



Contextuality and Noncommutative Geometry in Quantum Mechanics

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Abstract: Observable properties of a classical physical system can be modelled deterministically as functions from the space of pure states to outcome values; dually, states can be modelled as functions from the algebra of observables to outcome values. The probabilistic predictions of quantum physics are contextual in that they preclude this classical assumption of reality: noncommuting observables, which are not assumed to be jointly measurable, cannot be consistently ascribed deterministic values even if one enriches the description of a quantum state. Here, we consider the geometrically dual objects of noncommutative operator algebras of observables as being generalisations of classical (deterministic) state spaces to the quantum setting and argue that these generalised state spaces represent the objects of study of noncommutative operator geometry. By adapting the spectral presheaf of Hamilton–Isham–Butterfield, a formulation of quantum state space that collates contextual data, we reconstruct tools of noncommutative geometry in an explicitly geometric fashion. In this way, we bridge the foundations of quantum mechanics with the foundations of noncommutative geometry à la Connes et al. To each unital C^* -algebra \mathcal{A} we associate a geometric object—a diagram of topological spaces collating quotient spaces of the noncommutative space underlying \mathcal{A} —that performs the role of a generalised Gelfand spectrum. We show how any functor F from compact Hausdorff spaces to a suitable target category $\tilde{\mathcal{C}}$ can be applied directly to these geometric objects to automatically yield an extension \tilde{F} acting on all unital C^* -algebras. This procedure is used to give a novel formulation of the operator K_0 -functor via a finitary variant \tilde{K}_f of the extension \tilde{K} of the topological K -functor. We then delineate a C^* -algebraic conjecture that the extension of the functor that assigns to a topological space its lattice of open sets assigns to a unital C^* -algebra the Zariski topological lattice of its primitive ideal spectrum, i.e. its lattice of closed two-sided ideals. We prove the von Neumann algebraic analogue of this conjecture.

1. Introduction

The mathematical description of classical physical systems exhibits an elegant interplay between algebraic aspects of observables and geometric aspects of states. A system can be described in two (dually) equivalent ways depending on whether one takes states or observables as primary. Adopting a *realist* or *ontological* perspective, one starts with a space of states and constructs observables as (continuous) functions from states to scalar values. Conversely, adopting an *operational* or *epistemic* perspective, one starts with an algebra of observables and constructs states as (homomorphic) functions from observables to scalar values. Such state–observable dualities are manifestations of the duality between geometry and algebra that is a common thread running throughout mathematics. We are particularly interested in the interplay described by the Gel'fand–Naïmark duality [44] between the categories of unital commutative C^* -algebras (of observables) and compact Hausdorff spaces (of states).

Quantum systems are described by their C^* -algebra of observables which, by the Gel'fand–Naïmark–Segal construction [44, 87], can be represented as an algebra of Hilbert space operators. However, as quantum algebras are noncommutative, Gel'fand–Naïmark duality cannot be used to obtain a geometric description as in the classical case. Indeed, pure quantum states do not ascribe deterministic values to all observables; rather, a quantum state yields for each observable a probability distribution on the various outcomes possible upon measurement.

The inherently probabilistic nature of quantum mechanics has discomfited advocates of *physical realism* since the theory's inception. Einstein [37], in his famous foundational debates with Bohr, argued that the quantum state does not provide a 'complete description' of a system. These debates led to the study of *hidden variable models* of quantum theory: models in which quantum states are represented as probability distributions over a space of more fundamental *ontic states* that yield deterministic values for all observables. Motivated by a desire to hold onto realism, one may insist that a hidden variable model be *noncontextual*: that the values of the system's observable properties be independent of the precise method of observation, and, in particular, of which other observables are measured simultaneously. However, the no-go theorem of Bell–Kochen–Specker [12, 68] rules out hidden variable models of this kind, showing that *contextuality* is a necessary feature of any theory reproducing the highly-verified empirical predictions of quantum mechanics.

The primary motivation of this work is to study a candidate geometric notion of state space for quantum systems that maintains as closely as possible a realist perspective in the sense alluded to above. In pursuing this, we identify and explore a connection with the well-studied mathematical field of noncommutative geometry: our geometric notion of state space will be the geometric dual of a noncommutative algebra of observables. Our desired geometric construction must necessarily account for contextuality as an obstacle towards a naively ontological quantum state space.

Our starting point is the *spectral presheaf* formulation of the Bell–Kochen–Specker theorem. Hamilton, Isham, and Butterfield [52, 60] associate to a von Neumann algebra a presheaf of compact Hausdorff spaces, varying over *contexts* (commutative von Neumann subalgebras representing sets of jointly measurable observables). The Bell–Kochen–Specker theorem finds expression as the nonexistence of a global section of points (i.e. a global point in the generalised 'space'), whereas Gleason's theorem [47] can be expressed as a correspondence between quantum states and global sections of probability distributions (i.e. a global probability distribution on the generalised 'space'). These observations strongly suggest the role that the spectral presheaf might play as a

notion of quantum state space that fundamentally incorporates contextuality. Indeed, this idea forms the basis of a considerable body of research into topos-theoretic approaches to quantum physics by Isham, Butterfield, Döring, et al. [31–34, 52, 60–62] and by Heunen, Landsman, Spitters, et al. [54–56]. More recent developments [11, 28–30] pursue the idea of regarding spectral presheaves as providing a generalised notion of space dual to noncommutative von Neumann algebras.

We directly relate this body of research to the programme of noncommutative operator geometry of Connes et al. [20], in which mathematicians regard noncommutative operator algebras as generalised geometric spaces. The result is a plethora of generalisations of geometric tools to the noncommutative algebraic setting that are constructed indirectly via Gel'fand–Naimark duality.

Our contribution is to associate diagrams of topological spaces, akin to the spectral presheaf, to noncommutative algebras and to use them to give direct geometric formulations of notions from noncommutative geometry. We argue this is necessary for any concretely spatial object to be considered a quantum state space in the sense of being dual to a noncommutative algebra. Physically, the topological spaces in the diagram associated to an algebra can be thought of as state spaces for sets of compatible observables. Mathematically, they are precisely those quotient spaces of the ‘noncommutative space’ represented by the algebra that are tractable in the sense of being (classical) topological spaces.

The general scheme is as follows: given a concept defined on (compact Hausdorff) topological spaces (corresponding to unital commutative C^* -algebras), one lifts it from the contexts to a global concept by taking a limit, thus yielding a corresponding extension defined for all unital C^* -algebras. In order to support the connection between the global concepts defined via direct extension and those defined indirectly via Gel'fand–Naimark duality, we apply this template to extend two different concepts: K -theory and open sets. First, we show how a finitary variant of the extension \tilde{K} of the topological K -functor yields a novel formulation of the operator K_0 -functor. Secondly, we conjecture a correspondence between the extension of the notion of topologically open sets and closed two-sided ideals of the algebra, and prove the von Neumann algebraic version of this conjecture.

Notation. For simplicity, given a functor $F: \mathbf{A} \rightarrow \mathbf{B}$, we do not distinguish it notationally from the same map regarded as a functor $\mathbf{A}^{\text{op}} \rightarrow \mathbf{B}^{\text{op}}$. The same applies to $G: \mathbf{A}^{\text{op}} \rightarrow \mathbf{B}$ and $G: \mathbf{A} \rightarrow \mathbf{B}^{\text{op}}$, since we treat $(\mathbf{A}^{\text{op}})^{\text{op}}$ as equal to \mathbf{A} .

We shall also denote by F the lifting of a functor $F: \mathbf{A} \rightarrow \mathbf{B}$ to the categories of diagrams introduced in Sect. 3.1, mapping \mathbf{A} -valued diagrams to \mathbf{B} -valued diagrams (see the remarks at the end of that section for details).

Given functors $F, G: \mathbf{A} \rightarrow \mathbf{B}$, we write $F \simeq G$ to denote that F and G are naturally isomorphic.

For reference, Table 1 lists the categories mentioned throughout this article, and their duals where applicable (see Sect. 2.1).

Wherever we consider a unital sub- C^* -algebra \mathcal{A}' of a unital C^* -algebra \mathcal{A} , the intended meaning is that \mathcal{A}' is a subobject of \mathcal{A} in the category $\text{u}C^*$ of unital C^* -algebras; that is, \mathcal{A}' is not simply a sub- C^* -algebra of \mathcal{A} that happens to be unital, but additionally the unit of \mathcal{A}' must be the same as the unit of \mathcal{A} .

Summary of main results. We define the category $\text{Diag}(\mathbf{C})$ whose objects are diagrams in the category \mathbf{C} , i.e. functors from any small category to \mathbf{C} . We then introduce

$\varprojlim: \text{Diag}(\mathbf{C}) \rightarrow \mathbf{C}$ when \mathbf{C} is complete. This generalises the usual limit functors for diagrams of a fixed shape. These constructions have duals denoted $\varinjlim: \text{Diag}(\mathbf{C}) \rightarrow \mathbf{C}$ when \mathbf{C} is cocomplete.

The *spatial diagram functor* $G: \mathbf{uC}^{*\text{op}} \rightarrow \text{Diag}(\mathbf{KHaus})$ associates a diagram of compact Hausdorff spaces to each unital C^* -algebra: the objects in the diagram $G(\mathcal{A})$ are the Gelfand spectra of unital commutative sub- C^* -algebras of \mathcal{A} , while the morphisms arise from inner automorphisms of \mathcal{A} . We shall also consider some variations (finitary C^* -algebraic and von Neumann algebraic) of this construction.

For any functor $F: \mathbf{KHaus} \rightarrow \mathbf{C}$ to a complete target category, we define an *extension* $\tilde{F}: \mathbf{uC}^{*\text{op}} \rightarrow \mathbf{C}$ that acts on a unital C^* -algebra \mathcal{A} by applying F (lifted to diagrams) to the diagram $G(\mathcal{A})$ and then taking the limit:

$$\tilde{F} = \varprojlim \circ F \circ G : \mathbf{uC}^{*\text{op}} \rightarrow \text{Diag}(\mathbf{KHaus}) \rightarrow \text{Diag}(\mathbf{C}) \rightarrow \mathbf{C}.$$

The functor \tilde{F} extends F . By this, we mean that the two functors agree on unital commutative C^* -algebras:

$$\tilde{F}|_{\mathbf{ucC}^*} \simeq F \circ \Sigma,$$

where the functor $\Sigma: \mathbf{uC}^{*\text{op}} \rightarrow \mathbf{KHaus}$ maps a unital commutative C^* -algebra to its Gelfand spectrum.

We compare the extension of important topological concepts with their existing generalisation in noncommutative geometry. First, we consider the topological K -functor, $K: \mathbf{KHaus}^{\text{op}} \rightarrow \mathbf{Ab}$, and give a novel formulation of operator K -theory, $K_0: \mathbf{C}^* \rightarrow \mathbf{Ab}$ via a finitary variant \tilde{K}_f of \tilde{K} , the extension of the topological K -functor:¹

Theorem 5.2. $K_0 \simeq K_0 \circ \mathcal{K} \simeq \tilde{K}_f \circ \mathcal{K}$ as functors $\mathbf{uC}^* \rightarrow \mathbf{Ab}$. Consequently, $K_0: \mathbf{C}^* \rightarrow \mathbf{Ab}$ is naturally isomorphic to the extension via unitalisation of the functor $\tilde{K}_f \circ \mathcal{K}$.

In the statement above, $\mathcal{K}: \mathbf{uC}^* \rightarrow \mathbf{C}^*$ is the stabilisation functor and \tilde{K}_f is the finitary version of \tilde{K} , in the sense that the extension of K is obtained, for a unital C^* -algebra \mathcal{A} , via a diagram $G_f(\mathcal{A})$ of the Gelfand spectra of its unital *finite-dimensional* commutative sub- C^* -algebras. Since stable C^* -algebras are nonunital, this then needs to be extended to all C^* -algebras, which is done via unitalisation.

We then consider the functor $\mathcal{J}: \mathbf{KHaus} \rightarrow \mathbf{CMSLat}$ that maps a compact Hausdorff space to its lattice of closed sets ordered by reverse inclusion (which is isomorphic to the lattice of open sets ordered by inclusion) and a continuous function to its direct image map.² Let $\mathcal{I}: \mathbf{uC}^* \rightarrow \mathbf{CMSLat}$ be the functor mapping a unital C^* -algebra to its lattice of closed two-sided ideals (equivalently, the lattice of open sets of the C^* -algebra's primitive ideal spectrum) and a unital $*$ -homomorphism to its preimage map. We conjecture that $\tilde{\mathcal{J}} \simeq \mathcal{I}$ and prove the von Neumann algebraic analogue:

¹ The category \mathbf{Ab} of abelian groups is cocomplete. So, K can be seen as a functor from \mathbf{KHaus} to a complete category, \mathbf{Ab}^{op} . This can be extended as explained earlier to all unital C^* -algebras, yielding a functor $\mathbf{uC}^* \rightarrow \mathbf{Ab}$. Note that a limit in the category \mathbf{Ab}^{op} is a colimit in \mathbf{Ab} . This is then extended to the category \mathbf{C}^* of all (i.e. not-necessarily-unital) C^* -algebras via unitalisation, in the same fashion that K_0 is.

² Dealing with closed sets makes the action on morphisms easier to state, as it is given by the map taking a set to its image, whereas for open sets one would have the map taking a set to the complement of the image of its complement.

Theorem 6.2. *Let $\mathcal{T}_W: \mathbf{HStonean} \rightarrow \mathbf{CMSLat}$ be the functor assigning to a hyperstonean space its lattice of clopen sets ordered by reverse inclusion and to an open continuous function its direct image map, and let $\mathcal{I}_W: \mathbf{vNA}^{\text{op}} \rightarrow \mathbf{CMSLat}$ be the functor assigning to a von Neumann algebra its lattice of ultraweakly closed two-sided ideals and to an ultraweakly continuous (or normal) unital $*$ -homomorphism its inverse image map. Then $\tilde{\mathcal{T}}_W \simeq \mathcal{I}_W$, where $\tilde{\mathcal{T}}_W$ is the von Neumann algebraic extension of \mathcal{T} .*

Here, the extension $\tilde{\mathcal{T}}_W$ is obtained, for each von Neumann algebra \mathcal{A} , via a diagram $G_W(\mathcal{A})$ whose objects are the spectra of its commutative sub-von Neumann algebras and whose morphisms arise from inner automorphisms of \mathcal{A} .

Outline. The remainder of this article is organised as follows:

- Section 2 surveys the main aspects of state-observable dualities, quantum contextuality, the spectral presheaf, and noncommutative geometry, and expands on the motivation for this work;
- Section 3 introduces the necessary technical machinery for functorially associating diagrams of topological spaces to operator algebras;
- Section 4 defines the notion of an *extension* of a concept defined for compact Hausdorff topological spaces to one defined for all unital C^* -algebras;
- Section 5 considers the extension of topological K -theory and gives a novel geometric formulation of operator K -theory;
- Section 6 explains the conjectured correspondence between the extension of open sets and closed two-sided ideals, and proves the von Neumann algebraic analogue;
- Section 7 outlines future lines of research.

The appendices contain additional and background material:

- Appendix A presents an alternative explicit construction of the colimit functor of Sect. 4.1;
- Appendix B contains background material on topological and operator K -theory, expanding on the presentation in Sects. 5.1 and 5.2;
- Appendix C contains background material on the primitive ideal spectrum of a C^* -algebra and some facts about von Neumann algebras needed in Sect. 6.

This article is based on the doctoral dissertation of the first author [24]. Earlier versions of the main results have appeared in the unpublished manuscripts [23] (Sects. 3–5 and Conjecture 6.1) and [25] (Sect. 6).

2. Background and Motivation

We survey the main background topics to make the results accessible to both mathematicians and physicists and to expand on the motivation for our work.

2.1. (Classical) state-observable dualities. Observables, being representatives of quantities that vary with state, are generally endowed with algebraic structure capturing the arithmetic of quantities. States, on the other hand, are endowed with geometric structure: states are close to each other when they share similar physical properties.

Important examples are those classical systems that can be modelled in terms of Poisson geometry [75]. The collection of pure states is in fact a geometric space: a

Table 1. Glossary of categories and their duals

Notation	Objects	Morphisms	Dual
C^*	C^* -algebras	*-homomorphisms	
uC^*	Unital C^* -algebras	Unital *-homomorphisms	
ucC^*	Unital commutative C^* -algebras	Unital *-homomorphisms	KHaus
vNA	von Neumann algebras	Ultraweakly continuous (or normal) unital *-homomorphisms	
$cvNA$	Commutative von Neumann algebras	Ultraweakly continuous (or normal) unital *-homomorphisms	HStonean
BA	Boolean algebras	Boolean algebra homomorphism	Stone
cBA	Complete Boolean algebras	Complete Boolean algebra homomorphisms	Stonean
$caBA$	Complete atomic Boolean algebras	Complete Boolean algebra homomorphisms	Set
Set	Sets	Functions	$caBA$
Top	Topological spaces	Continuous functions	
KHaus	Compact Hausdorff spaces	Continuous functions	ucC^*
Stone	Stone spaces	Continuous functions	BA
Stonean	Stonean spaces	Open continuous functions	cBA
HStonean	Hyperstonean spaces	Open continuous functions	$cvNA$
Ab	Abelian groups	Group homomorphisms	
AbMon	Abelian monoids	Monoid homomorphisms	
CRng	Commutative rings	Ring homomorphisms	
CMSLat	Complete lattices	Complete meet-semilattice homomorphisms (meet-preserving functions)	
Cat	Small categories	Functors	

Poisson manifold. This justifies the use of the terminology *state space*. Any smooth real-valued map on this manifold can be taken to represent an observable quantity and, taken together, these maps form a commutative algebra with pointwise operations. In this case, the Poisson bracket provides the additional structure of a Lie algebra. Hence, we refer to the *algebra of observables*.

In the above example, predictions for the outcomes of experiments are deterministic and observables are explicitly represented as quantity-valued functions on the state space. However, the fact that a pairing of a state with an observable results in a quantity means that fixing a state yields a quantity-valued function on the collection of observables. Identifying a state with the function on observables it defines allows realising the state space as a space of functions from the algebra of observables to an algebra of quantities.

This perspective is common in duality theory. The simplest example is the Stone-type duality between the categories **Set** of sets and functions and $caBA$ of complete atomic Boolean algebras and complete Boolean algebra homomorphisms [97]. In one direction, a functor maps a set S to the Boolean algebra $\text{Hom}_{\text{Set}}(S, 2)$ of functions to $2 = \{0, 1\}$ and maps a function $f: S \rightarrow T$ to the $caBA$ -morphism $f^*: \text{Hom}_{\text{Set}}(T, 2) \rightarrow \text{Hom}_{\text{Set}}(S, 2)$ given by $f^*(g) = g \circ f$. Similarly, in the opposite direction, one can use the functor $\text{Hom}_{caBA}(-, 2)$, where 2 is the two-element Boolean algebra, to complete the duality of these categories. This establishes a (dual) equivalence between a category of geometric objects—sets can be seen as trivial geometries with no structure beyond cardinality—and a category of algebraic objects.

A duality of the same form—defined by Hom functors to a dualising object 2 —exists between the categories **Stone** of Stone spaces and continuous functions and **BA** of Boolean algebras and Boolean algebra homomorphisms [90,92,93]; see [46] as an elementary reference and [63] for more general forms of this duality. The geometric

nature of Stone spaces, which are particular kinds of topological spaces, is clearer in this instance. This example also demonstrates a logical form of duality between semantics and syntax: the algebraic category of Boolean algebras can be seen as the category of propositional theories whereas the geometric category of Stone spaces is the category of corresponding spaces of two-valued models [10,97].

A classic example of geometric-algebraic duality, which informs Sect. 6, is that between unital commutative rings and affine schemes [53]. Given such a ring R , one can define a topological space $\text{Spec}R$, called the prime spectrum (or just spectrum) of R , whose points are the prime ideals of R and whose open sets are indexed by the ideals of R . One can then define a sheaf of commutative rings on $\text{Spec}R$ such that the stalk at a prime ideal p is the localisation of R at p , turning $\text{Spec}R$ into a locally ringed space. The locally ringed spaces that arise in this way are called affine schemes. The commutative ring giving rise to an affine scheme can be recovered by taking the ring of global sections of the scheme. In this way, a geometric dual to the category of unital commutative rings is constructed and geometric tools and reasoning can be brought to bear on subjects that make use of commutative rings, such as number theory. Many other examples of geometric-algebraic dualities can be found; see [67].

The most important example for our purposes is the Gel'fand–Naimark duality between the category \mathbf{KHaus} of compact Hausdorff spaces and continuous functions and the category \mathbf{ucC}^* of unital commutative C^* -algebras and unital $*$ -homomorphisms [44]. Under this duality, a space X is mapped to the unital commutative C^* -algebra $C(X)$ of all the continuous complex-valued functions on X . The reversal of this process—going from a commutative algebra \mathcal{A} to the topological space whose algebra of functions is \mathcal{A} —is accomplished by the Gel'fand spectrum functor Σ . The points of the space $\Sigma(\mathcal{A})$ are the characters of \mathcal{A} , i.e. unital homomorphisms from \mathcal{A} to \mathbb{C} , with topology given by pointwise convergence (the weak- $*$ topology). So, similarly to the Stone dualities discussed above, Gel'fand–Naimark duality arises from Hom functors to a dualising object: in this case, \mathbb{C} .³ $\text{Hom}_{\mathbf{ucC}^*}(\mathcal{A}, \mathbb{C})$ is topologised by pointwise convergence, using the topology of \mathbb{C} ; $\text{Hom}_{\mathbf{Top}}(X, \mathbb{C})$ inherits (pointwise) the algebraic structure from \mathbb{C} and is given the uniform norm.⁴

Gel'fand–Naimark duality has a clear interpretation as a state-observable duality. The objects of the geometric category can be seen as state spaces of classical systems. Observables, in this analogy, are the continuous real-valued functions on the state space, i.e. the self-adjoint elements of the algebra of observables. The Gel'fand spectrum functor recovers the pure state space from the algebra of observables. We attribute a classical nature to these models since states are associated with well-defined values for all observables simultaneously.

Von Neumann algebras constitute an important special class of C^* -algebras. The topological spaces that arise as Gel'fand spectra of commutative von Neumann algebras are hyperstonean spaces [26,48]. These are extremally disconnected compact Hausdorff (or Stonean) spaces with sufficiently many positive normal measures; see [96, Definition III.14] for more details. The appropriate notion of morphism when dealing with von Neumann algebras is that of ultraweakly continuous (or normal) unital $*$ -homomorphisms. Corresponding to such morphisms between commutative von Neumann algebras are open continuous maps between their spectra. Thus, Gel'fand–Naimark duality restricts

³ There is also a real version of this duality, with \mathbb{R} as the dualising object [63].

⁴ Note that \mathbb{C} is not in fact a compact Hausdorff space, and thus does not live in \mathbf{KHaus} . However, this duality can be extended to one between locally compact Hausdorff spaces and (not-necessarily-unital) commutative C^* -algebras.

to a duality between the categories cvNA of commutative von Neumann algebras and ultraweakly continuous (or normal) unital $*$ -homomorphisms and HStonean of hyperstonean spaces and morphisms of Stonean spaces, i.e. open continuous functions. See e.g. [96, Chapter III.1] for the objects part of this duality and [71, Lecture 14] for the morphisms.

