Simplified polynomial formulation for the calculation of the Moment-Curvature diagram of RC rectangular sections

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ABSTRACT
The seismic response of multi-span RC bridges is often based on the response of the piers, provided that deck, bearings and foundations remain elastic. The seismic response of a RC bridge pier is influenced, in general, by different mechanisms (i.e. flexure, shear, lap-splice or buckling of the longitudinal reinforcement bars, second order effects). For mechanisms different from the flexural one, simplified formulations are available in literature. On the other hand, the flexural behaviour of the pier can be characterised by means of equivalent plastic hinge length and Moment-Curvature diagram of the base section, usually carried out with a computer software. In this paper, it is proposed a simplified analytical solution to obtain the Moment-Curvature relationship for rectangular RC sections, in both principal axes. The solution is based on adjusted polynomials, fitted against a database of 800 numerical Moment-Curvature analyses of rectangular RC sections. The proposed polynomials allow to define the cross-section capacity curve through the position of 6 characteristic points and they are based on 4 input parameters: depth-to-width ratio of the cross-section, axial force ratio, longitudinal reinforcement ratio, transversal reinforcement ratio. The solution is tested through the application to a RC rectangular section case study and comparison of the resulting capacity curves to the outcome of refined numerical Moment-Curvature analyses. The results show that the proposed analytical solution is a reliable method to characterise the flexural response of RC rectangular cross-sections.

1 INTRODUCTION

In this paper the Authors propose a simplified analytical procedure to determine the Moment-Curvature diagram for RC rectangular sections, in both the principal axes. The solution consists in adjusted polynomial functions fitted against a database of 800 numerical Moment-Curvature analyses which includes wide ranges of the geometrical and mechanical parameters that define the section itself. With the proposed solution it is possible to obtain a piecewise representation of the flexural response of RC sections by defining 4 dimensionless input parameters: section aspect ratio, axial load ratio, mechanical ratio of longitudinal reinforcement and volumetric ratio of transverse reinforcement.

This work is part of a wider project (including the work in Mezzina et al., 2012, Raffaele et al., 2013a,c, Raffaele et al., 2015c) in which the same approach was adopted to solid (Raffaele et al. 2015b) and hollow circular sections (Raffaele et al. 2015a). The main goal of the project is to construct a tool that allows to quickly estimate the Moment-Curvature response of RC sections using a very poor set of input data while keeping a fair level of accuracy/reliability.

The proposed simplified procedure is well suited for applications in which a large number of Moment-Curvature analyses is to be performed, and the available information is poor, as it usually happens in the regional-scale assessment of buildings (e.g. Del Gaudio et al., 2015, Uva et al., 2016) or bridges (e.g. Pinho et al., 2009).

A mechanically-based regional scale seismic assessment inventory of bridges portfolios is an example in which the proposed procedure could be particularly effective, since the piers play a major role in the behaviour of the bridges themselves. Such kind of inventories are paramount for the quantification of the seismic risk. In fact, the majority of the bridges was designed according to non-seismic philosophy, as pointed out in monitoring studies (e.g. Porco et al., 2013a, Uva et al., 2014b). The degradation of the materials (Porco et al., 2013b, 2014, Uva et al., 2014a) increases the risk, corroborating the need of such studies.

Considering that alternative failure modes can be accounted for with simplified formulations (Raffaele 2004, Raffaele et al. 2013b, Raffaele et al. 2014a, Raffaele et al. 2014b), the flexural behaviour the bridge pier is commonly evaluated with a numerically-based Moment-Curvature approach, for which a large number of parameters is needed. Instead, the proposed simplified analytical formulation is deemed to be more effective, especially if a low "knowledge level" is obtained with regard to the structural properties of the piers, and sensitivity analysis must be considered.

The optimisation of RC sections, with the aim of obtaining specified strength and deformation capacities with a minimum amount of reinforcement, is another example in which the simplified polynomial procedure is deemed to be particularly effective. In fact, it can be used to quickly test different configurations and individuate the optimum one.

In order to test its reliability, the proposed procedure was applied to a typical RC wall rectangular cross-section. The good match of the outcome of the simplified procedure with the results of a numerically-based Moment-Curvature approach demonstrate the reliability of the polynomial approach.

