THE STEKLOV SPECTRUM OF CUBOIDS

ALEXANDRE GIROUARD, JEAN LAGACÉ, IOSIF POLTEROVICH, AND ALESSANDRO SAVO

ABSTRACT. The paper is concerned with the Steklov eigenvalue problem on cuboids of arbitrary dimension. We prove a two-term asymptotic formula for the counting function of Steklov eigenvalues on cuboids in dimension $d \geq 3$. Apart from the standard Weyl term, we calculate explicitly the second term in the asymptotics, capturing the contribution of the $(d-2)$–dimensional facets of a cuboid. Our approach is based on lattice counting techniques. While this strategy is similar to the one used for the Dirichlet Laplacian, the Steklov case carries additional complications. In particular, it is not clear how to establish directly the completeness of the system of Steklov eigenfunctions admitting separation of variables. We prove this result using a family of auxiliary Robin boundary value problems. Moreover, the correspondence between the Steklov eigenvalues and lattice points is not exact, and hence more delicate analysis is required to obtain spectral asymptotics. Some other related results are presented, such as an isoperimetric inequality for the first Steklov eigenvalue, a concentration property of high frequency Steklov eigenfunctions and applications to spectral determination of cuboids.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Asymptotics of the Steklov spectrum.** The Steklov eigenvalues of a bounded Euclidean domain $\Omega \subset \mathbb{R}^d$ are the real numbers $\sigma \in \mathbb{R}$ for which there exists a nonzero harmonic function $u : \Omega \to \mathbb{R}$ such that $\partial_n u = \sigma u$ on the boundary $\partial\Omega$. Here ∂_n denotes the outward normal derivative, which exists almost everywhere provided the boundary $\partial\Omega$ is Lipschitz. Under this assumption, it is known that for $d \geq 2$ the Steklov spectrum is discrete (see [1]) and is given by the increasing sequence of eigenvalues $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \ldots \nearrow \infty$, where each eigenvalue is repeated according to its multiplicity. The counting function $N : \mathbb{R} \to \mathbb{N}$ is then defined by $N(\sigma) := \#\{j \in \mathbb{N} : \sigma_j < \sigma\}$. For domains with smooth boundary, one can show using pseudodifferential techniques that the counting function satisfies Weyl's law

$$N(\sigma) = \frac{\omega_{d-1}}{(2\pi)^{d-1}} \mathrm{Vol}_{d-1}(\partial\Omega)\sigma^{d-1} + O(\sigma^{d-2}) \quad \text{as } \sigma \nearrow +\infty, \tag{1.1.1}$$

where ω_{d-1} is the measure of the unit ball $B_1(0) \subset \mathbb{R}^{d-1}$. The remainder estimate in (1.1.1) is sharp and attained on a round ball. Moreover, a two-term asymptotic formula for the counting function holds under a non-periodicity condition of the geodesic flow on $\partial\Omega$ (see [14, formula (5.1.8)]).

Understanding precise asymptotics for Steklov eigenvalues on domains with singularities, such as corners and edges, is significantly more challenging, since pseudodifferential techniques do not work in this case (see [5, Section 3] for a discussion). Using variational methods, one can prove a one-term Weyl asymptotic formula that holds for any piecewise C^1 Euclidean domain (see [1]):

$$N(\sigma) = \frac{\omega_{d-1}}{(2\pi)^{d-1}} \mathrm{Vol}_{d-1}(\partial\Omega)\sigma^{d-1} + o(\sigma^{d-1}) \quad \text{as } \sigma \nearrow +\infty. \tag{1.1.2}$$

Key words and phrases. Steklov problem, cuboids, spectral asymptotics, lattice counting.

However, in order to get sharper asymptotics, one needs to understand the contribution of singularities to the counting function. In two dimensions, some results in this direction have been recently obtained in [12]. In the present paper we aim to explore the most basic higher-dimensional example: the Euclidean cuboids.

1.2. **Main result.** Given $d \in \mathbb{N}$, the *cuboid*[1] with parameters $a_1, \dots, a_d > 0$ is defined as a product of the intervals

$$\Omega = (-a_1, a_1) \times (-a_2, a_2) \times \dots \times (-a_d, a_d) \subset \mathbb{R}^d.$$

If $a_1 = a_2 = \cdots = a_d$ we say that Ω is a *cube*. The main result of this paper is the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be the cuboid with parameters $a_1, \dots, a_d > 0$. For $d \geq 3$, the counting function of Steklov eigenvalues satisfies a two-term asymptotic formula as $\sigma \to \infty$:*

$$N(\sigma) = C_1 \operatorname{Vol}_{d-1}(\partial\Omega)\sigma^{d-1} + C_2 \operatorname{Vol}_{d-2}(\partial^2\Omega)\sigma^{d-2} + O\left(\sigma^\eta\right), \tag{1.2.1}$$

where $\partial^2\Omega$ denotes the union of all the $(d-2)$-dimensional facets of Ω. Here $\eta = 2/3$ for $d = 3$ and $\eta = d - 2 - \frac{1}{d-1}$ for $d \geq 4$. The constants C_1 and C_2 are given by

$$C_1 = \frac{\omega_{d-1}}{(2\pi)^{d-1}}$$

and

$$C_2 = \frac{2^{\frac{d-2}{2}}\omega_{d-2}}{(2\pi)^{d-2}} - \frac{2G_{d-1,1}}{\pi^{d-1}} - \frac{\omega_{d-2}}{2(2\pi)^{d-2}}$$

where

$$G_{d-1,1} = \underbrace{\int_0^{\pi/2} \dots \int_0^{\pi/2}}_{d-2} \operatorname{arccot}\left(\prod_{j=1}^{d-2} \csc\theta_j\right) \prod_{k=1}^{d-2} \sin^k(\theta_k)\, \mathrm{d}\theta_1 \dots \mathrm{d}\theta_{d-2}.$$

