Pseudo-embeddings of the (point,k-spaces)-geometry of PG$(n, 2)$ and projective embeddings of $DW(2n-1, 2)$

Bart De Bruyn

February 14, 2017

Abstract

In this paper, we classify all homogeneous pseudo-embeddings of the point-line geometry defined by the points and k-dimensional subspaces of PG$(n, 2)$, and use this to study the local structure of homogeneous full projective embeddings of the dual polar space $DW(2n-1, 2)$. Our investigation allows us to distinguish n possible types for such homogeneous embeddings. For each of these n types, we shall construct a homogeneous full projective embedding of $DW(2n-1, 2)$.

Keywords: homogeneous projective embedding, (symplectic) dual polar space, pseudo-embedding, pseudo-hyperplane
MSC2010: 05B25, 51A45, 51A50

1 Introduction

In [15], we introduced *pseudo-hyperplanes* and *pseudo-embeddings* of point-line geometries. For geometries with three points per line, these notions coincide with the notions of hyperplanes and full projective embeddings. If pseudo-embeddings exist, then there exists a largest one which is called the *universal pseudo-embedding*. The vector dimension of the universal pseudo-embedding space will be called the *pseudo-embedding rank*. As we will see in Section 2.1 (Proposition 2.4), an important tool for determining whether a pseudo-embedding is universal is the notion of *pseudo-generating rank*.

If all automorphisms of the geometry lift to automorphisms of the pseudo-embedding space, then the pseudo-embedding is called *homogeneous*. In [14, 16], we classified all homogeneous pseudo-embeddings of the affine spaces AG$(n, 4)$, the projective spaces PG$(n, 4)$ and all generalized quadrangles of order $(3, t)$. The present paper achieves this goal for another family of geometries. The classification of the homogeneous pseudo-embeddings of these geometries will have implications for projective embeddings of symplectic dual polar spaces. In fact, this was also our initial motivation for studying this problem. Let us first define the geometries whose pseudo-embeddings will be investigated.

For all $n, k \in \mathbb{N}$ with $1 \leq k \leq n+1$, let $\mathcal{S}_{n,k}$ be the point-line geometry whose points and lines are the points and k-dimensional subspaces of $\mathrm{PG}(n, 2)$, with incidence being the one induced by $\mathrm{PG}(n, 2)$. If $k = 1$, then the geometry $\mathcal{S}_{n,k}$ is just $\mathrm{PG}(n, 2)$ (regarded as a point-line geometry). If $k = n + 1$, then $\mathcal{S}_{n,k}$ has $2^{n+1} - 1$ points, but no lines.

With n and k as above, let $\epsilon_{n,k}$ be the map from $\mathrm{PG}(n, 2)$ to $\mathrm{PG}(N_{n,k}, 2)$, where $N_{n,k} := -1 + \sum_{i=1}^{k} \binom{n+1}{i}$, mapping the point $(X_0, X_1, \ldots, X_n)$ of $\mathrm{PG}(n, 2)$ to the point

$$(X_0, X_1, \ldots, X_n, X_0X_1, X_0X_2, \ldots, X_{n-1}X_n, X_0X_1X_2, \ldots, X_{n-k+1}X_{n-k+2} \cdots X_n)$$

of $\mathrm{PG}(N_{n,k}, 2)$. The following is our first main result of this paper.

Theorem 1.1 (1) *The pseudo-generating and pseudo-embedding ranks of $\mathcal{S}_{n,k}$ are equal to $\sum_{i=1}^{k} \binom{n+1}{i}$.*

(2) *$\epsilon_{n,k}$ is isomorphic to the universal pseudo-embedding of $\mathcal{S}_{n,k}$.*

(3) *If $k \leq n$, then every homogeneous pseudo-embedding of $\mathcal{S}_{n,k}$ is isomorphic to $\epsilon_{n,k}$.*

The fact that the pseudo-embedding rank of $\mathcal{S}_{n,k}$ is equal to $\sum_{i=1}^{k} \binom{n+1}{i}$ implies (see Section 2.1) that the binary code of the points and k-dimensional subspaces of $\mathrm{PG}(n, 2)$ has dimension $|\mathrm{PG}(n, 2)| - \sum_{i=1}^{k} \binom{n+1}{i} = \sum_{i=k+1}^{n+1} \binom{n+1}{i} = \sum_{i=0}^{n-k} \binom{n+1}{i}$. This is precisely Corollary 5.3.2 of Assmus and Key [1]. General formulas for the dimension of the p-ary code of the points and k-dimensional subspaces of $\mathrm{PG}(n, q)$, $q = p^h$ with p prime and $h \in \mathbb{N} \setminus \{0\}$, can be found in Hamada [17, Theorem 1] or Inamdar and Sastry [19, Theorem 2.13]. These formulas are usually more complex. E.g., Theorem 2.13 of [19] tells us that the dimension of the above-mentioned binary code is also equal to $1 + \sum_{i=1}^{n-k} \sum_{s=0}^{i-1} (-1)^s \binom{n+1}{s} \binom{n+i-2s}{n}$.

Let $\Delta = DW(2n - 1, 2)$ with $n \in \mathbb{N} \setminus \{0, 1\}$ denote the symplectic dual polar space associated with a symplectic polarity of $\mathrm{PG}(2n - 1, 2)$.

Let x denote a point of Δ. For every convex subspace F through x, we denote by $\mathcal{L}_F$ the set of lines through x contained in F. Let $\mathcal{S}_x$ denote the point-line geometry whose points and lines are the lines and quads through x, with incidence being containment. Then $\mathcal{S}_x \cong \mathrm{PG}(n - 1, 2)$. The k-dimensional subspaces of $\mathcal{S}_x$ are then the sets $\mathcal{L}_F$ for convex subspaces F of diameter $k + 1$ through x. After having introduced coordinates in $\mathcal{S}_x$, we may identify $\mathrm{PG}(n - 1, 2)$ with $\mathcal{S}_x$ and assume that the map $\epsilon_{n-1,k}$ (as defined above) maps lines of Δ through x to points of $\mathrm{PG}(N_{n-1,k}, 2)$.

Suppose now that ϵ is a full projective embedding of $DW(2n - 1, 2)$ into a projective space Σ. The image of $x^\perp$ (i.e. the set of points collinear with x) generates a subspace Σ_x of Σ. The map ϵ naturally induces a map ϵ_x from the points of $\mathcal{S}_x$ to the points of the quotient space $\Sigma_x / \epsilon(x)$. The following is our second main result of this paper.

Theorem 1.2 *Suppose ϵ is a homogeneous full embedding of $DW(2n-1, 2)$ in a projective space Σ and x is a point of $DW(2n - 1, 2)$. Then there exists a unique $k \in \{1, 2, \ldots, n\}$ such that the maps ϵ_x and $\epsilon_{n-1,k}$ are isomorphic. For this value of k, we thus have $\dim(\Sigma_x) = \sum_{i=1}^{k} \binom{n}{i}$.*

A homogeneous full projective embedding ϵ of $DW(2n-1,2)$ is said to be of *type* $k \in \{1,2,\ldots,n\}$ if the maps ϵ_x and $\epsilon_{n-1,k}$ are isomorphic. The fact that ϵ is homogeneous implies that this definition is independent of the considered point x. Homogeneous full projective embeddings of $DW(2n-1,2)$ have not yet been intensively studied. Up to now, four such embeddings were known, the spin-embedding, the Grassmann embedding, the universal embedding and the full projective embedding of $DW(2n-1,2)$ induced by the universal embedding of the Hermitian dual polar space $DH(2n-1,4)$ into which $DW(2n-1,2)$ is fully and isometrically embeddable. We also prove the following.

Theorem 1.3 (1) *For every $i \in \{1,2,\ldots,n\}$, there exists a homogeneous full projective embedding of $DW(2n-1,2)$ that has type i.*

(2) *Every homogeneous full projective embedding of $DW(2n-1,2)$ that has type 1 is isomorphic to the spin-embedding.*

(3) *The Grassmann embedding of $DW(2n-1,2)$ has type 2.*

(4) *The full projective embedding of $DW(2n-1,2)$ induced by the universal embedding of $DH(2n-1,4)$ has type 2 if $n=2$ and type 3 if $n \geq 3$.*

(5) *The universal embedding of $DW(2n-1,2)$ has type n.*

We also show that among the homogeneous full projective embeddings of type i of $DW(2n-1,2)$, there exists a universal one from which all others of type i can be derived (by means of quotients).

In the area of full projective embeddings of geometries, there are several results of the following form:

> If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two point-line geometries admitting universal full projective embeddings $\widetilde{\epsilon_1}$ and $\widetilde{\epsilon_2}$ such that $\mathcal{S}_1$ is a full subgeometry of $\mathcal{S}_2$, then the full projective embedding of $\mathcal{S}_1$ induced by $\widetilde{\epsilon_2}$ is isomorphic to $\widetilde{\epsilon_1}$.

E.g., it is known that the full projective embedding of $Q(4,2) \cong DW(3,2)$ induced by the universal embedding of $Q^-(5,2) \cong DH(3,4)$ is also universal. In [10, Theorem 1.8], we showed that the full projective embedding of $DW(5,2)$ induced by the universal embedding of $DH(5,4)$ is also universal. Invoking parts (4) and (5) of Theorem 1.3, we now see that the above statement is false in case $\mathcal{S}_1 = DW(2n-1,2)$ and $\mathcal{S}_2 = DH(2n-1,4)$ with $n \geq 4$. Summarizing, we then have:

Corollary 1.4 *Let $n \in \mathbb{N} \setminus \{0,1\}$. Then the full projective embedding of $DW(2n-1,2)$ induced by the universal embedding of $DH(2n-1,4)$ is itself also universal if and only if $n \in \{2,3\}$.*

2 Preliminaries

2.1 Pseudo-embeddings and pseudo-hyperplanes

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a point-line geometry having the property that the number of points on each line is finite and at least three. If $X_1, X_2 \subseteq \mathcal{P}$, then $X_1 * X_2 = X_1 *_{\mathcal{P}} X_2$ denotes the complement $\mathcal{P} \setminus (X_1 \Delta X_2)$ of the symmetric difference $X_1 \Delta X_2$ of X_1 and X_2. The operator $*$ on the set $2^{\mathcal{P}}$ of all subsets of $\mathcal{P}$ is commutative and associative. Moreover, $X * X = \mathcal{P}$ and $X * \mathcal{P} = X$ for every $X \in 2^{\mathcal{P}}$. A map ϵ from $\mathcal{P}$ to the point set of a projective space Σ will often be denoted by $\epsilon : \mathcal{S} \to \Sigma$. Two such maps $\epsilon_1 : \mathcal{S} \to \Sigma_1$ and $\epsilon_2 : \mathcal{S} \to \Sigma_2$ are called *isomorphic* if there exists an isomorphism $\phi : \Sigma_1 \to \Sigma_2$ such that $\epsilon_2 = \phi \circ \epsilon_1$.

Let V be a (possibly infinite-dimensional) vector space over the finite field $\mathbb{F}_2$ of order 2. If W is a k-dimensional subspace of V with $2 \leq k < \infty$, then a *frame* of $\mathrm{PG}(W)$ is any set of points of the form $\{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \ldots, \langle \bar{e}_k \rangle, \langle \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_k \rangle\}$, where $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_k\}$ is a basis of W. A *pseudo-embedding* of $\mathcal{S}$ into the projective space $\mathrm{PG}(V)$ is a mapping $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$ for which the following hold:

(PE1) The image of ϵ generates the projective space $\mathrm{PG}(V)$.

(PE2) ϵ maps every line of $\mathcal{S}$ to a frame of some subspace of $\mathrm{PG}(V)$.

If ϵ is moreover injective, then the pseudo-embedding is called *faithful*. A pseudo-embedding $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$ is called *G-homogeneous*, where G is a group of automorphisms of $\mathcal{S}$, if for every $\theta \in G$, there exists a (necessarily unique) projectivity $\overline{\theta}$ of $\mathrm{PG}(V)$ such that $\epsilon(x^\theta) = \epsilon(x)^{\overline{\theta}}$ for every point x of $\mathcal{S}$. In case G consists of all automorphisms of $\mathcal{S}$, we will also talk about a *homogeneous pseudo-embedding*.