In all these instances, our algebraic categories consist of objects with commutative operations. In quantum theory, the model of a system is specified by a noncommutative algebra of observables. Understanding the geometric duals of these objects is essential to completing our understanding of how quantum mechanics revises the nature of classical theories and, in particular, notions of states of systems. It is also a fundamental question of purely mathematical interest.

2.2. Contextuality: the Bell–Kochen–Specker theorem. This theorem establishes that quantum mechanics is contextual in the sense that it does not admit a hidden variable model where (hidden) ontic states ascribe consistent values to all observables simultaneously, independent of the method of observation, i.e. of which other observables are measured together with some observable. In fact, it shows that it is not possible to construct even a single such consistent deterministic ontic state.

Suppose we have a quantum system modelled by a (noncommutative) von Neumann algebra. Observables, or measurements, are given by the self-adjoint operators. An ontic state as described above would then yield a valuation:

Definition 2.1. A valuation on a von Neumann algebra \mathcal{A} is a map v from the self-adjoint operators of \mathcal{A} to \mathbb{R} such that $v(1) = 1$ and for any pair of commuting observables A and B , $v(A + B) = v(A) + v(B)$ and $v(AB) = v(A)v(B)$.

These conditions are necessary for such a potential hidden state to be consistent with a quantum state in the sense that it does not predict the occurrence of any impossible events. Note that when A and B are two commuting observables, then $A + B$ and AB also commute with both A and B . Therefore, in this case, A , B , AB , and $A + B$ can all be measured together. Upon performing these measurements on any quantum state, the obtained joint outcomes satisfy the functional relations above.⁵

Theorem 2.2 (Bell–Kochen–Specker [12, 68]). *No valuations exist on $\mathcal{B}(\mathcal{H})$ if $\dim \mathcal{H} > 2$.*

Observe that, restricted to projections, a valuation is a map that takes the values 0 or 1 and is additive on sets of orthogonal projections. Kochen and Specker proved that such a valuation on projections is impossible to construct by providing a collection W of 117 vectors in a Hilbert space of dimension 3 such that no subset of W intersects each orthogonal triple in W precisely once.

The result is extended to all separable von Neumann algebras without summands of type I_1 or I_2 in [27], showing that no valuations exist for quantum systems described by algebras of observables of these kinds.

The study of contextuality has enjoyed a revival in recent times. A number of abstract formalisms to study contextuality in general physical theories have been developed

⁵ In some presentations (such as [81] for $\mathcal{B}(\mathcal{H})$ or [27, Lemma 6] for general von Neumann algebras), the sum and product rules in the definition of valuations are derived from a different assumption, the functional composition principle (or FUNC principle). This states the requirement that $v(f(A)) = f(v(A))$ for a class of functions $f: \mathbb{R} \rightarrow \mathbb{R}$, which in the case of valuations on general von Neumann algebras is taken to be that of Borel functions.

recently [1, 2, 17, 91]. Also, recent work suggests that it might be considered a resource conferring advantage in computational and information-processing tasks [59, 80].

2.3. The spectral presheaf. The research programme known as the topos approach to quantum theory aims to achieve a reformulation of quantum theory that resembles the classical picture as closely as possible, but taking contextuality as a central feature. The central idea is to study a quantum system via its *contexts* or *classical perspectives*.

Formally, a context may be taken to correspond to a commutative subalgebra of the algebra of observables. Physically, this represents a set of properties that can be simultaneously measured with one experimental procedure. The idea to consider quantum systems via classical contexts has a long history, in a sense going back to Bohr [15], and appearing explicitly in Edwards [35].

Regarding contextuality as a central aspect of quantum mechanics, Isham and Butterfield [52, 60–62] proposed the use of presheaves to assign data to these contexts and glue it together in a consistent way in order to achieve a full description of a quantum system via the pasting of all its partial classical perspectives. This idea was further developed by Döring and Isham [31–34] and, along somewhat different lines, by Heunen et al. [54–56]. The topos approach to quantum mechanics suggests a candidate geometric object to take the role of the state space in analogy to the classical case: the spectral presheaf. This object collects the classical partial state spaces of commutative subalgebras along with morphisms used to consistently relate data from different classical perspectives.

Definition 2.3. A *context* of a von Neumann algebra \mathcal{A} is a commutative sub-von Neumann algebra of \mathcal{A} . The *context category* $\mathbf{C}(\mathcal{A})$ is the subcategory of commutative von Neumann algebras whose objects are the contexts of \mathcal{A} and whose morphisms are the inclusion maps between them.

For every context V , the Gel'fand spectrum functor can be used to construct a sample space $\Sigma(V)$ whose points represent the possible outcomes for a measurement procedure jointly measuring all the observables in V . The elements o of $\Sigma(V)$ are functions assigning real numbers to the observables in V while preserving addition and multiplication. These conditions are easily justified on physical grounds and are sufficient to guarantee that o assigns to a self-adjoint operator a value from its spectrum. As explained in Sect. 2.1, this collection of functions comes equipped with an extremally disconnected compact Hausdorff topology (in fact, a hyperstonean topology) coming from pointwise convergence, which is discrete in the case that \mathcal{A} is finite-dimensional.

Definition 2.4 (Spectral presheaf). Let \mathcal{A} be a von Neumann algebra. Its *spectral presheaf* is the functor of type $\mathbf{C}(\mathcal{A})^{\text{op}} \rightarrow \mathbf{HStonean}$ that maps each object and morphism of $\mathbf{C}(\mathcal{A})$ to its image under the Gel'fand spectrum functor.

An inclusion map $\iota: V \hookrightarrow V'$ corresponds to a coarse-graining, i.e. the context V represents a procedure measuring a subset of the observables measured by the procedure represented by V' . The image under the Gel'fand spectrum functor of such an inclusion, $\Sigma(\iota): \Sigma(V') \rightarrow \Sigma(V)$, acts by restriction: an outcome map $o \in \Sigma(V')$ is taken to $o|_V$.

Accordingly, a global section of the spectral presheaf of \mathcal{A} is a choice of $o_V \in \Sigma(V)$ for all contexts V of \mathcal{A} such that $o_V = o_{V'}|_V$ whenever $V \subset V'$. Therefore, the Bell–Kochen–Specker theorem can be reformulated in terms of the spectral presheaf:

Theorem 2.5 ([27, 52]). *Suppose \mathcal{A} is a separable von Neumann algebra without type I_1 or I_2 summands. Then its spectral presheaf has no global sections.*

This is simply a reformulation of the Bell–Kochen–Specker theorem for these von Neumann algebras [27], since valuations on a von Neumann algebra \mathcal{A} correspond to global sections of its spectral presheaf.

Thus, the impossibility of providing a mathematical model in the classical sense for quantum theory is expressed by constructing a geometric object associated to a quantum system via collating the sample spaces associated to contexts, linked by a simple consistency condition related to coarse-graining, and by demonstrating that said object possesses no ‘global points’.

These geometries represented by spectral presheaves do, however, possess global probability distributions. Remarkably, these distributions are in correspondence with (possibly mixed) quantum states. Just as the lack of points of spectral presheaves is equivalent to a landmark theorem of quantum foundations, the Bell–Kochen–Specker theorem, the correspondence between distributions on spectral presheaves and quantum states is equivalent to Gleason’s theorem [18, 47, 100, 101] (see also [50, 73]). This observation was first made by de Groote [22] and is succinctly expressed using the framework described in this article (see Sect. 4.3).

2.4. The noncommutative geometry of C^ -algebras.* Noncommutative geometry is the mathematical study of noncommutative algebras by the extension of geometric tools that have been rephrased in the language of commutative algebra to the noncommutative setting [67]. Given a duality between geometric objects and commutative algebras, such as Gelfand–Naïmark duality, we can rephrase geometric concepts by expressing them algebraically in terms of functions. For example, if we wish to algebraically express the idea of an open set of a (locally compact Hausdorff) topological space X , we might think about the set of functions that vanish outside of it and note that this is an ideal of $C(X)$. In fact, there is a bijective correspondence between closed ideals of $C(X)$ and open sets of X . As a more complicated example, the Serre–Swan theorem [94] allows us to identify vector bundles over X with finitely generated projective $C(X)$ -modules. Remarkably, these algebraic descriptions of geometric concepts do not crucially rely on the commutativity of $C(X)$. Therefore, one can generalise geometric tools and intuition to noncommutative algebras \mathcal{A} by using these same algebraic descriptions. This justifies thinking of a noncommutative C^* -algebra as a *noncommutative (locally compact Hausdorff) topological space*. The elements of the C^* -algebra \mathcal{A} are thought of as continuous complex-valued functions on a metaphorical noncommutative space. Such a space defies explicit description by conventional mathematical ideas about what a space is; for example, it cannot be thought of as a collection of points, for such an object always has a commutative algebra of functions.

One of the best examples of an extension of a topological tool to the setting of noncommutative spaces is that of K -theory. The isomorphism classes of vector bundles over a compact Hausdorff space X form a semigroup under direct sum and $K(X)$ is defined to be the Grothendieck group of this semigroup. The K -functor is an important cohomological invariant in the study of topology. By using the geometry-to-algebra dictionary described above, one defines an extension of K to unital C^* -algebras \mathcal{A} in terms of equivalence classes of finitely generated projective \mathcal{A} -modules; in this way, the operator K_0 -functor is constructed. It is an extension of K in the sense that when \mathcal{A} is commutative, i.e. $\mathcal{A} \simeq C(X)$ for a space X , then $K_0(\mathcal{A}) \simeq K(X)$. In this way, one obtains a powerful invariant of C^* -algebras which forms the basis of a classification programme [38]. The modern account of operator K_0 uses an equivalent formulation in terms of equivalence classes of projections in matrix algebras over \mathcal{A} [86].

Table 2. Dictionary of concepts between geometry and algebra

Geometry	Algebra
Continuous function from a space to \mathbb{C}	Element of the algebra (operator)
Continuous function from a space to \mathbb{R}	Self-adjoint element of the algebra
Range of a function	Spectrum of an operator
Open set	Closed two-sided ideal
Vector bundle	Finitely generated projective module
Cartesian product	Minimal tensor product
Disjoint union	Direct sum
Infinitesimal	Compact operator
Regular Borel probability measure	State
Integral	Trace
1-point compactification	Unitisation

With considerable effort, this process of translation from geometry to algebra yields a conceptual dictionary covering a vast terrain within mathematics. It is not just topological concepts that can be translated into the language of algebra; there are also noncommutative extensions of measure theory, differential geometry, etc. [20]; see Table 2.

2.5. Motivation. The unreasonable effectiveness of topological tools and intuition in the study of C^* -algebras suggests the existence of a deeper principle at work. The method of translating geometric ideas into algebra in order to generalise them is powerful but can be somewhat ad hoc. Ideally, one may hope for a new conception of space, of which the commutative/topological situation would be a special case, and which would constitute (the objects of) a category dual to that of (noncommutative) C^* -algebras. That is, one would be able to extend the notion of Gel'fand spectrum of a commutative algebra to the noncommutative case by assigning to an algebra \mathcal{A} such a 'space', whose set of continuous functions would be, in some sense, \mathcal{A} . As pointed out above, an explicit description of (currently imaginary) noncommutative topological spaces is very difficult since such spaces defy most contemporary ideas about mathematical spaces. It is difficult to know how to begin defining such an object. However, we can imagine that equipped with such an explicit description, should it not depart too far from the commutative situation, one could find natural and intuitive methods of extending topological tools.

Thus, our criterion for a successful explicit manifestation of noncommutative space is that it naturally leads to extensions of topological concepts that agree with well-known and useful noncommutative geometric concepts. In effect, we aim to complete the conceptual diagram of Fig. 1. This diagram requires some explanation. The top row describes the two dually equivalent mathematical formalisms for encapsulating the operational content of a classical system: the topological picture, in which states are taken as the primitive concept, and the commutative C^* -algebraic picture, in which observables are taken as primitive.

The arrows represent methods for the translation and generalisation of concepts. The Gel'fand spectrum functor allows for any notion or theorem phrased in terms of the topological structure of spaces to be translated into algebraic terms; e.g. open sets of a space become closed two-sided ideals of an algebra. Once a concept has been phrased in terms of algebra, it can be applied without modification to the noncommutative case; e.g. finitely generated projective modules of a commutative algebra (the equivalent of vector bundles) become finitely generated projective modules of a not-necessarily-commutative algebra. Thus, the composition of the top and right arrows can be seen as the usual process of generating the basic entries of the noncommutative dictionary.

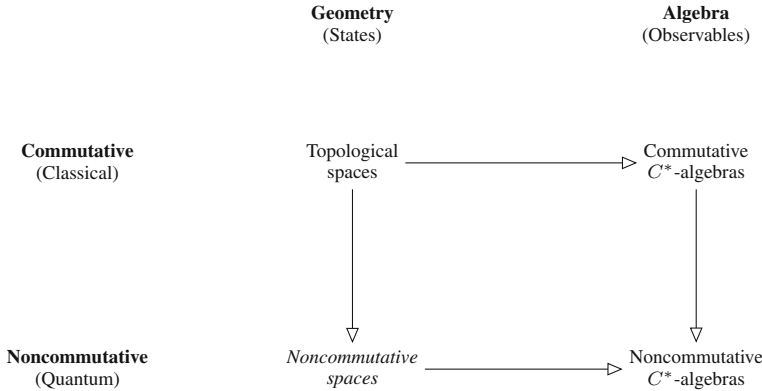


Fig. 1. Here, we give a (nonrigorous) high-level diagram representing the heuristic processes by which topological concepts are generalised to the noncommutative setting. The top and right arrows correspond to the usual method of translation: the top arrow represents translating a topological notion to an algebraic one via Gel'fand–Naïmark duality and the right arrow represents applying this algebraic definition in the noncommutative setting. Our aim is to generalise the Gel'fand spectrum functor Σ to a functor \mathbf{G} that assigns to each noncommutative algebra a *noncommutative space*. This requires proposing a candidate construction of noncommutative space that is manifestly geometric. Further, we ask that this notion of noncommutative space comes naturally equipped with processes corresponding to arrows completing the diagram—a left arrow corresponding to a way of generalising topological concepts to these noncommutatively spatial objects, and a bottom arrow corresponding to porting such (generalised) topological concepts to noncommutative algebraic ones via the new association \mathbf{G} —in such a way that reproduces the results of the usual translation process

Note, however, that there is some ambiguity in this translation process. A topological concept can be translated in several different ways, which means that intuition and judgement must be deployed when determining appropriate algebraic analogues. As a very simple example, open sets of a space X are in correspondence with both the closed left ideals of $C(X)$ and the closed two-sided ideals of $C(X)$ as these two collections are identical in the commutative case. Thus, finding a completely automatic method of translation that eliminates such ambiguities would in itself constitute an advance in the structural understanding of noncommutative geometry.

Akemann and Pedersen [79] proposed to replace the translation process by working directly with Giles–Kummer’s [45] and Akemann’s [5] noncommutative generalisations of the basic topological notions of open and closed sets. In contrast, we do not employ algebraic generalisations of basic topological notions. Instead, we work with objects that slightly generalise the notion of topological space and come readily equipped with an alternative to the translation process.

In addition to the work of Akemann–Pedersen and Giles–Kummer on noncommutative generalisations of Gel'fand–Naïmark duality, there have been a number of alternative approaches by authors including Alfsen [6], Bichteler et al. [13], Dauns–Hofmann [21], Fell [39], Heunen et al. [57], Kadison [64], Kruml et al. [69], Kruszyński–Woronowicz [70], Mulvey [78], Resende [82], Shultz [89], and Takesaki [95]. An excellent discussion of many of these works is contained in a paper by Fujimoto [43].

Our goal with this work is to find, to a first approximation, a way of completing the conceptual diagram of Fig. 1. That is, we aim to propose and study candidate definitions of a category $\mathbf{NCSpaces}$ of *noncommutative spaces* and a generalised Gel'fand spectrum functor $\mathbf{G}: \mathbf{uC}^{*\text{OP}} \rightarrow \mathbf{NCSpaces}$ acting on the category \mathbf{uC}^* of *all* unital C^* -algebras and unital $*$ -homomorphisms. The first motivation is to provide a geometric

manifestation for a notion of noncommutative space (namely, the quantum state space described above) whose existence is currently understood as being merely metaphorical. The second is to exploit this geometric manifestation to obtain a canonical method for importing concepts of topology to noncommutative algebra.

The primary desiderata of a guess for a notion of noncommutative space is that it comes equipped with: (i) a natural method of generalising notions from topology (that is, a left arrow in the informal diagram), and (ii) a functorial association of such a generalised ‘space’ $\mathbf{G}(\mathcal{A})$ to each noncommutative algebra \mathcal{A} , which provides a way of translating generalised topological concepts to noncommutative algebras by applying them to the corresponding noncommutative ‘space’ (i.e. a bottom arrow). That the composition of these two translations match the usual noncommutative dictionary would justify thinking of $\mathbf{G}(\mathcal{A})$ as the geometric manifestation of a noncommutative algebra \mathcal{A} .

Our proposal for a notion of noncommutative space and a functor \mathbf{G} , as inspired by Isham and Butterfield’s work, is to consider diagrams of topological spaces associated to contexts. Our primary mathematical objectives are two-fold. First, we argue that a necessary desideratum of a proposed geometric interpretation of a noncommutative algebra is a precise account of the relationship between topological concepts and their noncommutative analogues. Indeed, this requirement will be critical for fixing the structure of our diagrams—specifically, the class of morphisms that are included. Second, we aim to provide evidence that this is achievable.

We do not necessarily expect that this will immediately yield a full categorical duality, i.e. a concrete category equivalent to $\mathbf{uC}^{*\text{op}}$, but rather stimulate progress towards that goal—or towards a better understanding of the obstacles to that goal. Finding such a full duality would require characterising the objects and morphisms of $\mathbf{NCSpaces}$ that are in the image of a functor \mathbf{G} and establishing that that \mathbf{G} is faithful and injective on objects so that its image is a category and \mathbf{G} an equivalence onto it. Note that the particular first approximations to \mathbf{G} that we consider in this article—which associate diagrams of topological spaces to a noncommutative algebra—are faithful but not full, and in particular not essentially injective.⁶ Instances of noncommutative concepts that lack a commutative analogue could provide guidance on which additional data, such as a group action, one might need to take into account when defining a \mathbf{G} to achieve a categorical duality. The failure of a particular guess for \mathbf{G} to be essentially injective may also provide such guidance; however, one might also be open to the possibility that nonisomorphic algebras contain equivalent topological information and thus correspond to the same (or homeomorphic) noncommutative space.

Even without a complete categorical duality for C^* -algebras, the perspective outlined in this article may prove to be useful. The extent to which noncommutative geometry can be understood directly in topological terms is a wide open—and, in our estimation, interesting—question.

The framework of extensions, developed in Sect. 4, formalises how certain ways of associating diagrams of topological spaces to noncommutative algebras come with left and bottom arrows, and in this way yield a noncommutative counterpart for every topological concept. In Sect. 5, we determine the appropriate \mathbf{G} such that the associated extension of topological K -theory essentially matches the established noncommutative K -theory. In Sect. 6, as a verification of this candidate construction of \mathbf{G} , we conjecture that it can be used to extend the topological notion of open set to that of closed two-

⁶ This follows from the existence of a C^* -algebra \mathcal{A} nonisomorphic to its opposite algebra [19]. As both \mathcal{A} and \mathcal{A}^{op} have the same commutative sub- C^* -algebras (contexts), both will be assigned identical diagrams of topological spaces.

sided ideal, and we prove the analogue of this conjecture in the setting of von Neumann algebras.

3. Spatial Diagrams

We introduce the technical machinery necessary for contravariantly functorially associating diagrams of topological spaces, describing quotient spaces of a ‘noncommutative space’, to noncommutative operator algebras. We consider functors that associate to a unital C^* -algebra a diagram whose objects are spectra of contexts and whose morphisms are chosen to yield a natural method, described in the next section, of extending functors that act on compact Hausdorff spaces to functors acting on all unital C^* -algebras. An analogous method is applicable to extending functors that act on hyperstonean spaces to functors acting on all von Neumann algebras.

3.1. The categories of all diagrams in \mathbf{C} . We propose to associate to each unital C^* -algebra \mathcal{A} a diagram of topological spaces whose objects are the spectra of the unital commutative sub- C^* -algebras of \mathcal{A} . Given that this association should generalise the Gelfand spectrum functor, we would naturally expect it to be contravariantly functorial.

Typically, one thinks of a diagram $D: \mathbf{A} \rightarrow \mathbf{C}$ in a category \mathbf{C} as living inside the functor category $\mathbf{C}^{\mathbf{A}}$. This is inadequate for our purposes as different algebras have different sets of commutative subalgebras and will thus be mapped to diagrams of different shapes. We introduce a general construction that allows considering diagrams of different shapes on the same footing.

Definition 3.1. For any category \mathbf{C} , $\underline{\text{Diag}}(\mathbf{C})$, the covariant category of all diagrams in \mathbf{C} has as objects all the functors from any small category to \mathbf{C} ; and the morphisms from a diagram $D: \mathbf{A} \rightarrow \mathbf{C}$ to a diagram $E: \mathbf{B} \rightarrow \mathbf{C}$ are pairs (f, η) where $f: \mathbf{A} \rightarrow \mathbf{B}$ is a functor and η is a natural transformation from D to $E \circ f$.

The composition $(g, \mu) \circ (f, \eta)$ of two $\underline{\text{Diag}}(\mathbf{C})$ -morphisms

$$(f, \eta): D_1 \rightarrow D_2 \quad \text{and} \quad (g, \mu): D_2 \rightarrow D_3$$

is given by $(g \circ f, (\mu_f) \circ \eta)$ where μ_f is the natural transformation from $D_2 \circ f$ to $D_3 \circ g \circ f$ given by $(\mu_f)_a = \mu_{f(a)}$.

The contravariant category of all diagrams in \mathbf{C} , $\overleftarrow{\text{Diag}}(\mathbf{C})$, is the category $\underline{\text{Diag}}(\mathbf{C}^{\text{op}})^{\text{op}}$. Its objects are all contravariant functors from a small category to \mathbf{C} ; and the morphisms from a diagram $D: \mathbf{A}^{\text{op}} \rightarrow \mathbf{C}$ to a diagram $E: \mathbf{B}^{\text{op}} \rightarrow \mathbf{C}$ are pairs (f, η) where $f: \mathbf{B} \rightarrow \mathbf{A}$ is a functor and η is a natural transformation from $D \circ f$ to E .

The categories defined above can be constructed by considering the colax-slice and lax-slice 2-categories Cat/\mathbf{C} [88] and forgetting the 2-categorical structure.