For $d = 2$, the counting function admits a one-term asymptotics

$$N(\sigma) = \pi^{-1} \operatorname{Vol}_1(\partial\Omega)\sigma + O(1).$$

Remark 1.2. It can be shown that $C_2 > 0$ for all $d \geq 3$, see Appendix B. The constants $G_{d,1}$ are special cases of constants $G_{p,q}$ which will be introduced in Section 3. The constants $G_{2,1}$ and $G_{3,1}$ can be computed explicitly as

$$G_{2,1} = \frac{1}{2}\left(-1 + \sqrt{2}\right)\pi$$
$$G_{3,1} = \frac{1}{8}(-2 + \pi)\pi.$$

Remark 1.3. For $d = 2$, the above asymptotics also follows from [12, Corollary 1.6.1]. In dimensions $d \geq 3$ we do not expect the error estimates obtained in Theorem 1.1 to be sharp (see also Remark 3.18), and it is an interesting open problem to determine the optimal value of the exponent η in (1.2.1).

[1] Cuboids are also often referred to as *boxes*, *d-orthotopes* or *hyperrectangles*. The term “cuboids” appears to be more common in recent literature on spectral geometry (see [6, 18]).

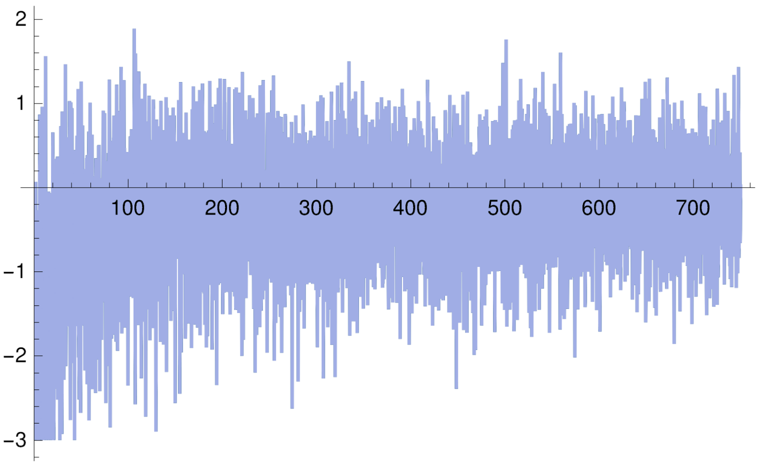

FIGURE 1. $\frac{N(\sigma)-C_1\,\mathrm{Vol}_2(\partial\Omega)\sigma^2-C_2\,\mathrm{Vol}_1(\partial^2\Omega)\sigma}{\sigma^{2/3}}$ for $\sigma < 750$.

Remark 1.4. For $d = 3$, Theorem 1.1 implies that

$$\mathscr{R}(\sigma) = \frac{N(\sigma) - C_1\,\mathrm{Vol}_2(\partial\Omega)\sigma^2 - C_2\,\mathrm{Vol}_1(\partial^2\Omega)\sigma}{\sigma^{2/3}}$$

is a bounded function of σ. In order to validate the expression for the constant C_2 obtained in Theorem 1.1, we have checked numerically that this claim holds, using the approximate eigenvalues introduced in Section 3 on a cube with side lengths 2. Figure 1 shows that $|\mathscr{R}(\sigma)| \le 3$ for $\sigma < 750$ which corresponds to approximately a million eigenvalues.

1.3. **Outline of the proof.** The proof of Theorem 1.1 is given in Section 3. The outline of the argument is as follows. First, we show that the Steklov eigenvalue problem on a cuboid admits separation of variables, see Lemma 2.1 below. Separation of variables yields eigenfunctions that are products of trigonometric, hyperbolic and possibly linear factors. One can check that the number of eigenvalues corresponding to eigenfunctions containing linear terms is at most finite, see Theorem 2.6. The same theorem also shows that the eigenvalue counting problem can be reduced to a family of approximate lattice counting problems. More specifically, given $1 \le p \le d$, we consider the counting function N_p of eigenvalues corresponding to eigenfunctions with exactly p trigonometric factors. It turns out that for each $p > 1$, the counting function N_p satisfies a two-term asymptotic formula, see Proposition 3.1. The functions N_p for $p = d-1$ and $p = d-2$ are the dominant ones. In particular, the main term in (1.2.1) corresponds to the main term in the asymptotics for N_{d-1}. The second term in (1.2.1) is obtained as as a sum of the main term in the asymptotics of N_{d-2} and the second term in N_{d-1}. The latter also splits into two parts: one is the standard contribution of overcounted lattice points (see Lemma 3.17), and the other has to do with the geometry of the domain E_σ defined by (3.4.12) arising in the lattice counting problem. While this domain E_σ converges to a ball as $\sigma \to \infty$, the approximation produces an error that contributes to the second term of (1.2.1). This explains why the coefficient C_2 is represented by a sum of three constants. Note that while two of these constants are negative, the coefficient C_2 is always positive, see Appendix B.