2 RESEARCH METHODOLOGY

A database of 800 rectangular RC cross-sections was created, considering wide ranges for geometrical and mechanical parameters (Section 2.1), which are deemed to be appropriate to represent typical bridge piers cross sections (and frame members, with some approximation). Basing on dimensional analysis, 4 dimensionless parameters were selected to completely define the properties of each case. A numerically-based Moment-Curvature (M-ϕ) analysis was carried out for each entry of the database, in both principal directions of the cross-section.

Each numerical (M-ϕ) curve was post-processed, individuating 6 "characteristic" points (Section 2.2) which are deemed to be sufficient to represent the flexural behaviour of the cross-section considering a piecewise linear curve.

A set of polynomial functions, fitted through a least squares method linear regression, was defined to represent the position of the characteristic points as a function of the selected dimensionless parameters (Equations (4) to (7) define the input parameters of the polynomials).

2.1 Database of numerical analyses

A high number of geometrical and mechanical parameters is needed, in general, to define a RC rectangular cross-section, including dimensions, B and H, clear cover, c, axial load, N, concrete unconfined compressive strength, tensile strength and Young's modulus, (f_c, f_{ct}, E_c), longitudinal and transverse reinforcement pattern, including yield stress, (f_{ys},f_{yh}), modulus of elasticity (E_s, E_h). Considering the typical design practice, a number of assumptions was adopted in the definition of the analyses of the database, in order to reduce the number of the involved parameters.

A fixed value of the shortest dimension of the cross-section $B = 0.5m$ was adopted in carrying out the (M-ϕ) analyses. This is in line with the aim of calibrating the simplified (M-ϕ) analytical formulation for the evaluation of bridge piers. Nonetheless, the results are deemed to be valid also for beam/column members with a slight intrinsic approximation. Clear cover was assumed to be equal to $c = 0.05B$. Concrete tensile strength was related to the compressive one through Eq. (1) (NTC2008).

$$f_t = 0.2(0.3f_c^{2/3}) \qquad (1)$$

Equal properties were assigned to longitudinal and transverse reinforcement, hence Eq. (2) holds. Two different bar sizes (with areas $A_{s,b}^{(1)}, A_{s,h}^{(1)}$) were used in the cross-section, as shown in Figure 1. The two sizes were constrained in a way that the amount of reinforcement per unit length of the perimeter of the confined core of the section was constant, Eq.(3), in which n_b and n_h are the number of rebars located on each short or long side of the section, excluding one corner bar (Figure 1). For each entry of the database, a given value of the longitudinal reinforcement ratio ω was selected, calculating the bar dimensions using Equations (3) and (6).

$$f_{yh} = f_{ys} \quad and \quad E_h = E_s \qquad (2)$$

$$A_{s,b}^{(1)} = \frac{(n_h - 1)(B - 2c)}{(n_b + 1)(H - 2c)} A_{s,h}^{(1)} \qquad (3)$$

The transverse reinforcement pattern was assumed to be composed by a perimeter hoop and a tie for each longitudinal bar on the long side of the cross-section (Figure 1), with the same bar size $A_{sp}^{(1)}$. Therefore, the number of ties parallel to the short side of the section is equal to $n_{tie,b} = n_h - 1$, while $n_{tie,b} = 0$ since there is no tie in this direction. The spacing, s, was considered to be constant in the whole database. Therefore, by fixing a value of the transverse reinforcement ratio ρ_{sp}, Eq. (7) was adopted to determine the value of $A_{sp}^{(1)}$ to be used in the analyses.

It is deemed that less-efficient patterns of the stirrups are equivalent, in terms of confined concrete performance, to the above-mentioned one with a lower value of the stirrup size $A_{sp}^{(1)}$. For this reason, the assumed stirrups configuration is not deemed to jeopardise the generality of the final polynomial formulations.

The remaining parameters involved in the definition of the cross-section properties were grouped basing on dimensional analysis (also used in Peruš et al., 2006): section aspect ratio, Eq. (4), axial load ratio, Eq. (5), mechanical ratio of longitudinal reinforcement, Eq. (6) and volumetric ratio of transverse reinforcement, Eq. (7).