If $\epsilon : \mathcal{S} \to \Sigma$ is a pseudo-embedding and π is a subspace of Σ that is disjoint from the image of ϵ and every subspace of the form $\langle \epsilon(x_1), \epsilon(x_2), \ldots, \epsilon(x_k) \rangle$, where $x_1, x_2, \ldots, x_k$ are all the points of a line of $\mathcal{S}$, then a new pseudo-embedding $\epsilon/\pi : \mathcal{S} \to \Sigma/\pi$ can be defined which maps each point x of $\mathcal{S}$ to the point $\langle \pi, \epsilon(x) \rangle$ of the quotient projective space Σ/π. This pseudo-embedding ϵ/π is called a *quotient* of ϵ. We will write $\epsilon_1 \geq \epsilon_2$ if ϵ_2 is isomorphic to a quotient of ϵ_1. A pseudo-embedding $\widetilde{\epsilon} : \mathcal{S} \to \widetilde{\Sigma}$ is called *universal* if $\widetilde{\epsilon} \geq \epsilon$ for any pseudo-embedding ϵ of $\mathcal{S}$. If this is the case, then there exists a unique subspace π of $\widetilde{\Sigma}$ for which ϵ is isomorphic to $\widetilde{\epsilon}/\pi$. If $\mathcal{S}$ has a pseudo-embedding, then by [15, Theorem 1.2(1)] we know that $\mathcal{S}$ has a universal pseudo-embedding, which is moreover unique (up to isomorphism) and homogeneous ([16, Theorem 2.4]). If $\mathcal{S}$ has a faithful pseudo-embedding, then its universal pseudo-embedding is also faithful. If $\widetilde{\epsilon} : \mathcal{S} \to \mathrm{PG}(\widetilde{V})$ is the universal pseudo-embedding of $\mathcal{S}$, then the dimension of the $\mathbb{F}_2$-vector space $\widetilde{V}$ is called the *pseudo-embedding rank* of $\mathcal{S}$ and denoted by $er(\mathcal{S})$. In case $\mathcal{S}$ is finite, we know from [15, Theorem 1.2(2)] that $er(\mathcal{S}) = |\mathcal{P}| - rank_{\mathbb{F}_2}(M)$, where M is any point-line incidence matrix of $\mathcal{S}$.

A *pseudo-hyperplane* of $\mathcal{S}$ is a proper subset H of $\mathcal{P}$ such that every line of $\mathcal{S}$ contains an even number of points of $\mathcal{P} \setminus H$. Note that if all lines have even size, then $\emptyset$ is a pseudo-hyperplane. If H_1 and H_2 are two distinct pseudo-hyperplanes of $\mathcal{S}$, then $H_1 * H_2$ is again a pseudo-hyperplane of $\mathcal{S}$. If $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$ is a pseudo-embedding, then $\mathcal{H}_\epsilon$ denotes the set of all subsets of the form $\epsilon^{-1}(\epsilon(\mathcal{P}) \cap \Pi)$, where Π is a hyperplane of $\mathrm{PG}(V)$. By [15, Theorem 1.1] every element of $\mathcal{H}_\epsilon$ is a pseudo-hyperplane of $\mathcal{S}$, a so-called *pseudo-hyperplane arising from* ϵ. By [15] (Theorem 1.3 and page 79, part (d)), we know the following.

Proposition 2.1 ([15]) *If $\mathcal{S}$ has a pseudo-embedding and $\widetilde{\epsilon} : \mathcal{S} \to \widetilde{\Sigma}$ denotes its universal pseudo-embedding, then $\mathcal{H}_{\widetilde{\epsilon}}$ is the set of all pseudo-hyperplanes of $\mathcal{S}$. Moreover, if ϵ is a pseudo-embedding of $\mathcal{S}$ such that $\mathcal{H}_\epsilon$ coincides with the set of all pseudo-hyperplanes of $\mathcal{S}$, then ϵ is isomorphic to $\widetilde{\epsilon}$.*

The following result is a special case of Lemma 2.2 of [15].

Proposition 2.2 ([15]) *Let ϵ be a pseudo-embedding of $\mathcal{S}$, L a line of $\mathcal{S}$ and x_1, x_2 two distinct points of L. Then there exists a pseudo-hyperplane $H \in \mathcal{H}_\epsilon$ containing all points of L, except x_1 and x_2.*

The following result is precisely Corollary 2.7 of [16].

Proposition 2.3 ([16]) *Let G be a group of automorphisms of $\mathcal{S}$.*

- *If ϵ is a G-homogeneous pseudo-embedding of $\mathcal{S}$, then the set $\mathcal{H} = \mathcal{H}_\epsilon$ satisfies the following:*
  - (a) *$\mathcal{H}$ can be written as a disjoint union $\bigcup_{i \in I} \mathcal{H}_i$, where each $\mathcal{H}_i$, $i \in I$, is a G-orbit of pseudo-hyperplanes of $\mathcal{S}$.*
  - (b) *If $H_1, H_2 \in \mathcal{H}$ with $H_1 \neq H_2$, then $H_1 * H_2 \in \mathcal{H}$.*
  - (c) *If L is a line of $\mathcal{S}$ containing an odd number of points, then for every point x of L, there exists a pseudo-hyperplane of $\mathcal{H}$ having only the point x in common with L.*
  - (d) *If L is a line of $\mathcal{S}$ containing an even number of points, then for every two distinct points x_1 and x_2 of L, there exists a pseudo-hyperplane of $\mathcal{H}$ having only the points x_1 and x_2 in common with L.*
  - (e) *For every point x of $\mathcal{S}$, there exists a pseudo-hyperplane of $\mathcal{H}$ not containing x.*
- *Conversely, if $\mathcal{H}$ is a finite set of pseudo-hyperplanes of $\mathcal{S}$ satisfying the conditions (a), (b), (c), (d) and (e) above, then there exists a pseudo-embedding ϵ of $\mathcal{S}$ such that $\mathcal{H} = \mathcal{H}_\epsilon$. The pseudo-embedding ϵ is uniquely determined, up to isomorphism, and is G-homogeneous.*

Observe that condition (e) in Proposition 2.3 follows from conditions (c) and (d) if there is at least one line incident with x.

A *pseudo-subspace* of $\mathcal{S}$ is a set X of points of $\mathcal{S}$ such that no line of $\mathcal{S}$ has a unique point in common with $\mathcal{P} \setminus X$. Every set X of points of $\mathcal{S}$ is contained in a unique smallest pseudo-subspace, namely the intersection of all pseudo-subspaces of $\mathcal{S}$ containing the set X. If this smallest pseudo-subspace coincides with $\mathcal{P}$, then X is called a *pseudo-generating set*. The smallest size of a pseudo-generating set of $\mathcal{S}$ is called the *pseudo-generating rank* of $\mathcal{S}$ and denoted by $gr(\mathcal{S})$. The following proposition is precisely Theorem 1.5 of [15]. This proposition often allows to determine whether a given pseudo-embedding is universal.

Proposition 2.4 ([15]) *If $\mathcal{S}$ has a pseudo-embedding, then the following hold:*

(1) $er(\mathcal{S}) \leq gr(\mathcal{S})$.

(2) *If there exists a pseudo-embedding $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$ and a pseudo-generating set X such that $|X| = \dim(V) < \infty$, then $er(\mathcal{S}) = gr(\mathcal{S}) = \dim(V)$ and ϵ is isomorphic to the universal pseudo-embedding of $\mathcal{S}$.*

2.2 The symplectic dual polar space $DW(2n-1, 2)$

Let ζ be a symplectic polarity of the projective space $\mathrm{PG}(2n-1, 2)$, $n \geq 2$. Then associated with ζ, there is the following point-line geometry $DW(2n-1, 2)$:

- The points of $DW(2n-1, 2)$ are the $(n-1)$-dimensional subspaces that are totally isotropic with respect to ζ.
- The lines of $DW(2n-1, 2)$ are the $(n-2)$-dimensional subspaces that are totally isotropic with respect to ζ,
- Incidence is reverse containment.

The point-line geometry $DW(2n-1, 2)$ belongs to the family of dual polar spaces. Every line of $DW(2n-1, 2)$ is incident with precisely three points. If x and y are two points of $DW(2n-1, 2)$, then $\mathrm{d}(x, y)$ denotes the distance between x and y in the collinearity graph Γ of $DW(2n-1, 2)$. This collinearity graph Γ has diameter n. The dual polar space $DW(2n-1, 2)$ is a *near polygon*, meaning that for every point x and every line L, there exists a unique point on L nearest to x.

If α is a totally isotropic subspace of $\mathrm{PG}(2n-1, 2)$ of dimension $n-1-k$, $k \in \{0, 1, \ldots, n\}$, then the set of all $(n-1)$-dimensional totally isotropic subspaces containing α is a convex subspace of diameter k of $DW(2n-1, 2)$, and every convex subspace of diameter k of $DW(2n-1, 2)$ is obtained in this way. Convex subspaces of diameter 2 are also called *quads*. If F is a convex subspace of diameter $k \geq 2$ of $DW(2n-1, 2)$, then the point-line geometry $\widetilde{F}$ induced on F by those lines that have all their points in F is isomorphic to $DW(2k-1, 2)$. If x and y are two points of $DW(2n-1, 2)$ at distance $k \in \{0, 1, \ldots, n\}$ from each other, then x and y are contained in a unique convex subspace $\langle x, y \rangle$ of diameter k. The maximal distance of a point of $DW(2n-1, 2)$ to a given convex subspace of diameter k is equal to $n-k$. If x is a point of $DW(2n-1,2)$ at distance l from a convex subspace F of diameter k, then the smallest convex subspace $\langle x, F\rangle$ containing x and F has diameter $k+l$. If F is a convex subspace and x is a point, then F contains a unique point $\pi_F(x)$ nearest to x, and $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_F(x)) + \mathrm{d}(\pi_F(x),y)$ for every point $y \in F$.

Since every line of $DW(2n-1,2)$ has three points, the pseudo-hyperplanes of $DW(2n-1,2)$ are precisely the hyperplanes of $DW(2n-1,2)$, i.e. the proper sets of points of $DW(2n-1,2)$ intersecting each line in either one or three points. We give two constructions for hyperplanes of $DW(2n-1,2)$.

- For every point x of $DW(2n-1,2)$, the set H_x of points at distance at most $n-1$ from x is a hyperplane of $DW(2n-1,2)$, called a *singular hyperplane with center* x.

- Suppose F is a convex subspace of diameter $\delta \geq 1$ of $DW(2n-1,2)$ and G is a hyperplane of $\widetilde{F}$. Denote by $\overline{G}$ the set of points at distance at most $n-\delta-1$ from F, together with all points x at distance $n-\delta$ from F for which $\pi_F(x) \in G$. Then $\overline{G}$ is a hyperplane of $DW(2n-1,2)$, called the *extension* of G.

Since every line of $DW(2n-1,2)$ has three points, the pseudo-embeddings of $DW(2n-1,2)$ are precisely the (possibly non-injective) full projective embeddings (meaning that lines are mapped to lines). The universal pseudo-embedding will then also be called the *universal embedding.* A full projective embedding of $DW(2n-1,2)$ is called *polarized* if every singular hyperplane arises from it. Up to now, four homogeneous full projective embeddings of $DW(2n-1,2)$ were known.

- The Grassmann embedding of $DW(2n-1,2)$ is a homogeneous full embedding in the projective space $\mathrm{PG}(\binom{2n}{n} - \binom{2n}{n-2} - 1, 2)$, see Cooperstein [8, Proposition 5.1].

- The spin-embedding of $DW(2n-1,2)$ is a homogeneous full embedding in the projective space $\mathrm{PG}(2^n-1,2)$, see Buekenhout & Cameron [5, Section 7].

- The universal embedding of $DW(2n-1,2)$ is homogeneous. This embedding is not so well-understood but its vector dimension (i.e. (pseudo-)embedding rank of $DW(2n-1,2)$) has been determined. This embedding rank is equal to $\frac{(2^n+1)(2^{n-1}+1)}{3}$ as was proved by Yoshiara [28] for $n=3$, by Cooperstein [7] for $n \in \{4,5\}$ and for general n independently by Blokhuis & Brouwer [4] and Li [20].