1.4. **Discussion.** The second term in Weyl asymptotics (1.2.1) for cuboids could be compared with the corresponding term in the asymptotic expression [14, formula (5.1.8)] mentioned earlier, which holds on smooth manifolds with boundary, satisfying a non-periodicity condition. Recall that in the smooth case, the second term is proportional to the integral of the mean curvature of the boundary. Note that a cuboid could be viewed as a limit of smooth hypersurfaces with all the curvatures concentrated in small neighborhoods shrinking to the union of the $(d-2)$-dimensional facets of the cuboid. This naive observation provides some intuition regarding the nature of the second term in asymptotic formula (1.2.1).

It would be very interesting to establish an analogue of Theorem 1.1 for arbitrary Euclidean polyhedra and, more generally, for Riemannian manifolds with edges, satisfying certain non-periodicity assumptions. While the present paper was in the final stages of preparation, V. Ivrii [10] informed us on his work in progress in this direction.

Another promising direction of further research in the subject is to explore the asymptotic expansion for the Steklov heat trace on Euclidean polyhedra, as well as on arbitrary Riemannian manifolds with edges. In particular, one could ask whether the Steklov spectral asymptotics contains information on the lower-dimensional facets of polyhedra. While the Weyl asymptotics does not appear to be accurate enough for that purpose, the Steklov heat trace asymptotics is likely to give a positive answer to this question. We intend to explore it elsewhere.

Remark 1.5. The existence of a two-term asymptotic formula for the counting function of Steklov eigenvalues on a cube was claimed earlier in [13]. However, the proof of this claim contained a miscalculation invalidating the argument. Indeed, in the beginning of [13, Section 3], the authors write down the boundary condition at $x_i = 0$ in case $\beta_i < 0$ and get $c_1\sqrt{|\beta_i|} = \lambda c_2$, while it should be $-c_1\sqrt{|\beta_i|} = \lambda c_2$, since the normal derivative at $x_i = 0$ is $-\partial_i$. Due to this missing minus sign, the authors obtain the equation $\sin(\sqrt{\beta_i}) = 0$ leading to an exact correspondence between Steklov eigenvalues and lattice points. However, in reality this correspondence is only approximate (see subsection 2.3), and therefore counting eigenvalues is a significantly more difficult task. Note also that the completeness of eigenfunctions admitting separation of variables was not justified in [13].

1.5. **An isoperimetric inequality for the first Steklov eigenvalue.** Given a cuboid $\Omega \subset \mathbb{R}^d$ with parameters $a_1, \dots, a_d > 0$, let $\Omega^\star$ and $\Omega^\sharp$ be the cubes such that

$$\operatorname{Vol}_{d-1}\partial\Omega^\star = \operatorname{Vol}_{d-1}\partial\Omega \qquad \text{and} \qquad \operatorname{Vol}_d\Omega^\sharp = \operatorname{Vol}_d\Omega.$$

Theorem 1.6. *For any cuboid* Ω*,*

- $\sigma_1(\Omega^\star) \ge \sigma_1(\Omega)$*, with equality if and only if* $\Omega^\star = \Omega$*;*
- $\sigma_1(\Omega^\sharp) \ge \sigma_1(\Omega)$*, with equality if and only if* $\Omega^\sharp = \Omega$*.*

The proof of the theorem is presented in Section 4.3. In a way, it is not surprising that the cube, being the most symmetric of all cuboids, maximizes σ_1 under both volume and surface area restrictions. Theorem 1.6 could be compared with the well-known Weinstock's inequality [19] stating that the disk is a unique maximizer for σ_1 among planar simply connected domains with a given perimeter (see also a recent generalization of this result for convex domains in higher dimensions obtained in [4]), as well as with Brock's result [3] which states that balls are unique maximizers among Euclidean domains $\Omega \subset \mathbb{R}^d$ with prescribed d–volume.

It follows from Theorem 1.6 that any cube is spectrally determined among all cuboids.

Corollary 1.7. *Let $\Omega \subset \mathbb{R}^d$ be a cuboid which is isospectral to the cube $\Omega_a \subset \mathbb{R}^m$ with side lengths $2a > 0$. Then $d = m$ and $\Omega = \Omega_a$.*

Proof. It follows from Theorem 1.1 that $d = m$ and $\mathrm{Vol}_{d-1}(\partial\Omega) = \mathrm{Vol}_{d-1}(\partial\Omega_a)$. Moreover, since $\sigma_1(\Omega) = \sigma_1(\Omega_a)$, the conclusion follows from the uniqueness of the maximizer in Theorem 1.6. □

Note that a similar corollary with an almost identical proof holds for planar simply-connected domains, among which the disk is spectrally determined, using the case of equality in Weinstock's theorem [19].

Is still unknown whether there exist nonisometric Steklov isospectral Euclidean domains. Our results imply that if two rectangles are Steklov isospectral, they are isometric.