Note that a functor $F: \mathbf{C} \rightarrow \mathbf{C}'$ naturally induces a functor from $\underline{\text{Diag}}(\mathbf{C})$ to $\underline{\text{Diag}}(\mathbf{C}')$, which we will also denote by F . Explicitly, if $D: \mathbf{A} \rightarrow \mathbf{C}$, then $F(D)$ is simply $F \circ D$, while a $\underline{\text{Diag}}(\mathbf{C})$ -morphism (f, η) is sent to the $\underline{\text{Diag}}(\mathbf{C}')$ -morphism $(f, F\eta)$ where $(F\eta)_a = F(\eta_a)$. In a similar fashion, the functor F also induces a functor $F: \overleftarrow{\text{Diag}}(\mathbf{C}) \rightarrow \overleftarrow{\text{Diag}}(\mathbf{C}')$. Note that, for contravariant functors, we get the following: a functor $G: \mathbf{C}'^{\text{op}} \rightarrow \mathbf{C}$ induces functors $G: \overleftarrow{\text{Diag}}(\mathbf{C})^{\text{op}} \rightarrow \overleftarrow{\text{Diag}}(\mathbf{C}')$

and $G: \underline{\text{Diag}}(\mathbf{C})^{\text{op}} \longrightarrow \underline{\text{Diag}}(\mathbf{C}')$, since $\underline{\text{Diag}}(\mathbf{C}^{\text{op}}) = \underline{\text{Diag}}(\mathbf{C})^{\text{op}}$ and $\underline{\text{Diag}}(\mathbf{C}^{\text{op}}) = \underline{\text{Diag}}(\mathbf{C})^{\text{op}}$.

3.2. Semispectral functors. Having defined a category that simultaneously accommodates diagrams of varying shape, we are ready to begin defining our contravariantly functorial associations of diagrams of topological spaces to unital C^* -algebras. We will define a class of such functorial associations. What all these contravariant functors from the category of unital C^* -algebras to diagrams of compact Hausdorff spaces have in common is that they will associate to each unital C^* -algebra a diagram (i.e. a functor) whose domain is a subcategory of the category of unital commutative C^* -algebras. In fact, in each case, the objects of the domain subcategory of the diagram associated to a C^* -algebra are its unital commutative sub- C^* -algebras. The class of morphisms in this domain subcategory, however, will be allowed to vary. There is an analogous version for von Neumann algebras where one considers only their commutative sub-von Neumann algebras; this will also be of interest to us.

Our motivating example is the spectral presheaf (see Definition 2.4). The recipe for its construction, which we aim to generalise, is as follows:

1. take a von Neumann algebra \mathcal{A} ;
2. consider the subcategory $\mathbf{C}(\mathcal{A})$ of cvNA whose objects are the commutative sub-von Neumann algebras (contexts) of \mathcal{A} and whose morphisms are the inclusions between such subalgebras;
3. consider the inclusion functor $i_{\mathbf{C}(\mathcal{A})}$ of $\mathbf{C}(\mathcal{A})$ in cvNA : this is an object of $\underline{\text{Diag}}(\text{cvNA})$;
4. compose the (von Neumann) Gel'fand spectrum functor $\Sigma: \text{cvNA}^{\text{op}} \longrightarrow \text{HStonean}$ with this inclusion functor to yield an object of $\underline{\text{Diag}}(\text{HStonean})$, i.e. a functor $\mathbf{C}(\mathcal{A})^{\text{op}} \longrightarrow \text{HStonean}$, called the spectral presheaf of \mathcal{A} .

This association of spectral presheaves to von Neumann algebras can be made functorial in a natural way. Given an ultra-weakly continuous, or normal, unital $*$ -homomorphism $\phi: \mathcal{A} \longrightarrow \mathcal{B}$, we can define a $\underline{\text{Diag}}(\text{cvNA})$ -morphism (f, η) as follows: $f: \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{B})$ sends a commutative sub-von Neumann algebra V of \mathcal{A} to the commutative sub-von Neumann algebra $\phi(V)$ of \mathcal{B} , and an inclusion $V \hookrightarrow V'$ to the inclusion $\phi(V) \hookrightarrow \phi(V')$; while $\eta: i_{\mathbf{C}(\mathcal{A})} \longrightarrow i_{\mathbf{C}(\mathcal{B})} \circ f$ is the natural transformation with components η_V defined to be $\phi|_V: V \longrightarrow \phi(V)$. This yields a functor $s: \text{vNA} \longrightarrow \underline{\text{Diag}}(\text{cvNA})$, which will be our first example of a (von Neumann algebraic) semispectral functor. The Gel'fand spectrum functor for von Neumann algebras, $\Sigma: \text{cvNA}^{\text{op}} \longrightarrow \text{HStonean}$, lifts to a functor from $\Sigma: \underline{\text{Diag}}(\text{cvNA})^{\text{op}} \longrightarrow \underline{\text{Diag}}(\text{HStonean})$, mapping (f, η) to a $\underline{\text{Diag}}(\text{HStonean})$ -morphism between the spectral presheaves of \mathcal{B} and of \mathcal{A} . Overall, this yields a functor $\Sigma \circ s: \text{vNA}^{\text{op}} \longrightarrow \underline{\text{Diag}}(\text{HStonean})$.

We will generalise this recipe to unital C^* -algebras. However, we will also want to consider other choices of morphisms to include in our diagrams. In the next section, we see that certain ways of associating diagrams of spaces to algebras automatically yield a method for extending topological functors to functors that act on all unital C^* -algebras. The family of morphisms we include in our diagrams determines the resulting method of extensions. Thus, we will vary the family of morphisms in order to determine the one whose method of extending functors matches up with the canonical generalisation

process of noncommutative geometry. This was the motivation behind the reconstruction of the definition of operator K -theory.

Definition 3.2. A functor $\sigma : \mathbf{uC}^* \rightarrow \underline{\text{Diag}}(\mathbf{uC}^*)$ is called *semispectral* if:

1. For any unital C^* -algebra \mathcal{A} , $\sigma(\mathcal{A})$ is the inclusion functor of a subcategory $\text{dom}(\sigma(\mathcal{A}))$ of \mathbf{uC}^* whose objects are unital commutative sub- C^* -algebras of \mathcal{A} ;
2. For a unital $*$ -homomorphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$, $\sigma(\phi)$ is the $\underline{\text{Diag}}(\mathbf{uC}^*)$ -morphism $(f, \eta) : \sigma(\mathcal{A}) \rightarrow \sigma(\mathcal{B})$, where $f : \text{dom}(\sigma(\mathcal{A})) \rightarrow \text{dom}(\sigma(\mathcal{B}))$ takes a unital commutative sub- C^* -algebra $V \subset \mathcal{A}$ to the unital commutative sub- C^* -algebra $\phi(V) \subset \mathcal{B}$, and η is the natural transformation with components η_V being the unital $*$ -homomorphisms $\phi|_V : V \rightarrow \phi(V)$;
3. If \mathcal{A} is commutative, then it is the terminal object of $\text{dom}(\sigma(\mathcal{A}))$.

Similarly, a functor $\sigma : \mathbf{vNA} \rightarrow \underline{\text{Diag}}(\mathbf{cvNA})$ is called *semispectral* if the analogous conditions hold, with “sub-von Neumann algebras” and “normal unital $*$ -homomorphisms” substituted as appropriate.

The third condition will be required below to ensure agreement in the commutative case between a functor and its extension.

3.3. Spatial diagrams. Our primary objects of study will be spatial diagrams, which are ways of associating diagrams of topological spaces to unital C^* -algebras (or von Neumann algebras), determined by a semispectral functor. Given a semispectral functor, the corresponding spatial diagram functor is obtained via Gel'fand–Naiřmark duality:

Definition 3.3. Given a semispectral functor $\sigma : \mathbf{uC}^* \rightarrow \underline{\text{Diag}}(\mathbf{uC}^*)$, its corresponding *spatial diagram functor* $G_\sigma : \mathbf{uC}^{*\text{op}} \rightarrow \underline{\text{Diag}}(\mathbf{KHaus})$ is given as

$$G_\sigma = \Sigma \circ \sigma : \mathbf{uC}^* \rightarrow \underline{\text{Diag}}(\mathbf{uC}^*) \rightarrow \underline{\text{Diag}}(\mathbf{KHaus})^{\text{op}},$$

where Σ is the Gel'fand spectrum functor $\mathbf{uC}^{*\text{op}} \rightarrow \mathbf{KHaus}$ lifted to diagrams.

Analogously, given a (von Neumann) semispectral functor $\sigma : \mathbf{vNA} \rightarrow \underline{\text{Diag}}(\mathbf{cvNA})$, its corresponding spatial diagram functor $G_\sigma : \mathbf{vNA}^{\text{op}} \rightarrow \underline{\text{Diag}}(\mathbf{HStonean})$ is given as

$$G_\sigma = \Sigma \circ \sigma : \mathbf{vNA} \rightarrow \underline{\text{Diag}}(\mathbf{cvNA}) \rightarrow \underline{\text{Diag}}(\mathbf{HStonean})^{\text{op}},$$

where Σ is the (von Neumann) Gel'fand spectrum functor $\mathbf{cvNA}^{\text{op}} \rightarrow \mathbf{HStonean}$ lifted to diagrams.

As explained in the previous section, the first example (for von Neumann algebras) of a spatial diagram functor is the spectral presheaf functor, which is obtained from the semispectral functor $s : \mathbf{vNA} \rightarrow \underline{\text{Diag}}(\mathbf{cvNA})$ that sends a von Neumann algebra to the diagram consisting of its von Neumann subalgebras and inclusions between them. An analogous definition of spectral presheaf can also be given for C^* -algebras.

For our main results, however, we will need to consider other semispectral functors (and corresponding spatial diagram functors), which also take into account unitary equivalences between subalgebras. We now give these definitions.

Definition 3.4. Given a unital C^* -algebra \mathcal{A} , its *unitary subcategory* $\mathbf{S}(\mathcal{A})$ of \mathbf{ucC}^* has as objects the unital commutative sub- C^* -algebras of \mathcal{A} , and as morphisms the restrictions of inner automorphisms of \mathcal{A} . That is, the morphisms between two unital commutative sub- C^* -algebras $V, V' \subset \mathcal{A}$ are precisely those $\text{Ad}_u|_{V'}^V: V \rightarrow V'$ of the form $\text{Ad}_u|_{V'}^V(v) = uvu^*$ for some unitary $u \in \mathcal{A}$ such that $uVu^* \subset V'$.

The composition of two such morphisms is given as conjugation by the product of their respective unitaries, which is also a unitary, and so $\mathbf{S}(\mathcal{A})$ is indeed a subcategory of \mathbf{ucC}^* . Note that any morphism $\text{Ad}_u|_{V'}^V: V \rightarrow V'$ can be decomposed as $\iota \circ r$ where r is the isomorphism $\text{Ad}_u|_V^{uVu^*}$ between V and uVu^* defined by conjugation by u and where ι is the inclusion $uVu^* \hookrightarrow V'$.

Definition 3.5. The *unitary semispectral functor* $g: \mathbf{ucC}^* \rightarrow \underline{\text{Diag}}(\mathbf{ucC}^*)$ sends a unital C^* -algebra \mathcal{A} to the inclusion functor $i_{\mathbf{S}(\mathcal{A})}: \mathbf{S}(\mathcal{A}) \rightarrow \mathbf{ucC}^*$. The action of g on unital $*$ -morphisms is fixed by Condition 2 in Definition 3.2: given a unital $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$, its image $g(\phi)$ is (f, η) where $f: \mathbf{S}(\mathcal{A}) \rightarrow \mathbf{S}(\mathcal{B})$ is the functor taking a unital commutative sub- C^* -algebra $V \subset \mathcal{A}$ to $\phi(V) \subset \mathcal{B}$ and η is the natural transformation whose component at V is the unital $*$ -homomorphism $\phi|_V: V \rightarrow \phi(V)$.

We denote by G the corresponding spatial diagram functor $G_g = \Sigma \circ g: \mathbf{ucC}^{*\text{op}} \rightarrow \underline{\text{Diag}}(\mathbf{KHaus})$.

Note that when \mathcal{A} is commutative, the morphisms in $\mathbf{S}(\mathcal{A})$ are simply the inclusions, which is why Condition 3 of Definition 3.2 holds.

The topological spaces in the diagram $G(\mathcal{A})$ should be thought of as being those that arise as quotient spaces of the hypothetical noncommutative space underlying \mathcal{A} . To see this, note that a sub- C^* -algebra V of $C(X)$ yields an inclusion $\iota: V \hookrightarrow C(X)$ which corresponds to a continuous surjection $\Sigma(\iota): X \rightarrow \Sigma(V)$. This surjection is a quotient map since both spaces are compact and Hausdorff [98, p. 12]. Thus, in accordance with the central tenet of noncommutative geometry, unital sub- C^* -algebras of a unital noncommutative algebra \mathcal{A} are to be understood as having an underlying noncommutative space that is a quotient space of the noncommutative space underlying \mathcal{A} . By considering only the commutative subalgebras, we are restricting our attention to the tractable quotient spaces: those that are genuine topological spaces. The morphisms of the diagram serve to track how these quotient spaces fit together inside the larger noncommutative space.

We will require in our analysis of operator K_0 a slight modification of the unitary subcategory:

Definition 3.6. Given a unital C^* -algebra \mathcal{A} , its *finitary unitary subcategory* $\mathbf{S}_f(\mathcal{A})$ of \mathbf{ucC}^* has as objects the unital *finite-dimensional* commutative sub- C^* -algebras of \mathcal{A} , and as morphisms the restrictions of inner automorphisms of \mathcal{A} .

This is used to define a functor g_f , which is a finitary version of the unitary semispectral functor g :

Definition 3.7. The *finitary version of the unitary semispectral functor* $g_f: \mathbf{ucC}^* \rightarrow \underline{\text{Diag}}(\mathbf{ucC}^*)$ sends a unital C^* -algebra \mathcal{A} to the inclusion functor $i_{\mathbf{S}_f(\mathcal{A})}: \mathbf{S}_f(\mathcal{A}) \rightarrow \mathbf{ucC}^*$. For a unital $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$, its image $g_f(\phi)$ is defined to be

(f, η) where $f: \mathbf{S}_f(\mathcal{A}) \rightarrow \mathbf{S}_f(\mathcal{B})$ is the functor taking a unital finite-dimensional commutative sub- C^* -algebra $V \subset \mathcal{A}$ to $\phi(V) \subset \mathcal{B}$ and η is the natural transformation whose component at V is the unital $*$ -homomorphism $\phi|_V: V \rightarrow \phi(V)$.

We denote by G_f the corresponding spatial diagram functor

$$G_{g_f} = \Sigma \circ g_f: \mathbf{uC}^{*\text{op}} \rightarrow \overleftarrow{\text{Diag}}(\mathbf{KHaus}).$$

Note that we are abusing terminology here as the functor g_f is not semispectral. In particular, it fails to obey condition 3 of Definition 3.2. Thus, strictly speaking, G_f is not a spatial diagram functor as in Definition 3.3. However, in Sect. 5, we will employ G_f just as if it were a spatial diagram functor in order to reproduce operator K -theory.

For the main result of Sect. 6, we will be dealing with von Neumann algebras only, and as such we require an analogous version of the spatial diagram functor G in this setting:

Definition 3.8. Given a von Neumann algebra \mathcal{A} , its *unitary subcategory* $\mathbf{S}_W(\mathcal{A})$ of \mathbf{cvNA} has as objects the commutative sub-von Neumann algebras of \mathcal{A} and as morphisms the restrictions of inner automorphisms of \mathcal{A} .

The (von Neumann) unitary semispectral functor $g_W: \mathbf{vNA} \rightarrow \overrightarrow{\text{Diag}}(\mathbf{cvNA})$ sends a von Neumann algebra \mathcal{A} to the inclusion functor $i_{\mathbf{S}_W(\mathcal{A})}: \mathbf{S}_W(\mathcal{A}) \rightarrow \mathbf{cvNA}$ and is defined on a normal unital $*$ -homomorphism $f: \mathcal{A} \rightarrow \mathcal{B}$ in a manner analogous to Definition 3.5.

We denote by G_W the corresponding spatial diagram functor

$$G_{g_W} = \Sigma \circ g_W: \mathbf{vNA}^{\text{op}} \rightarrow \overleftarrow{\text{Diag}}(\mathbf{HStonean}).$$

4. Extensions of Topological Functors

We give a generalisation of limit and colimit functors that act on certain functor categories to ones that act on categories of diagrams. This allows us to define the extension of a topological functor to a noncommutative algebraic one, given a semispectral functor as described in the previous section. The extension process is interpreted as decomposing a noncommutative space into tractable quotient spaces, applying a topological functor to each one, and pasting the results together. We illustrate this construction by presenting formulations of (generalised versions of) the Bell–Kochen–Specker and Gleason’s theorems in this framework.

4.1. The generalised limit and colimit functors. When a category \mathbf{C} is cocomplete, it admits a colimit functor $\overrightarrow{\text{lim}}: \mathbf{C}^A \rightarrow \mathbf{C}$ for diagrams over any fixed shape A .

A key feature of the construction of $\overrightarrow{\text{Diag}}(\mathbf{C})$ in the case where \mathbf{C} is cocomplete is the existence of a generalised colimit functor $\overrightarrow{\text{lim}}: \overrightarrow{\text{Diag}}(\mathbf{C}) \rightarrow \mathbf{C}$. It assigns to a diagram $D: A \rightarrow \mathbf{C}$ the same \mathbf{C} -object that is assigned to D by the usual colimit functor for A -shaped diagrams, of type $\mathbf{C}^A \rightarrow \mathbf{C}$. If η is a natural transformation between D and a diagram $D': A \rightarrow \mathbf{C}$ of the same shape, i.e. a \mathbf{C}^A -morphism, then the generalised $\overrightarrow{\text{lim}}$ functor maps the $\overrightarrow{\text{Diag}}(\mathbf{C})$ -morphism (id_A, η) between D and D' to the same \mathbf{C} -morphism that is assigned to η by the usual colimit functor $\mathbf{C}^A \rightarrow \mathbf{C}$. What is novel is the ability to assign \mathbf{C} -morphisms between colimits of diagrams of different shapes to $\overrightarrow{\text{Diag}}(\mathbf{C})$ -morphisms between these diagrams.

In this section, we give a concise description of the generalised colimit functor; in Appendix A, we present a more direct and explicit construction in terms of coequalisers and coproducts. Everything in this section applies equally well—that is, all dual statements hold true—when \mathbf{C} is complete, in which case we have a generalised limit functor $\overleftarrow{\lim} : \underline{Diag}(\mathbf{C}) \rightarrow \mathbf{C}$.

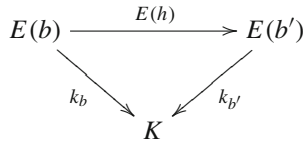
First, note that a $\underline{Diag}(\mathbf{C})$ -morphism (f, η) between diagrams $D : \mathbf{A} \rightarrow \mathbf{C}$ and $E : \mathbf{B} \rightarrow \mathbf{C}$ can be decomposed as

$$(f, \eta) = (f, id_{E \circ f}) \circ (id_{\mathbf{A}}, \eta) : D \rightarrow E \circ f \rightarrow E \tag{1}$$

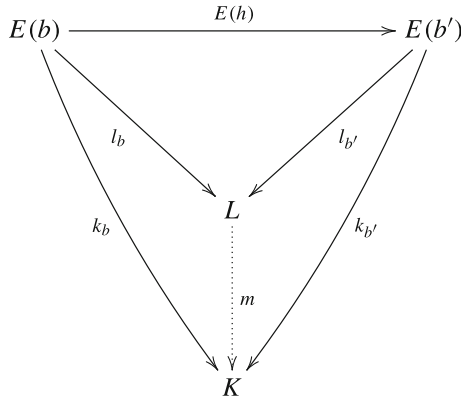
where $id_{E \circ f}$ is the identity natural transformation from the diagram $E \circ f : \mathbf{A} \rightarrow \mathbf{C}$ to itself.

We shall show in detail how to define the (functorial) action of $\overrightarrow{\lim} : \underline{Diag}(\mathbf{C}) \rightarrow \mathbf{C}$ on morphisms of the form $(f, id_{E \circ f}) : E \circ f \rightarrow E$ for $f : \mathbf{A} \rightarrow \mathbf{B}$ and $E : \mathbf{B} \rightarrow \mathbf{C}$. The action on morphisms of the form $(id_{\mathbf{A}}, \eta)$ for η a natural transformation between diagrams $D, E : \mathbf{A} \rightarrow \mathbf{C}$ simply reduces to that of the usual colimit functor $\overrightarrow{\lim} : \mathbf{C}^{\mathbf{A}} \rightarrow \mathbf{C}$ on the $\mathbf{C}^{\mathbf{A}}$ -morphism η . The action for a general morphism (f, η) is then determined by decomposition (1) and the need to obey functoriality.

Recall that a cocone of a diagram $E : \mathbf{B} \rightarrow \mathbf{C}$ is a pair (K, k) consisting of a \mathbf{C} -object K together with a natural transformation k from E to the constant diagram to the fixed object K . In other words, a cocone is an association to each \mathbf{B} -object b of a \mathbf{C} -morphism $k_b : E(b) \rightarrow K$ such that for every \mathbf{B} -morphism $h : b \rightarrow b'$ we have that $k_b = k_{b'} \circ E(h)$, i.e. the following diagram commutes:



A cocone (L, l) of E is a colimit of E if, for any other cocone (K, k) of E , there is a unique \mathbf{C} -morphism $m : L \rightarrow K$ satisfying $k_b = m \circ l_b$ for every \mathbf{B} -object b . The situation is summarised in the following diagram:



When a colimit of a diagram exists (as is the case for E since we are assuming \mathbf{C} to be cocomplete), it is unique up to isomorphism. We can therefore speak of *the* colimit of E and write $\overrightarrow{\lim}(E)$ for the object L above.

Given a $\underline{Diag}(\mathbf{C})$ -morphism of the form $(f, id_{E \circ f}): E \circ f \longrightarrow E$, we must define a \mathbf{C} -morphism from $\underline{\lim}(E \circ f)$ to $\underline{\lim}(E)$. The fact that $(\underline{\lim}(E), l)$ is a cocone for E implies that $(\underline{\lim}(E), l_f)$ is a cocone for $E \circ f$, where for an object a of \mathbf{A} , $(l_f)_a = l_{f(a)}$. The definition of colimit for the diagram $E \circ f$ provides a uniquely determined \mathbf{C} -morphism $m: \underline{\lim}(E \circ f) \longrightarrow \underline{\lim}(E)$ that maps the cocone constituting the colimit of $E \circ f$ to the cocone $(\underline{\lim}(E), l_f)$. We can therefore define $\underline{\lim}(f, id_{E \circ f})$ to be this morphism m . The universal property is then used to show functoriality of $\underline{\lim}$ on the class of morphisms of the form $(f, id_{E \circ f})$.