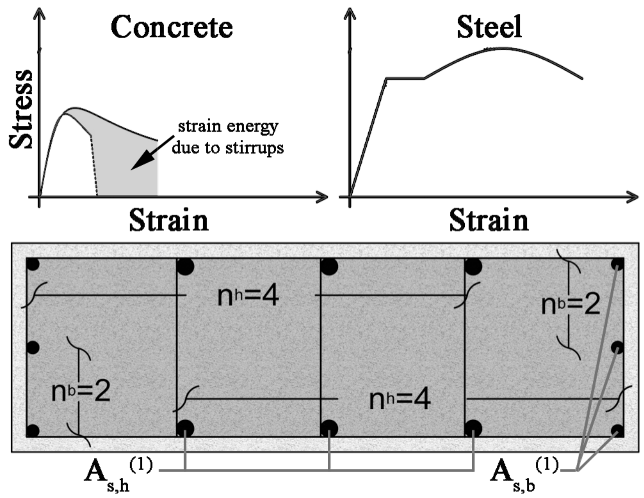

Figure 1. Definition of the cross-section parameters considered in the database.

$$\beta = H/B \tag{4}$$

$$\upsilon = \frac{N}{BHf_c} \tag{5}$$

$$\omega = \frac{(2n_h A_{s,h}^{(1)} + 2n_b A_{s,b}^{(1)})f_{ys}}{BHf_c} \tag{6}$$

$$\rho_{sp} = \frac{A_{sp}^{(1)}[2B + 2H - 4c + n_{tie,h}(H - 2c)n_{tie,b}(B - 2c)]}{(H - 2c)(B - 2c)s} \tag{7}$$

Table 1 shows the selected values for the input dimensionless parameters. It is worth mentioning that the range of the input parameters was carefully selected to include the most probable configurations of RC rectangular sections that might be used in engineering practice. For each combination of the parameters, two Moment-Curvature analyses were conducted with the software KSU-RC (Esmaeily and Peterman, 2007), considering the two principal axes of the rectangular sections. A total of 800 combinations of the input parameters was considered, leading to 1600 numerical analyses.

Table 1. Input values for the construction of the database.

Parameter	Range		Number of samples
	Min	Max	
β	1	8	4
ν	0.1	1	10
ϖ	0.05	0.4	4
ρ	0.001	0.04	5

The obtained Moment-Curvature diagrams (M-ϕ) were expressed in dimensionless form (m-χ), according to Eqs. (8) and (9), specifying the formulation for each direction of the analysis.

Longitudinal (strong)	Transverse (weak)	
$m = \frac{M}{BH^2 fc}$	$m = \frac{M}{HB^2 fc}$	(8)
$\chi = \varphi H$	$\chi = \varphi B$	(9)

2.2 *Individuation of the characteristic points*

Each Moment-Curvature diagram in the database was post-processed using a Matlab-based code, in order to "extract" 6 characteristic points, which are deemed to be sufficient to represent the flexural behaviour of the cross-section by means of a piecewise linearisation (Figure 2).

The mechanically-based definition of each characteristic point is defined as follows:

- **Cracking** is the point for which the furthermost core concrete fibre is subjected to a tensile strain equal to $\varepsilon_{ct} = f_{ct}/E_c$;
- **First Yielding** is associated to a compressive strain equal to $\varepsilon_{ct} = 0.002$ for the furthermost core concrete fibre or to the

tensile yield of the first rebar, whichever occurs first (Priestley et al., 2007);

- **Concrete Peak** is related to the reaching of the strain corresponding to the peak stress in the furthermost compressed fibre in the confined core of the section;
- **Spalling** is the point on the (M-ϕ) diagram which is related to the ultimate compressive strain of the unconfined concrete (0.0045), registered in the furthermost fibre in the cover. This limit state is typically associated to a degrading branch in the (M-ϕ) diagram;
- **Post Spalling** corresponds to the end of the degrading branch of the (M-ϕ) diagram, related to the expulsion of the concrete cover;
- The **Ultimate** point of the curve is associated to either the ultimate compressive strain for confined concrete (according to Mander et al, 1988) or ultimate tensile strain for steel (assumed to be 0.06, according to NZSEE 2017).