Corollary 1.8. *The Steklov spectrum of a rectangle uniquely determines its side lengths.*

The proof of this corollary is presented in Section 4.4. Let us conclude the introduction with the following conjecture:

Conjecture 1.9. *Any two Steklov isospectral cuboids are isometric.*

Plan of the paper. In Section 2, we explore the structure of Steklov eigenvalues and eigenfunctions on cuboids. In particular, in subsection 2.1 we describe separation of variables and prove that it yields a complete system of Steklov eigenfunctions. In subsection 2.2 a classification of eigenfunctions is presented based on the number of linear, trigonometric and hyperbolic terms, which is later used in subsection 2.3 to reduce the problem of counting eigenvalues to counting approximate lattice points. Theorem 1.1 is proved in Section 3. This is the most technically involved part of the paper, involving tools from analytic number theory and Fourier analysis. Other results of the paper are proved in Section 4. In particular, a somewhat surprising observation that Steklov eigenfunctions may concentrate on lower dimensional facets of cuboids is presented in subsection 4.1. Subsections 4.3 and 4.4 provide the proofs of Theorem 1.6 and Corollary 1.8. Appendix A contains the proof of an auxiliary Lemma A.1 used in subsection 3.4. In Appendix B we justify the positivity of the constant C_2 as stated in Remark 1.2.

Remark 1.10. Right before submitting our paper on the archive, we learned of the preprint [17] which discusses Steklov eigenvalues of rectangles and cuboids of dimension 3. Note that [17, Conjecture 3.1] immediately follows from our Proposition 4.2.

Acknowledgments. We would like to thank Leonid Parnovski and Dmitry Faifman for useful discussions. In particular, they lead us to the results presented in subsection 4.1. Part of this work was accomplished in 2016 when A.G. visited the Institut de mathématiques de Neuchâtel, and its hospitality is gratefully acknowledged. Research of A.G. is supported by NSERC and FRQNT. Research of J.L. is supported by the Alexander Graham Bell Canada Graduate Scholarship. Research of I.P. is supported by NSERC, FRQNT and Canada Research Chairs Program. Research of A.S. is supported by PRIN 2015.

2. Eigenfunctions and separation of variables

2.1. **Separation of variables.** The following lemma shows that the method of separation of variables is applicable to the computation of the Steklov spectrum of a product of compact manifolds with boundary. In particular, we justify completeness of the system of Steklov eigenfunctions admitting separation of variables.

Lemma 2.1. *Let M_1 and M_2 be smooth compact Riemannian manifolds with boundary. Let $\sigma \geq 0$ be a Steklov eigenvalue of the product manifold $M = M_1 \times M_2$ with the eigenspace $F_\sigma \subset L^2(M)$. There exists a basis $(u^{(1)}, \dots, u^{(m)})$ of F_σ such that each $u^{(j)} : M_1 \times M_2 \to \mathbb{R}$ is separable:*

$$u^{(j)}(x_1, x_2) = u_1^{(j)}(x_1) u_2^{(j)}(x_2), \qquad 1 \leq j \leq m,$$

where $u_1^{(j)} : M_1 \to \mathbb{R}$ and $u_2^{(j)} : M_2 \to \mathbb{R}$.

Proof. Consider the Robin problem with parameter $\sigma \geq 0$ on M

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } M, \\ \partial_n u = \sigma u & \text{on } \partial M. \end{cases}$$

It is well known that the Robin problem on M admits separation of variables, since $L^2(M) = L^2(M_1) \otimes L^2(M_2)$ is a product space, see *e.g.* [16, Section 11.5]. The number $\sigma \geq 0$ is a Steklov eigenvalue of M if and only if 0 is an eigenvalue of the Robin problem with parameter σ, and the corresponding eigenspace is the same for both problems. Since one can find a separated eigenbasis for F_σ by virtue of it being a Robin eigenspace on M, it then suffices to use the same basis for F_σ when we consider it as a Steklov eigenspace. □

Remark 2.2. It is not easy to show directly that the traces of all separable Steklov eigenfunctions form a basis in $L^2(\partial M)$, since the boundary ∂M of a product manifold is not itself a product manifold.

Remark 2.3. Lemma 2.1 yields completeness of the system of separable Steklov eigenfunctions on cuboids. Surprisingly, a complete proof of this result has not appeared in the literature even in the case of rectangles. Note that the completeness argument for the square presented in [5, Section 3] does not extend to arbitrary rectangles, contrary to the claim made in [2, Section 4] and in [17]. Indeed, the proof given in [5] uses in a crucial way the diagonal symmetries of the square, which allow to use a connection to the vibrating beam problem via mixed Steklov-Neumann-Dirichlet problems on an isosceles right triangle.

Let $d \in \mathbb{N}$ and consider the cuboid Ω with parameters $a_1, \dots, a_d > 0$. Because Ω is a product of compact intervals, it follows from Lemma 2.1 that there exists a complete set $\{u_j\}_{j \in \mathbb{N}_0}$ of separated Steklov eigenfunctions on Ω. Consider a function $u : \Omega \to \mathbb{R}$ given by the product $u(x) = u_1(x_1) \dots u_d(x_d)$, where $u_j : [-a_j, a_j] \to \mathbb{R}$. Requiring u to be a Steklov eigenfunction with eigenvalue $\sigma \geq 0$ leads to numbers $\lambda_1, \lambda_2, \dots, \lambda_d \in \mathbb{R}$ such that

$$\begin{cases} u_j'' + \lambda_j u_j = 0 \quad \text{on} \quad (-a_j, a_j), \\ u_j'(a_j) = \sigma u_j(a_j), \\ -u_j'(-a_j) = \sigma u_j(-a_j), \end{cases} \tag{2.1.1}$$

subject to the harmonicity condition

$$\sum_{j=1}^d \lambda_j = 0. \tag{2.1.2}$$

The following lemma describes the eigenvalues and eigenfunctions of the auxilary one-dimensional Steklov spectral problem (2.1.1) with a parameter $\lambda \in \mathbb{R}$.