- Similarly as a symplectic polarity of $\mathrm{PG}(2n-1,2)$ defines the dual polar space $DW(2n-1,2)$, a unitary polarity of $\mathrm{PG}(2n-1,4)$ will define a dual polar space $DH(2n-1,4)$ with three points per line. By De Bruyn [9], we know that the dual polar space $DW(2n-1,2)$ can be fully and isometrically embedded into $DH(2n-1,4)$ such that every automorphism of $DW(2n-1,2)$ lifts to an automorphism of $DH(2n-1,4)$. The latter implies that every homogeneous full projective embedding, in particular[1] the universal embedding, of $DH(2n-1,4)$ will induce a homogeneous full projective embedding of $DW(2n-1,2)$. Also the universal embedding of $DH(2n-1,4)$ is not so well-understood. Its vector dimension $\frac{4^n+2}{3}$ has been determined by Yoshiara [28] for $n=3$ and by Li [21] for general n.

3 A pseudo-generating set of the geometry $\mathcal{S}_{n,k}$

The proof of the following lemma provides an inductive way to construct pseudo-generating sets of $\mathcal{S}_{n,k}$. Later, we shall see that the pseudo-generating sets that arise in this way have the smallest possible size.

Lemma 3.1 *For all $k,n \in \mathbb{N}\setminus\{0\}$ with $k \leq n+1$, the geometry $\mathcal{S}_{n,k}$ has a pseudo-generating set of size $\binom{n+1}{1}+\binom{n+1}{2}+\cdots+\binom{n+1}{k}$.*

Proof. We first prove the claim in the case $k=1$. If $k=1$, then $\mathcal{S}_{n,k}$ is isomorphic to the point-line system of $\mathrm{PG}(n,2)$ and in this case a pseudo-generating set of $\mathcal{S}_{n,k}$ is just a generating set of $\mathrm{PG}(n,2)$. The smallest size of such a generating set is $n+1=\binom{n+1}{1}$.

Next, we prove the claim in the cases where $k=n$ or $k=n+1$. If $k=n+1$, then $\mathcal{S}_{n,k}$ contains $2^{n+1}-1$ points but no lines, and so $\mathcal{S}_{n,k}$ has a pseudo-generating set of size $2^{n+1}-1=\binom{n+1}{1}+\binom{n+1}{2}+\cdots+\binom{n+1}{n+1}$. If $k=n$, then $\mathcal{S}_{n,k}$ is isomorphic to a line of size $2^{n+1}-1$ and such a line has a pseudo-generating set of size $2^{n+1}-2=\binom{n+1}{1}+\binom{n+1}{2}+\cdots+\binom{n+1}{n}$.

Let us now turn to the general case. We shall prove the claim by induction on n. The base case is the case where $n=1$ (and so $k\in\{1,2\}$), but then we already know that the claim is valid. We may therefore suppose that $n\geq 2$ and $1<k<n$. (In fact, we then have that $n\geq 3$ since $n>k\geq 2$.)

Let (p,π) be a non-incident point-hyperplane pair of $\mathrm{PG}(n,2)$. The points of π and the k-dimensional subspaces contained in π define a subgeometry $\mathcal{S}_1$ of $\mathcal{S}_{n,k}$ isomorphic to $\mathcal{S}_{n-1,k}$. By the induction hypothesis, the geometry $\mathcal{S}_1$ has a pseudo-generating set X_1 of size $\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{k}$. The points of π and the $(k-1)$-dimensional subspaces contained in π define a point-line geometry $\mathcal{S}_2$ isomorphic to $\mathcal{S}_{n-1,k-1}$. By the induction hypothesis, the geometry $\mathcal{S}_2$ has a pseudo-generating set Y_2 of size $\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{k-1}$. For every $y\in\pi$, let y' denote the unique point on the line py distinct from p and y, and put $X_2 := \{y' \mid y\in Y_2\}$.

We claim that the set $X := \{p\}\cup X_1\cup X_2$ is a pseudo-generating set of $\mathcal{S}_{n,k}$. Let $\overline{X}$ denote the smallest pseudo-subspace containing X. Since X_1 is a pseudo-generating set of $\mathcal{S}_1$, we have $\pi\subseteq\overline{X}$ and hence $\{p\}\cup\pi\subseteq\overline{X}$. Let π' denote the set of points of $\mathrm{PG}(n,2)$ not contained in $\pi\cup\{p\}$. Let $\mathcal{S}_3$ be the subgeometry of $\mathcal{S}_{n,k}$ determined by the points of π' and the k-dimensional subspaces through p. The map $y\mapsto y'$ determines an isomorphism between $\mathcal{S}_2$ and $\mathcal{S}_3$, implying that X_2 is a pseudo-generating set of the geometry $\mathcal{S}_3$.

[1] Another homogeneous full projective embedding of $DH(2n-1,4)$ is known, namely the Grassmann embedding, but the induced embedding is the Grassmann embedding of $DW(2n-1,2)$, see [12, Theorem 1.1].

We claim that $\overline{X} \cap \pi'$ is a pseudo-subspace of $\mathcal{S}_3$. Suppose that this is not the case. Then there exists a k-dimensional subspace α through p such that all but one point of $\alpha \cap \pi'$ belong to $\overline{X} \cap \pi'$. As $\{p\} \cup \pi \subseteq \overline{X}$, this implies that all but one point of α belong to $\overline{X}$, in contradiction with the fact that $\overline{X}$ is a pseudo-subspace of $\mathcal{S}_{n,k}$.

As $\overline{X} \cap \pi'$ is a pseudo-subspace of $\mathcal{S}_3$ containing the pseudo-generating set X_2 of $\mathcal{S}_3$, we have that the point set π' of $\mathcal{S}_3$ is contained in $\overline{X} \cap \pi'$. Hence, $\overline{X}$ contains $\{p\} \cup \pi \cup \pi'$, i.e. all points of PG$(n, 2)$. So, $X = \{p\} \cup X_1 \cup X_2$ is a pseudo-generating set. Applying Pascal's rule a number of times, we find that its size is equal to

$$1+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{k}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{k-1} = \binom{n+1}{1}+\binom{n+1}{2}+\cdots+\binom{n+1}{k}.$$

■

4 A 2^{n+1}-dimensional $GL(n+1, 2)$-module

In this section, we define a 2^{n+1}-dimensional module for the group $GL(n+1, 2)$, $n \in \mathbb{N}\setminus\{0\}$, and classify all its submodules. This will be useful later when we want to classify all homogeneous pseudo-embeddings of $\mathcal{S}_{n,k}$, $k \in \{1, 2, \ldots, n+1\}$.

Consider the polynomial ring $\mathbb{F}_2[X_0, X_1, \ldots, X_n]$ and let $\mathcal{I}$ be the ideal $(X_0^2 - X_0, X_1^2 - X_1, \ldots, X_n^2 - X_n)$ of $\mathbb{F}_2[X_0, X_1, \ldots, X_n]$. Put $f_1^* := 1$ and $f_2^* := \sum_{M \in \mathcal{M}} M$, where $\mathcal{M}$ denotes the set of all monomials of the form $X_{i_1} X_{i_2} \cdots X_{i_l}$ where $l \in \{1, 2, \ldots, n+1\}$ and $i_1, i_2, \ldots, i_l \in \{0, 1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_l$. The polynomial ring $\mathbb{F}_2[X_0, X_1, \ldots, X_n]$ can be regarded as an infinite-dimensional vector space over $\mathbb{F}_2$. For every $k \in \{1, 2, \ldots, n+1\}$, let $\mathcal{F}_k$ denote the subspace of dimension $\binom{n+1}{1}+\binom{n+1}{2}+\cdots+\binom{n+1}{k}$ generated by all monomials of $\mathcal{M}$ having degree at most k. Also, put $\mathcal{F}_0 := \{0\}$ and $\mathcal{F} := \langle \mathcal{F}_{n+1}, f_1^* \rangle$. We note the following.

(P1) If $f \in \mathcal{F}$ and every element of $\mathbb{F}_2^{n+1}$ is a root of f, then $f = 0$.

(P2) For every $f \in \mathbb{F}_2[X_0, X_1, \ldots, X_n]$, there exists a unique $g \in \mathcal{F}$ such that $f - g \in \mathcal{I}$.

(P3) $\mathcal{I}$ consists of those elements of $\mathbb{F}_2[X_0, X_1, \ldots, X_n]$ for which each element of $\mathbb{F}_2^{n+1}$ is a root.

Consider the group $GL(n+1, 2)$ whose elements are the nonsingular $(n+1) \times (n+1)$ matrices over the field $\mathbb{F}_2$. Every $A \in GL(n+1, 2)$ determines a permutation ϕ_A of $\mathbb{F}_2[X_0, X_1, \ldots, X_n]$ if one performs the following substitutions to the elements of $\mathbb{F}_2[X_0, X_1, \ldots, X_n]$:

$$[X_0, X_1, \cdots, X_n]^T \mapsto A \cdot [X_0, X_1, \cdots, X_n]^T.$$

Then the following hold:

(1) $(f_1 + f_2)^{\phi_A} = f_1^{\phi_A} + f_2^{\phi_A}$ for all $f_1, f_2 \in \mathbb{F}_2[X_0, X_1, \ldots, X_n]$ and all $A \in GL(n+1, 2)$.

(2) For all $A, B \in GL(n+1, 2)$, $\phi_{AB} = \phi_A \phi_B$ (where permutations are composed from left to right).

(3) If $A \in GL(n+1,2)$, then ϕ_A is the identical permutation if and only if A is the identity matrix.

So, we obtain a faithful representation $A \mapsto \phi_A$ of the group $GL(n+1,2)$ on the $\mathbb{F}_2$-vector space $\mathbb{F}_2[X_0, X_1, \ldots, X_n]$, allowing us to regard $\mathbb{F}_2[X_0, X_1, \ldots, X_n]$ as a $GL(n+1,2)$-module.

Let $A \in GL(n+1,2)$ and $f \in \mathcal{F}$. Then f^{ϕ_A} not necessarily belongs to $\mathcal{F}$. (E.g., the element X_1X_2 is mapped to $X_1X_2 + X_2^2$ by the substitutions $X_1 \mapsto X_1 + X_2$, $X_2 \mapsto X_2$). By Property (P2) however, there exists a unique $g \in \mathcal{F}$ such that $f^{\phi_A} - g \in \mathcal{I}$. We define $f^{\phi'_A} := g$. By Property (P3), the element ϕ_A stabilizes the ideal $\mathcal{I}$ and so permutes the elements of the quotient space $\mathbb{F}_2[X_0, X_1, \ldots, X_n]/\mathcal{I}$. We thus see that ϕ'_A permutes the elements of $\mathcal{F}$. We obtain a faithful representation $A \mapsto \phi'_A$ of the group $GL(n+1,2)$ on the $\mathbb{F}_2$-vector space $\mathcal{F}$, allowing us to regard the 2^{n+1}-dimensional vector space $\mathcal{F}$ as a $GL(n+1,2)$-module. Our aim is now to determine all submodules of this $GL(n+1,2)$-module.

A coordinate transformation cannot increase the degree of a polynomial $f \in \mathcal{F}$, nor introduce a constant term that was not originally there, implying that the subspaces $\{0\} = \mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n, \mathcal{F}_{n+1}, \langle \mathcal{F}_0, f_1^* \rangle = \langle f_1^* \rangle, \langle \mathcal{F}_1, f_1^* \rangle, \ldots, \langle \mathcal{F}_n, f_1^* \rangle, \langle \mathcal{F}_{n+1}, f_1^* \rangle = \mathcal{F}$ are submodules.

Lemma 4.1 *$\langle f_2^* \rangle$ is also a submodule.*

Proof. In order to show that $\langle f_2^* \rangle$ is a submodule, we need to show that the polynomial $f_2^* = \sum_{M \in \mathcal{M}} M$ remains invariant under the following coordinate transformations:

(i) $X_0 \mapsto X_{\sigma(0)}$, $X_1 \mapsto X_{\sigma(1)}$, …, $X_n \mapsto X_{\sigma(n)}$, where σ is a permutation of $\{0, 1, \ldots, n\}$;

(ii) $X_0 \mapsto X_0 + X_1$, $X_1 \mapsto X_1$, …, $X_n \mapsto X_n$.