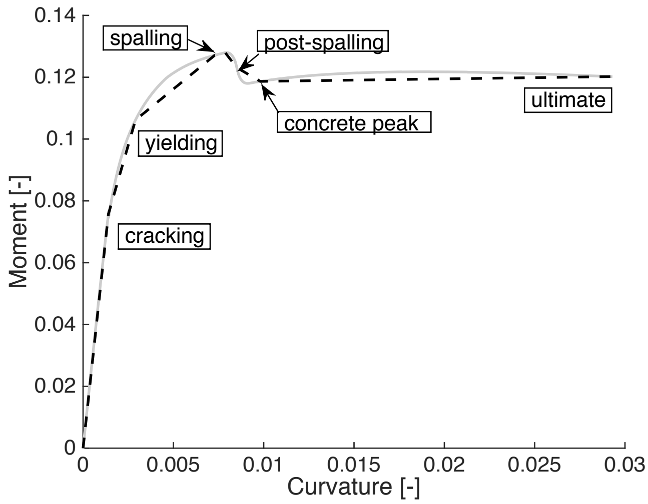

Figure 2. Individuation of the characteristic points.

2.3 *Relationship between unconfined concrete compressive strength and the ultimate curvature*

It is worth mentioning that the compressive strength of the unconfined concrete (f_c) was not explicitly considered as an input parameter for the definition of the entries of the database. However, this parameter has an indirect effect on the ultimate curvature of a cross-section.

In fact, according to Mander et al., 1988, each amount of transverse reinforcement is associated to a level of strain energy-storing capacity (shaded area in Figure 1) that defines the ultimate strain of the confined concrete law. Therefore, fixing the amount of transverse reinforcement, the confined concrete ultimate strain is higher when the unconfined concrete compressive strength is lower. Consequently, a similar behaviour affects the ultimate curvature.

To account for this effect, an ultimate curvature correction factor was calibrated basing on a smaller database (10 cases), in which the only varying parameter is f_c (20:50 MPa). Its use is defined in Section 3.2.

3 RESULTS

3.1 *Major trends*

For each entry of the database, defined by a combination of the parameters in Equations (4) to (7), the moment and curvature of the characteristic points were selected and analysed in order to investigate their trends, clearly confirming the basic knowledge regarding RC sectional analysis.

Figure 3 shows an abacus depicting the variability of the ultimate moment and curvature for a fixed value of the section aspect ratio ($\beta = 4$), the minimum considered value of the mechanical ratio of longitudinal reinforcement ($\omega = 0.05$) and the full range of variation for the volumetric ratio of transverse reinforcement. Figure 4 differs from the previous one since it refers to the maximum value of the mechanical ratio of longitudinal reinforcement ($\omega = 0.4$). For a graphically-based use of the abacuses, a linear interpolation might be applied for intermediate values of the mechanical ratio of longitudinal reinforcement.

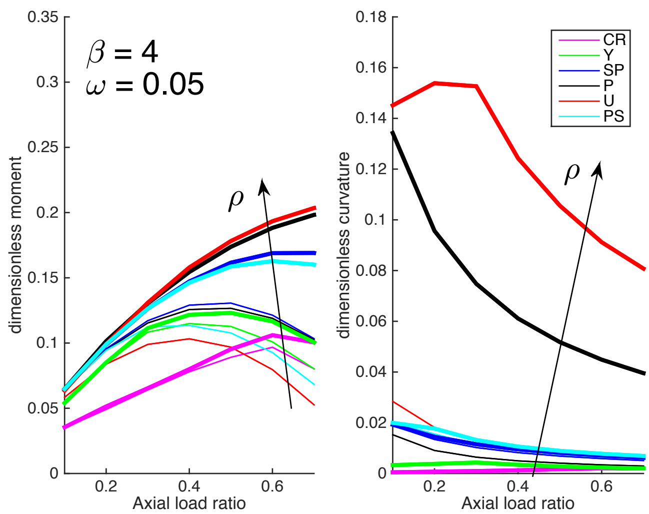

Figure 3. Ultimate moment and curvature trends: section aspect ratio β=4, longitudinal reinforcement ratio ω=0.05.