If we additionally have a natural transformation $\mu: E \longrightarrow E'$, by a similar argument, any cocone (K, k) of E' yields a cocone $(K, (k \circ \mu)_f)$ of $E \circ f$. In particular, this is the case for the colimit of E' , the cocone $(\underline{\lim}(E'), l')$. Universality then implies that there is a unique morphism from $\underline{\lim}(E \circ f)$ to $\underline{\lim}(E')$ that maps the cocone constituting the colimit of $E \circ f$ to the cocone $(\underline{\lim}(E'), (l' \circ \mu)_f)$ obtained as we just described. We thus conclude that

$$\underline{\lim}(id_B, \mu) \circ \underline{\lim}(f, id_{E \circ f}) = \underline{\lim}(f, id_{E' \circ f}) \circ \underline{\lim}(id_A, \mu_f), \quad (2)$$

where $\underline{\lim}$ is only being applied to morphisms of each of the two classes from decomposition (1), for which it has already been defined. Together with functoriality of the colimit on each of these two classes, this suffices to demonstrate that the entire action of the colimit is functorial: if for diagrams $D_i: \mathbf{A}_i \longrightarrow \mathbf{C}$ with $i \in \{1, 2, 3\}$ we have morphisms $(f, \eta): D_1 \longrightarrow D_2$ and $(g, \mu): D_2 \longrightarrow D_3$, then

$$\begin{aligned} & \underline{\lim}(g, \mu) \circ \underline{\lim}(f, \eta) \\ = & \quad \{\text{by definition of } \underline{\lim} \text{ on a general morphism}\} \\ & \underline{\lim}(g, id_{D_3 \circ g}) \circ \underline{\lim}(id_{A_2}, \mu) \circ \underline{\lim}(f, id_{D_2 \circ f}) \circ \underline{\lim}(id_{A_1}, \eta) \\ = & \quad \{\text{by (2)}\} \\ & \underline{\lim}(g, id_{D_3 \circ g}) \circ \underline{\lim}(f, id_{D_3 \circ g \circ f}) \circ \underline{\lim}(id_{A_1}, \mu_f) \circ \underline{\lim}(id_{A_1}, \eta) \\ = & \quad \{\text{by functoriality of } \underline{\lim} \text{ on each class}\} \\ & \underline{\lim}(g \circ f, id_{D_3 \circ g \circ f}) \circ \underline{\lim}(id_{A_1}, \mu_f \circ \eta) \\ = & \quad \{\text{by definition of } \underline{\lim} \text{ on a general morphism}\} \\ & \underline{\lim}(g \circ f, \mu_f \circ \eta) \\ = & \quad \{\text{by definition of composition in } \underline{Diag}(\mathbf{C})\} \\ & \underline{\lim}((g, \mu) \circ (f, \eta)). \end{aligned}$$

We give an explicit description of this generalised colimit construction for the case of diagrams of abelian groups, which will be needed in Sect. 5. This is computed as the instantiation of the construction of colimits in terms of coequalisers and coproducts given in Appendix A to the category \mathbf{Ab} of abelian groups and group homomorphisms.

Let $D: \mathbf{A} \longrightarrow \mathbf{Ab}$ and $E: \mathbf{B} \longrightarrow \mathbf{Ab}$ be two diagrams of abelian groups and (f, η) be a $\underline{Diag}(\mathbf{Ab})$ -morphism from D to E .

First, we describe the colimit of D in \mathbf{Ab} , and thus its image under the functor $\underline{\lim}: \underline{Diag}(\mathbf{Ab}) \longrightarrow \mathbf{Ab}$. Consider the direct sum of the groups $D(a)$ over all objects a

in \mathbf{A} . If g is an element of the group $D(a)$, we use the notation $(g)_a$ to indicate the element of this direct sum with g in the a -th component and 0 in all the others. The colimit of D is this direct sum modulo the identifications along the morphisms in the diagram D ; more precisely, modulo the subgroup generated by the elements $(g)_a - (D(h)(g))_{a'}$ where $g \in D(a)$ and $h: a \rightarrow a'$ is an \mathbf{A} -morphism.

We now describe the image of (f, η) under the functor $\lim: \underline{Diag}(\mathbf{Ab}) \rightarrow \mathbf{Ab}$. It is enough to indicate how the group homomorphism $\lim(f, \eta)$ acts on elements of the colimit of D of the form $[(g)_a]$. It does so by mapping $[(g)_a]$ to the element $[(\eta_a(g))_{f(a)}]$ of the colimit of E . This is well-defined for if an \mathbf{A} -morphism $h: a \rightarrow a'$ identifies, over D , $(g)_a$ with $(D(h)(g))_{a'}$, then the \mathbf{B} -morphism $f h: f(a) \rightarrow f(a')$ identifies, over E , $(\eta_a(g))_{f(a)}$ with $(E(fh)(\eta_a(g)))_{f(a')}$, which is equal to $(\eta_{a'}(D(h)(g)))_{f(a')}$ by naturality of η , i.e. by $(E \circ f)(h) \circ \eta_a = \eta_{a'} \circ D(h)$.

4.2. Extensions of functors. For a fixed semispectral functor σ , we define a natural method for extending functors $F: \mathbf{KHaus} \rightarrow \mathbf{C}$ when \mathbf{C} is complete. The idea is to use σ to turn an algebra \mathcal{A} into a diagram of commutative algebras, apply the Gel'fand spectrum functor contextwise to this diagram to yield a diagram of topological spaces, apply F contextwise to yield a diagram in \mathbf{C} , and finally, apply the extended limit functor $\lim: \underline{Diag}(\mathbf{C}) \rightarrow \mathbf{C}$. This procedure is also described for the von Neumann algebraic case.

Intuitively, one should think of the extension process as decomposing a noncommutative space into its quotient spaces, retaining those which are genuine topological spaces, applying the topological functor to each one of them, and pasting together the results. Varying the semispectral functor effectively changes the precise method of gluing together the topological data into a single \mathbf{C} -object.

Definition 4.1. Given a semispectral functor $\sigma: \mathbf{uC}^* \rightarrow \underline{Diag}(\mathbf{uC}^*)$, a complete category \mathbf{C} , and a functor $F: \mathbf{KHaus} \rightarrow \mathbf{C}$, the σ -extension of F , denoted $\tilde{F}_\sigma: \mathbf{uC}^{*\text{op}} \rightarrow \mathbf{C}$, is given by

$$\begin{aligned} \tilde{F}_\sigma &= \lim \circ F \circ G_\sigma \\ &: \mathbf{uC}^{*\text{op}} \rightarrow \underline{Diag}(\mathbf{KHaus}) \rightarrow \underline{Diag}(\mathbf{C}) \rightarrow \mathbf{C}. \end{aligned}$$

Note that F in the right-hand side of the above expression stands for the functor from $\underline{Diag}(\mathbf{KHaus})$ to $\underline{Diag}(\mathbf{C})$ induced by the given $F: \mathbf{KHaus} \rightarrow \mathbf{C}$ (see the last paragraph of Sect. 3.1), while G_σ is the spatial diagram functor corresponding to σ (Definition 3.3).

Extensions are analogously defined with respect to von Neumann algebraic semispectral functors $\sigma: \mathbf{vNA} \rightarrow \underline{Diag}(\mathbf{cvNA})$, with a functor $F: \mathbf{HStonean} \rightarrow \mathbf{C}$ to a complete category \mathbf{C} yielding a σ -extension $\tilde{F}_\sigma: \mathbf{vNA}^{*\text{op}} \rightarrow \mathbf{C}$.

In some applications (notably, in the case of the topological K -functor that we shall consider in the next section), we are interested in extending a contravariant functor F from \mathbf{KHaus} to a cocomplete category \mathbf{C} . This is naturally covered by the definition above by applying it to \mathbf{C}^{op} as the target category. The process yields an extension $\tilde{F}_\sigma: \mathbf{uC}^* \rightarrow \mathbf{C}$, whose last step—taking a limit in \mathbf{C}^{op} —corresponds to taking a colimit in \mathbf{C} . Explicitly, in this instance, we have

$$\begin{aligned} \tilde{F}_\sigma &= \varprojlim_{\mathbf{C}^{\text{op}}} F \circ G_\sigma \\ &: \mathbf{uC}^{*\text{op}} \longrightarrow \underline{\text{Diag}}(\mathbf{KHaus}) \longrightarrow \underline{\text{Diag}}(\mathbf{C}^{\text{op}}) \longrightarrow \mathbf{C}^{\text{op}}, \end{aligned}$$

but we could also write

$$\begin{aligned} \tilde{F}_\sigma &= \varinjlim_{\mathbf{C}} F \circ G_\sigma \\ &: \mathbf{uC}^* \longrightarrow \underline{\text{Diag}}(\mathbf{KHaus})^{\text{op}} \longrightarrow \underline{\text{Diag}}(\mathbf{C}) \longrightarrow \mathbf{C}, \end{aligned}$$

with F in this expression standing for the contravariant functor from $\underline{\text{Diag}}(\mathbf{KHaus})$ to $\underline{\text{Diag}}(\mathbf{C})$ induced by $F: \mathbf{KHaus}^{\text{op}} \rightarrow \mathbf{C}$ (see the last paragraph of Sect. 3.1).

The third property in the definition of semispectral functor—that the category picked out by $\sigma(\mathcal{A})$ has \mathcal{A} as a terminal object when \mathcal{A} is commutative—is crucial in ensuring that \tilde{F}_σ does indeed extend F . As a consequence of this condition, the diagram $(F \circ \Sigma \circ \sigma)(\mathcal{A})$ has $F(\Sigma(\mathcal{A}))$ as an initial object, which is therefore equal to its limit (up to isomorphism). Hence, we have that $\tilde{F}_\sigma(\mathcal{A}) \simeq (F \circ \Sigma)(\mathcal{A})$ for every commutative \mathcal{A} . The second property then ensures that given a homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ between commutative algebras, $\tilde{F}_\sigma(\phi)$ completes the commutative square formed by these isomorphisms and $(F \circ \Sigma)(\phi)$, i.e.

$$\begin{array}{ccc} \tilde{F}_\sigma(\mathcal{A}) & \xrightarrow{\cong} & (F \circ \Sigma)(\mathcal{A}) \\ \tilde{F}_\sigma(\phi) \downarrow & & \downarrow (F \circ \Sigma)(\phi) \\ \tilde{F}_\sigma(\mathcal{B}) & \xrightarrow{\cong} & (F \circ \Sigma)(\mathcal{B}) \end{array}$$

commutes. Thus, these isomorphisms define a natural equivalence between $\tilde{F}_\sigma|_{\mathbf{uC}^{*\text{op}}}$ and $F \circ \Sigma$. We have thus proved that:

Theorem 4.2. *For a semispectral functor σ , a complete category \mathbf{C} , and a functor $F: \mathbf{KHaus} \rightarrow \mathbf{C}$, $\tilde{F}_\sigma|_{\mathbf{uC}^{*\text{op}}} \simeq F \circ \Sigma$.*

We are primarily interested in the case that σ is the unitary semispectral functor g (Definition 3.5): we shall reserve the notation \tilde{F} to denote the g -extension \tilde{F}_g of F . Similarly, we will write \tilde{F}_f for the finitary restriction of the g -extension of F , i.e. for its g_f -extension, which uses the finitary version of the unitary semispectral functor g_f (Definition 3.7). Finally, we write \tilde{F}_W for the g_W -extension of a functor $F: \mathbf{HStonean} \rightarrow \mathbf{C}$, using the unitary semispectral functor for von Neumann algebras g_W (Definition 3.8).

4.3. Theorems of quantum foundations.

Having established the framework of extensions, we demonstrate how they can be used to succinctly express two fundamental theorems of quantum foundations: the Bell–Kochen–Specker theorem [12, 68] and Gleason’s theorem [47].

The first of these reformulations is due to Hamilton, Isham, and Butterfield [52], here similarly stated for the generalised version of the Bell–Kochen–Specker theorem

by Döring [27]. The second was given by de Groot [22], and is a reformulation of the generalised version of Gleason's theorem to most von Neumann algebras, due to Christensen [18], Yeadon [100, 101], and others (see [50, 73]).

In this section, we consider the spectral presheaf functor $\Sigma \circ s: \mathbf{vNA}^{\text{op}} \rightarrow \underline{\text{Diag}}(\mathbf{HStonean})$ obtained from the semispectral functor $s: \mathbf{vNA} \rightarrow \underline{\text{Diag}}(\mathbf{cvNA})$ as described after Definition 3.3. We write $\tilde{F}_s = \underline{\text{lim}} \circ F \circ \Sigma \circ s$ for the s -extension of a functor $F: \mathbf{HStonean} \rightarrow \mathbf{C}$ whose target \mathbf{C} is a complete category. We also restrict our extensions to the full subcategory of those von Neumann algebras that are separable and contain no type I_2 direct summands. In the statement of the Bell–Kochen–Specker theorem, we also have to exclude abelian (type I_1) summands.

Theorem 4.3 (Reformulation of the Bell–Kochen–Specker theorem [27, 52]). *The s -extension \tilde{U}_s of the forgetful functor $U: \mathbf{HStonean} \rightarrow \mathbf{Set}$ yields the empty set on separable von Neumann algebras without type I_1 or I_2 summands.*

This can be interpreted as saying that the notion of ‘points’ cannot be extended (in our sense) from the commutative to the noncommutative world, or that a ‘noncommutative space’ is not a geometry in the usual sense: a collection of ‘points’ with some additional structure.

Let $D: \mathbf{HStonean} \rightarrow \mathbf{Set}$ be the regular Borel probability distribution functor which assigns to a hyperstonean topological space X the set of all regular⁷ Borel probability measures on X and to a continuous function f the corresponding pushforward map f_* on measures, defined by $f_*\mu(e) = \mu(f^{-1}(e))$.

Theorem 4.4 (Reformulation of the Gleason–Christensen–Yeadon theorem [22]). *The s -extension \tilde{D}_s of the regular Borel probability distribution functor is naturally isomorphic, on the full subcategory of separable von Neumann algebras without type I_2 summands, to the contravariant functor mapping a von Neumann algebra to its set of states (positive linear functionals of unit norm) and a normal unital $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ to the corresponding pullback that takes a state ρ of \mathcal{B} to the state $\rho \circ \phi$ of \mathcal{A} .*

These two theorems can be read as indicating that while the ‘noncommutative space’ $\Sigma \circ s(\mathcal{A})$ has no points, it nonetheless admits globally consistent probability distributions, and that these distributions correspond to quantum states.

5. Reconstructing Operator K -Theory

Topological K -theory, defined in terms of vector bundles, is an extraordinary cohomology theory. Its C^* -algebraic generalisation, operator K -theory, is similarly defined in terms of the noncommutative analogue of vector bundles, i.e. finitely generated projective modules, and plays an important role in the study of C^* -algebras, e.g. as a classifying invariant [38].

⁷ A Borel measure μ is said to be *regular* if, for any Borel set X ,

$$\mu(X) = \inf \{ \mu(U) \mid X \subset U, U \text{ open} \} = \sup \{ \mu(K) \mid K \subset X, K \text{ compact} \}.$$

Note that for compact spaces, these coincide with the (in general, weaker) notion of Radon measure [42, Corollary 7.6]. By the Riesz–Markov–Kakutani representation theorem [66, 74, 85] (see e.g. [42, Theorem 7.2]), regular Borel measures on a compact Hausdorff space X are in one-to-one correspondence with positive linear functionals of unit norm, i.e. states, of the commutative C^* -algebra $C(X)$.

In this section, we consider the extension of the topological K -theory functor, $K : \mathbf{KHaus}^{\text{op}} \rightarrow \mathbf{Ab}$. The most natural conjecture is that this extension yields operator K -theory on unital C^* -algebras, $K_0 : \mathbf{u}C^* \rightarrow \mathbf{Ab}$:⁸

Conjecture 5.1. ([23]) $K_0 \simeq \tilde{K} : \mathbf{u}C^* \rightarrow \mathbf{Ab}$.

The extension via unitalisation of \tilde{K} would then yield operator K -theory on arbitrary C^* -algebras, $K_0 : C^* \rightarrow \mathbf{Ab}$, as this is the way the latter is obtained from $K_0 : \mathbf{u}C^* \rightarrow \mathbf{Ab}$ (see Definition 5.10 below).

As detailed in Sect. 5.2, operator K -theory is stable. That is, we have $K_0 \simeq K_0 \circ \mathcal{K}$, where \mathcal{K} is the stabilisation functor (see Definition 5.11 below). Therefore, one could think of operator K -theory as being defined only for the stabilisations of unital C^* -algebras. The K -theory of unital C^* -algebras can be obtained by first stabilising the algebra and then applying the K_0 -functor restricted to this class of algebras; moreover, the K -theory of arbitrary C^* -algebras can then be obtained as usual via unitalisation.

This justifies weakening the conjecture to require only that K_0 and \tilde{K} agree on the stabilisations of unital C^* -algebras (and of unital $*$ -homomorphisms). Note that, since stable algebras are nonunital, this necessitates extending \tilde{K} to nonunital C^* -algebras, which can be done via unitalisation, following the same procedure used to extend K_0 from unital to all C^* -algebras. This weakened conjecture would require that $K_0 \simeq K_0 \circ \mathcal{K} \simeq \tilde{K} \circ \mathcal{K}$ as functors $\mathbf{u}C^* \rightarrow \mathbf{Ab}$ (and consequently, that the extension via unitalisation of $\tilde{K} \circ \mathcal{K}$ be naturally isomorphic to the functor $K_0 : C^* \rightarrow \mathbf{Ab}$).

In fact, we encounter the need to further modify this conjecture by limiting our spatial diagrams of stable C^* -algebras to include only the finite quotient spaces. The main result proven in this section is:

Theorem 5.2. $K_0 \simeq K_0 \circ \mathcal{K} \simeq \tilde{K}_f \circ \mathcal{K}$ as functors $\mathbf{u}C^* \rightarrow \mathbf{Ab}$. Consequently, $K_0 : C^* \rightarrow \mathbf{Ab}$ is naturally isomorphic to the extension via unitalisation of the functor $\tilde{K}_f \circ \mathcal{K}$.

Here, \tilde{K}_f is defined for unital C^* -algebras as the g_f -extension of K , where g_f is the finitary version of the unitary semispectral functor given in Definition 3.7; see also Definition 4.1 of the general extension process and the subsequent remarks regarding contravariant functors. Explicitly,

$$\begin{aligned} \tilde{K}_f &= \varinjlim \circ K \circ G_f \\ &: \mathbf{u}C^* \rightarrow \underline{\text{Diag}}(\mathbf{KHaus})^{\text{op}} \rightarrow \underline{\text{Diag}}(\mathbf{Ab}) \rightarrow \mathbf{Ab}. \end{aligned}$$

Note that as stable C^* -algebras are not unital, we need to extend \tilde{K}_f to all C^* -algebras via unitalisation (see Definition 5.17 below) to yield $\tilde{K}_f : C^* \rightarrow \mathbf{Ab}$.

We thus find that operator K -theory, K_0 , can be defined as a colimit of topological vector bundles over *finite* quotient spaces of a noncommutative space. This result suggests fixing the appropriate class of morphisms in our ansatz semispectral functor, i.e. the diagrams associated to C^* -algebras, to be the restrictions of inner automorphisms.

⁸ This would be sufficient to recover operator K -theory on arbitrary C^* -algebras, since the functor $K_0 : C^* \rightarrow \mathbf{Ab}$ is obtained from its unital version via unitalisation, as we shall see.

5.1. Technical background: topological K -theory. We introduce the basic background on topological and operator K -theory, focusing on the minimum required to follow the proof of Theorem 5.2. A slightly more detailed presentation can be found in Appendix B; or see e.g. [40, 86, 99].

Definition 5.3. For a compact Hausdorff space X , its *vector bundle monoid*, $V(X)$, is the set of isomorphism classes of complex vector bundles over X with the abelian addition operation of fibrewise direct sum: $[E] + [F] = [E \oplus F]$. A continuous function $f: X \rightarrow Y$ yields a monoid homomorphism $V(f): V(Y) \rightarrow V(X)$ by the pullback of bundles, $V(f)([E]) = [f^*E]$. This defines a functor $V: \mathbf{KHaus}^{\text{op}} \rightarrow \mathbf{AbMon}$, where \mathbf{AbMon} is the category of abelian monoids and monoid homomorphisms.

Definition 5.4. For an abelian monoid M , its *Grothendieck group*, $\mathcal{G}(M)$, is the abelian group $(M \times M)/\sim$ where \sim is the equivalence relation given by

$$(a, b) \sim (c, d) \quad \text{iff} \quad \exists e \in M, \quad a + d + e = b + c + e.$$

For a monoid homomorphism $\phi: M \rightarrow N$, the group homomorphism $\mathcal{G}(\phi): \mathcal{G}(M) \rightarrow \mathcal{G}(N)$ is given by $\mathcal{G}(\phi)([(a, b)]) = [(\phi(a), \phi(b))]$. This defines a functor $\mathcal{G}: \mathbf{AbMon} \rightarrow \mathbf{Ab}$.

Intuitively, an element $[(a, b)]$ of $\mathcal{G}(M)$ can be thought of as a formal difference $a - b$ of elements of M . With this interpretation in mind, it is easy to see that $\mathcal{G}(M)$ is indeed a group, with addition given componentwise, neutral element $[(0, 0)]$, and the inverse of $[(a, b)]$ equal to $[(b, a)]$. Moreover, there is a monoid homomorphism $i: M \rightarrow \mathcal{G}(M)$ given by $a \mapsto [(a, 0)]$. As an example, the Grothendieck group of the additive monoid of natural numbers (including zero) is the additive group of integers.

The Grothendieck group functor \mathcal{G} is an explicit presentation of the group completion functor, the left adjoint to the forgetful functor from \mathbf{Ab} to \mathbf{AbMon} . This means that $\mathcal{G}(M)$ is the ‘most general’ group containing a homomorphic image of M , in the sense that it satisfies the universal property that any monoid homomorphism from M to an abelian group factors uniquely through the monoid homomorphism $i: M \rightarrow \mathcal{G}(M)$.

Definition 5.5. The *topological K -functor* $K: \mathbf{KHaus}^{\text{op}} \rightarrow \mathbf{Ab}$ is $\mathcal{G} \circ V$.

5.2. Technical background: Operator K -theory. Following the usual method of noncommutative geometry, in order to generalise a topological concept to the noncommutative case, one must begin with a characterisation of the topological concept in question in terms of commutative algebra. In the case of K -theory, this requires phrasing the notion of a *complex vector bundle over X* in terms of the algebra $C(X)$ of continuous complex-valued functions on X :

Theorem 5.6 (Serre–Swan [94]). *The category of complex vector bundles over a compact Hausdorff space X is equivalent to the category of finitely generated projective $C(X)$ -modules.*

Finitely generated projective \mathcal{A} -modules can be identified with (equivalence classes of) projections in matrix algebras $M_n(\mathcal{A})$ over the C^* -algebra \mathcal{A} , which are more convenient to work with. We are now ready to define operator K_0 for unital C^* -algebras.

Definition 5.7. Let \mathcal{A} be a C^* -algebra. Two projections $p \in M_n(\mathcal{A})$ and $q \in M_m(\mathcal{A})$ with $n, m \in \mathbb{N}$ are *Murray–von Neumann equivalent*, denoted $p \sim_M q$, whenever there is a partial isometry $v \in M_{m,n}(\mathcal{A})$ such that $p = vv^*$ and $q = v^*v$.

Definition 5.8 (*The Murray–von Neumann semigroup for unital \mathcal{A}*). Let \mathcal{A} be a unital C^* -algebra. Its *Murray–von Neumann semigroup*, $V_0(\mathcal{A})$, is the set of Murray–von Neumann equivalence classes of projections in matrices over \mathcal{A} :

$$\bigsqcup_{n \in \mathbb{N}} \{p \in M_n(\mathcal{A}) \mid p \text{ is a projection}\} / \sim_M.$$

It is equipped with the abelian addition operation

$$[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right],$$

for which the equivalence class of the zero projection is a neutral element. Therefore, $V_0(\mathcal{A})$ is an abelian monoid. A unital $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ yields a monoid homomorphism $V_0(\phi): V_0(\mathcal{A}) \rightarrow V_0(\mathcal{B})$ given by $[p] \mapsto [M_n(\phi)(p)]$ for each $n \in \mathbb{N}$ and p a projection in $M_n(\mathcal{A})$, where $M_n(\phi)$ acts on elements of $M_n(\mathcal{A})$ by entrywise application of ϕ . This defines a functor $V_0: \mathbf{uC}^* \rightarrow \mathbf{AbMon}$.