Then $f_2^* = \sum_{M \in \mathcal{M}} M$ remains invariant under all coordinate transformations. Obviously, $f_2^* = \sum_{M \in \mathcal{M}} M$ remains invariant under the coordinate transformations mentioned in (i). Under the coordinate transformations mentioned in (ii), the polynomial $f_2^* = \sum_{M \in \mathcal{M}} M$ becomes $\sum_{M \in \mathcal{M}} M + \sum_{M \in \mathcal{M}} M'$, where M' with $M \in \mathcal{M}$ is the following monomial:

- If the variable X_0 does not occur in M, then $M' = 0$.
- If the variable X_0 occurs in M, but not X_1, then M' is the monomial obtained from M by replacing X_0 by X_1.
- If both the variables X_0, X_1 occur in M, then M' is obtained from M by removing the variable X_0.

Obviously, every M' with $M \in \mathcal{M}$ is either zero or is nonzero with degree contained in the set $\{1, 2, \ldots, n\}$. In the latter case, the variable X_1 occurs in M', but the variable X_0 does not.

Now, let $X_{i_1}X_{i_2} \cdots X_{i_l}$ be a monomial of $\mathcal{M}$ with $l \in \{1, 2, \ldots, n\}$ and $1 = i_1 < i_2 < \cdots < i_l$. Then there are precisely two $M \in \mathcal{M}$ for which $M' = X_{i_1}X_{i_2} \cdots X_{i_l}$, namely

$M = X_0 X_{i_2} \cdots X_{i_l}$ and $M = X_0 X_{i_1} X_{i_2} \cdots X_{i_l}$. We conclude that $\sum_{M \in \mathcal{M}} M' = 0$. So, under the coordinate transformation mentioned in (ii), the polynomial $f_2^* = \sum_{M \in \mathcal{M}} M$ remains invariant. ∎

As also $\langle f_1^* \rangle$ is a submodule, we thus have:

Corollary 4.2 *Each subspace of* $\{0, f_1^*, f_2^*, f_1^* + f_2^*\}$ *is a submodule.*

Theorem 4.3 *Every submodule of* $\mathcal{F}$ *has the form* $V_1 \oplus V_2$*, where* V_1 *is a submodule of* $\{0, f_1^*, f_2^*, f_1^* + f_2^*\}$ *and* V_2 *is one of* $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n$.

Proof. Let $\mathcal{B}$ be a submodule of the $GL(n+1,2)$-module $\mathcal{F}$. If all polynomials of $\mathcal{B}$ are constants, then $\mathcal{B}$ is equal to either $\{0\}$ or $\langle f_1^* \rangle$. So, we may suppose that $\mathcal{B}$ contains a polynomial f_1 that is not a constant. We choose this nonconstant f_1 such that its degree $d \in \{1, 2, \ldots, n+1\}$ is as big as possible. We distinguish two cases.

Case I: $1 \leq d \leq n$

Let $X_{i_1} X_{i_2} \ldots X_{i_d}$ with $i_1 < i_2 < \cdots < i_d$ be a monomial occurring in f_1. As $d \leq n$, there exists a $j_1 \in \{0, 1, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_d\}$. The substitutions $X_{i_1} \mapsto X_{i_1} + X_{j_1}$, $X_i \mapsto X_i$ for all $i \in \{0, 1, \ldots, n\} \setminus \{i_1\}$ turn f_1 in another polynomial g that must belong to $\mathcal{B}$. The difference $g - f_1$ also belongs to $\mathcal{B}$ and has the form $X_{j_1} f_2$, where f_2 is a polynomial of degree $d-1$ in the variables of the set $\{X_0, X_1, \ldots, X_n\} \setminus \{X_{i_1}, X_{j_1}\}$. In fact, f_2 is obtained as follows:

- Remove in f_1 all monomials that do not contain the variable X_{i_1}.
- In all monomials of f_1 where the variables X_{i_1} and X_{j_1} occur, remove the factor X_{i_1}.
- In all monomials of f_1 where the variable X_{i_1} occurs but not X_{j_1}, replace X_{i_1} by X_{j_1}.

In case $d \geq 2$, we can apply the above reasoning on f_2 to see that $\mathcal{B}$ contains an element of the form $X_{j_1} X_{j_2} f_3$, with j_1, j_2 distinct elements of $\{0, 1, \ldots, n\}$ and f_3 a polynomial of degree $d-2$ in the set $\{X_0, X_1, \ldots, X_n\} \setminus \{X_{j_1}, X_{j_2}\}$ of variables. An inductive argument then shows that $\mathcal{B}$ contains a monomial of the form $X_{j_1} X_{j_2} \cdots X_{j_d}$ with $j_1, j_2, \ldots, j_d$ distinct elements of $\{0, 1, \ldots, n\}$. Further substitutions then show that $\mathcal{B}$ contains all monomials of degree d.

We now show by downwards induction on $i \in \{1, 2, \ldots, d\}$ that $\mathcal{B}$ contains all monomials of degree i. As we have shown above, this is true for $i = d$. Consider now an $i \in \{1, 2, \ldots, d-1\}$ and suppose the claim is valid for $i+1$. Let $j_1, j_2, \ldots, j_i$ be distinct elements of $\{0, 1, \ldots, n\}$ and let j_{i+1} be still another element of this set. By the induction hypothesis, the monomial $f = X_{j_1} \cdots X_{j_i} X_{j_{i+1}}$ belongs to $\mathcal{B}$. The substitutions $X_{j_1} \mapsto X_{j_1}, \ldots, X_{j_i} \mapsto X_{j_i}, X_{j_{i+1}} \mapsto X_{j_{i+1}} + X_{j_i}$ turn f into the polynomial $X_{j_1} \cdots X_{j_i} X_{j_{i+1}} + X_{j_1} \cdots X_{j_i}$. This polynomial as well as f belong to $\mathcal{B}$, implying that also $X_{j_1} X_{j_2} \cdots X_{j_i}$ belongs to $\mathcal{B}$. Further substitutions then show that $\mathcal{B}$ contains all monomials of degree i.

By the above, we then have that $\mathcal{F}_d \subseteq \mathcal{B} \subseteq \langle \mathcal{F}_d, f_1^* \rangle$, implying that either $\mathcal{B} = \mathcal{F}_d$ or $\mathcal{B} = \langle \mathcal{F}_d, f_1^* \rangle$.

Case II: $d = n + 1$

Since $\mathcal{F}_n \cap \mathcal{B}$ is a submodule, we know from Case I that $\mathcal{F}_n \cap \mathcal{B} = \mathcal{F}_l$ for some $l \in \{0, 1, 2, \dots, n\}$. If $l = n$, then since $\mathcal{F} = \mathcal{F}_n \oplus V_2$, $\mathcal{B}$ must be among the possibilities that have been listed.

Suppose therefore that $l \leq n - 1$. Let f be an element of $\mathcal{B}$ of degree $d = n + 1$. Then $X_0 X_1 \dots X_n$ is a monomial occurring in f. We show that for every $m \in \{l+1, l+2, \dots, n+1\}$, all monomials of degree m occur in f. We prove this by downwards induction on m, the case $m = n + 1$ being obvious as $X_0 X_1 \cdots X_n$ is a monomial occurring in f. Before we proceed, we note the following: if f' is a polynomial obtained from f by permuting the variables $X_0, X_1, \dots, X_n$, then $f - f' \in \mathcal{F}_n \cap \mathcal{B}$ belongs to $\mathcal{F}_l$, implying that if a monomial of degree $m \in \{l + 1, l + 2, \dots, n + 1\}$ occurs in f, then all monomials of that degree occur in f. Suppose now that the monomial $X_0 X_1 \cdots X_m$ occurs in f for a certain $m \in \{l + 1, l + 2, \dots, n\}$. Let f' be the polynomial obtained from f after performing the substitutions:

$$X_0 \mapsto X_0 + X_1, \quad X_1 \mapsto X_1, \quad X_2 \mapsto X_2, \dots, X_n \mapsto X_n.$$

As $f - f' \in \mathcal{F}_n \cap \mathcal{B}$ belongs to $\mathcal{F}_l$, the monomial $X_1 X_2 \cdots X_m$ cannot occur in $f - f'$. This is only possible if the monomial $X_0 X_2 \cdots X_m$ occurs in f. Hence, all monomials of degree m occur in f. As $\mathcal{F}_l \subseteq \mathcal{B}$ and all monomials of degree $m \in \{l + 1, l + 2, \dots, n + 1\}$ occur in f, we see that at least one of the following occurs:

- $f_2^* \in \mathcal{B}$ and $\langle \mathcal{F}_l, f_2^* \rangle \subseteq \mathcal{B} \subseteq \langle \mathcal{F}_l, f_1^*, f_2^* \rangle$.
- $f_1^* + f_2^* \in \mathcal{B}$ and $\langle \mathcal{F}_l, f_1^* + f_2^* \rangle \subseteq \mathcal{B} \subseteq \langle \mathcal{F}_l, f_1^*, f_2^* \rangle$.

So, $\mathcal{B}$ is equal to either $\langle \mathcal{F}_l, f_2^* \rangle$, $\langle \mathcal{F}_l, f_1^* + f_2^* \rangle$ or $\langle \mathcal{F}_l, f_1^*, f_2^* \rangle$. ∎

5 The universal pseudo-embedding of $\mathcal{S}_{n,k}$

Let $n, k \in \mathbb{N} \setminus \{0\}$ with $k \leq n + 1$. Choose a basis $(\bar{e}_0, \bar{e}_1, \dots, \bar{e}_n)$ in $\mathrm{PG}(n, 2)$ such that $\langle X_0 \bar{e}_0 + X_1 \bar{e}_1 + \cdots + X_n \bar{e}_n \rangle$ is the point $(X_0, X_1, \dots, X_n)$ of $\mathrm{PG}(n, 2)$. The number of nonzero X_i's is called the *weight* of the point $(X_0, X_1, \dots, X_n)$. Choose a basis $(\bar{f}_0, \bar{f}_1, \dots, \bar{f}_N)$ in $\mathrm{PG}(N, 2)$ with $N := N_{n,k}$ such that $\langle Y_0 \bar{f}_0 + Y_1 \bar{f}_1 + \cdots + Y_N \bar{f}_N \rangle$ is the point $(Y_0, Y_1, \dots, Y_N)$ of $\mathrm{PG}(N, 2)$. The base vector $\bar{f}_i$ with $i \in \{0, 1, \dots, N\}$ is said to be of *type* $j \in \{1, 2, \dots, k\}$ if

$$\sum_{s=1}^{j-1} \binom{n+1}{s} \leq i \leq -1 + \sum_{i=1}^{j} \binom{n+1}{s}.$$

The corresponding 1-space $\langle \bar{f}_i \rangle$ will be called a *point of type* j of $\mathrm{PG}(N, 2)$.

Lemma 5.1 *(a) The image of the map $\epsilon_{n,k}$ generates* $\mathrm{PG}(N,2)$.

(b) *For every projectivity θ of $\mathrm{PG}(n,2)$, there exists a projectivity $\overline{\theta}$ of $\mathrm{PG}(N,2)$ such that $\epsilon_{n,k}(x^\theta) = [\epsilon_{n,k}(x)]^{\overline{\theta}}$ for every point x of $\mathrm{PG}(n,2)$.*

(c) *$\epsilon_{n,k}$ is a homogeneous pseudo-embedding of $\mathcal{S}_{n,k}$.*

Proof. (a) Let Σ denote the subspace of $\mathrm{PG}(N,2)$ generated by the image of $\epsilon_{n,k}$. It easily follows by induction on $j \in \{1,2,\ldots,k\}$ that all points of type j belong to Σ:

- By considering the images of the points of weight 1 of $\mathrm{PG}(n,2)$, we see that all points of type 1 of $\mathrm{PG}(N,2)$ are contained in Σ.
- By considering the images of the points of weight $w \in \{2,3,\ldots,k\}$ of $\mathrm{PG}(n,2)$ and invoking the fact that all points of type $l \in \{1,2,\ldots,w-1\}$ of $\mathrm{PG}(N,2)$ belong to Σ, we see that also all points of type w of $\mathrm{PG}(N,2)$ belong to Σ.