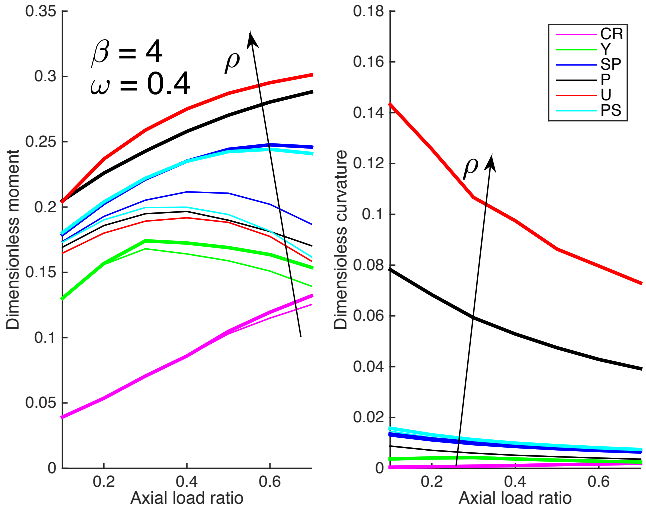

Figure 4. Ultimate moment and curvature trends: section aspect ratio β=4, longitudinal reinforcement ratio ω=0.4.

3.2 *Characteristic polynomials*

The values of the characteristic moments and curvatures, extracted for each analysis in the database, were used to fit polynomial functions using a least square method linear regression. The polynomials were defined on the basis of the 4 dimensionless input parameters defined in Section 2.1.

In each direction, and for each characteristic moment or curvature, a polynomial function was created, Eq. (10). This process led to 24 different polynomials which allow to quickly construct the Moment-Curvature diagram of rectangular sections which are characterised by mechanical and geometrical parameters within the scope of the database created for this study.

The degree of the polynomials, with respect to the 4 selected input parameters, was chosen basing on the analysed trends of the data (Section 3.1). Although this is not a statistical analysis, it is worth mentioning that each of the fitted functions is characterised by an $adjusted - R^2$ value which is greater than 95%. Moreover, a $p - value$ check was carried out for each term of the functions, eliminating the terms with a $p - value$ smaller than 5%.

$$m_{char}(\chi_{char}) = a_1 + a_2\beta^2 + a_3\beta\omega + a_4\beta\rho + a_5\omega\rho + a_6\beta\upsilon + a_7\omega\upsilon + a_8\rho\upsilon + a_9\upsilon^2 + a_{10}\beta^3 + a_{11}\beta^2\omega + a_{12}\beta^2\rho + a_{13}\beta\omega\rho + a_{14}\beta^2\upsilon + a_{15}\beta\omega\upsilon + a_{16}\beta\rho\upsilon + a_{17}\omega\rho\upsilon + a_{18}\beta\upsilon^2 + a_{19}\omega\upsilon^2 + a_{20}\rho\upsilon^2 + a_{21}\upsilon^3. \quad (10)$$

It is worth repeating that unconfined concrete compressive strength was not considered an input parameter for the polynomials. The variability of the ultimate curvature with respect to this parameter is captured with the ultimate curvature correction factor, Eq. (11), which is multiplied to the ultimate curvature calculated in with the appropriate version of Eq. (10).

$$CF = 2.3541 - 0.06f_c + 0.000553f_c^2 \quad (11)$$

The complete set of polynomial functions developed in this study was implemented in an Excel spreadsheet. The spreadsheet allows the construction of Moment-Curvature diagrams for RC sections in a quick and easy way, with no effort for the user.

4 APPLICATION TO A CASE STUDY

The proposed simplified analytical formulation was applied to a RC wall cross-section case study, not included in the database, and compared to a numerically-based Moment-Curvature analysis, in order to demonstrate its reliability. The mechanical and geometrical characteristics of the RC rectangular section case study are shown in Table 2 (see also Figure 1).

It is worth mentioning that a Moment-Curvature approach might be, in general, theoretically inappropriate for a wall cross-section, since the “plane-sections” hypothesis is demonstrated to be invalid for such cases. However, this approach is widely accepted, at least to obtain a first estimation of the flexural behaviour of the section, which might be appropriate for all practical purposes, unless detailed evaluations are needed.

In Figure 5 the numerically-based and the simplified polynomial Moment–Curvature diagrams are shown. It is clear that with the simplified approach it is possible to obtain a reliable estimation of the curve, for both principal directions, since the comparisons show very good match, both qualitatively and quantitatively. The simplified procedure somehow fails to capture the softening behaviour in the last branch of the (M-ϕ) diagram in the longitudinal direction. Anyway, it is deemed that the most meaningful points of the curve, namely the yielding moment and curvature, maximum moment and ultimate curvature, are very well estimated.

Table 2. Mechanical characteristics of the case study.