Definition 5.9. The *operator K_0 -functor for unital C^* -algebras*, $K_0: \mathbf{uC}^* \rightarrow \mathbf{Ab}$, is $\mathcal{G} \circ V_0$.

We now describe the extension of K_0 to all C^* -algebras. The same recipe will later be used to extend other functors from unital to all C^* -algebras.

Let \mathcal{A} be a C^* -algebra (which may or may not be unital). By minimally adjoining a unit element to \mathcal{A} , one obtains the unitalisation \mathcal{A}^+ (see Definition B.10) and a short exact sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{\iota} \mathcal{A}^+ \xrightarrow{\pi} \mathbb{C} \longrightarrow 0.$$

Moreover, $(-)^+$ is a functor from \mathbf{C}^* to \mathbf{uC}^* .

Definition 5.10. The K_0 group of a C^* -algebra \mathcal{A} is the subgroup of $K_0(\mathcal{A}^+)$ given by the kernel of $K_0(\pi)$. A $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ yields a homomorphism from $\ker K_0(\mathcal{A}^+ \xrightarrow{\pi} \mathbb{C})$ to $\ker K_0(\mathcal{B}^+ \xrightarrow{\pi} \mathbb{C})$ by restriction of $K_0(\phi^+)$ to the kernel of $K_0(\mathcal{A}^+ \xrightarrow{\pi} \mathbb{C})$. This defines the *operator K_0 -functor*, $K_0: \mathbf{C}^* \rightarrow \mathbf{Ab}$.

We now consider stability, a key property of the operator K_0 -functor.

Definition 5.11. The *stabilisation functor* $\mathcal{K}: \mathbf{C}^* \rightarrow \mathbf{C}^*$ maps a C^* -algebra \mathcal{A} to the C^* -algebra $\mathcal{A} \otimes \mathcal{K}$ where (in an abuse of notation) \mathcal{K} is the C^* -algebra of compact operators on a separable infinite-dimensional Hilbert space (see Definition B.14). A $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ is mapped to $\phi \otimes id_{\mathcal{K}}$.

We also denote the restriction of the stabilisation functor to the category of unital C^* -algebras by $\mathcal{K}: \mathbf{uC}^* \rightarrow \mathbf{C}^*$.

Since the C^* -algebra $\mathcal{K} \otimes \mathcal{K}$ is isomorphic to \mathcal{K} , the stabilisation functor is an idempotent operation, i.e. $\mathcal{K} \circ \mathcal{K} \simeq \mathcal{K}$.

Definition 5.12. A C^* -algebra \mathcal{A} is called *stable* if it is isomorphic to its stabilisation, i.e. $\mathcal{A} \simeq \mathcal{K}(\mathcal{A}) = \mathcal{A} \otimes \mathcal{K}$.

Theorem 5.13. *Operator K -theory is stable. That is, $K_0 \simeq K_0 \circ \mathcal{K}$.*

Consequently, the operator K_0 -functor is determined by its restriction to stable C^* -algebras (in fact, by its restriction to stabilisations of unital C^* -algebras and of unital $*$ -homomorphisms).

The Murray–von Neumann semigroup, and thus the K_0 -group, of a unital C^* -algebra \mathcal{A} can be expressed in a rather simple fashion in terms of projections of its stabilisation [86, Exercise 6.6]. We require this definition in the proof of Theorem 5.2 and thus describe it in explicit detail.

Definition 5.14. Let \mathcal{A} be a C^* -algebra. Two projections p and q in \mathcal{A} are *unitarily equivalent*, denoted by $p \sim_u q$, whenever there is a unitary $u \in \mathcal{A}^+$ such that $p = uqu^*$. We write $[p]$ for the unitary equivalence class of p .

Given projections $p_1, \dots, p_k \in \mathcal{K}(\mathcal{A})$, one can find pairwise orthogonal representatives of their unitary equivalence classes, i.e. there exist projections $q_1, \dots, q_k \in \mathcal{K}(\mathcal{A})$ such that $p_i \sim_u q_i$ ($i \in \{1, \dots, k\}$) and all the q_i are pairwise orthogonal [86, Exercise 6.6].

The Murray–von Neumann semigroup for unital C^* -algebras admits the following alternative characterisation:

Definition 5.15 (*The Murray–von Neumann semigroup for unital \mathcal{A} , alternative definition*). Let \mathcal{A} be a unital C^* -algebra. The elements of $V_0(\mathcal{A})$ are the unitary equivalence classes of projections in $\mathcal{K}(\mathcal{A})$. The abelian addition operation is given by orthogonal addition. That is, if p and p' are two projections in $\mathcal{K}(\mathcal{A})$, then $[p] + [p'] = [q + q']$ where q and q' are orthogonal representatives of $[p]$ and $[p']$, respectively (i.e. $p \sim_u q$, $p' \sim_u q'$, and $q \perp q'$). The equivalence class of the zero projection is a neutral element for this operation, making $V_0(\mathcal{A})$ an abelian monoid. A unital $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ yields a monoid homomorphism $V_0(\phi): V_0(\mathcal{A}) \rightarrow V_0(\mathcal{B})$ by $[p] \mapsto [\mathcal{K}(\phi)(p)]$. This defines a functor $V_0: \mathbf{uC}^* \rightarrow \mathbf{AbMon}$.

Through this reformulation of the Murray–von Neumann semigroup functor V_0 , we automatically get a new description of K_0 . Recall that this is obtained by composition with the Grothendieck group functor, as $K_0 = \mathcal{G} \circ V_0$. Then, $K_0(\mathcal{A})$ is simply the collection of formal differences

$$[p] - [q]$$

of elements of $V_0(\mathcal{A})$ with

$$[p] - [q] = [p'] - [q']$$

precisely when there exists $[r]$ such that

$$[p] + [q'] + [r] = [p'] + [q] + [r].$$

Composing the action on morphisms of the Grothendieck group functor after the action of V_0 just defined, we find that a unital $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ between unital C^* -algebras yields a group homomorphism between the K_0 groups of \mathcal{A} and \mathcal{B} given by

$$[p] - [q] \mapsto [\mathcal{K}(\phi)(p)] - [\mathcal{K}(\phi)(q)].$$

5.3. *Main theorem.* We now prove the main theorem of this section (Theorem 5.2): that the functors $K_0 : \mathbf{uC}^* \rightarrow \mathbf{Ab}$ and $\tilde{K}_f \circ \mathcal{K} : \mathbf{uC}^* \rightarrow \mathbf{Ab}$ are naturally isomorphic. It follows that the $K_0 : \mathbf{C}^* \rightarrow \mathbf{Ab}$ functor can be reconstructed in terms of \tilde{K}_f , the stabilisation functor of unital C^* -algebras $\mathcal{K} : \mathbf{uC}^* \rightarrow \mathbf{C}^*$, and unitalisation.

Recall that the functor \tilde{K}_f is defined on the category of unital C^* -algebras as

$$\begin{aligned} \tilde{K}_f : \mathbf{uC}^* \rightarrow \mathbf{Ab} &= \varinjlim \circ K \circ G_f \\ &= \varinjlim \circ K \circ \Sigma \circ g_f. \end{aligned}$$

We will prove the main theorem in three steps:

1. Give a simple presentation of the \tilde{K}_f group of a unital C^* -algebra \mathcal{A} in terms of its unitary equivalence classes of projections (Lemma 5.16).
2. Extend the domain category of \tilde{K}_f from \mathbf{uC}^* to the category \mathbf{C}^* of all C^* -algebras and $*$ -homomorphisms via unitalisation in the same way that K_0 is extended from unital to all C^* -algebras, and give a similar presentation of the \tilde{K}_f group of an arbitrary C^* -algebra. This is necessary to make sense of the composition $\tilde{K}_f \circ \mathcal{K}$ as all stable C^* -algebras are nonunital.
3. Construct a natural isomorphism between $K_0 : \mathbf{uC}^* \rightarrow \mathbf{Ab}$ and $\tilde{K}_f \circ \mathcal{K} : \mathbf{uC}^* \rightarrow \mathbf{Ab}$ (Theorem 5.2).

Lemma 5.16. *For a unital C^* -algebra \mathcal{A} ,*

$$\tilde{K}_f(\mathcal{A}) = \langle [p]_u \mid [p]_u = [p_1]_u + [p_2]_u \text{ whenever } p = p_1 + p_2 \rangle$$

is the group generated by the unitary equivalence classes of projections in \mathcal{A} modulo the relations coming from addition of orthogonal projections. Moreover, for a unital $$ -homomorphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ between unital C^* -algebras \mathcal{A} and \mathcal{B} ,*

$$\tilde{K}_f(\phi)([p]_u) = [\phi(p)]_u.$$

Proof. We will first compute the action of \tilde{K}_f on objects before computing its action on unital $*$ -homomorphisms. Recall Definitions 3.6 and 3.7. The objects of the finitary unitary subcategory $\mathbf{S}_f(\mathcal{A})$ are the unital finite-dimensional commutative sub- C^* -algebras of \mathcal{A} . The morphisms are given by the restrictions of inner automorphisms. These morphisms are all of the form $\iota \circ r$ where ι is an inclusion between subalgebras and r is an isomorphism $\text{Ad}_u|_V^{uVu^*} : V \rightarrow uVu^*$ between subalgebras given by conjugation by a unitary $u \in \mathcal{A}$.

Under the Gel'fand spectrum functor, the image of such an object V is a finite discrete space $\Sigma(V)$ whose points are in correspondence with the atomic projections of the sub-algebra V . The images of the inclusions $\iota : V \hookrightarrow V'$ are surjections $\Sigma(\iota) : \Sigma(V') \rightarrow \Sigma(V)$ with the property that whenever a point $s \in \Sigma(V)$ corresponds to a projection p atomic in V , then p is the sum of the atomic projections in V' that correspond to the points of $(\Sigma(\iota))^{-1}(s)$. In turn, an isomorphism r , arising from conjugation by a unitary u , is sent by Σ to a bijection that connects points whose corresponding projections are related by conjugation by u .

Under the topological K -functor, each object $\Sigma(V)$ of the diagram of $G_f(\mathcal{A})$ yields a direct sum of copies of \mathbb{Z} , one for each point. That is, one gets a trivial vector bundle of every possible dimension (and formal inverses) over each point. Each of these copies of \mathbb{Z} is generated by the trivial bundle of dimension 1. Taking the colimit of the diagram

$K \circ G_f(\mathcal{A}) = K \circ \Sigma \circ g_f(\mathcal{A})$ then yields, as described at the end of Sect. 4.1, a direct sum of the abelian groups $K \circ \Sigma(V)$ indexed by the objects V of $\mathbf{S}_f(\mathcal{A})$ modulo the relations generated by the morphisms of $\mathbf{S}_f(\mathcal{A})$. In our case, this is a quotient of the direct sum of copies of \mathbb{Z} , one for each pair (V, p) where V is a finite-dimensional unital commutative sub- C^* -algebra of \mathcal{A} and p is an atomic projection in V .

The images under $K \circ \Sigma$ of the inclusions result in identifying the generator of the copy of \mathbb{Z} associated to a pair (V, p) with the sum of the generators associated to pairs $\{(V', p_i)\}_i$ whenever $V \subset V'$ and $\sum_i p_i = p$. Every nonzero projection $p \in \mathcal{A}$ is an atomic projection in the subalgebra $\mathbb{C}p + \mathbb{C}(1 - p)$, which is included in every subalgebra that contains p . Consequently, the generators associated to the same projection p atomic in different subalgebras are all identified in the colimit. Hence, we may speak of the element of the colimit group $[(p)]$ associated to p without reference to which subalgebra it appears in. Thus, the abelian group $\tilde{K}_f(\mathcal{A})$ can be viewed as a quotient of the free abelian group generated by the elements $[(p)]$. The isomorphisms in the diagram ensure that the elements associated to unitarily equivalent projections are identified. We may thus denote the elements of the colimit group associated to projections unitarily equivalent to p by $[p]_u$. The second class of identifications consists of those between elements $[p]_u$ and $\sum_i [p_i]_u$ whenever the p_i (are mutually orthogonal and) sum to p .

We now consider the action of $\tilde{K}_f = \varinjlim \circ K \circ \Sigma \circ g_f$ on a unital $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$. By Definition 3.7, $g_f(\phi)$ is defined to be (f, η) where $f: \mathbf{S}_f(\mathcal{A}) \rightarrow \mathbf{S}_f(\mathcal{B})$ is the functor taking an object $V \subset \mathcal{A}$ to $\phi(V) \subset \mathcal{B}$ and η is the natural transformation whose component at V is the unital $*$ -homomorphism $\phi|_V: V \rightarrow \phi(V)$.

Suppose $[p]_u \in \tilde{K}_f(\mathcal{A})$ with p a projection in \mathcal{A} . The element $[p]_u$ of the colimit can be identified with a trivial vector bundle B_p of dimension one over the point corresponding to p in the space associated by $\Sigma \circ g_f(\mathcal{A})$ to the subalgebra $V_p = \mathbb{C}p + \mathbb{C}(1 - p)$ of \mathcal{A} . The natural transformation η of the morphism of diagrams $g_f(\phi)$ includes a component $\eta_{V_p} = \phi|_{V_p}: V_p \rightarrow V_{\phi(p)}$ that maps p to $\phi(p)$ and $1 - p$ to $1 - \phi(p)$.

Under the image of the lifting of Σ to diagrams, this component becomes $\Sigma(\phi|_{V_p}): \Sigma(V_{\phi(p)}) \rightarrow \Sigma(V_p)$ that maps the point corresponding to $\phi(p)$ to the one corresponding to p (and the point corresponding to $1 - \phi(p)$ to the one corresponding to $1 - p$).

Then, under the topological K -functor, this becomes $K \circ \Sigma(V_{\phi(p)}): K \circ \Sigma(V_p) \rightarrow K \circ \Sigma(V_{\phi(p)})$, which pulls back vector bundles along the map $\Sigma(\phi|_{V_p})$. The bundle B_p is pulled back to the trivial vector bundle of dimension one over the point corresponding to $\phi(p)$ in the discrete space $\Sigma(V_{\phi(p)})$. The pulled-back bundle is identified with $[\phi(p)]_u$ in the colimit $\tilde{K}_f(\mathcal{A})$ and we conclude that $\tilde{K}_f(\phi)([p]_u) = [\phi(p)]_u$. \square

We now extend the functor \tilde{K}_f to all C^* -algebras via unitalisation in the same way that K_0 is extended.

Definition 5.17. The \tilde{K}_f group of a C^* -algebra \mathcal{A} (unital or not) is the subgroup of $\tilde{K}_f(\mathcal{A}^+)$ given by the kernel of $\tilde{K}_f(\pi)$, where $\pi: \mathcal{A}^+ \rightarrow \mathbb{C}$ is the projection map in the unitalisation short exact sequence (c.f. Definition 5.10). A $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ yields a homomorphism from $\ker \tilde{K}_f(\mathcal{A}^+ \xrightarrow{\pi} \mathbb{C})$ to $\ker \tilde{K}_f(\mathcal{B}^+ \xrightarrow{\pi} \mathbb{C})$ by restriction of $\tilde{K}_f(\phi^+)$ to the kernel of $\tilde{K}_f(\mathcal{A}^+ \xrightarrow{\pi} \mathbb{C})$. This defines the \tilde{K}_f functor, $\tilde{K}_f: C^* \rightarrow \text{Ab}$.

To check that this extended map is well-defined on morphisms, we must show that for any $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$, the unital $*$ -homomorphism $\tilde{K}_f(\phi^+) : \tilde{K}_f(\mathcal{A}^+) \rightarrow \tilde{K}_f(\mathcal{B}^+)$ carries $\ker \tilde{K}_f(\mathcal{A}^+ \xrightarrow{\pi_{\mathcal{A}}} \mathbb{C})$ into $\ker \tilde{K}_f(\mathcal{B}^+ \xrightarrow{\pi_{\mathcal{B}}} \mathbb{C})$. Functoriality then follows immediately from that of $\tilde{K}_f \circ (-)^+ : \mathbf{C}^* \rightarrow \mathbf{uC}^* \rightarrow \mathbf{Ab}$. This is done by noting that the following diagram in \mathbf{uC}^* commutes:

$$\begin{array}{ccc} \mathcal{A}^+ & \xrightarrow{\pi} & \mathbb{C} \\ \phi^+ \downarrow & \nearrow \pi & \\ \mathcal{B}^+ & & \end{array}$$

and, therefore, so does its image under $\tilde{K}_f : \mathbf{uC}^* \rightarrow \mathbf{Ab}$:

$$\begin{array}{ccc} \tilde{K}_f(\mathcal{A}^+) & \xrightarrow{\tilde{K}_f(\pi)} & \mathbb{Z} \\ \tilde{K}_f(\phi^+) \downarrow & \nearrow \tilde{K}_f(\pi) & \\ \tilde{K}_f(\mathcal{B}^+) & & \end{array}$$

A particular consequence of the following lemma is that the new functor $\tilde{K}_f : \mathbf{C}^* \rightarrow \mathbf{Ab}$ agrees with the original functor $\tilde{K}_f : \mathbf{uC}^* \rightarrow \mathbf{Ab}$ on unital C^* -algebras. This justifies not distinguishing notationally between them.

Lemma 5.18 *For an arbitrary C^* -algebra \mathcal{A} ,*

$$\tilde{K}_f(\mathcal{A}) = \langle [p]_u \mid [p]_u = [p_1]_u + [p_2]_u \text{ whenever } p = p_1 + p_2 \rangle$$

is the group generated by the unitary equivalence classes of projections in \mathcal{A} modulo the relations coming from addition of orthogonal projections. Moreover, for a $$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ between C^* -algebras \mathcal{A} and \mathcal{B} ,*

$$\tilde{K}_f(\phi)([p]_u) = [\phi(p)]_u.$$

Proof. Let \mathcal{A} be a C^* -algebra. We need to determine the kernel of $\tilde{K}_f(\pi)$ with π the canonical projection from \mathcal{A}^+ to \mathbb{C} . Note that \tilde{K}_f in the previous sentence refers to the functor $\tilde{K}_f : \mathbf{uC}^* \rightarrow \mathbf{Ab}$ defined for unital C^* -algebras. Therefore, we can use Lemma 5.16 to perform this calculation.

All projections in \mathcal{A}^+ are of the form p or $1 - p$ for p a projection in \mathcal{A} . From the lemma, the colimit group $\tilde{K}_f(\mathcal{A}^+)$ is generated by elements of the form $[p]_u$ and $[1 - p]_u$ for projections $p \in \mathcal{A}$. As

$$[1 - p]_u = [1]_u - [p]_u,$$

we see that all elements of $\tilde{K}_f(\mathcal{A}^+)$ can be expressed as \mathbb{Z} -linear combinations of elements of the form $[p]_u$ with p a projection in \mathcal{A} or $[1]_u$. Such an element is in the kernel of $\tilde{K}_f(\pi)$ if and only if the coefficient for $[1]_u$ is 0. Hence, $\ker \tilde{K}_f(\pi)$ is the subgroup of $\tilde{K}_f(\mathcal{A}^+)$ generated by the elements $[p]_u$ for p a projection in \mathcal{A} .

The action for a $*$ -homomorphism ϕ is clear as its image is defined as a restriction of $\tilde{K}_f(\phi^+)$. □

Theorem 5.2. $K_0 \simeq K_0 \circ \mathcal{K} \simeq \tilde{K}_f \circ \mathcal{K}$ as functors $\mathbf{uC}^* \rightarrow \mathbf{Ab}$. Consequently, $K_0: \mathbf{C}^* \rightarrow \mathbf{Ab}$ is naturally isomorphic to the extension via unitalisation of the functor $\tilde{K}_f \circ \mathcal{K}$.

Proof. We are now ready to define the natural isomorphism $\eta: K_0 \rightarrow \tilde{K}_f \circ \mathcal{K}$ as functors $\mathbf{uC}^* \rightarrow \mathbf{Ab}$. For a unital C^* -algebra \mathcal{A} , the component $\eta_{\mathcal{A}}$ of this natural transformation sends $[p] - [q] \in K_0(\mathcal{A})$, where p and q are projections in $\mathcal{K}(\mathcal{A})$ (see Definition 5.15), to $[p]_u - [q]_u \in \tilde{K}_f(\mathcal{K}(\mathcal{A}))$, i.e. in the kernel of $\tilde{K}_f(\pi: \mathcal{K}(\mathcal{A})^+ \rightarrow \mathbb{C})$. This is well-defined, for if $[p] - [q] = [p'] - [q']$, i.e. (by Definition 5.15) if there is an r such that

$$[p] + [q'] + [r] = [p'] + [q] + [r],$$

then we can find pairwise orthogonal representatives of all these equivalence classes of projections by the remark after Definition 5.14, and show that

$$[p]_u - [q]_u = [p']_u - [q']_u.$$

Preservation of addition follows by a similar argument.

We define an inverse map to demonstrate bijectivity of $\eta_{\mathcal{A}}$. A generator $[p]_u$ of $\tilde{K}_f(\mathcal{K}(\mathcal{A}))$ is sent by $\eta_{\mathcal{A}}^{-1}$ to $[p]$. Since the relations from Lemma 5.18 (between $[p]_u$ and $\sum_i [p_i]_u$ whenever $p = \sum_i p_i$ and between $[p]_u$ and $[q]_u$ whenever p and q are unitarily equivalent) are also satisfied by the elements $[p]$ in the K_0 group of \mathcal{A} , $\eta_{\mathcal{A}}^{-1}$ is a well-defined map.

To demonstrate the naturality of these isomorphisms, we show that for any unital $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$, the following diagram commutes:

$$\begin{CD} K_0(\mathcal{A}) @>\eta_{\mathcal{A}}>> (\tilde{K}_f \circ \mathcal{K})(\mathcal{A}) \\ @V K_0(\phi) VV @VV (\tilde{K}_f \circ \mathcal{K})(\phi) V \\ K_0(\mathcal{B}) @>\eta_{\mathcal{B}}>> (\tilde{K}_f \circ \mathcal{K})(\mathcal{B}) \end{CD}$$

Suppose $[p] - [q]$ is an arbitrary element of $K_0(\mathcal{A})$.

$$\begin{aligned} ((\tilde{K}_f \circ \mathcal{K})(\phi) \circ \eta_{\mathcal{A}})([p] - [q]) &= ((\tilde{K}_f \circ \mathcal{K})(\phi))([p]_u - [q]_u) \\ &= \tilde{K}_f(\mathcal{K}(\phi))([p]_u - [q]_u) \\ &= [\mathcal{K}(\phi)(p)]_u - [\mathcal{K}(\phi)(q)]_u \\ &= \eta_{\mathcal{B}}([\mathcal{K}(\phi)(p)] - [\mathcal{K}(\phi)(q)]) \\ &= (\eta_{\mathcal{B}} \circ K_0(\phi))([p] - [q]) \end{aligned}$$

This calculation can be seen diagrammatically:

$$\begin{CD} [p] - [q] @>\eta_{\mathcal{A}}>> [p]_u - [q]_u \\ @V K_0(\phi) VV @VV (\tilde{K}_f \circ \mathcal{K})(\phi) V \\ [\mathcal{K}(\phi)(p)] - [\mathcal{K}(\phi)(q)] @>\eta_{\mathcal{B}}>> [\mathcal{K}(\phi)(p)]_u - [\mathcal{K}(\phi)(q)]_u \end{CD}$$

□

We have thus shown that $K_0: \mathbf{uC}^* \rightarrow \mathbf{Ab}$ and $\tilde{K}_f \circ \mathcal{K}: \mathbf{uC}^* \rightarrow \mathbf{Ab}$ are naturally isomorphic functors. Consequently, the extension of $\tilde{K}_f \circ \mathcal{K}$ to all C^* -algebras via unitalisation yields a functor $\tilde{K}_f \circ \mathcal{K}: \mathbf{C}^* \rightarrow \mathbf{Ab}$ that is naturally isomorphic to $K_0: \mathbf{C}^* \rightarrow \mathbf{Ab}$. Therefore, the complete operator K_0 -functor is reconstructed solely in terms of topological K -theory, the finitary version of the unitary semispectral functor, stabilisation, and unitalisation.