Claim (a) of the lemma then follows from the fact that $\mathrm{PG}(N,2)$ is generated by the points of types 1, 2, ..., k.

(b) Claim (b) of the lemma follows from the definition of the map $\epsilon_{n,k}$ and the fact that $A \mapsto \phi'_A$ defines a faithful representation of $GL(n+1,2)$ on $\mathcal{F}_k$.

(c) In order to prove (c), it suffices by (a) and (b) to show that for any $m = 2^{k+1}-1$ points $p_1, p_2, \ldots, p_m$ forming a k-dimensional subspace π of $\mathrm{PG}(n,2)$, the points $\epsilon_{n,k}(p_1), \epsilon_{n,k}(p_2), \ldots, \epsilon_{n,k}(p_{m-1})$ are linearly independent, and $\epsilon_{n,k}(p_m) \in \langle \epsilon_{n,k}(p_1), \epsilon_{n,k}(p_2), \ldots, \epsilon_{n,k}(p_{m-1})\rangle$. By part (b), we may assume that π has equation $X_{k+1} = X_{k+2} = \cdots = X_n = 0$ and that $p_m = (1,1,\ldots,1,0,0,\ldots,0)$ has weight $k+1$. Let Ω denote the subspace of dimension $2^{k+1}-3$ of $\mathrm{PG}(N,2)$ determined by the $2^{k+1}-2$ coordinate positions corresponding to the monomials M that involve only the variables $X_0, X_1, \ldots, X_k$ and for which $1 \leq \deg(M) \leq k$. Obviously, $\epsilon_{n,k}(p_i) \in \Omega$ for every $i \in \{1,2,\ldots,m\}$. By a similar reasoning as in (a), we can see that all $m-1 = 2^{k+1}-2$ points of weight 1 of Ω are all contained in the subspace $\langle \epsilon_{n,k}(p_1), \epsilon_{n,k}(p_2), \ldots, \epsilon_{n,k}(p_{m-1})\rangle$, implying that the points $\epsilon_{n,k}(p_1), \epsilon_{n,k}(p_2), \ldots, \epsilon_{n,k}(p_{m-1})$ are linearly independent and $\epsilon_{n,k}(p_m) \in \Omega = \langle \epsilon_{n,k}(p_1), \epsilon_{n,k}(p_2), \ldots, \epsilon_{n,k}(p_{m-1})\rangle$. ■

Parts (1) and (2) of Theorem 1.1 are now immediate consequences of Proposition 2.4 and Lemmas 3.1, 5.1. We now also see that the pseudo-generating set constructed in Lemma 3.1 has the smallest possible size.

For every $f \in \mathcal{F}$, we denote by X_f the set of all points $(X_0, X_1, \ldots, X_n)$ of $\mathrm{PG}(n,2)$ for which $f(X_0, X_1, \ldots, X_n) = 0$. Theorem 1.1(2) and Proposition 2.1 then imply the following.

Theorem 5.2 *The pseudo-hyperplanes of $\mathcal{S}_{n,k}$ are precisely the sets X_f for elements $f \in \mathcal{F}_k \setminus \{0\}$. The map $f \mapsto X_f$ defines a bijective correspondence between $\mathcal{F}_k \setminus \{0\}$ and the set of pseudo-hyperplanes of $\mathcal{S}_{n,k}$.*

6 The homogeneous pseudo-embeddings of $\mathcal{S}_{n,k}$

In this section, we classify all homogeneous pseudo-embeddings of $\mathcal{S}_{n,k}$, where $n, k \in \mathbb{N} \setminus \{0\}$ with $k \leq n+1$. The case $k = n+1$ is straightforward. In this case, the geometry $\mathcal{S}_{n,k} = \mathcal{S}_{n,n+1}$ has no lines and so the automorphism group is the symmetric group on $2^{n+1}-1$ letters. It is then easily seen that there are up to isomorphism three homogeneous pseudo-embeddings, the universal one whose image is a basis in a projective space of dimension $2^{n+1}-2$, the pseudo-embedding whose image is a frame in a projective space of dimension $2^{n+1}-3$, and an unfaithful pseudo-embedding where all points have the same image. In the sequel, we suppose that $k \leq n$.

Lemma 6.1 *If f_1 and f_2 are two elements of $\mathcal{F}_k$, then $X_{f_1+f_2} = X_{f_1} * X_{f_2}$.*

Proof. If $p = (X_0, X_1, \ldots, X_n)$ is a point of $\mathrm{PG}(n,2)$, then $(f_1+f_2)(X_0, X_1, \ldots, X_n) = 0$ if and only if $f_1(X_0, X_1, \ldots, X_n) = f_2(X_0, X_1, \ldots, X_n)$, i.e. if and only if either $p \in X_{f_1} \cap X_{f_2}$ or $p \notin X_{f_1} \cup X_{f_2}$. This happens precisely when $p \in X_{f_1} * X_{f_2}$. ■

Lemma 6.2 *Let A be a nonsingular $(n+1)\times(n+1)$ matrix over $\mathbb{F}_2$ and let $f \in \mathcal{F}_k \setminus \{0\}$. Put $g = f^{\phi'_A}$ and let η be the projectivity of $\mathrm{PG}(n,2)$ mapping $(X_0, X_1, \ldots, X_n)$ to $(Y_0, Y_1, \ldots, Y_n)$, where $[Y_0\ Y_1\ \cdots\ Y_n]^T = A^{-1}\cdot[X_0\ X_1\ \cdots\ X_n]^T$. Then $X_g = X_f^\eta$.*

Proof. Let $\widetilde{g}$ denote the polynomial obtained from f by performing the substitutions $[X_0, X_1, \cdots, X_n]^T \mapsto A\cdot[X_0, X_1, \cdots, X_n]^T$. By definition of ϕ'_A, we know that $g - \widetilde{g} \in \mathcal{I}$. So, the set $X_{\widetilde{g}}$ consisting of all points $(X_0, X_1, \ldots, X_n)$ of $\mathrm{PG}(n,2)$ satisfying $\widetilde{g}(X_0, X_1, \ldots, X_n)$ coincides with X_g. Suppose $Y_0, Y_1, \ldots, Y_n, X_0, X_1, \ldots, X_n$ are variables related by the equation $[Y_0, Y_1, \cdots, Y_n]^T = A^{-1}\cdot[X_0, X_1, \cdots, X_n]^T$, then $A\cdot[Y_0, Y_1, \cdots, Y_n]^T = [X_0, X_1, \cdots, X_n]^T$ and $f(X_0, X_1, \ldots, X_n) = \widetilde{g}(Y_0, Y_1, \ldots, Y_n)$, from which it follows that $X_{\widetilde{g}} = X_f^\eta$. ■

Now, suppose $\epsilon : \mathcal{S}_{n,k} \to \Sigma$ is a homogeneous pseudo-embedding of $\mathcal{S}_{n,k}$. Let $\mathcal{F}_\epsilon$ denote the set of all polynomials $f \in \mathcal{F}_k$ such that either $f \neq 0$ and $X_f \in \mathcal{H}_\epsilon$, or $f = 0$, see Theorem 5.2. By Proposition 2.3(a)+(b) and Lemmas 6.1, 6.2, the following properties hold:

(1) If $f_1, f_2 \in \mathcal{F}_\epsilon$, then $f_1 + f_2 \in \mathcal{F}_\epsilon$.

(2) If $f \in \mathcal{F}_\epsilon$ and $A \in GL(n+1,2)$, then $f^{\phi'_A} \in \mathcal{F}_\epsilon$.

We conclude that $\mathcal{F}_\epsilon$ is a submodule of the $GL(n+1,2)$-module $\mathcal{F}_k$. From the classification of the submodules of the $GL(n+1,2)$-module $\mathcal{F}$, see Theorem 4.3, it then follows that $\mathcal{F}_\epsilon = \mathcal{F}_l$ for some $l \in \{1, 2, \ldots, k\}$.

We show that $l = k$. Suppose to the contrary that $l < k$. Consider the k-dimensional subspace α of $\mathrm{PG}(n,2)$ with equations $X_{k+1} = X_{k+2} = \cdots = X_n = 0$, and consider the following points of α:

$$p_1 = (1, 1, \ldots, 1, 0, 0, \ldots, 0), \qquad p_2 = (1, 1, \ldots, 1, 1, 0, \ldots, 0),$$

where p_1 has weight k and p_2 has weight $k+1$. By Proposition 2.2, there exists an $f \in \mathcal{F}_\epsilon \setminus \{0\}$ such that X_f contains all points of α, except for p_1 and p_2.

Put $f = \sum_{M \in \mathcal{M}'} k_M \cdot M$, where $\mathcal{M}'$ is the set of all monomials of degree at most k of $\mathcal{M}$ and $k_M \in \mathbb{F}_2$ for all $M \in \mathcal{M}'$. As all points of α with weight $w \in \{1, 2, \ldots, k-1\}$ belong to X_f, an inductive argument easily shows that $k_M = 0$ for all $M \in \mathcal{M}' \cap \mathbb{F}_2[X_0, X_1, \ldots, X_k]$ for which $\deg(M) \in \{1, 2, \ldots, k-1\}$. Since the point $p_1 = (1, 1, \ldots, 1, 0, \ldots, 0)$ does not belong to X_f, this then implies that $k_M = 1$ if $M = X_0 X_1 \ldots, X_{k-1}$. As $f \in \mathcal{F}_l$, this implies that $l \geq k$, i.e. $l = k$.

So, $\mathcal{F}_\epsilon = \mathcal{F}_k$ and $\mathcal{H}_\epsilon$ consists of all pseudo-hyperplanes of $\mathcal{S}_{n,k}$ by Theorem 5.2. Proposition 2.1 then implies that ϵ is isomorphic to the universal pseudo-embedding of $\mathcal{S}_{n,k}$. This finishes the proof of Theorem 1.1(3).

7 Application to homogeneous full projective embeddings of $DW(2n-1, 2)$

7.1 Proof of Theorem 1.2

Suppose $\epsilon : DW(2n-1, 2) \to \Sigma$ is a homogeneous full embedding of $DW(2n-1, 2)$ in a projective space Σ. Let $G \cong Sp(2n, 2)$ denote the full automorphism group of $DW(2n-1, 2)$. For every point x of $DW(2n-1, 2)$, $\mathcal{S}_x \cong \mathrm{PG}(n-1, 2)$ and so we may identify $\mathrm{PG}(n-1, 2)$ with $\mathcal{S}_x$. Every element of the stabilizer G_x of x inside G then determines an automorphism of $\mathcal{S}_x = \mathrm{PG}(n-1, 2)$, and every automorphism of $\mathcal{S}_x$ is induced by an element of G_x. Let $\mathcal{L}$ denote the set of all lines through x, and for every convex subspace F through x, let $\mathcal{L}_F$ denote the set of lines through x contained in F. Then ϵ defines a map ϵ_x from the point set $\mathcal{L}$ of $\mathcal{S}_x = \mathrm{PG}(n-1, 2)$ to the set of points of the quotient space $\Sigma_x / \epsilon(x)$. If the image of $\mathcal{L}$ is a collection of linearly independent points of $\Sigma_x / \epsilon(x)$, then ϵ_x defines a pseudo-embedding of $\mathcal{S}_{n-1,n}$ into $\Sigma_x / \epsilon(x)$ which is isomorphic to the universal pseudo-embedding $\epsilon_{n-1,n}$. Suppose therefore that the image of $\mathcal{L}$ is not a collection of linearly independent points of $\Sigma_x / \epsilon(x)$. Then let $\delta \geq 2$ be the smallest positive integer such that there exists a convex subspace F of diameter δ through x for which the image of $\mathcal{L}_F$ is not a linearly independent set of points of $\Sigma_x / \epsilon(x)$. As δ is the smallest diameter for which this is possible, we then know that $\epsilon_x(\mathcal{L}_F)$ is a frame of a subspace of $\Sigma_x / \epsilon(x)$. Since ϵ is homogeneous and G_x acts transitively on the set of subspaces of dimension $\delta - 1$ of $\mathcal{S}_x = \mathrm{PG}(n-1, 2)$, we then see that for every convex subspace F' of diameter δ through x, the set $\epsilon_x(\mathcal{L}_{F'})$ is a frame of a subspace $\Sigma_x / \epsilon(x)$. So, ϵ_x defines a pseudo-embedding of the geometry $\mathcal{S}_{x,\delta-1}$. Since G_x induces the full group of automorphisms of $\mathcal{S}_x = \mathrm{PG}(n-1, 2)$ and ϵ is homogeneous, the pseudo-embedding ϵ_x of $\mathcal{S}_{x,\delta-1}$ should also be homogeneous, i.e. isomorphic to $\epsilon_{n-1,\delta-1}$ by Theorem 1.1(3).