H [m]	B [m]	c [m]	n_h [m]	d_h [mm]	n_b [-]	d_b [mm]	d_{st} [mm]	s [m]	f_c [MPa]	f_{ct} [MPa]	f_{ys} [MPa]	N [MN]
7	1.8	0.05	7	26	7	12	14	0.19	25	3	390	211

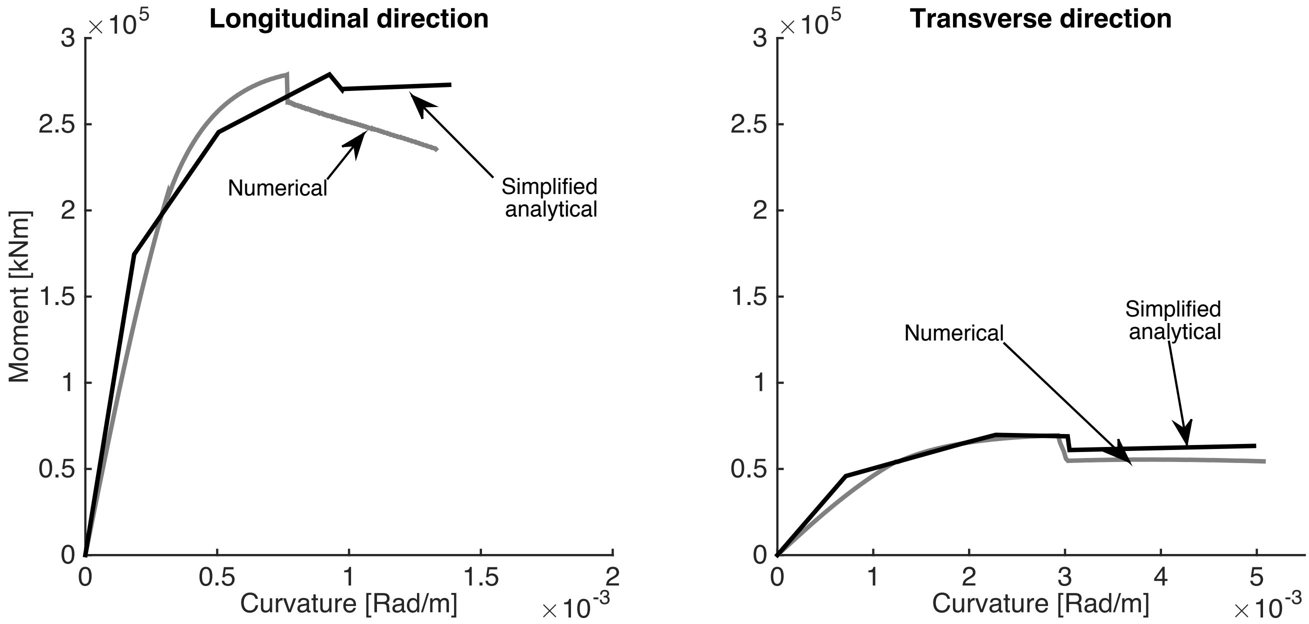

Figure 5. Case study rectangular cross section.

5 CONCLUSIONS

A simplified analytical procedure to determine the Moment-Curvature diagram for RC rectangular sections, in both the principal axes, it is proposed in this paper. The solution consists in adjusted polynomial functions fitted against a database of 800 numerical Moment-Curvature analyses which includes wide ranges of the geometrical and mechanical parameters that define the section itself. Only 4 input parameters are needed: section aspect ratio, axial load ratio, mechanical ratio of longitudinal reinforcement and volumetric ratio of transverse reinforcement. Hence the amount of needed data about the RC member is very limited.

The proposed procedure was applied to a RC wall case study with rectangular cross section, comparing the results with a traditional numerically-based Moment-Curvature approach. The good match between the simplified and the numerical approach demonstrates the reliability of the procedure in the estimation of the flexural response of rectangular RC sections.

It is worth mentioning that this work is part of a wider project in which similar solutions were developed for different shapes of the cross-section, including the circular and hollow circular one. The final goal of the project is to develop a quick-yet-reliable tool to derive in a simplified way the full behaviour of RC members, with particular focus to bridge piers, also including alternative failure modes (shear, lap-splice or buckling of the longitudinal bars, second order effects).

Regional scale mechanically-based first level assessments of bridges or the optimisation of RC sections in the pre-design phase are the situations in which this tool is deemed to be particularly effective.

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