6. Noncommutative Topology

A natural step in using extensions to directly obtain noncommutative analogues from basic topological concepts would be to establish the conjecture that extending the topological notion of closed subset leads to its algebraic generalisation: closed two-sided ideal. Background information and definitions for this section can be found in Appendix C.

We now formalise this idea. Write \mathbf{CMSLat} for the category of complete meet-semilattices: its objects are complete lattices and its morphisms are complete meet-semilattice homomorphisms, i.e. functions that preserve arbitrary meets. Let $\mathcal{T}: \mathbf{KHaus} \rightarrow \mathbf{CMSLat}$ be the functor that assigns to a compact Hausdorff space its complete lattice of closed sets ordered by reverse inclusion (with $C_1 \leq C_2$ if and only if $C_1 \supset C_2$) and to a continuous function the complete meet-semilattice homomorphism mapping a closed set to its image under the function.⁹ Let $\tilde{\mathcal{T}}$ be its g -extension, using the unitary semispectral functor g of Definition 3.5. Moreover, let $\mathcal{I}: \mathbf{uC}^{*\text{op}} \rightarrow \mathbf{CMSLat}$ be the functor that sends a unital C^* -algebra to its complete lattice of closed two-sided ideals and a unital $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ to the homomorphism of complete meet-semilattices $\mathcal{I}(\phi): \mathcal{I}(\mathcal{B}) \rightarrow \mathcal{I}(\mathcal{A})$ mapping an ideal $I \subset \mathcal{B}$ to the ideal $\phi^{-1}(I)$ of \mathcal{A} . In the commutative case, there is a correspondence between closed sets and closed ideals via Gel'fand duality: $\mathcal{I}|_{\mathbf{ucC}^{*\text{op}}} \simeq \mathcal{T} \circ \Sigma$. This suggests the following conjecture:

Conjecture 6.1. ([23]) $\tilde{\mathcal{T}} \simeq \mathcal{I}$.

The principal theorem proved in this section is the von Neumann algebraic analogue of this conjecture:

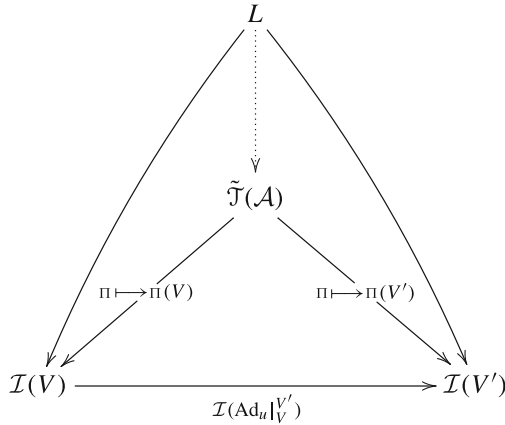
Theorem 6.2. *Let $\mathcal{T}_W: \mathbf{HStonean} \rightarrow \mathbf{CMSLat}$ be the functor assigning to a hyperstonean space its lattice of clopen sets ordered by reverse inclusion and to an open continuous function its direct image map, and let $\mathcal{I}_W: \mathbf{vNA}^{\text{op}} \rightarrow \mathbf{CMSLat}$ be the functor assigning to a von Neumann algebra its lattice of ultraweakly closed two-sided ideals and to an ultraweakly continuous (or normal) unital $*$ -homomorphism its inverse image map. Then $\tilde{\mathcal{T}}_W \simeq \mathcal{I}_W$, where $\tilde{\mathcal{T}}_W$ is the von Neumann algebraic extension of \mathcal{T} .*

Note that by von Neumann algebraic extension here we mean the g_W -extension, using the (von Neumann) unitary semispectral functor of Definition 3.8 that assigns to a von Neumann algebra the diagram consisting of its abelian von Neumann subalgebras and restrictions of inner automorphisms.

We begin by recasting Conjecture 6.1 in purely algebraic terms as a correspondence between what we call *total* and *invariant partial* ideals of unital C^* -algebras. We then formulate the analogous correspondence for von Neumann algebraic ideals, obtaining a statement equivalent to our principal theorem, and prove it.

⁹ Note that a continuous function between compact Hausdorff spaces is closed, hence the direct image map preserves arbitrary meets of closed sets $\bigwedge A_i = \text{cl}(\bigcup A_i)$.

6.1. *Partial and total ideals.* To prove Conjecture 6.1 would essentially be to demonstrate a bijective correspondence between closed two-sided ideals of a unital C^* -algebra \mathcal{A} and certain functions Π that map unital commutative sub- C^* -algebras V of a unital C^* -algebra \mathcal{A} to closed ideals of V . To see this, note that the limit lattice $\tilde{\mathcal{I}}(\mathcal{A}) = \varprojlim \circ \mathcal{I} \circ G(\mathcal{A})$ is the terminal cone over the diagram $(\mathcal{I} \circ G)(\mathcal{A})$.



Moreover, the category \mathbf{CMSLat} is monadic over \mathbf{Set} , as it is the Eilenberg–Moore category of algebras of the powerset monad [3, Examples 20.5(3) and 20.10(3)]. Consequently, the forgetful functor $U: \mathbf{CMSLat} \rightarrow \mathbf{Set}$ creates limits [3, Proposition 20.12(10)]. This means that the limit of a diagram in \mathbf{CMSLat} can be obtained by taking its limit in \mathbf{Set} —where it is given as a subset of a Cartesian product defined by equations corresponding to compatibility conditions—and equipping it with the componentwise partial order or componentwise lattice operations.

Hence, the elements of $\tilde{\mathcal{I}}(\mathcal{A})$ are precisely what we will call the *invariant partial ideals* of \mathcal{A} : choices of elements $\Pi(V)$ from each $\mathcal{I}(V)$ subject to the condition of Eq. (4) below. We can thus recast Conjecture 6.1 (and, analogously, Theorem 6.2) in terms of a correspondence between total ideals and invariant partial ideals.

6.1.1. *Partial and total ideals of C^* -algebras.* By a *total ideal* of a C^* -algebra \mathcal{A} , we mean a norm-closed two-sided ideal of \mathcal{A} .

Definition 6.3. A *partial ideal* of a unital C^* -algebra \mathcal{A} is a map Π that assigns to each unital commutative sub- C^* -algebra V of \mathcal{A} a closed ideal of V such that $\Pi(V) = \Pi(V') \cap V$ whenever $V \subset V'$.

Note that the last conditions can be rephrased as requiring that for any inclusion morphism $\iota: V \hookrightarrow V'$, we have $\Pi(V) = \mathcal{I}(\iota)(\Pi(V'))$, i.e. the following diagram commutes:

$$\begin{array}{ccc}
 V' & & \{*\} \xrightarrow{*\mapsto \Pi(V')} \mathcal{I}(V') \\
 \uparrow \iota & & \searrow \text{\scriptsize } *\mapsto \Pi(V) \quad \downarrow \mathcal{I}(\iota)::I \mapsto I \cap V \\
 V & & \mathcal{I}(V)
 \end{array} \tag{3}$$

The concept of partial ideal was introduced by Reyes [83] in the more general context of partial C^* -algebras. His definition differs slightly but is equivalent in our case: a subset P of normal elements of \mathcal{A} such that $P \cap V$ is a closed ideal of V for all unital commutative sub- C^* -algebras V of \mathcal{A} .

Partial ideals exist in abundance: every closed left (or right) ideal I of \mathcal{A} gives rise to a partial ideal Π_I in a natural way by choosing $\Pi_I(V)$ to be $I \cap V$.

For example, in a matrix algebra $M_n(\mathbb{C})$, the right ideal $pM_n(\mathbb{C})$, for $p \in M_n(\mathbb{C})$ a nontrivial projection, yields a nontrivial partial ideal of $M_n(\mathbb{C})$ in this way. As matrix algebras are simple, it cannot be the case that these nontrivial partial ideals also arise as Π_I from a total ideal I . This raises a natural question:

Question 6.4. Which partial ideals of unital C^* -algebras arise from total ideals?

Some partial ideals do not even arise from left or right ideals: for example, choosing arbitrary nontrivial ideals from every nontrivial unital commutative sub- C^* -algebra of $M_2(\mathbb{C})$ yields, in nearly all cases, nontrivial partial ideals of $M_2(\mathbb{C})$. However, a hint towards identifying those partial ideals that arise from total ideals is given by a simple observation. If u is a unitary of \mathcal{A} , then $uIu^* = I$ for any total ideal $I \subset \mathcal{A}$. This imposes a strong condition on the partial ideal Π_I that arises from I .

Definition 6.5. An *invariant partial ideal* Π of a unital C^* -algebra \mathcal{A} is a partial ideal of \mathcal{A} such that, for each unital commutative sub- C^* -algebra $V \subset \mathcal{A}$ and any unitary $u \in \mathcal{A}$, the conjugation by u of the ideal associated to V is the ideal associated to the conjugation by u of V . That is,

$$u\Pi(V)u^* = \Pi(uVu^*)$$

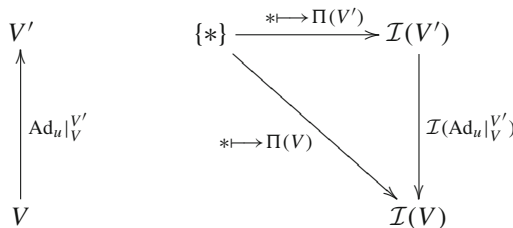
Equivalently, if we write $\text{Ad}_u : \mathcal{A} \rightarrow \mathcal{A}$ for the inner automorphism given by conjugation by u , i.e. $a \mapsto uau^*$, the condition above reads

$$\text{Ad}_u(\Pi(V)) = \Pi(\text{Ad}_u(V)).$$

Imposing this invariance condition on partial ideals is equivalent to extending the requirement on Π of Diagram (3) from inclusions $\iota : V \rightarrow V'$ to all $*$ -homomorphisms $\text{Ad}_u|_V^{V'} : V \rightarrow V'$ arising as a restriction of the domain and codomain of an inner automorphism. An invariant partial ideal is precisely a choice of $\Pi(V) \in \mathcal{I}(V)$ for each unital commutative sub- C^* -algebra V of \mathcal{A} such that whenever there is a morphism $\text{Ad}_u|_V^{V'} : V \rightarrow V'$ as above, then

$$\Pi(V) = \mathcal{I}(\text{Ad}_u|_V^{V'}) (\Pi(V')) = \text{Ad}_{u^*}(\Pi(V')) \cap V = u^* \Pi(V') u \cap V; \tag{4}$$

i.e. the following diagram commutes.



Thus, we arrive at the following reformulation of our original C^* -algebraic conjecture:

Conjecture 6.6 (Reformulation of Conjecture 6.1). A partial ideal of a unital C^* -algebra arises from a total ideal if and only if it is an invariant partial ideal. Consequently, the map $I \mapsto \Pi_I$ is a bijective correspondence between total ideals and invariant partial ideals.

Note that the first part of the statement says that the map $I \mapsto \Pi_I$ is surjective onto the invariant partial ideals. The second part of the statement follows immediately, since injectivity of this map is easily demonstrated: the left inverse is given by mapping an invariant partial ideal of the form Π_I to the linear span of $\bigcup_V \Pi(V)$, which is equal to I itself.

6.1.2. Partial and total ideals of von Neumann algebras. A total ideal of a von Neumann algebra is, as in Definition C.15, an ultraweakly closed two-sided ideal. One may define *partial ideal* (resp. *invariant partial ideal*) for a von Neumann algebra by replacing in Definition 6.3 (resp. Definition 6.5) the occurrences of “unital commutative sub- C^* -algebra” with “commutative sub-von Neumann algebra” and “closed ideal” with “ultraweakly closed ideal”. As before, a total ideal I determines an invariant partial ideal Π_I in the same way, and the map $I \mapsto \Pi_I$ is injective.

The rest of this section is devoted to establishing the following von Neumann algebraic analogue of Conjecture 6.6:

Theorem 6.7 (Reformulation of Theorem 6.2). *A partial ideal of a von Neumann algebra arises from a total ideal if and only if it is an invariant partial ideal. Consequently, the map $I \mapsto \Pi_I$ is a bijective correspondence between total ideals and invariant partial ideals.*

Besides its intrinsic interest, establishing this theorem provides some measure of evidence for the verity of Conjecture 6.6. It may be the case that the proof of the theorem may be adapted to show that the original conjecture holds for a large class of—or perhaps all— C^* -algebras.

Total ideals of a von Neumann algebra \mathcal{A} are in bijective correspondence with central projections z of \mathcal{A} : every total ideal I is of the form $z\mathcal{A}$ for a unique z (Theorem C.17). This allows us to rephrase the theorem in terms of projections, which are vastly more convenient to work with.

Definition 6.8. A *consistent family of projections* of a von Neumann algebra \mathcal{A} is a map Φ that assigns to each commutative sub-von Neumann algebra V of \mathcal{A} a projection $\Phi(V)$ in V such that:

1. for any V and V' such that $V \subset V'$, $\Phi(V)$ is the largest projection in V which is less than or equal to $\Phi(V')$, i.e.

$$\Phi(V) = \sup \{q \text{ is a projection in } V \mid q \leq \Phi(V')\}.$$

An *invariant family of projections* is such a map further satisfying:

2. for any unitary element $u \in \mathcal{A}$, $\Phi(uVu^*) = u\Phi(V)u^*$.

The correspondence between total ideals and central projections yields correspondences between partial ideals (resp. invariant partial ideals) and consistent (resp. invariant) families of projections. Therefore, we shall establish Theorem 6.7 by proving the equivalent statement below. Just as was the case for ideals, any projection p determines

a consistent family of projections Φ_p defined by choosing $\Phi_p(V)$ to be the largest projection p in V which is less than or equal to p . For a central projection z , Φ_z turns out to be an invariant family. In the opposite direction, any consistent family of projections Φ gives a central projection $\Phi(\mathcal{Z}(\mathcal{A}))$ where $\mathcal{Z}(\mathcal{A})$ is the centre of \mathcal{A} .

Theorem 6.9 (Reformulation of Theorems 6.2 and 6.7). *A consistent family of projections of a von Neumann algebra arises from a central projection if and only if it is an invariant family of projections. Consequently, the maps $z \mapsto \Phi_z$ and $\Phi \mapsto \Phi(\mathcal{Z}(\mathcal{A}))$ define a bijective correspondence between central projections and invariant families of projections.*

6.2. Technical preliminaries.

6.2.1. Little lemmata. In proving our main result, we shall make use of some simple properties of consistent families of projections which we record here as lemmata for clarity.

Lemma 6.10. *Let \mathcal{A} be a von Neumann algebra and Φ be a consistent family of projections in \mathcal{A} . Suppose V and V' are commutative sub-von Neumann algebras of \mathcal{A} with $V \subset V'$. Then:*

- (i) $\Phi(V) \leq \Phi(V')$;
- (ii) if $p \in V$ and $p \leq \Phi(V')$, then $p \leq \Phi(V)$;
- (iii) in particular, if $\Phi(V') \in V$, then $\Phi(V') = \Phi(V)$.

Proof. Properties (i) and (ii) are simple consequences of the requirement in the definition of consistent family of projections that $\Phi(V)$ is the largest projection in V smaller than $\Phi(V')$. Property (iii) is a particular case of (ii). \square

Given a commutative subset X of a von Neumann algebra \mathcal{A} , denote by V_X the commutative sub-von Neumann algebra of \mathcal{A} generated by X and the centre $\mathcal{Z}(\mathcal{A})$, i.e. $V_X = (X \cup \mathcal{Z}(\mathcal{A}))''$. Note that $V_\emptyset = \mathcal{Z}(\mathcal{A})$. Given a nonempty finite commutative set of projections $\{p_1, \dots, p_n\}$, we write V_{p_1, \dots, p_n} for $V_{\{p_1, \dots, p_n\}}$.

Lemma 6.11. *Let \mathcal{A} be a von Neumann algebra and Φ a consistent family of projections in \mathcal{A} . Let M be a commutative set of projections in \mathcal{A} and let s be the supremum of the projections in M . If $\Phi(V_m) \geq m$ for all $m \in M$, then $\Phi(V_s) \geq s$.*

Proof. For all $m \in M$, since $V_m \subseteq V_M$, we have

$$\Phi(V_M) \geq \Phi(V_m) \geq m$$

by Lemma 6.10-(i) and the assumption that $\Phi(V_m) \geq m$. Hence, $\Phi(V_M)$ is at least the supremum of the projections in M , i.e. $\Phi(V_M) \geq s$. Now, note that $s \in V_M$ as it is a supremum of projections in V_M , hence $V_s \subset V_M$. From this and $s \in V_s$, we conclude by Lemma 6.10-(ii) that $s \leq \Phi(V_s)$. \square

6.2.2. Partial orthogonality. We introduce the following notion, which will prove useful in establishing our main result. Note that, given a projection p , we write p^\perp for the projection $1 - p$.

Definition 6.12. Two projections p and q in a von Neumann algebra are *partially orthogonal* whenever there exists a central projection z such that zp and zq are equal while $z^\perp p$ and $z^\perp q$ are orthogonal.

A set of projections is said to be *partially orthogonal* whenever any pair of projections in the set is partially orthogonal.

Note that partially orthogonal projections necessarily commute. Moreover, if p_1 and p_2 are partially orthogonal, then so is the pair zp_1 and zp_2 for any central projection z . We will require in the sequel the following simple lemma:

Lemma 6.13. *Let p_1 and p_2 be projections and z be a central projection in a von Neumann algebra such that zp_1 and zp_2 are partially orthogonal and $z^\perp p_1$ and $z^\perp p_2$ are partially orthogonal. Then p_1 and p_2 are partially orthogonal.*

Proof. As zp_1 and zp_2 are partially orthogonal, there exists a central projection y such that

$$yzp_1 = yzp_2 \quad \text{and} \quad y^\perp zp_1 \perp y^\perp zp_2.$$

Similarly, as $z^\perp p_1$ and $z^\perp p_2$ are partially orthogonal, there exists a central projection x such that

$$xz^\perp p_1 = xz^\perp p_2 \quad \text{and} \quad x^\perp z^\perp p_1 \perp x^\perp z^\perp p_2.$$

Summing both statements above, we conclude that

$$(yz + xz^\perp)p_1 = (yz + xz^\perp)p_2 \quad \text{and} \quad (y^\perp z + x^\perp z^\perp)p_1 \perp (y^\perp z + x^\perp z^\perp)p_2,$$

where $yz + xz^\perp$ is a central projection and $(yz + xz^\perp)^\perp = y^\perp z + x^\perp z^\perp$. So, p_1 and p_2 are partially orthogonal. \square

6.2.3. Main lemma. When comparing projections, we write \leq to denote the usual order on projections, \preceq_M for the order up to Murray–von Neumann equivalence, and \preceq_u for the order up to unitary equivalence.

The following lemma is one of the main steps of the proof. The idea is to start with a projection q in a von Neumann algebra and to cover, as much as possible, its central carrier $C(q)$ by a commutative subset of the unitary orbit of q . The lemma states that, in order to cover $C(q)$ with projections from the unitary orbit of q , it suffices to take a commutative subset, M , and (at most) one other projection, uqu^* , which is above the remainder $C(q) - \sup M$. In other words, the remainder from what can be covered by a commutative set M is smaller than or equal to q up to unitary equivalence.

Lemma 6.14. *Let q be a projection in a von Neumann algebra \mathcal{A} . There exists a set M of projections in \mathcal{A} such that:*

- (i) $q \in M$;
- (ii) M is a subset of the unitary orbit of q ;
- (iii) M is a commutative set;
- (iv) the supremum s of M satisfies $s^R \preceq_u q$ where $s^R = C(s) - s = C(q) - s$.

Proof. Let O be the unitary orbit of q . The partially orthogonal subsets of O that contain q form a poset under inclusion. Given a chain in this poset, its union is partially orthogonal: any two projections in the union must appear together somewhere in one subset in the chain and are thus partially orthogonal. Hence, by Zorn’s lemma, we can construct a maximal partially orthogonal subset M of the unitary orbit of q such that $q \in M$. Clearly, M satisfies conditions (i)–(iii).

Denote by s the supremum of the projections in M . Its central carrier $C(s)$ is equal to the central carrier $C(q)$ of q . This is because $C(-)$ is constant on unitary orbits and $C(\sup_{m \in M} m) = \sup_{m \in M} C(m)$. We now need to show that $s^R \preceq_u q$.

By the comparison theorem for projections in a von Neumann algebra (Theorem C.14), there is a central projection y such that

$$ys^R \succ_M yq \quad \text{and} \quad y^\perp s^R \preceq_M y^\perp q. \tag{5}$$

We show that this central projection can be taken to be below $C(q)$. Consider the central projection $z = yC(q)$. Then, for any projection $r \leq C(q)$, i.e. $C(q)r = r$, we have

$$zs = yC(q)r = yr \quad \text{and} \quad z^\perp r = r - zr = r - yr = y^\perp r.$$

Since both $s^R, q \leq C(q)$, one can rewrite (5) as

$$zs^R \succ_M zq \quad \text{and} \quad z^\perp s^R \preceq_M z^\perp q,$$

where $z = yC(q) \leq C(q)$.

By Proposition C.12, as q and s^R are orthogonal, there are unitaries that witness these order relationships. That is, there are unitaries u and v such that

$$zs^R \geq z(uqu^*) \quad \text{and} \quad z^\perp s^R \leq z^\perp (vqv^*). \tag{6}$$

We will show that z vanishes and thus conclude that $s^R \leq vqv^*$.

Define u_z to be the unitary $zu + z^\perp 1$ which acts as u within the range of z and as the identity on the range of z^\perp . We first establish that $u_z qu_z^*$ and m are partially orthogonal for every $m \in M$.

Let $m \in M$. As M was defined to be a partially orthogonal set of projections and $q \in M$, we know that q and m are partially orthogonal, and thus that $z^\perp q$ and $z^\perp m$ are partially orthogonal. However, as $z^\perp u_z = z^\perp$, we may express this as: $z^\perp (u_z qu_z^*)$ and $z^\perp m$ are partially orthogonal. Additionally, on the range of z , we have that

$$z(u_z qu_z^*) = z(uqu^*) \leq zs^R \quad \text{and} \quad zm \leq zs,$$

implying that $z(u_z qu_z^*)$ and zm are orthogonal, hence partially orthogonal. Putting both parts together, we have that $z^\perp u_z qu_z^*$ and $z^\perp m$ are partially orthogonal and that $z(u_z qu_z^*)$ and zm are partially orthogonal. We may thus apply Lemma 6.13 and conclude that $u_z qu_z^*$ and m are partially orthogonal as desired.