7.2 Homogeneous full projective embeddings of types 1 and n

Proposition 7.1 *The universal embedding of $DW(2n-1, 2)$ has type n.*

Proof. Let x be a point of $DW(2n-1,2)$ and $\widetilde{\epsilon} : DW(2n-1,2) \to \widetilde{\Sigma}$ the universal embedding of $DW(2n-1,2)$. By McClurg [22] and Li [20], we then know that $\dim(\widetilde{\Sigma}_x) = 2^n - 1 = \sum_{i=1}^{n} \binom{n}{i}$. This implies by Theorem 1.2 that $\widetilde{\epsilon}$ has type n. ■

Proposition 7.2 *Suppose ϵ is a homogeneous full projective embedding of $DW(2n-1,2)$ of type 1. Then ϵ is isomorphic to the spin-embedding of $DW(2n-1,2)$.*

Proof. As ϵ is homogeneous, we know from Blok et al. [3, Theorem 1.1] that ϵ is polarized.

Suppose now that Q is a quad, x a point of Q and L_1, L_2, L_3 three lines of Q through x. Since ϵ has type 1, there exists a plane α of the embedding space that contains $\epsilon(x)$ such that $\epsilon(L_1)$, $\epsilon(L_2)$, $\epsilon(L_3)$ are the three lines of α through $\epsilon(x)$. Note that $\widetilde{Q} \cong DW(3,2) \cong Q(4,2) \cong W(2)$. From the classification of the full projective embeddings of the generalized quadrangle $Q(4,2) \cong W(2)$, it then follows that $\langle \epsilon(Q) \rangle$ should have dimension 3. Together with the fact that ϵ is polarized, it then follows from [11, Theorem 1.6] that ϵ is isomorphic to the spin-embedding of $DW(2n-1,2)$. ■

7.3 The existence of homogeneous embeddings for each type $i \in \{1,2,\ldots,n\}$

Let x be a point of $DW(2n-1,2)$. Then $\mathcal{S}_x \cong \mathrm{PG}(n-1,2)$ and as before we may identify $\mathrm{PG}(n-1,2)$ with $\mathcal{S}_x$. The points of $\mathrm{PG}(n-1,2)$ are then the lines of $DW(2n-1,2)$ through x. For every hyperplane H containing x, let $\mathcal{L}_H$ denote the set of lines through x contained in H. The hyperplane H arises from the universal embedding $\widetilde{e} : DW(2n-1,2) \to \widetilde{\Sigma}$ of $DW(2n-1,2)$, implying by Proposition 7.1 and Theorem 1.2 that $\mathcal{L}_H$ is either the whole set of lines through x or a pseudo-hyperplane of $\mathcal{S}_{n-1,n}$ arising from its universal pseudo-embedding $\widetilde{\epsilon}_{n-1,n}$. Let f_H be the unique element of $\mathcal{F}_n$ such that $\mathcal{L}_H = X_{f_H}$ (see Theorem 5.2). We say that H has *type* $i \in \{-\infty, 1, 2, \ldots, n\}$ *with respect to* x if i is the degree of f_H. A collection $\mathcal{H}$ of hyperplanes of $DW(2n-1,2)$ is said to be of *type* $i \in \{-\infty, 1, 2, \ldots, n\}$ *with respect to* x if there exists a hyperplane H of $\mathcal{H}$ containing x and if i is the maximal degree of the polynomials f_H, where H is a hyperplane of $\mathcal{H}$ through x. The following is an immediate consequence of Theorems 1.1(2), 1.2, 5.2 and Proposition 2.1.

Proposition 7.3 *A homogeneous full projective embedding ϵ of $DW(2n-1,2)$ has type $i \in \{1,2,\ldots,n\}$ if and only if $\mathcal{H}_\epsilon$ has type i with respect to one (and hence all) points of $DW(2n-1,2)$.*

We shall use Proposition 7.3 to prove that homogeneous full projective embeddings of $DW(2n-1,2)$ exist for all types $i \in \{1,2,\ldots,n\}$. To achieve this goal, we shall need the following notion and invoke the following two lemmas.

A set $\mathcal{H}$ of hyperplanes of $DW(2n-1,2)$ is called *complete* if $H_1 * H_2 \in \mathcal{H}$ for all $H_1, H_2 \in \mathcal{H}$ with $H_1 \neq H_2$. For every set $\mathcal{H}$ of hyperplanes of $DW(2n-1,2)$, we denote by $\langle \mathcal{H} \rangle$ the smallest complete set of hyperplanes containing $\mathcal{H}$.

Lemma 7.4 *Let F_1 and F_2 be two convex subspaces of diameter $\delta \in \{1, 2, \ldots, n\}$ of $DW(2n-1, 2)$. Let H_i with $i \in \{1, 2\}$ be a hyperplane of $\widetilde{F_i}$ and let $\overline{H_i}$ be the hyperplane of $DW(2n-1, 2)$ that arises by extending H_i. If x is a point of $DW(2n-1, 2)$ contained in $\overline{H_1} * \overline{H_2}$ but not in $\overline{H_1} \cup \overline{H_2}$, then there exists a hyperplane G_1 of $\widetilde{F_1}$, a hyperplane G_2 of $\widetilde{F_2}$ and points $y_1, y_2, \ldots, y_k$ of $DW(2n-1, 2)$ for some $k \in \mathbb{N}$ such that*

- *$x \in \overline{G_1} \cap \overline{G_2}$, where $\overline{G_i}$ with $i \in \{1, 2\}$ denotes the extension of G_i;*
- *$d(x, y_i) = n - 1$ for every $i \in \{1, 2, \ldots, k\}$;*
- *$\overline{H_1} * \overline{H_2} = \overline{G_1} * \overline{G_2} * H_{y_1} * H_{y_2} * \cdots * H_{y_k}$.*

Proof. As $x \notin \overline{H_i}$, we have $\mathrm{d}(x, F_i) = n - \delta$ for every $i \in \{1, 2\}$ and so there exists a point $u_i \in F_i$ not contained in H_x. As $H_x \cap F_i$ is the singular hyperplane of $\widetilde{F_i}$ with center $\pi_{F_i}(x)$, and every subspace containing $F_i \setminus (F_i \cap H_x)$ contains the whole of F_i, we can take u_i in such a way that $u_i \notin H_x \cup H_i$. Put $G_i := H_i *' (H'_{u_i})$, where H'_{u_i} is the singular hyperplane of $\widetilde{F_i}$ with center u_i, and the operator $*'$ is defined on the subsets of F_i. As $H_i \neq H'_{u_i}$, the set G_i is a hyperplane of $\widetilde{F_i}$. Moreover, $H_i = G_i *' (H'_{u_i})$. By the definition of extensions of hyperplanes, we then know that $\overline{H_i} = \overline{G_i} * H_{u_i}$ and $\overline{G_i} = \overline{H_i} * H_{u_i}$. Hence, $\overline{H_1} * \overline{H_2} = \overline{G_1} * \overline{G_2} * H_{u_1} * H_{u_2}$, $x \notin H_{u_1} \cup H_{u_2}$ and $x \in \overline{G_1} \cap \overline{G_2}$. If $u_1 = u_2$, then $\overline{H_1} * \overline{H_2} = \overline{G_1} * \overline{G_2}$ and we are done. We suppose therefore that $u_1 \neq u_2$. By Shult [25, Lemma 6.1] or Blok and Brouwer [2, Theorem 7.3], there exists a path $u_1 = z_0, z_1, \ldots, z_k = u_2$ in the complement of H_x that connects u_1 with u_2. For every $i \in \{1, 2, \ldots, k\}$, let y_i be the unique point of the line $z_{i-1}z_i$ distinct from z_{i-1} and z_i. Since $\mathrm{d}(x, z_{i-1}) = \mathrm{d}(x, z_i) = n$, we have $\mathrm{d}(x, y_i) = n - 1$. Also, $H_{u_1} * H_{u_2} = (H_{z_0} * H_{z_1}) * (H_{z_1} * H_{z_2}) * \cdots * (H_{z_{k-1}} * H_{z_k}) = H_{y_1} * H_{y_2} * \cdots * H_{y_k}$ and $\overline{H_1} * \overline{H_2} = \overline{G_1} * \overline{G_2} * H_{y_1} * H_{y_2} * \cdots * H_{y_k}$. ■

Lemma 7.5 *Let G be a hyperplane of a convex subspace F of diameter $\delta \geq 1$ of $DW(2n-1, 2)$. Let x be a point of $DW(2n-1, 2)$ at distance $n - \delta$ from F such that $\pi_F(x) \in G$. Then x belongs to the hyperplane $\overline{G}$ of $DW(2n-1, 2)$ that extends G, and the type of the hyperplane $\overline{G}$ of $DW(2n-1, 2)$ with respect to x equals the type of the hyperplane G of $\widetilde{F}$ with respect to $\pi_F(x)$.*

Proof. Let F'' be the convex subspace $\langle x, \pi_F(x) \rangle$ of diameter $n - \delta$, and let F' be a convex subspace of diameter δ through x such that $F' \cap F'' = \{x\}$. In $\mathcal{S}_x = \mathrm{PG}(n-1, 2)$, let α' and α'' be the complementary subspaces corresponding to respectively F' and F''. Let $\mathcal{L}$ denote the set of lines through x contained in $\overline{G}$, and let $\mathcal{L}'$ denote the set of lines through x contained in $F' \cap \overline{G}$. Let X, respectively X', denote the set of points of $\mathcal{S}_x = \mathrm{PG}(n-1, 2)$ corresponding to $\mathcal{L}$, respectively $\mathcal{L}'$. Then $\mathcal{L}$ is the cone with vertex α'' and basis $X' \subseteq \alpha'$. With respect to a suitable basis of $\mathcal{S}_x = \mathrm{PG}(n-1, 2)$, we thus have $f_{\overline{G}} = f_{\overline{G} \cap F'}$. So, the type of the hyperplane $\overline{G}$ of $DW(2n-1, 2)$ with respect to x coincides with the type of the hyperplane $\overline{G} \cap F'$ of $\widetilde{F'}$ with respect to x. Now, every point of F' has distance $n - \delta$ from F and the map $y \mapsto \pi_F(y)$ defines an isomorphism between $\widetilde{F'}$ and $\widetilde{F}$ mapping $\overline{G} \cap F'$ to G. So, the type of the hyperplane $\overline{G} \cap F'$ of $\widetilde{F'}$ with respect to x coincides with the type of the hyperplane G of $\widetilde{F}$ with respect to $\pi_F(x)$. ■

Now, for every $i \in \{1, 2, \ldots, n\}$, let $\mathcal{H}_i$ denote the set of hyperplanes of $DW(2n-1,2)$ that arise by extending a hyperplane of a convex subspace of diameter i of $DW(2n-1,2)$. Then $\mathcal{H}_i$ contains all singular hyperplanes (as extensions of singular hyperplanes are again singular hyperplanes). The set $\langle \mathcal{H}_i \rangle$ of hyperplanes of $DW(2n-1,2)$ satisfies the properties (a) and (b) mentioned in Proposition 2.3. It also satisfies Property (c): for every flag (x, L) of $DW(2n-1,2)$, there exists a point y at distance $n-1$ from L such that x is the unique point of L nearest to y, and for each such point y, the singular hyperplane $H_y \in \mathcal{H}_i \subseteq \langle \mathcal{H}_i \rangle$ intersects L in the singleton $\{x\}$. Proposition 2.3 tells us now that there exists a unique homogeneous full projective embedding ϵ of $DW(2n-1,2)$ for which $\langle \mathcal{H}_i \rangle = \mathcal{H}_\epsilon$. If $i = n$, then $\langle \mathcal{H}_i \rangle = \mathcal{H}_i$ consists of all hyperplanes of $DW(2n-1,2)$, implying by Proposition 2.1 that ϵ is isomorphic to the universal embedding of $DW(2n-1,2)$ which has type n by Proposition 7.1.