Lemma 2.4. *Let $\lambda \in \mathbb{R}$. The non-zero solutions $\varphi : [-a,a] \to \mathbb{R}$ of the differential equation $\varphi'' + \lambda\varphi = 0$ subject to the boundary conditions*

$$\varphi'(a) = \sigma\varphi(a) \quad \text{and} \quad -\varphi'(-a) = \sigma\varphi(-a)$$

for some constant $\sigma \geq 0$, are constant multiples of one the following functions:

(i) For $\lambda = 0$, $\varphi(t) \equiv 1$ and $\sigma = 0$ or $\varphi(t) = t$ and $\sigma = a^{-1}$.

(ii) For $\lambda = \alpha^2 > 0$, one of

$$\varphi(t) = \sin(\alpha t) \qquad \text{with } \sigma = \alpha\cot(\alpha a),$$
$$\varphi(t) = \cos(\alpha t) \qquad \text{with } \sigma = -\alpha\tan(\alpha a).$$

In other words, for each $\ell \in \{0,1\}$, $\sigma = \alpha\cot\left(\alpha a + \ell\frac{\pi}{2}\right)$ is an eigenvalue.

(iii) For $\lambda = -\beta^2 < 0$, one of

$$\varphi(t) = \sinh(\beta t) \qquad \text{with } \sigma = \beta\coth(\beta a)$$
$$\varphi(t) = \cosh(\beta t) \qquad \text{with } \sigma = \beta\tanh(\beta a).$$

In other words, for each $j \in \{-1,1\}$, $\sigma = \beta\tanh(\beta a)^j$ is an eigenvalue.

It will be useful to introduce a uniform notation for these eigenvalues. Given $a > 0$ and $\ell \in \{0,1\}$, let

$$T_{a,\ell}(x) = x\cot\left(ax + \ell\frac{\pi}{2}\right) = \begin{cases} x\cot(ax) & \text{for } \ell = 0, \\ -x\tan(ax) & \text{for } \ell = 1, \end{cases}$$

and

$$H_{a,\ell}(x) = \begin{cases} x\coth(ax) & \text{for } \ell = 0, \\ x\tanh(ax) & \text{for } \ell = 1. \end{cases}$$

It follows from Lemma 2.4 that separable eigenfunctions are products of linear factors, trigonometric factors (the function sin for $\ell = 0$, and cos for $\ell = 1$) and hyperbolic factors (the function sinh for $\ell = 0$, and cosh for $\ell = 1$). A careful accounting of these will be presented.

2.2. Classification of eigenfunctions.

It follows from the previous paragraph that there is a complete set of Steklov eigenfunctions given by products of linear, trigonometric and hyperbolic factors. They are of the form

$$u(x_1,\dots,x_d) = \prod_{i\in\tau_0} x_i \prod_{j\in\tau_1} \mathrm{Trig}_j(\alpha_j x_j) \prod_{k\in\tau_2} \mathrm{Hyp}_k(\beta_k x_k) \tag{2.2.1}$$

where τ_0, τ_1, τ_2 are disjoint subsets of $S_d := \{1,2,\dots,d\}$ such that $\tau_0 \cup \tau_1 \cup \tau_2 = S_d$, and each $\mathrm{Trig}_j \in \{\sin,\cos\}$ and $\mathrm{Hyp}_k \in \{\sinh,\cosh\}$. In order for this function to be a Steklov eigenfunction corresponding to the eigenvalue $\sigma > 0$, the function u must be harmonic. This amounts to the following restatement of condition (2.1.2) in terms of the constants α_j and β_k:

$$\sum_{j\in\tau_1} \alpha_j^2 = \sum_{k\in\tau_2} \beta_k^2. \tag{2.2.2}$$

This equation will be called the *harmonicity condition.* Moreover, the spectral parameter σ has to be the same on each face of the cuboid. By Lemma 2.4 this translates into the following equations, called the *compatibility conditions*:

$$\sigma = \begin{cases} a_i^{-1} & \text{for } i \in \tau_0, \\ T_{a_i,\ell(i)}(\alpha_i) & \text{for } i \in \tau_1, \\ H_{a_i,\ell(i)}(\beta_i) & \text{for } i \in \tau_2. \end{cases} \tag{2.2.3}$$

Here the function $\ell : S_d \to \{0,1\}$ is used to specify which trigonometric and hyperbolic functions are used, according to the convention introduced in Lemma 2.4. The corresponding eigenfunction (2.2.1) is then given precisely by the product of the factors $u_i : [-a_i, a_i] \to \mathbb{R}$ which are specified by

$$u_i(x_i) = \begin{cases} \mathrm{Trig}_{\ell(i)}(\alpha_i x_i) & \text{for } i \in \tau_1, \\ \mathrm{Hyp}_{\ell(i)}(\beta_i x_i) & \text{for } i \in \tau_2, \\ x_i & \text{otherwise,} \end{cases} \tag{2.2.4}$$

where $\mathrm{Trig}_0 = \sin$, $\mathrm{Trig}_1 = \cos$, $\mathrm{Hyp}_0 = \sinh$ and $\mathrm{Hyp}_1 = \cosh$.