Having established that $u_z qu_z^*$ is partially orthogonal to all the projections in M , it follows by maximality of M that $u_z qu_z^* \in M$. Hence,

$$zu_z qu_z^* \leq u_z qu_z^* \leq \sup M = s.$$

Yet, by construction,

$$zu_z qu_z^* = zuqu^* \leq zs^R \leq s^R,$$

and so $zu_zqu_z^*$ must be orthogonal to s . Being both contained within and orthogonal to s , $zu_zqu_z^*$ must vanish. Therefore, the unitarily equivalent projection zq must also vanish. Now, $zq = 0$ implies that z^\perp covers q . But since z^\perp is a central projection, it must also cover the central carrier of q , i.e. $C(q) \leq z^\perp$. We thus have $z \leq C(q) \leq z^\perp$, forcing z to be zero.

We may finally conclude, by (6), that $s^R \leq vqv^*$. \square

6.3. Main theorem. Theorem 6.9, and thus our principal result, Theorem 6.7 or Theorem 6.2, will follow as an immediate corollary of the following theorem:

Theorem 6.15. *In a von Neumann algebra \mathcal{A} , any invariant family of projections Φ arises from a central projection, i.e. Φ is equal to Φ_z for the central projection $z = \Phi(\mathcal{Z}(\mathcal{A}))$.*

Proof. Let Φ be an invariant family of projections. Suppose W is a commutative sub-von Neumann algebra of \mathcal{A} which contains the centre $\mathcal{Z}(\mathcal{A})$, and let q be the projection $\Phi(W)$. We claim that q is, in fact, equal to its own central carrier $C(q)$, and thus central. As $q \leq C(q)$ is true by definition, we must show that $q \geq C(q)$.

We start by applying Lemma 6.14 to q . Let M be the resulting commuting set of projections in the unitary orbit of q , s be the supremum of the projections in M , and s^R be $C(q) - s = C(s) - s$. From the lemma, we know that $s^R \preceq_u q$, i.e. there exists a unitary u such that $s^R \leq uqu^*$.

First note that, since $V_q \subset W$ and $q \in V_q$, by Lemma 6.10-(iii), we have that $\Phi(V_q) = q$. Then, by unitary invariance of the family of projections, for every $m \in M$ we have that $\Phi(V_m) = m$. Hence, we can apply Lemma 6.11 to conclude that $\Phi(V_s) \geq s$. We also conclude, again by unitary invariance of Φ , that $\Phi(V_{uqu^*}) = uqu^* \geq s^R$.

Now, note that uqu^* and s^R commute. Moreover, $V_s = V_{s^R}$ since $s^R = C(q) - s$, hence s^R is in the algebra generated by s and the centre, and vice-versa. So, there is a commutative sub-von Neumann algebra $V_{s,uqu^*} \supseteq V_s, V_{uqu^*}$. By Lemma 6.10-(i) and the two conclusions of the preceding paragraph, we then have

$$\Phi(V_{s,uqu^*}) \geq \Phi(V_s) \vee \Phi(V_{uqu^*}) \geq s \vee s^R = C(q).$$

But, since $C(q) \in V_{uqu^*}$ by virtue of it being contained in the centre, we can apply Lemma 6.10-(ii) to find that $\Phi(V_{uqu^*}) \geq C(q)$, i.e. $uqu^* \geq C(q)$. Finally, by unitary invariance,

$$q = u^*(uqu^*)u \geq u^*C(q)u = C(q),$$

concluding the proof that q is central.

We have shown that the projection $\Phi(W)$ is central for every commutative sub-von Neumann algebra W containing the centre $\mathcal{Z}(\mathcal{A})$. By Lemma 6.10-(iii), this means that $\Phi(W)$ is equal to $\Phi(\mathcal{Z}(\mathcal{A}))$, the projection chosen at the centre, for all such W . In turn, this determines the image of Φ on all commutative sub-von Neumann algebras W' as

$$\Phi(W') = \sup \{ p \text{ is a projection in } W' \mid p \leq \Phi(V_{W'}) = \Phi(\mathcal{Z}(\mathcal{A})) \},$$

and we find that Φ must be equal to $\Phi_{\Phi(\mathcal{Z}(\mathcal{A}))}$. \square

7. Conclusions

In this work, we have argued that a nonstandard—but nevertheless foundationally important—notation of quantum state space is the dual object of a noncommutative algebra with respect to state-observable duality, and noted that such spaces are the objects of study within noncommutative geometry of C^* -algebras. In noncommutative geometry [20], Gel'fand–Naïmark duality justifies interpreting noncommutative C^* -algebras as representing the algebra of observables on a hypothetical geometric space; it further provides a heuristic method of translating topological concepts into algebraic language. We further argued that an explicitly geometric construction of this notion of quantum state space should provide simple means for the direct extension of topological concepts to noncommutative generalisations in a manner coinciding with the constructions of noncommutative geometry.

Our ansatz for a geometric space associated to a noncommutative algebra comes from the *spectral presheaf* construction of Hamilton–Isham–Butterfield [52], which accounts for the essential nonclassicality of quantum theory expressed by the Bell–Kochen–Specker theorem [12,68] by associating a classical state space to each context (commutative subalgebra) of the algebra of observables. We offer an alternative interpretation of this collection of classical state spaces as the collection of tractable quotient spaces of the noncommutative space represented by the algebra of observables. We show how functorial associations of spatial diagrams to algebras yields automatic methods of extending functors defined on topological spaces to ones defined on C^* -algebras. After modifying the spectral presheaf by including data related to inner automorphisms, we consider the extensions of two functors: the topological K -functor [8] and the functor \mathcal{T} that assigns to a space its lattice of closed subsets.

In the former case, we give a novel definition of operator K -theory, K_0 , in terms of a colimit of vector bundles over the finite quotient spaces of stable noncommutative spaces. This formally aligns very closely with the extension \tilde{K} of the topological K -functor. Specifically, we have shown that operator K_0 of a C^* -algebra \mathcal{A} corresponds to the extension of the topological K -functor from the spaces corresponding to finite-dimensional subalgebras of the stabilisation of \mathcal{A} . While $K_0 \simeq \tilde{K}$ holds for finite-dimensional C^* -algebras, whether it holds in general (or whether $K_0 \simeq \tilde{K} \circ \mathcal{K}$) remains an open question.

In the latter case, we establish a bijective correspondence between ideals of a von Neumann algebra and what could be thought of as clopen subsets of its associated spatial diagram. More formally, we display a natural isomorphism $\mathcal{I}_W \simeq \tilde{\mathcal{T}}_W$ between the functor \mathcal{I}_W that assigns to a von Neumann algebra its lattice of ultraweakly closed two-sided ideals and the extension of the functor $\tilde{\mathcal{T}}_W$ mapping a (hyperstonean) topological space to its lattice of clopen sets. This theorem is the von Neumann algebraic analogue of the conjecture that $\mathcal{I} \simeq \tilde{\mathcal{T}}$ where \mathcal{I} is the ideal lattice functor and $\tilde{\mathcal{T}}$ the functor mapping a topological space to its lattice of closed sets.

As a consequence of the von Neumann algebraic theorem, the C^* -algebraic conjecture holds for all finite-dimensional C^* -algebras. The question of whether it holds for all C^* -algebras remains open. An immediate question is whether the conjecture holds for AF-algebras, i.e. those that arise as limits of finite-dimensional C^* -algebras [16]. This would follow immediately from a proof of the continuity of $\tilde{\mathcal{T}}$. Another approach would be to prove the whole conjecture directly by using the proof of the von Neumann algebraic version as a guide. Indeed, one might still be able to reduce the question to one about projections by working in the enveloping von Neumann algebra \mathcal{A}^{**} of a C^* -algebra

\mathcal{A} . In this setting, the total ideals of a C^* -algebra \mathcal{A} correspond to certain total ideals of the enveloping algebra \mathcal{A}^{**} [7]: those that correspond to open central projections. One would have to find a correspondence between open central projections of \mathcal{A}^{**} and certain families of open projections obeying a restricted form of unitary invariance. This might also be formulated directly at the C^* -algebraic level by considering approximate identities for ideals as playing the role of central projections.

Proving that $\mathcal{I} \simeq \tilde{\mathcal{I}}$ would establish a strong relationship between the topologies of the geometric object $G(\mathcal{A})$ and $\text{Prim}(\mathcal{A})$, the primitive ideal space of \mathcal{A} : we would be able to recover the lattice of the hull-kernel topology on $\text{Prim}(\mathcal{A})$ as the limit of the topological lattices of the object $G(\mathcal{A})$. Establishing this conjecture would allow considering G to be an enrichment of Prim . Prim is a C^* -algebraic variant of the ring-theoretic spectrum whose hull-kernel topology provides the basis for sheaf-theoretic methods in ring theory.

Our proposal for a notion of noncommutative spectrum bears structural similarity with related functional and order-theoretic constructions that represent noncommutative algebras as a commutative fragment augmented with a unitary group action [11, 28, 51, 58, 76]. Novel abstract frameworks for understanding noncommutative algebras in terms of commutative subalgebras have appeared since our starting to work on this line of research, notably [41, 58], which include some results with a similar flavour to ours. Understanding the relationships between these approaches, and the question of whether some synthesis of them might better clarify noncommutative geometry is an important line of future work.

As discussed in more detail in Sect. 2.5, to establish a concrete duality, it would be important to characterise which diagrams of spaces arise as spatial diagrams of a noncommutative algebra and which maps between them correspond to unital $*$ -homomorphisms. To facilitate computations, some notion of a sub-spatial diagram ‘cover’, analogous to a tractable choice of charts for a manifold, may be needed. Another key step would be to recover a noncommutative algebra from its spatial diagram together with some extra data.

There are some topological concepts that are usually understood to be inextendable to the commutative setting. The simplest example is provided by the notion of points. We have seen that the forgetful functor to the category Set , which associates to a space its set of points, has a trivial extension to the noncommutative setting, and that this fact corresponds to the Bell–Kochen–Specker theorem from quantum foundations. So, the topological notion of points does not survive our process of translation, in agreement with the common intuition that noncommutative spaces have *no points*. On the other hand, there are interesting concepts in the noncommutative setting that become trivial when restricted to the commutative case, e.g. Tomita–Takesaki theory. Clearly, such intrinsically noncommutative concepts cannot be obtained by the process of extension outlined in this article, via diagrams of topological spaces corresponding to commutative subalgebras. However, they can provide valuable guidance in determining what extra data must be adjoined to the diagram of topological spaces in order to recover a noncommutative algebra, since that data must be used in an essential way to define those concepts.

It would also be interesting to calculate the spatial diagram explicitly for some special examples. One promising possibility is the canonical commutation relations algebra, which is closely connected with quantisation and thus physically very significant. In this case, the Krichever–Mulase classification [77] of certain commutative subalgebras of $\mathbb{C}[[x]][[\partial]]$ provides a potentially highly useful roadmap. Another possibly tractable class

of algebras for computations are those that arise as crossed product algebras, wherein a group action on a C^* -algebra is embedded in a larger C^* -algebra where the action is realised as a group of inner automorphisms. This class includes within it the important example of noncommutative tori, the computation of whose K -theory was considered a very difficult problem [84].

The idea of looking at commutative quotients is a very general one and could perhaps be applied to analyse other sorts of noncommutative algebras other than C^* -algebras. The ideas outlined above might be applied to any duality involving a category of geometric objects and a category of commutative algebras.

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A. Concrete Colimit Construction

Here, we give a concrete construction of the generalised colimit functor $\varinjlim: \underline{Diag}(\mathbf{C}) \rightarrow \mathbf{C}$ in terms of coequalisers of coproducts.

Recall that the colimit of a functor D from a category \mathbf{A} to a cocomplete category \mathbf{C} can be expressed as a coequaliser of two coproducts [72, p. 355]:

$$\coprod_{u: i \rightarrow j} D(\text{dom}u) \begin{matrix} \xrightarrow{\theta} \\ \xrightarrow{\tau} \end{matrix} \coprod_i D(i)$$

The first coproduct is over all morphisms $u: i \rightarrow j$ of \mathbf{A} and the second is over all objects i of \mathbf{A} . We denote the canonical injections for these coproducts by

$$\lambda_v: D(\text{dom}v) \rightarrow \coprod_{u: i \rightarrow j} D(\text{dom}u)$$

and

$$\kappa_j: D(j) \rightarrow \coprod_i D(i).$$

The morphisms θ and τ can be defined by specifying their compositions with the λ_v :

$$\theta \circ \lambda_v = \kappa_{\text{dom}v} \quad \text{and} \quad \tau \circ \lambda_v = \kappa_{\text{cod}v} \circ D(v)$$

The advantage of this coequaliser presentation of the colimit is that we may determine a \mathbf{C} -morphism between the colimits of two functors $D: \mathbf{A} \rightarrow \mathbf{C}$ and $E: \mathbf{B} \rightarrow \mathbf{C}$ by specifying a natural transformation between their coequaliser diagrams. That is, by giving its components, \mathbf{C} -morphisms N and M such that the following diagrams commute:

$$\begin{array}{ccc} \coprod_{u: i \rightarrow j} D(\text{dom}u) & \xrightarrow{\theta} & \coprod_i D(i) \\ \downarrow N & & \downarrow M \\ \coprod_{u': i' \rightarrow j'} E(\text{dom}u') & \xrightarrow{\theta'} & \coprod_{i'} E(i') \end{array}$$

$$\begin{array}{ccc} \coprod_{u: i \rightarrow j} D(\text{dom}u) & \xrightarrow{\tau} & \coprod_i D(i) \\ \downarrow N & & \downarrow M \\ \coprod_{u': i' \rightarrow j'} E(\text{dom}u') & \xrightarrow{\tau'} & \coprod_{i'} E(i') \end{array}$$

where i' and u' range over all objects and all morphisms of \mathbf{B} , respectively, while i and u are as above. We denote the canonical injections into the coproducts for E by $\lambda'_{i'}$ and $\kappa'_{j'}$.

Given a $\text{Diag}(\mathbf{C})$ -morphism (f, η) between D and E we define N and M by giving their compositions with the canonical injections:

$$N \circ \lambda_v = \lambda'_{f(v)} \circ \eta_{\text{dom}v} \quad \text{and} \quad M \circ \kappa_j = \kappa'_{f(j)} \circ \eta_j,$$

It is straightforward to verify that the above diagrams commute, that is, that $\theta' \circ N = M \circ \theta$ and that $\tau' \circ N = M \circ \tau$, by computing the composition of these maps with the λ_v . The \mathbf{C} -morphism assigned by $\xrightarrow{\text{lim}}$ to (f, η) is then defined to be the morphism that is induced by the natural transformation (whose components are N and M) between the coequaliser diagrams for the colimits of D and E .

Functoriality of $\xrightarrow{\text{lim}}$ is then straightforwardly verified by computing the compositions of the components of the natural transformations induced by (f, η) and (g, μ) and seeing that the resulting natural transformation is the same as the one induced by $(g \circ f, \mu_f \circ \eta)$.

B. Topological and C^* -Algebraic K -Theory

B.1. Topological K -theory. Topological K -theory, invented by Atiyah–Hirzebruch [8] after Grothendieck [49], is an extraordinary cohomology theory, i.e. satisfies the Eilenberg–Steenrod axioms [36] except the dimension axiom. It is determined by a sequence of contravariant functors from \mathbf{KHaus} to \mathbf{Ab} . After early successes, including the solution to the classical problem of determining how many linearly independent vector fields can be constructed on \mathbf{S}^n [4], the subject blossomed to include algebraic and analytic versions. The core idea is to describe the geometry of a space by algebraic

information about the possible vector bundles over it. Here, we briefly review its definition. Its generalisation to C^* -algebras, operator K_0 , is a key tool of noncommutative geometry and will be outlined in the next subsection.

Definition B.1. For a compact Hausdorff space X , its *vector bundle monoid*, $V(X)$, is the set of isomorphism classes of complex vector bundles over X with the abelian addition operation of fibrewise direct sum: $[E] + [F] = [E \oplus F]$. A continuous function $f: X \rightarrow Y$ yields a monoid homomorphism $V(f): V(Y) \rightarrow V(X)$ by the pullback of bundles. That is, if $p: E \rightarrow Y$ is a bundle over Y , the bundle f^*E is a bundle over X given by the projection to X of

$$\{(x, v) \in X \times E \mid f(x) = p(v)\}$$

and $V(f)([E]) = [f^*E]$. This defines a functor $V: \mathbf{KHaus}^{\text{op}} \rightarrow \mathbf{AbMon}$.

Definition B.2. For an abelian monoid M , its *Grothendieck group*, $\mathcal{G}(M)$, is the abelian group $(M \times M)/\sim$ where \sim the equivalence relation given by

$$(a, b) \sim (c, d) \text{ iff } \exists e \in M, a + d + e = b + c + e.$$

For a monoid homomorphism $\phi: M \rightarrow N$, the group homomorphism $\mathcal{G}(\phi): \mathcal{G}(M) \rightarrow \mathcal{G}(N)$ is given by $\mathcal{G}(\phi)([(a, b)]) = [(\phi(a), \phi(b))]$. This defines a functor $\mathcal{G}: \mathbf{AbMon} \rightarrow \mathbf{Ab}$.

Intuitively, an element $[(a, b)]$ of $\mathcal{G}(M)$ can be thought of as a formal difference $a - b$ of elements of M . With this interpretation in mind, it is easy to see that $\mathcal{G}(M)$ is indeed a group, with addition given componentwise, neutral element $[(0, 0)]$, and the inverse of $[(a, b)]$ equal to $[(b, a)]$. Moreover, there is a monoid homomorphism $i: M \rightarrow \mathcal{G}(M)$ given by $a \mapsto [(a, 0)]$. As an example, the Grothendieck group of the additive monoid of natural numbers (including zero) is the additive group of integers.

The Grothendieck group functor \mathcal{G} is an explicit presentation of the group completion functor, the left adjoint to the forgetful functor from \mathbf{Ab} to \mathbf{AbMon} . This means that $\mathcal{G}(M)$ is the ‘most general’ group containing a homomorphic image of M , in the sense that it satisfies the universal property that any monoid homomorphism from M to an abelian group factors uniquely through the monoid homomorphism $i: M \rightarrow \mathcal{G}(M)$.

Definition B.3. The *topological K -functor* $K: \mathbf{KHaus}^{\text{op}} \rightarrow \mathbf{Ab}$ is $\mathcal{G} \circ V$.

From the topological K -functor, one can easily construct the full sequence of functors K_n for $n \in \mathbb{N}$.

Definition B.4. The *suspension functor* $S: \mathbf{KHaus} \rightarrow \mathbf{KHaus}$ maps a space X to the quotient space

$$X \times [0, 1] / \{(x, 0) \sim (x', 0) \text{ and } (x, 1) \sim (x', 1) \text{ for all } x, x' \in X\}$$

and a continuous function $f: X \rightarrow Y$ to the map $[(x, t)] \mapsto [(f(x), t)]$.

Definition B.5. *Topological K -theory* is the sequence of functors $K_n: \mathbf{KHaus}^{\text{op}} \rightarrow \mathbf{Ab}$ with $n \in \mathbb{N}$ defined by $K_n = K \circ S^n$.

Bott periodicity [9] provides natural isomorphisms $K_n \cong K_{n+2}$. We are left with $K_0 = K$ and $K_1 = K \circ S$. Note that topological K -theory additionally possesses a ring structure which does not survive in the noncommutative case.

B.2. Operator K -theory. Here, we outline the generalisation of topological K -theory to operator K -theory by the canonical method of noncommutative geometry. We provide the definition and properties of the operator K_0 -functor which we will use in our analysis of the extension of the topological K -functor. These are basic facts found in any introduction to the subject, e.g. [40,86,99]. We start by defining K_0 for unital C^* -algebras, and then extend it to the nonunital case via unitalisation.

B.2.1. Operator K -theory for unital C^* -algebras. In order to generalise a topological concept to the noncommutative case, one must begin with a characterisation in terms of commutative algebra of the topological concept in question. In the case of K -theory, this requires phrasing the notion of a *complex vector bundle over X* in terms of the algebra $C(X)$ of continuous complex-valued functions on X . This rephrasing is provided by the Serre–Swan theorem:

Theorem B.6 (Serre–Swan [94]). *The category of complex vector bundles over a compact Hausdorff space X is equivalent to the category of finitely generated projective $C(X)$ -modules.*

Recall that a projective \mathcal{A} -module is the direct summand of a free \mathcal{A} -module. Roughly, the module associated to a vector bundle E over X is the set of continuous global sections of E with the obvious operations. This justifies considering a finitely generated projective (left) \mathcal{A} -module to represent a complex vector bundle over the noncommutative space underlying the C^* -algebra \mathcal{A} .

The canonical translation process of noncommutative geometry suggests, having now in our possession an algebraic characterisation in terms of $C(X)$ of the topological notion of complex vector bundle, that we use it to define its noncommutative generalisation. That is, define the Murray–von Neumann semigroup of a C^* -algebra to be the abelian monoid of its finitely generated projective modules (up to the appropriate notion of equivalence and with an appropriate addition operation). It turns out to be more convenient to work with an algebraic gadget which is equivalent to finitely generated projective \mathcal{A} -modules: namely, projections in a matrix algebra $M_n(\mathcal{A})$ over \mathcal{A} . If μ is such a finitely generated projective module μ , then there exists another module μ^\perp such that $\mu \oplus \mu^\perp \simeq \mathcal{A}^n$. We thus identify the module μ with the projection $p: \mathcal{A}^n \rightarrow \mu$, or rather, with the canonical representation of that projection as an element of the matrix algebra $M_n(\mathcal{A})$.

Equipped with our algebraic characterisation of vector bundles, we are ready to begin defining operator K -theory in a manner directly analogous with the construction of topological K -theory.

Definition B.7. Let \mathcal{A} be a C^* -algebra. Two projections $p \in M_n(\mathcal{A})$ and $q \in M_m(\mathcal{A})$ with $n, m \in \mathbb{N}$ are *Murray–von Neumann equivalent*, denoted $p \sim_M q$, whenever there is a partial isometry v in the C^* -algebra $M_{m,n}(\mathcal{A})$ of $m \times n$ matrices over \mathcal{A} such that $p = vv^*$ and $q = v^*v$.

Definition B.8 (The Murray–von Neumann semigroup for unital \mathcal{A}). Let \mathcal{A} be a unital C^* -algebra. Its *Murray–von Neumann semigroup*, $V_0(\mathcal{A})$, is the set of Murray–von Neumann equivalence classes of projections in matrices over \mathcal{A} :

$$\bigsqcup_{n \in \mathbb{N}} \{p \in M_n(\mathcal{A}) \mid p \text{ is a projection}\} / \sim_M .$$

It is equipped with the abelian addition operation

$$[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right],$$

for which the equivalence class of the zero projection is a neutral element. Therefore, $V_0(\mathcal{A})$ is an abelian monoid. A unital $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ yields a monoid homomorphism $V_0(\phi): V_0(\mathcal{A}) \rightarrow V_0(\mathcal{B})$ given by $[p] \mapsto [M_n(\phi)(p)]$ for each $n \in \mathbb{N}$ and p a projection in $M_n(\mathcal{A})$, where $M_n(\phi)$ acts on elements of $M_n(\mathcal{A})$ by entrywise application of ϕ . This defines a functor $V_0: \mathbf{uC}^* \rightarrow \mathbf{AbMon}$.