Theorem 7.6 *ϵ is a homogeneous full projective embedding of $DW(2n-1,2)$ of type i.*

Proof. We will rely on Proposition 7.3. Let x be an arbitrary point of $DW(2n-1,2)$. We need to prove the following two facts:

(1) There exists a hyperplane $H \in \mathcal{H}_\epsilon$ through x that has type i with respect to x.

(2) The type with respect to x of any hyperplane $H \in \mathcal{H}_\epsilon$ through x is at most i.

We first prove (1). Let F be a convex subspace of diameter i for which $\mathrm{d}(x, F) = n - i$. Let G be a hyperplane of $\widetilde{F}$ through $\pi_F(x)$ such that G has maximal possible type i with respect to x. By Propositions 7.1 and 7.3, such a hyperplane can be taken in the set $\mathcal{H}_{\widetilde{\epsilon}}$ where $\widetilde{\epsilon}$ is the universal embedding of $\widetilde{F}$. Let H be the hyperplane of $DW(2n-1,2)$ that extends G. By Lemma 7.5, the hyperplane H through x has type i with respect to x.

We next prove (2). We first deal with the case where H is the extension of a hyperplane G of a convex subspace F of diameter i. If $\mathrm{d}(x, F) \leq n - i - 2$ or ($\mathrm{d}(x, F) = n - i - 1$ and $\pi_F(x) \in G$), then $x^\perp \subseteq H$ and so $f_H = 0$ has degree $-\infty$. If $\mathrm{d}(x, F) = n - i - 1$ and $\pi_F(x) \notin G$, then $\mathcal{L}_H$ consists of all lines through x contained in the convex subspace $\langle x, F \rangle$ of diameter $n-1$, implying that $\mathcal{L}_H$ is a hyperplane of $\mathcal{S}_x$ and that f_H has degree 1. (Note that every line of $\mathcal{L}_H$ can only contain points at distance at most $n-i-1$ from F.) Finally, suppose that $\mathrm{d}(x, F) = n - i$. Then $\pi_F(x) \in G$, and the degree of G with respect to $\pi_F(x)$ is at most i. By Lemma 7.5, it then follows that H has degree at most i with respect to x.

In the general case, we know that $H = H_1' * H_2' * \cdots * H_h'$ for suitable $H_1', H_2', \ldots, H_h' \in \mathcal{H}_i$ (as $H \in \mathcal{H}_\epsilon = \langle \mathcal{H}_i \rangle$). Since $x \in H$, an even number of the hyperplanes $H_1', H_2', \ldots, H_h'$ do not contain x. If two of these hyperplanes do not contain x, then Lemma 7.4 implies that they can be replaced by a number of others that do contain x. We conclude that H can be written in the form $H_1 * H_2 * \cdots * H_k$, where each H_j, $j \in \{1, 2, \ldots, k\}$, contains x and is the extension of a hyperplane G_j of a convex subspace F_j of diameter i. We have $f_H = f_{H_1} + f_{H_2} + \cdots + f_{H_k}$ by Lemma 6.1. Since each polynomial f_{H_j}, $j \in \{1, 2, \ldots, k\}$, has degree at most i, we see that f_H should also have degree at most i. ∎

7.4 Embeddings induced by full projective embeddings of $DH(2n-1,4)$

Throughout this subsection, δ denotes an arbitrary element of $\mathbb{F}_4\setminus\{0,1\}$. Then $\delta^2+\delta=1$. In the following two propositions, we construct two homogeneous pseudo-embeddings of $\mathrm{PG}(n,4)$, $n\geq 1$. Note that these two pseudo-embeddings coincide when $n=1$.

Proposition 7.7 ([15, Proposition 4.2]) *The map e which maps every point $(X_0,X_1,\ldots,X_n)$ of $\mathrm{PG}(n,4)$ to the point $(X_0^3,X_1^3,\ldots,X_n^3,X_iX_j^2+X_jX_i^2,\delta X_iX_j^2+\delta^2X_jX_i^2\,|\,0\leq i<j\leq n)$ of $\mathrm{PG}(n^2+2n,2)$ is a homogeneous pseudo-embedding of $\mathrm{PG}(n,4)$.*

Proposition 7.8 ([14, Theorem 1.1]) *Let $\widetilde{e}$ be a map from $\mathrm{PG}(n,4)$ to $\mathrm{PG}(k,2)$, $k=\frac{n^3+3n^2+5n}{3}$, mapping the point $p=(X_0,X_1,\ldots,X_n)$ of $\mathrm{PG}(n,4)$ to the point $\widetilde{e}(p)=(Y_0,Y_1,\ldots,Y_k)$ of $\mathrm{PG}(k,2)$, where*

- *$n+1$ coordinates of $\widetilde{e}(p)$ are of the form X_i^3, where $i\in\{0,1,\ldots,n\}$;*
- *$\binom{n+1}{2}$ coordinates of $\widetilde{e}(p)$ are of the form $X_iX_j^2+X_i^2X_j$, where $i,j\in\{0,1,\ldots,n\}$ and $i<j$;*
- *$\binom{n+1}{2}$ coordinates of $\widetilde{e}(p)$ are of the form $\delta X_iX_j^2+\delta^2X_i^2X_j$, where $i,j\in\{0,1,\ldots,n\}$ and $i<j$;*
- *$\binom{n+1}{3}$ coordinates of $\widetilde{e}(p)$ are of the form $X_iX_jX_k+X_i^2X_j^2X_k^2$, where $i,j,k\in\{0,1,\ldots,n\}$ and $i<j<k$;*
- *$\binom{n+1}{3}$ coordinates of $\widetilde{e}(p)$ are of the form $\delta X_iX_jX_k+\delta^2X_i^2X_j^2X_k^2$, where $i,j,k\in\{0,1,\ldots,n\}$ and $i<j<k$.*

Then $\widetilde{e}$ is a pseudo-embedding of $\mathrm{PG}(n,4)$ which is isomorphic to the universal pseudo-embedding of $\mathrm{PG}(n,4)$.

The pseudo-embedding e of $\mathrm{PG}(n,4)$ constructed in Proposition 7.7 is called the *Hermitian Veronese embedding* of $\mathrm{PG}(n,4)$. The determination of the universal pseudo-embedding of $\mathrm{PG}(n,4)$ obtained in [14] relied on the classification of the pseudo-hyperplanes of $\mathrm{PG}(n,4)$ which itself was realized in the papers [26, 27] (for $n=2$), [18] (for $n=3$) and [24] (for general n). In [14, Theorem 1.4], we also obtained a complete classification of all homogeneous pseudo-embeddings of $\mathrm{PG}(n,4)$.

Proposition 7.9 *Every homogeneous pseudo-embedding of $\mathrm{PG}(n,4)$ is isomorphic to the Hermitian Veronese embedding or to the universal pseudo-embedding of $\mathrm{PG}(n,4)$.*

Regard now $\mathrm{PG}(n,2)$ as a Baer subgeometry of $\mathrm{PG}(n,4)$. Then any map $\epsilon:\mathrm{PG}(n,4)\to\Sigma$ from the point set of $\mathrm{PG}(n,4)$ to the point set of a projective space Σ will induce a map $\epsilon':\mathrm{PG}(n,2)\to\Sigma'$ from $\mathrm{PG}(n,2)$ to the subspace Σ' of Σ generated by the image of $\mathrm{PG}(n,2)$. Using the fact that $\delta^2+\delta=1$ and $X^2=X$ for every $X\in\mathbb{F}_2$, the following is then easily seen to be true.

Proposition 7.10 (1) *If $\epsilon:\mathrm{PG}(n,4)\to\Sigma$ is the Hermitian Veronese embedding of $\mathrm{PG}(n,4)$, then the induced map $\epsilon':\mathrm{PG}(n,2)\to\Sigma'$ is isomorphic to $\epsilon_{n,2}$.*

(2) *If* $\epsilon : \mathrm{PG}(n,4) \to \Sigma$ *is the universal pseudo-embedding of* $\mathrm{PG}(n,4)$ *with* $n \geq 2$*, then the induced map* $\epsilon' : \mathrm{PG}(n,2) \to \Sigma'$ *is isomorphic to* $\epsilon_{n,3}$.

We now discuss some applications of the above to homogeneous full projective embeddings of $DH(2n-1,4)$, $n \geq 2$. Recall that $DH(2n-1,4)$ is the dual polar space associated with a unitary polarity of $\mathrm{PG}(2n-1,4)$. For every point x of $DH(2n-1,4)$, let $\mathcal{S}_x$ denote the point-line geometry whose points and lines are the lines and quads of $DH(2n-1,4)$ through x, with incidence being containment. Then $\mathcal{S}_x \cong \mathrm{PG}(n-1,4)$. If ϵ is a full embedding of $DH(2n-1,4)$ into a projective space Σ, then the image of $x^\perp$ generates a subspace Σ_x of Σ. The embedding ϵ induces a map $\epsilon_x : \mathcal{S}_x \to \Sigma_x/\epsilon(x)$ of the point set of $\mathcal{S}_x$ to the set of points of the quotient space $\Sigma_x/\epsilon(x)$.

Proposition 7.11 *Suppose* ϵ *is a homogeneous full embedding of* $DH(2n-1,4)$ *in a projective space* Σ*. Then* ϵ_x *is isomorphic to the Hermitian Veronese embedding or to the universal pseudo-embedding of* $\mathcal{S}_x \cong \mathrm{PG}(n-1,4)$.

Proof. We first show that ϵ_x is a pseudo-embedding of $\mathcal{S}_x$. Let Q be a quad through x and let L_1, L_2, L_3, L_4, L_5 denote the five lines through x contained in Q. The embedding ϵ induces a full projective embedding of $\widetilde{Q} \cong DH(3,4) \cong Q^-(5,2)$, which is isomorphic to the standard embedding of that generalized quadrangle in $\mathrm{PG}(5,2)$. By a known property of this standard embedding, the set $\{\epsilon(L_1), \epsilon(L_2), \epsilon(L_3), \epsilon(L_4), \epsilon(L_5)\}$ defines a frame of a 3-dimensional subspace of the quotient space $\Sigma_x/\epsilon(x)$. This implies that ϵ_x is a pseudo-embedding of $\mathcal{S}_x$.

In view of Proposition 7.9, it suffices to show that the pseudo-embedding ϵ_x is homogeneous. But similarly as in the case of the dual polar space $DW(2n-1,2)$, this follows from the fact that ϵ is homogeneous and that the stabilizer of x inside the full automorphism group of $DH(2n-1,4)$ induces the full group of automorphisms of $\mathcal{S}_x \cong \mathrm{PG}(n-1,4)$. ■

If ϵ is a homogeneous full projective embedding of $DH(2n-1,4)$, then ϵ is said to be of *type 1* if ϵ_x is isomorphic to the Hermitian Veronese embedding of $\mathcal{S}_x \cong \mathrm{PG}(n-1,4)$, and of *type 2* if $n \geq 3$ and ϵ_x is isomorphic to the universal pseudo-embedding of $\mathcal{S}_x \cong \mathrm{PG}(n-1,4)$. As ϵ is homogeneous, these definitions do not depend on the considered point x. By [6, Corollary 1.5] (see also [23, Theorem 9.3]), the Grassmann-embedding of $DH(2n-1,4)$ has type 1, and by [13, Theorem 1.4], the universal embedding of $DH(2n-1,4)$ has type 2 if $n \geq 3$. Note that the universal embedding of $DH(3,4) \cong Q^-(5,2)$ has type 1 as this embedding is isomorphic to the Grassmann embedding.