Note that any separated eigenfunction that has a linear factor $u_j(x_j) = x_j$ contributes the eigenvalue $\sigma = a_j^{-1}$ to the spectrum. Since the multiplicity of each eigenvalue is finite, this can occur at most a finite number of times. We summarize the above mentioned facts in the following theorem.

Theorem 2.5. *Let $p \in \{1,\dots,d-1\}$, and let $\mathscr{T}_p$ be the set of all ordered bipartitions $\tau = (\tau_1, \tau_2)$ of $\{1,\dots,d\}$ in the sets of cardinality p and $q = d-p$. For each $\tau \in \mathscr{T}_p$ and any $\ell : \tau_1 \cup \tau_2 \to \{0,1\}$, let $S_{\tau,\ell}$ be the set of all numbers $\sigma > 0$ for which there exist positive numbers α_i for $i \in \tau_1$ and β_j, for $j \in \tau_2$, which solve*

$$\sigma = T_{a_i,\ell(i)}(\alpha_i) = H_{a_j,\ell(j)}(\beta_j) \qquad \forall i \in \tau_1, j \in \tau_2$$

subject to the constraint

$$\sum_{i \in \tau_1} \alpha_i^2 = \sum_{j \in \tau_2} \beta_j^2.$$

Denote also by S_0 the collection of Steklov eigenvalues corresponding to separated eigenfunctions having a linear factor. Then the Steklov spectrum of a cuboid Ω is given by the union of S_0 which contains at most finitely many elements, and the families $S_{\tau,l}$ for all possible choices of τ and ℓ.

2.3. **Reduction to approximate lattice counting.** We will now give a more precise description of the spectrum by constructing a correspondence between the Steklov eigenvalues of cuboids and the vertices of certain lattices.

Let Ω be a cuboid with parameters $a_1,\dots,a_d$. Let $p \in \{1,\dots,d-1\}$ represent the number of trigonometric factors of a separated eigenfunction without linear factors. Each bipartition $\tau = (\tau_1,\tau_2) \in \mathscr{T}_p$ then corresponds to a separated eigenfunction of the form

$$u(x_1,\dots,x_d) = \prod_{j \in \tau_1} \mathrm{Trig}_j(\alpha_j x_j) \prod_{k \in \tau_2} \mathrm{Hyp}_k(\beta_k x_k) \tag{2.3.1}$$

Let $\mathbb{N}_0 = \{0,1,2,\dots\}$ be the set of nonnegtive integers. Given $\mathbf{n} \in \mathbb{N}_0^p$, let

$$I_{\mathbf{n}} = I_{\mathbf{n},p,\tau} := \prod_{i \in \tau_1} \left(\frac{n_i \pi}{2a_i}, \frac{(n_i+1)\pi}{2a_i} \right] \subset \mathbb{R}^p.$$

The boxes $I_{\mathbf{n}}$ are fundamental domains of a lattice. The following theorem shows that each box gives rise to a cluster of at most 2^q eigenvalues and, moreover, the boxes $I_{\mathbf{n}}$ with $\mathbf{n} \in \mathbb{N}^p$ and $|\mathbf{n}|$ large enough correspond to precisely 2^q eigenvalues.

Theorem 2.6. *Given $p \in \{1,\dots,d-1\}$, and $q = d-p$, let $\tau \in \mathcal{T}_p$ specify the position of trigonometric and hyperbolic factors of eigenfunctions of the form* (2.3.1). *The following assertions hold:*

(i) *Eigenfunctions of the form* (2.3.1) *form a complete system of Steklov eigenfunctions on a cuboid up to a finite number of eigenfunctions containing linear factors.*

(ii) *For each $\mathbf{n} \in \mathbb{N}^p$, there exist at most 2^q eigenfunctions of the form* (2.3.1) *with $\boldsymbol{\alpha} \in I_{\mathbf{n}}$.*

(iii) *There exists a number $N \in \mathbb{N}$, such that for every $\mathbf{n} \in \mathbb{N}^p$ with $|\mathbf{n}| > N$, there are* exactly 2^q *eigenfunctions of the form* (2.3.1) *with $\boldsymbol{\alpha} \in I_{\mathbf{n}}$. The corresponding eigenvalues $\sigma_{\mathbf{n}}^{(k)}$, with $k \in \{1,\dots,2^q\}$, satisfy*

$$\sigma_{\mathbf{n}}^{(k)} = \frac{|\boldsymbol{\alpha}_{\mathbf{n}}|}{\sqrt{q}} + O\left(|\mathbf{n}|^{-\infty}\right) \tag{2.3.2}$$

for some $\boldsymbol{\alpha}_{\mathbf{n}} \in I_{\mathbf{n}}$, where $f(x) = O(|x|^{-\infty})$ means that $f(x) = O\left(|x|^{-N}\right)$ for N arbitrarily large.

(iv) *There exist only finitely many eigenfunctions of the form* (2.3.1) *such that $\mathbf{n} \in \mathbb{N}_0^p \setminus \mathbb{N}^p$. For each $\mathbf{n} \in \mathbb{N}_0^p \setminus \mathbb{N}^p$, there are at most 2^q eigenfunctions of the form* (2.3.1) *with $\boldsymbol{\alpha} \in I_{\mathbf{n}}$.*

Assertions (ii) and (iii) essentially say that up to a finite number of boxes, there is always exactly 2^q solutions in the box $I_{\mathbf{n}}$, while assertion (iv) says that while some boxes touching the coordinate hyperplanes $\{x_j = 0\}$ might contain solutions, this will only happen a finite number of times. This means that while all the three cases are needed to fully describe the spectrum, asymptotically we can only count eigenvalues described by (iii), up to a $O(1)$ error.