Definition B.9. The operator K_0 -functor for unital C^* -algebras, $K_0: \mathbf{uC}^* \rightarrow \mathbf{Ab}$, is $\mathcal{G} \circ V_0$.

B.2.2. Operator K -theory for (nonunital) C^ -algebras.* So far, we have defined operator K -theory only for the unital case. We describe the extension of K_0 to all C^* -algebras.

Definition B.10. The *minimal unitalisation* of a C^* -algebra \mathcal{A} (which itself may or may not be unital) is defined as the unital C^* -algebra \mathcal{A}^+ with underlying set $\mathcal{A} \times \mathbb{C}$, componentwise addition and scalar multiplication, and multiplication and involution given by

$$(a, z)(a', z') = (aa' + z'a + za', zz'), \quad (a, z)^* = (a^*, \bar{z}).$$

There exists a unique C^* -norm on \mathcal{A}^+ , whose definition we omit, extending the norm on \mathcal{A} .

Note that $(-)^+$ is a functor from \mathbf{C}^* to \mathbf{uC}^* : a $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ gets mapped to the unital $*$ -homomorphism $\phi^+: \mathcal{A}^+ \rightarrow \mathcal{B}^+$ given by $(a, z) \mapsto (\phi(a), z)$.

A copy of \mathcal{A} lives canonically inside \mathcal{A}^+ in the first component. Indeed, the unitalisation of a C^* -algebra yields a short exact sequence

$$0 \rightarrow \mathcal{A} \xrightarrow{\iota} \mathcal{A}^+ \xrightarrow{\pi} \mathbb{C} \rightarrow 0$$

with ι being the injection into the first component and π the projection to the second component. Exactness justifies identifying \mathcal{A} with $\ker \pi$.

Definition B.11. The K_0 group of a C^* -algebra \mathcal{A} is the subgroup of $K_0(\mathcal{A}^+)$ given by the kernel of $K_0(\pi)$. A $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ yields a homomorphism from $\ker K_0(\mathcal{A}^+ \xrightarrow{\pi} \mathbb{C})$ to $\ker K_0(\mathcal{B}^+ \xrightarrow{\pi} \mathbb{C})$ by restriction of $K_0(\phi^+)$ to the kernel of $K_0(\mathcal{A}^+ \xrightarrow{\pi} \mathbb{C})$. This defines the *operator K_0 -functor*, $K_0: \mathbf{C}^* \rightarrow \mathbf{Ab}$.

B.2.3. Higher operator K -groups. As in the topological case, one can easily construct the full sequence of functors K_n for $n \in \mathbb{N}$.

Definition B.12. The *suspension functor* $S: \mathbf{C}^* \rightarrow \mathbf{C}^*$ maps a C^* -algebra \mathcal{A} to a sub- C^* -algebra of $C(\mathbb{T}, \mathcal{A})$, the C^* -algebra of continuous \mathcal{A} -valued functions on the complex unit circle \mathbb{T} , consisting of those functions $f: \mathbb{T} \rightarrow \mathcal{A}$ satisfying $f(1) = 0$. (Alternatively, $S(\mathcal{A}) = \mathcal{A} \otimes C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ is the C^* -algebra of continuous complex-valued functions on \mathbb{R} vanishing at infinity.) A $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ is mapped to the $*$ -homomorphism $S(\phi): S(\mathcal{A}) \rightarrow S(\mathcal{B})$ defined by $f \mapsto \phi \circ f$.

Note that this suspension functor is an extension of the one defined on topological spaces in the sense that for algebras $\mathcal{A} = C(X)$, we have that $S(\mathcal{A}) = C(S(X))$.

Definition B.13. *Operator K-theory* is the sequence of functors $K_n : \mathbf{C}^* \rightarrow \mathbf{Ab}$ with $n \in \mathbb{N}$ defined by $K_n = K_0 \circ S^n$

Generalised Bott periodicity provides natural isomorphisms $K_n \simeq K_{n+2}$. We are left with K_0 and $K_1 = K_0 \circ S$.

B.2.4. Stability.

Definition B.14. The *compact operators* \mathcal{K} is the sub- C^* -algebra of $\mathcal{B}(\mathcal{H})$, with \mathcal{H} a Hilbert space of countable dimension, which is generated by the finite rank operators.

Alternatively, it is defined as the colimit (direct limit) in the category \mathbf{C}^* of the sequence of matrix algebras

$$M_1(\mathbb{C}) \hookrightarrow M_2(\mathbb{C}) \hookrightarrow M_3(\mathbb{C}) \hookrightarrow \dots$$

where the injections are inclusion into the upper left corner: $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$.

The C^* -algebra \mathcal{K} is *nuclear*, which means that, for any C^* -algebra \mathcal{A} , there is a unique C^* -norm on the algebraic tensor product $\mathcal{A} \otimes_{\text{alg}} \mathcal{K}$ and thus we may speak unambiguously of the C^* -algebra $\mathcal{A} \otimes \mathcal{K}$.

Definition B.15. The *stabilisation functor* $\mathcal{K} : \mathbf{C}^* \rightarrow \mathbf{C}^*$ maps a C^* -algebra \mathcal{A} to the C^* -algebra $\mathcal{K}(\mathcal{A}) = \mathcal{A} \otimes \mathcal{K}$. A $*$ -homomorphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is mapped to $\mathcal{K}(\phi) : \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{B})$ defined by $\mathcal{K}(\phi) = \phi \otimes id_{\mathcal{K}}$.

This is alternatively defined as the colimit of matrix algebra functors. That is, $\mathcal{K}(\mathcal{A})$ is the colimit (direct limit) in \mathbf{C}^* of the sequence

$$M_1(\mathcal{A}) \hookrightarrow M_2(\mathcal{A}) \hookrightarrow M_3(\mathcal{A}) \hookrightarrow \dots$$

where the morphisms are inclusions into the upper left corner. A $*$ -homomorphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ determines $*$ -homomorphisms $M_n(\phi) : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ by entrywise application of ϕ which form the components of a natural transformation yielding $\mathcal{K}(\phi)$ as colimit.

Since the C^* -algebra $\mathcal{K} \otimes \mathcal{K}$ is isomorphic to \mathcal{K} , the stabilisation functor is an idempotent operation, i.e. $\mathcal{K} \circ \mathcal{K} \simeq \mathcal{K}$.

Definition B.16. A C^* -algebra \mathcal{A} is called *stable* or a *stabilisation* if it is fixed (up to isomorphism) by the \mathcal{K} functor, i.e. $\mathcal{A} \simeq \mathcal{K}(\mathcal{A}) = \mathcal{A} \otimes \mathcal{K}$.

Note that no stable C^* -algebra can be unital. Two C^* -algebras \mathcal{A} and \mathcal{B} are *stably equivalent* when $\mathcal{K}(\mathcal{A}) \simeq \mathcal{K}(\mathcal{B})$. Among stable C^* -algebras, stable equivalence reduces to ordinary isomorphism equivalence. As we shall see, operator K -theory doesn't distinguish between stably equivalent algebras.

Theorem B.17. *Operator K-theory is matrix stable. That is, $K_0 \simeq K_0 \circ M_n$ and $K_1 \simeq K_1 \circ M_n$, where M_n is the functor that forms $(n \times n)$ -matrix algebras over C^* -algebras.*

Theorem B.18. *Operator K -theory is continuous. That is, if*

$$\mathcal{A}_1 \longrightarrow \mathcal{A}_2 \longrightarrow \mathcal{A}_3 \longrightarrow \dots$$

is a direct sequence of C^ -algebras and $*$ -homomorphisms,*

$$K_0(\mathcal{A}_1) \longrightarrow K_0(\mathcal{A}_2) \longrightarrow K_0(\mathcal{A}_3) \longrightarrow \dots$$

is its image under the K_0 -functor, and $\mathcal{A} = \varinjlim \mathcal{A}_n$, then,

$$K_0(\mathcal{A}) \simeq \varinjlim K_0(\mathcal{A}_n)$$

via the obvious homomorphism induced between cones. A similar statement holds for K_1 .

As a consequence of the preceding two theorems, and the alternative definition of the compact operators as the limit of a direct sequence of matrix algebras, we obtain:

Theorem B.19. *Operator K -theory is stable. That is, $K_0 \simeq K_0 \circ \mathcal{K}$ and $K_1 \simeq K_1 \circ \mathcal{K}$.*

Consequently, the operator K -theory functors are determined by their restrictions to stable C^* -algebras.

B.3. Alternative definition of operator K_0 -functor. The Murray–von Neumann semi-group, and thus the K_0 -group, of a unital C^* -algebra \mathcal{A} can be expressed in a rather simple fashion in terms of projections of its stabilisation [86, Exercise 6.6]. We require this definition in the proof of Theorem 5.2 and thus describe it in explicit detail.

Definition B.20. Let \mathcal{A} be a C^* -algebra. Two projections p and q in \mathcal{A} are *unitarily equivalent*, denoted by $p \sim_u q$, whenever there is a unitary $u \in \mathcal{A}^+$ such that $p = uqu^*$. We write $[p]$ for the unitary equivalence class of p .

Given projections $p_1, \dots, p_k \in \mathcal{K}(\mathcal{A})$, one can find pairwise orthogonal representatives of their unitary equivalence classes, i.e. there exist projections $q_1, \dots, q_k \in \mathcal{K}(\mathcal{A})$ such that $p_i \sim_u q_i$ ($i \in \{1, \dots, k\}$) and all the q_i are pairwise orthogonal [86, Exercise 6.6].

The Murray–von Neumann semigroup for unital C^* -algebras admits the following alternative characterisation:

Definition B.21 (*The Murray–von Neumann semigroup for unital \mathcal{A} , alternative definition*). Let \mathcal{A} be a unital C^* -algebra. The elements of $V_0(\mathcal{A})$ are the unitary equivalence classes of projections in $\mathcal{K}(\mathcal{A})$. The abelian addition operation is given by orthogonal addition. That is, if p and p' are two projections in $\mathcal{K}(\mathcal{A})$, then $[p] + [p'] = [q + q']$ where q and q' are orthogonal representatives of $[p]$ and $[p']$, respectively (i.e. $p \sim_u q$, $p' \sim_u q'$, and $q \perp q'$). The equivalence class of the zero projection is a neutral element for this operation, making $V_0(\mathcal{A})$ an abelian monoid. A unital $*$ -homomorphism $\phi: \mathcal{A} \longrightarrow \mathcal{B}$ yields a monoid homomorphism $V_0(\phi): V_0(\mathcal{A}) \longrightarrow V_0(\mathcal{B})$ by $[p] \longmapsto [\mathcal{K}(\phi)(p)]$. This defines a functor $V_0: \mathbf{uC}^* \longrightarrow \mathbf{AbMon}$.

Through this reformulation of the Murray–von Neumann semigroup functor V_0 , we automatically get a new description of K_0 by composition with the Grothendieck group functor, as $K_0 = \mathcal{G} \circ V_0$. Then, $K_0(\mathcal{A})$ is simply the collection of formal differences

$$[p] - [q]$$

of elements of $V_0(\mathcal{A})$ with

$$[p] - [q] = [p'] - [q']$$

precisely when there exists $[r]$ such that

$$[p] + [q'] + [r] = [p'] + [q] + [r].$$

Composing the action on morphisms of the Grothendieck group functor after the action of V_0 just defined, we find that a unital $*$ -homomorphism $\phi: \mathcal{A} \rightarrow \mathcal{B}$ between unital C^* -algebras yields a group homomorphism between the K_0 groups of \mathcal{A} and \mathcal{B} given by

$$[p] - [q] \mapsto [\mathcal{K}(\phi)(p)] - [\mathcal{K}(\phi)(q)].$$

C. Ideals of Operator Algebras

C.1. The primitive ideal space. Here, we include some basic facts on the prime ideal spectrum of rings and on its C^* -algebraic analogue, the primitive ideal space. These are required for our explication of the motivation for considering the extension of the closed-set lattice functor.

C.1.1. The spectrum of commutative rings. In commutative ring theory and algebraic geometry, the starting point for the application of geometrical methods is the association of topological spaces to rings [53, p. 70]. These are, in fact, locally ringed spaces; however, we will not be considering this additional structure.

Definition C.1. A *prime ideal* J of a commutative ring R is a ideal $J \subsetneq R$ such that whenever $a, b \in R$ with $ab \in J$ then $a \in J$ or $b \in J$.

The canonical examples of prime ideals come from the ideals of the ring of integers generated by prime numbers.

Definition C.2. Let R be a commutative ring and let $I \subset R$ be an ideal of R . Then $\text{hull}(I)$ is the set of prime ideals that contain I .

Definition C.3. The *spectrum functor* $\text{Spec}: \mathbf{CRng} \rightarrow \mathbf{Top}$ from the category of commutative rings and ring homomorphisms to the category of topological spaces and continuous functions is defined as follows.

Given a commutative ring R , $\text{Spec}(R)$ is the set of prime ideals of R , equipped with the *hull-kernel* (or *Zariski*, or *Jacobson*) *topology*, whose closed sets are of the form $\text{hull}(I)$ for some ideal $I \subset R$.

Given a ring homomorphism $h: R \rightarrow S$, $\text{Spec}(h): \text{Spec}(S) \rightarrow \text{Spec}(R)$ is the continuous function that maps a prime ideal J to its preimage $h^{-1}(J)$ under h .

C.1.2. The primitive ideal space. These definitions and theorems can be found in [7, p. 208] and [14, p. 118].

Definition C.4. A *primitive ideal* J of a C^* -algebra \mathcal{A} is an ideal that is the kernel of an irreducible representation of \mathcal{A} .

Recall that an irreducible representation of a C^* -algebra \mathcal{A} is a $*$ -representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ such that no nontrivial closed subspaces $S \subset \mathcal{H}$ satisfy $\pi(a)S \subset S$ for all $a \in \mathcal{A}$. Every pure state of \mathcal{A} gives rise to an irreducible representation \mathcal{A} by the Gel'fand–Naimark–Segal construction.

Definition C.5. Let \mathcal{A} be a C^* -algebra and let $I \subset \mathcal{A}$ be a closed two-sided ideal of \mathcal{A} . Then $\text{hull}(I)$ is the set of primitive ideals containing I .

Definition C.6. Let \mathcal{A} be a C^* -algebra. The *primitive ideal space* $\text{Prim}(\mathcal{A})$ is the set of primitive ideals of \mathcal{A} , equipped with the *hull-kernel* (or *Zariski*, or *Jacobson*) *topology* whose closed sets are of the form $\text{hull}(I)$ for some closed two-sided ideal $I \subset \mathcal{A}$.

Theorem C.7. *The map hull is an order preserving bijection between the set of closed two-sided ideals of a C^* -algebra \mathcal{A} and the closed sets of the hull-kernel topology on $\text{Prim}(\mathcal{A})$.*

Definition C.8. The *spectrum* $\hat{\mathcal{A}}$ of a C^* -algebra \mathcal{A} is the set of unitary equivalence classes of irreducible representations of \mathcal{A} . It is equipped with the coarsest topology with respect to which the map $[\pi] \mapsto \ker \pi$ is continuous.

The topology on $\hat{\mathcal{A}}$ is thus also order isomorphic to the partially ordered set of closed two-sided ideals of \mathcal{A} .

C.2. Von Neumann algebras. We briefly outline some required elementary facts about von Neumann algebras [7, Chapter 3].

Definition C.9. A *von Neumann algebra* \mathcal{A} is a $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$, for some Hilbert space \mathcal{H} , which contains the identity and is closed in the weak (operator) topology.

Recall that a net of operators (T_α) in $\mathcal{B}(\mathcal{H})$ converges to T in the *weak topology* if and only if, for every vector $v \in \mathcal{H}$ and linear functional $\phi \in \mathcal{H}^*$, we have that $(\phi(T_\alpha(v)))$ converges to $\phi(T(v))$. As convergence of a net of operators in norm implies its weak convergence, we see that von Neumann algebras are examples of C^* -algebras. We could equally well have defined von Neumann algebras to be $*$ -subalgebras of $\mathcal{B}(\mathcal{H})$ which contain the identity and are closed in the strong, ultraweak, or ultrastrong topologies as the closures of $*$ -subalgebras of $\mathcal{B}(\mathcal{H})$ in these topologies all coincide. Von Neumann proved that taking any of these closures of unital $*$ -subalgebras of $\mathcal{B}(\mathcal{H})$ coincides also with taking the double commutant (though he did not know of the ultrastrong topology).

We will primarily require facts about projections and ideals of von Neumann algebras and the relationship between the two notions.

C.2.1. Projections. The *projections* of \mathcal{A} are operators p such that $p = p^* = p^2$. They are orthogonal projections onto closed subspaces of \mathcal{H} . This yields a natural partial order on projections induced by the inclusion relation on their corresponding subspaces. Alternatively, this order can be defined by:

$$p \leq q \quad \text{iff} \quad pq = p \quad \text{iff} \quad qp = p.$$

We denote the partially ordered set of projections in \mathcal{A} by $\mathcal{P}(\mathcal{A})$. In von Neumann algebras, the collection of projections forms a complete lattice: the infimum $\inf_{\alpha} p_{\alpha}$ of an arbitrary collection of projections $\{p_{\alpha}\}_{\alpha}$ is given by the orthogonal projection onto $\bigcap_{\alpha} p_{\alpha} \mathcal{H}$ whereas the supremum $\sup_{\alpha} p_{\alpha}$ is the orthogonal projection onto the closed linear span of $\bigcup_{\alpha} p_{\alpha} \mathcal{H}$. The orthogonal complement map $(-)^{\perp}$ that sends p to $1 - p$ makes this lattice complemented in the sense that $p \vee p^{\perp} = 1$, $p \wedge p^{\perp} = 0$, and $p^{\perp\perp} = p$.

The set of projections in \mathcal{A} is also equipped with several other preorders which arise from the canonical partial order and certain compatible equivalence relations. We will require, in particular, the notions of Murray–von Neumann equivalence of projections and unitary equivalence of projections.

The intuition behind Murray–von Neumann equivalence is to identify projections whose corresponding image subspaces are of the same dimension. That is, there should be an operator $v \in \mathcal{A}$ mapping the Hilbert space \mathcal{H} to itself which isometrically maps the subspace of one projection to the subspace of another, thereby witnessing the equality of their dimension.

Definition C.10. Two projections p and q in a von Neumann algebra \mathcal{A} are *Murray–von Neumann equivalent*, denoted $p \sim_{\mathbf{M}} q$, if and only if there exists $v \in \mathcal{A}$ such that

$$p = v^*v \quad \text{and} \quad q = vv^*.$$

The partial order on $\mathcal{P}(\mathcal{A})$ induces a partial order on the set of Murray–von Neumann equivalence classes of projections. We write $p \preceq_{\mathbf{M}} q$ to denote that $p \sim_{\mathbf{M}} q'$ for some $q' \leq q$.

Definition C.11. Two projections p and q in a von Neumann algebra \mathcal{A} are *unitarily equivalent*, denoted $p \sim_{\mathbf{u}} q$, if and only if there exists a unitary element $u \in \mathcal{A}$ such that $p = uqu^*$.

Similarly, the partial order on $\mathcal{P}(\mathcal{A})$ induces a partial order on the set of unitary equivalence classes of projections. We write $p \preceq_{\mathbf{u}} q$ to denote that $p \sim_{\mathbf{u}} q'$ for some $q' \leq q$.

Unitary equivalence (resp. ordering) implies Murray–von Neumann equivalence (resp. ordering) for arbitrary pairs of projections. We will require the following partial converse for orthogonal projections.

Proposition C.12. *Let p and q be projections in a von Neumann algebra. If p and q are orthogonal, then $p \sim_{\mathbf{M}} q$ iff $p \sim_{\mathbf{u}} q$, and, moreover, $p \preceq_{\mathbf{M}} q$ iff $p \preceq_{\mathbf{u}} q$.*

For a proof of the statement concerning equivalences, see [7, Proposition 6.38]. The second statement is an easy consequence of this: $p \preceq_{\mathbf{M}} q$ means that $p \sim_{\mathbf{M}} q'$ for some $q' \leq q$; but if p and q are orthogonal then so are p and q' ; and so by the first statement one obtains $p \sim_{\mathbf{u}} q'$, meaning that $p \preceq_{\mathbf{u}} q$.

Definition C.13. The central carrier $C(p)$ of a projection $p \in \mathcal{A}$ is the smallest central projection above p :

$$C(p) = \inf \{z \in \mathcal{P}(\mathcal{A}) \cap \mathcal{Z}(\mathcal{A}) \mid p \leq z\}.$$

It is immediate from this definition that a projection p and a unitary rotation upu^* have the same central carrier for $p \leq z$ if and only if $upu^* \leq uzu^* = zuu^* = z$. It is also immediate that if $S \subset \mathcal{P}(\mathcal{A})$ is a set of projections, then $C(\sup S) = \sup \{C(p) \mid p \in S\}$.

One of the basic technical tools we will require is the comparison theorem of projections in a von Neumann algebra [65]. The intuitive idea is best understood in a factor (a von Neumann algebra with trivial centre) which can be thought of as an elementary direct summand. Here, the dimension of two projections can be compared; either they are of equal dimension, or the dimension of one exceeds the dimension of the other.

Theorem C.14 (Comparison theorem). *Let p and q be projections in a von Neumann algebra \mathcal{A} . There exists a central projection z in \mathcal{A} such that*

$$zp \succ_M zq \quad \text{and} \quad z^\perp p \preccurlyeq_M z^\perp q.$$

C.2.2. Ideals. Ideals of operator algebras must satisfy both the usual algebraic conditions as well as an additional topological condition.

It turns out that the appropriate notion of morphism for von Neumann algebras is not weakly continuous $*$ -homomorphism but rather ultraweakly continuous $*$ -homomorphism. The ultraweak topology is stronger than the weak topology.

Definition C.15. A left (resp. right) ideal I of a von Neumann algebra \mathcal{A} is a left (resp. right) ring ideal $I \subset \mathcal{A}$ that is closed in the ultraweak topology.

A total ideal or two-sided ideal I of a von Neumann algebra \mathcal{A} is a two-sided ring ideal $I \subset \mathcal{A}$ that is closed in the ultraweak topology.

Left, right, and total ideals correspond with projections. Examples of left (resp. right) ideals are the sets given by $\mathcal{A}p$ (resp. $p\mathcal{A}$). These are the kernels of morphisms given by right (resp. left) multiplication by p^\perp .

Theorem C.16 ([96, Proposition 3.12], [71, Lecture 9, Corollary 6]). *Every (ultraweakly closed) left ideal $L \subset \mathcal{A}$ of a von Neumann algebra \mathcal{A} is of the form $L = \mathcal{A}p$ for a projection $p \in \mathcal{P}(\mathcal{A})$. Further, the projection p is uniquely determined by L .*

Under this correspondence, total ideals are precisely those left or right ideals corresponding to central projections.

Theorem C.17 ([96, Proposition 3.12], [71, Lecture 9, Corollary 8]). *Every total ideal $I \subset \mathcal{A}$ of a von Neumann algebra \mathcal{A} is of the form $I = z\mathcal{A}z = z\mathcal{A} = \mathcal{A}z$ for a unique central projection $z \in \mathcal{P}(\mathcal{A}) \cap \mathcal{Z}(\mathcal{A})$.*

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