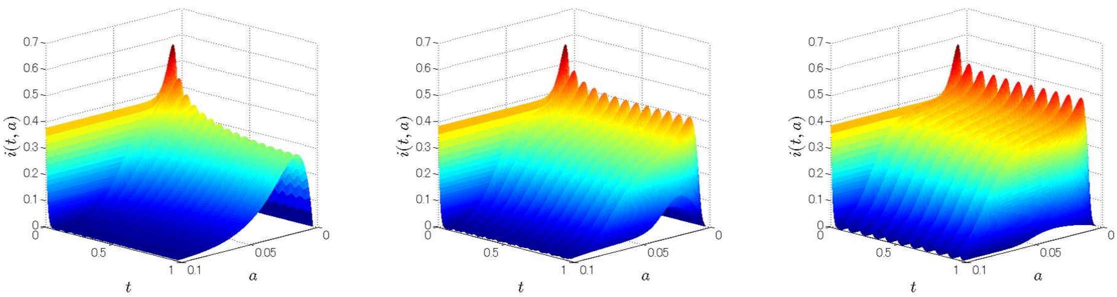
References

[1] W.O. Kermack, A.G. McKendrick, A contribution to the mathematical theory of epidemics, Proc. Roy. Soc. Lond. A 115 (1927) 700–721.



(a) $a^* = 0.04$ (b) $a^* = 0.05$ (c) $a^* = 0.06$

Figure 3: Time variation of $Y(t) = i(t, a^*)$ for $t \in [0, 2]$ and $a^* = 0.04$, 0.05 and 0.06.



(a) $a^* = 0.04$ (b) $a^* = 0.05$ (c) $a^* = 0.06$

Figure 4: Time variation of $i(t, a)$ for $(t, a) \in [0, 2] \times [0, 0.1]$ and $a^* = 0.04$, 0.05 and 0.06.

[2] C.C. McCluskey, Complete global stability for an SIR epidemic model with delay - distributed or discrete, Nonlinear Anal. RWA 11 (2010) 55–59.
[3] H. Guo, M.Y. Li, Z. Shuai, Global stability of the endemic equilibrium of multigroup SIR epidemic models, Canad. Appl. Math. Quart. 14 (2006) 259–284.
[4] H. Inaba, Threshold and stability results for an age-structured epidemic model, J. Math. Biol. 28 (1990) 411–434.
[5] O. Diekmann, J.A.J. Heesterbeek, J.A.J. Metz, On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, J. Math. Biol. 28 (1990) 365–382.
[6] H. Inaba, Age-Structured Population Dynamics in Demography and Epidemiology, Springer, Singapore, 2017.
[7] A.V. Melnik, A. Korobeinikov, Lyapunov functions and global stability for SIR and SEIR models with age-dependent susceptibility, Math. Biosci. Eng. 10 (2013) 369–378.
[8] T. Kuniya, Stability analysis of an age-structured SIR epidemic model with a reduction method to ODEs, Mathematics 6 (2018). DOI:10.3390/math6090147
[9] H.R. Thieme, Stability change of the endemic equilibrium in age-structured models for the spread of S-I-R type infectious diseases, in: S.N. Busenberg, M. Iannelli (Eds.), Differential Equations Models in Biology, Epidemiology and Ecology, Springer, Berlin, 1991, pp. 139–158.
[10] V. Andreasen, Instability in an SIR-model with age-dependent susceptibility, in: O. Arino, D. Axelrod, M. Kimmel, M. Langlais (Eds.), Mathematical Population Dynamics: Analysis of Heterogeneity, in: Theory of Epidemics, Wuerz Publ., Winnipeg, 1995, pp. 3–14.
[11] Y. Cha, M. Iannelli, F.A. Milner, Stability change of an epidemic model, Dyn. Syst. Appl. 9 (2000) 361–376.
[12] A. Franceschetti, A. Pugliese, Threshold behaviour of a SIR epidemic model with age structure and immigration, J. Math. Biol. 57 (2008) 1–27.
[13] E. Beretta, Y. Kuang, Geometric stability switch criteria in delay differential systems with delay dependent parameters, SIAM J. Math. Anal. 33 (2002) 1144–1165.
[14] F. Brauer, P. van den Driessche, L. Wang, Oscillations in a patchy environment disease model, Math. Biosci. 215 (2008) 1–10.

[15] X. Song, X. Zhou, X. Zhao, Properties of stability and Hopf bifurcation for a HIV infection model with time delay, Appl. Math. Model. 34 (2010) 1511–1523.

[16] T. Zhang, J. Liu, Z. Teng, Stability of Hopf bifurcation of a delayed SIRS epidemic model with stage structure, Nonlinear Anal. RWA 11 (2010) 293–306.

[17] Y. Yuan, J. Bélair, Threshold dynamics in an SEIRS model with latency and temporary immunity, J. Math. Biol. 69 (2014) 875–904.

[18] N. Ali, G. Zaman, O. Algahtani, Stability analysis of HIV-1 model with multiple delays, Adv. Diff. Equat. 88 (2016). DOI: 10.1186/s13662-016-0808-4

[19] J. Yang, M. Martcheva, Y. Chen, Imitation dynamics of vaccine decision-making behaviours based on the game theory, J. Biol. Dyn. 10 (2016) 31–58.

[20] X.C. Duan, J.F. Yin, X.Z. Li, Global Hopf bifurcation of an SIRS epidemic model with age-dependent recovery, Chaos, Solitons and Fractals 104 (2017) 613–624.