Proof of Theorem 2.6. Assertion (i) is a direct consequence of Lemmas 2.1 and 2.4. In order to prove assertion (ii), for each $\ell : S_d \to \{0,1\}$ and $\mathbf{n} \in \mathbb{N}^p$ we will show that there exists at most one eigenfunction. Up to a small error, the corresponding eigenvalue will be equal to the norm of a point which is located in the box $I_{2\mathbf{n}+\mathbf{m},p,\tau}$, where $\mathbf{m} \in \{0,1\}^p$ is determined by the restriction of ℓ to τ_1. Together with the choice of ℓ on τ_2, this will account for clusters of at most 2^q eigenvalues corresponding to each of the boxes $I_{\mathbf{n}}$.

Construction of an eigenfunction. For each $i \in \tau_2$, the function $\beta_i \mapsto H_{a_i,\ell(i)}(\beta_i)$, is increasing and positive for $\beta_i > 0$. It satisfies $H_{a_i,\ell(i)}(\beta_i) = \beta_i + O(\beta_i^{-\infty})$ as $\beta_i \to \infty$ and

$$\lim_{\beta_i \to 0} H_{a_i,\ell(i)}(\beta_i) = \begin{cases} \frac{1}{a_i} & \text{if } \ell(i) = 0, \\ 0 & \text{if } \ell(i) = 1. \end{cases}$$

This implies that the equations

$$H_{a_i,\ell(i)}(\beta_i) = H_{a_j,\ell(j)}(\beta_j) \qquad \forall i,j \in \tau_2 \tag{2.3.3}$$

define a connected curve $C_H = C_{H,p,\tau} \subset \mathbb{R}^q$ (the index H stands for "hyperbolic") which behaves like the diagonal

$$\{\boldsymbol{\beta} \in \mathbb{R}^q : \beta_i = \beta_j \text{ for each } i,j \in \tau_2\}$$

to infinite order as $|\boldsymbol{\beta}| \to \infty$. The common value given by equation (2.3.3) increases monotonically from some $c \geq 0$ to infinity along the curve C_H as it moves away from the origin. In fact, this non-negative constant is

$$c_\ell = \max\{0, a_i^{-1} : i \in \tau_2, \ell(i) = 0\}.$$

On the other hand, for each $i \in \tau_1$ the restricted function

$$T_{a_i,\ell(i)} : \left(\frac{n_i\pi}{a_i} + \frac{\ell(i)\pi}{2a_i}, \frac{n_i\pi}{a_1} + \frac{(\ell(i)+1)\pi}{2a_i} \right] \longrightarrow [0,\infty), \tag{2.3.4}$$

is decreasing and surjective. Hence, for each point $\boldsymbol{\beta} \in C_H \subset \mathbb{R}^q$, there exist unique numbers

$$\alpha_i(\boldsymbol{\beta}) \in \left(\frac{n_i\pi}{a_i} + \frac{\ell(i)\pi}{2a_i}, \frac{n_i\pi}{a_1} + \frac{(\ell(i)+1)\pi}{2a_i} \right] \qquad (\text{ for each } i \in \tau_1)$$

such that

$$T_{a_i,\ell(i)}(\alpha_i) = H_{a_j,\ell(j)}(\beta_j) \qquad \forall i \in \tau_1, j \in \tau_2. \tag{2.3.5}$$

This defines an image curve $C_T \subset \mathbb{R}^p$ given by

$$C_T = \{\alpha_i(\boldsymbol{\beta}) : i \in \tau_1, \boldsymbol{\beta} \in C_H\}.$$

In other words, we have defined a continuous map $\boldsymbol{\alpha} : C_H \longrightarrow C_T$ between these two curves. It follows from (2.3.4) that the curve C_T is contained in the box $I_{2\mathbf{n}+\mathbf{m}}$, where $\mathbf{m} \in \{0,1\}^p$ is determined by the restriction of ℓ to τ_1. In particular, as the value of $|\boldsymbol{\beta}|$ increases from its minimal value to $+\infty$ along the curve C_H, the value of $|\boldsymbol{\alpha}(\boldsymbol{\beta})|$ is contained in the compact interval

$$\left[\inf_{\mathbf{x}\in I_{2\mathbf{n}+\mathbf{m}}} |\mathbf{x}|, \sup_{\mathbf{x}\in I_{2\mathbf{n}+\mathbf{m}}} |\mathbf{x}| \right] \subset (0,\infty).$$

Hence, if $\inf_{\mathbf{x}\in I_{2\mathbf{n}+\mathbf{m}}} |\mathbf{x}| > c_\ell$ there will be a point $\boldsymbol{\beta} \in C_H$ such that $\boldsymbol{\alpha} = \alpha(\boldsymbol{\beta})$ satisfy $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$. This amounts to saying that any of the common values given by (2.3.5) is a Steklov eigenvalue of the cuboid. It follows from monotonicity of each factor in Equation (2.3.5) that this solution $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is unique.