Proposition 7.12 *Suppose* $DW(2n-1,2)$ *is fully and isometrically embedded in* $DH(2n-1,4)$*. Let* ϵ *be a homogeneous full projective embedding of* $DH(2n-1,4)$ *and let* ϵ' *be the homogeneous full projective embedding of* $DW(2n-1,2)$ *induced by* ϵ*. Then the following hold:*

- *If* ϵ *has type* 1*, then* ϵ' *has type* 2*. In particular, this holds if* ϵ *is isomorphic to the Grassmann embedding of* $DH(2n-1,4)$.

- *If ϵ has type 2, then ϵ' has type 3. In particular, this holds if $n \geq 3$ and ϵ is isomorphic to the universal embedding of $DH(2n-1,4)$.*

Proof. Let x be a point of $DW(2n-1,2)$. The lines and quads of $DH(2n-1,4)$ through x define a point-line geometry $\mathcal{S}_x \cong \mathrm{PG}(n-1,4)$, and the lines of $DW(2n-1,2)$ through x define a Baer subgeometry of $\mathcal{S}_x \cong \mathrm{PG}(n-1,4)$. The claim now follows from Proposition 7.10. ∎

By [12, Theorem 1.1], we know that the full projective embedding of $DW(2n-1,2)$ induced by the Grassmann embedding of $DH(2n-1,4)$ is isomorphic to the Grassmann embedding of $DW(2n-1,2)$. So, we see that parts (3) and (4) of Theorem 1.3 are also valid. The fact that the Grassmann embedding of $DW(2n-1,2)$ has type 2 also follows from [6, Corollary 1.5] and Pasini [23, Theorem 9.3].

7.5 Universal homogeneous projective embeddings of type i

Let $\widetilde{\epsilon} : DW(2n-1,2) \to \widetilde{\Sigma}$ denote the universal embedding of $DW(2n-1,2) = (\mathcal{P},\mathcal{L},\mathrm{I})$. Let $G \cong Sp(2n,2)$ denote the full automorphism group of $DW(2n-1,2)$. As $\widetilde{\epsilon}$ is homogeneous, G lifts to a group $\widetilde{G}$ of projectivities of $\widetilde{\Sigma}$. If x is a point of $DW(2n-1,2)$, then there exists a unique hyperplane Π_x of $\widetilde{\Sigma}$ such that $H_x = \widetilde{\epsilon}^{-1}(\widetilde{\epsilon}(\mathcal{P}) \cap \Pi_x)$. Put $I := \bigcap_{x\in\mathcal{P}} \Pi_x$. Let $\mathcal{L}_x$ denote the set of lines of $DW(2n-1,2)$ through x. As $\mathcal{S}_x \cong \mathrm{PG}(n-1,2)$, we may identify $\mathcal{L}_x$ with the point set of $\mathrm{PG}(n-1,2)$. By Proposition 7.1 and Theorem 1.2, the full projective embedding $\widetilde{\epsilon}$ induces a map $\widetilde{\epsilon}_x$ from $\mathcal{L}_x$ to $\widetilde{\Sigma}_x/\widetilde{\epsilon}(x)$ which is isomorphic to the universal embedding $\epsilon_{n-1,n}$ of $\mathcal{S}_{n-1,n}$.

Now, fix a certain $i \in \{1,2,\dots,n\}$. Then $\epsilon_{n-1,i}$ is also a pseudo-embedding of $\mathcal{S}_{n-1,n}$ (as there are no lines), and so is isomorphic to a quotient of $\epsilon_{n-1,n}$. Let β_i be the unique subspace of $\widetilde{\Sigma}_x$ through $\epsilon(x)$ such that the quotient of $\widetilde{\epsilon}_x$ defined by the subspace $\beta_i/\epsilon(x)$ of $\widetilde{\Sigma}_x/\epsilon(x)$ is isomorphic to $\epsilon_{n-1,i}$. Now, let π_i denote the smallest $\widetilde{G}$-invariant subspace of $\widetilde{\Sigma}$ containing $\beta_i \cap I$.

Theorem 7.13 *The following hold:*

(1) *π_i is disjoint from the image of $\widetilde{\epsilon}$;*

(2) *$\widetilde{\epsilon}/\pi_i$ is a homogeneous full projective embedding of type i of $DW(2n-1,2)$;*

(3) *if ϵ is a homogeneous full projective embedding of type i of $DW(2n-1,2)$ then $\widetilde{\epsilon}/\pi_i \geq \epsilon$.*

Proof. Obviously, the subspace I is $\widetilde{G}$-invariant. For every point y of $DW(2n-1,2)$, there exists a point z at maximal distance n from y, and for each such point z, we have $y \notin H_z$ and $\widetilde{\epsilon}(y) \notin \Pi_z$. So, I is disjoint from the image of $\widetilde{\epsilon}$. As $\beta_i \cap I \subseteq I$ and I is $\widetilde{G}$-invariant, we have that $\pi_i \subseteq I$ is disjoint from the image of $\widetilde{\epsilon}$. So, $\widetilde{\epsilon}/\pi_i$ defines a full projective embedding of $DW(2n-1,2)$. As π_i is $\widetilde{G}$-invariant, $\widetilde{\epsilon}/\pi_i$ is a homogeneous full projective embedding of $DW(2n-1,2)$.

Suppose that ϵ is a homogeneous full projective embedding of type j of $DW(2n-1,2)$. Without loss of generality, we may suppose that $\epsilon = \widetilde{\epsilon}/\alpha$, where α is some suitable subspace of $\widetilde{\Sigma}$ disjoint from the image of $\widetilde{\epsilon}$. Since ϵ is homogeneous, we know from Blok et al. [3, Theorem 1.1] that ϵ is also polarized, implying that $\alpha \subseteq \Pi_y$ for every $y \in \mathcal{P}$, i.e. $\alpha \subseteq I$. As ϵ is homogeneous, we also know from [3] that α is $\widetilde{G}$-invariant. Put $\beta := \langle \widetilde{\epsilon}(x), \alpha \cap \widetilde{\Sigma}_x \rangle$. As ϵ is a homogeneous embedding of type j, the quotient of the map $\widetilde{\epsilon}_x$ defined by the subspace $\beta/\epsilon(x)$ of $\widetilde{\Sigma}_x/\widetilde{\epsilon}(x)$ must be isomorphic to the universal pseudo-embedding $\epsilon_{n-1,j}$ of $\mathcal{S}_{n-1,j}$. This implies that $\beta = \beta_j$. Since $\alpha \subseteq I$, we have $I \cap \beta_j = I \cap \beta = \alpha \cap \widetilde{\Sigma}_x \subseteq \alpha$. As α is $\widetilde{G}$-invariant, α contains the subspace π_j. This implies that $\epsilon = \widetilde{\epsilon}/\alpha$ is isomorphic to a quotient of $\widetilde{\epsilon}/\pi_j$. The above reasoning in combination with Theorem 1.2 also shows that if homogeneous full projective embeddings of type j exist, then the homogeneous full projective embedding $\widetilde{\epsilon}/\pi_j$ should have type j. But such homogeneous projective embeddings indeed exist by Theorem 1.3(1). ■

References

[1] E. F. Assmus and J. D. Key. *Designs and their codes.* Cambridge Tracts in Mathematics 103, Cambridge University Press, 1992.

[2] R. J. Blok and A. E. Brouwer. The geometry far from a residue. pp. 29–38 in *Groups and geometries* (Siena, 1996), Trends Math., Birkhäuser, 1998.

[3] R. J. Blok, I. Cardinali, B. De Bruyn and A. Pasini. Polarized and homogeneous embeddings of dual polar spaces. *J. Algebraic Combin.* 30 (2009), 381–399.

[4] A. Blokhuis and A. E. Brouwer. The universal embedding dimension of the binary symplectic dual polar space. *Discrete Math.* 264 (2003), 3–11.

[5] F. Buekenhout and P. J. Cameron. *Projective and affine geometry over division rings.* Chapter 2 of the "Handbook of Incidence Geometry" (ed. F. Buekenhout), North-Holland, 1995.

[6] I. Cardinali and B. De Bruyn. The structure of full polarized embeddings of symplectic and Hermitian dual polar spaces. *Adv. Geom.* 8 (2008), 111–137.

[7] B. N. Cooperstein. On the generation of some dual polar spaces of symplectic type over GF(2). *European J. Combin.* 18 (1997), 741–749.

[8] B. N. Cooperstein. On the generation of dual polar spaces of symplectic type over finite fields. *J. Combin. Theory Ser. A* 83 (1998), 221–232.

[9] B. De Bruyn. Isometric full embeddings of $DW(2n-1,q)$ into $DH(2n-1,q^2)$. *Finite Fields Appl.* 14 (2008), 188–200.

[10] B. De Bruyn. Two new classes of hyperplanes of the dual polar space $DH(2n-1,4)$ not arising from the Grassmann embedding. *Linear Algebra Appl.* 429 (2008), 2030–2045.

[11] B. De Bruyn. The structure of the spin-embeddings of dual polar spaces and related geometries. *European J. Combin.* 29 (2008), 1242–1256.

[12] B. De Bruyn. On isometric full embeddings of symplectic dual polar spaces into Hermitian dual polar spaces. *Linear Algebra Appl.* 430 (2009), 2541–2552.

[13] B. De Bruyn. On sets of odd type of $\mathrm{PG}(n,4)$ and the universal embedding of the dual polar space $DH(2n-1,4)$. *Discrete Math.* 312 (2012), 554–560.

[14] B. De Bruyn. The pseudo-hyperplanes and homogeneous pseudo-embeddings of $\mathrm{AG}(n,4)$ and $\mathrm{PG}(n,4)$. *Des. Codes Cryptogr.* 65 (2012), 127–156.

[15] B. De Bruyn. Pseudo-embeddings and pseudo-hyperplanes. *Adv. Geom.* 13 (2013), 71–95.

[16] B. De Bruyn. The pseudo-hyperplanes and homogeneous pseudo-embeddings of the generalized quadrangles of order $(3,t)$. *Des. Codes Cryptogr.* 68 (2013), 259–284.

[17] N. Hamada. The rank of the incidence matrix of points and d-flats in finite geometries. *J. Sci. Hiroshima Univ. Ser. A-I Math.* 32 (1968), 381–396.

[18] J. W. P. Hirschfeld and X. Hubaut. Sets of even type in $\mathrm{PG}(3,4)$, alias the binary $(85,24)$ projective geometry code. *J. Combin. Theory Ser. A* 29 (1980), 101–112.

[19] S. P. Inamdar and N. S. N. Sastry. Codes from Veronese and Segre embeddings and Hamada's formula. *J. Combin. Theory Ser. A* 96 (2001), 20–30.

[20] P. Li. On the universal embedding of the $Sp_{2n}(2)$ dual polar space. *J. Combin. Theory Ser. A* 94 (2001), 100–117.

[21] P. Li. On the universal embedding of the $U_{2n}(2)$ dual polar space. *J. Combin. Theory Ser. A* 98 (2002), 235–252.

[22] P. McClurg. On the universal embedding of dual polar spaces of type $Sp_{2n}(2)$. *J. Combin. Theory Ser. A* 90 (2000), 104–122.

[23] A. Pasini. Embeddings and expansions. *Bull. Belg. Math. Soc. Simon Stevin* 10 (2003), 585–626.

[24] B. Sherman. On sets with only odd secants in geometries over GF(4). *J. London Math. Soc. (2)* 27 (1983), 539–551.

[25] E. Shult. On Veldkamp lines. *Bull. Belg. Math. Soc. Simon Stevin* 4 (1997), 299–316.

[26] M. Tallini Scafati. $\{k, n\}$-archi di un piano grafico finito, con particolare riguardo a quelli con due caratteria. *Rend. Acc. Naz. Lincei* 40 (1966), 812–818 and 1020–1025.

[27] M. Tallini Scafati. Caratterizzazione grafica delle forme hermitiane di un $S_{r,q}$. *Rend. Mat. e Appl.* 26 (1967), 273–303.

[28] S. Yoshiara. Embeddings of flag-transitive classical locally polar geometries of rank 3. *Geom. Dedicata* 43 (1992), 121–165.