Remark 2.7. Let $d = 4$, $a_1 = a_2 = a_3 = a_4 = 1$, $p = 2$ and $\tau_1 = (1,2)$. In this case, Figure 2 shows the intersections of the four different curves C_T with the boxes $I_{2\mathbf{n}+\mathbf{m}} \subset \mathbb{R}^2$ for $\mathbf{n} = (12,2)$ and $\mathbf{m} \in \{0,1\}^2$. The corresponding curve C_H for the particular choice of the hyperbolic factor given by $\ell(3) = 1$ and $\ell(4) = 0$, is shown on Figure 3. On each of these curves, the marked point corresponds to the solution of the compatibility equations. Note that the curves C_T intersect two of the boxes, and the functions $T_{a_i,\ell(i)}$ defined on them are positive in one box and negative in the other. The solutions of the compatibility equations lie on the positive side.

We now turn to assertion (iii). Observe first that there is a uniform bound on c_ℓ hence there is a N such that if $|\mathbf{n}| > N$ then

$$\inf_{\mathbf{x}\in I_\mathbf{n}} |\mathbf{x}| > c_\ell.$$

From the previous discussion this ensures that there are exactly 2^q solutions in the box $I_\mathbf{n}$. We proceed in two steps for the more quantitative part of the statement. First, we prove that eigenvalues do take the form (2.3.2), and then we show that for all $k \in \{1,2\ldots,2^q\}$ the same $\boldsymbol{\alpha}_\mathbf{n}$ works.

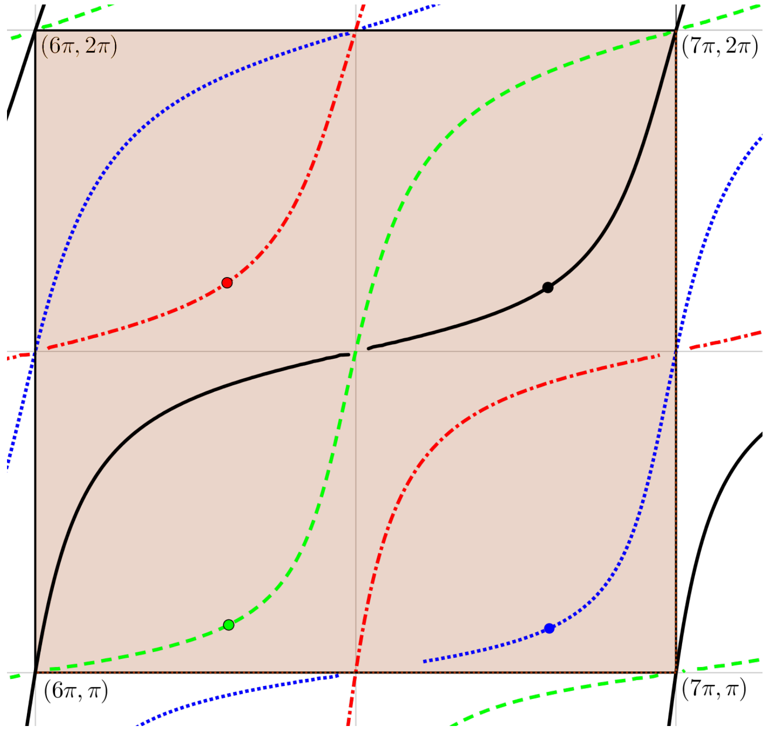

FIGURE 2. Various C_T curves in the situation where $d = 3$, $p = 2$ and $\tau_1 = \{1,2\}$.

Localisation. Fix the restriction $\ell : \tau_2 \to \{0,1\}^q$ for the moment. The various choices of trigonometric factors (represented by the choice of $\ell : \tau_1 \to \{0,1\}$) gives rises to exactly one solution $\boldsymbol{\alpha}_{2\mathbf{n}+\mathbf{m}}$ in each of the 2^p boxes $I_{2\mathbf{n}+\mathbf{m}}$, where $\mathbf{m}$ runs over all choices of $\mathbf{m} \in \{0,1\}^p$. For each of these $\mathbf{m}$, the corresponding eigenvalue is given by any of the functions appearing in Equation (2.3.5) evaluated on any of the coordinates of $(\boldsymbol{\alpha}_{2\mathbf{n}+\mathbf{m}}, \boldsymbol{\beta}_{2\mathbf{n}+\mathbf{m}}) \in \mathbb{R}^p \times \mathbb{R}^q$. It follows that for each $j \in \tau_2$, and $\mathbf{n} \in \mathbb{N}^q$

$$|\boldsymbol{\beta}_\mathbf{n}|^2 = \sum_{i \in \tau_2} \beta_{\mathbf{n},i}^2 = q\beta_{\mathbf{n},j}^2 + O\left(|\mathbf{n}|^{-\infty}\right).$$

Hence for each $j \in \tau_2$,

$$\beta_{\mathbf{n},j} = \frac{|\boldsymbol{\beta}_\mathbf{n}|}{\sqrt{q}} + O\left(|\mathbf{n}|^{-\infty}\right).$$

The corresponding eigenvalue is therefore given, for any $j \in \tau_2$, by

$$\sigma_\mathbf{n} = H_{a_j,\ell(j)}(\beta_{\mathbf{n},j}) = \frac{|\boldsymbol{\beta}_\mathbf{n}|}{\sqrt{q}} + O\left(|\mathbf{n}|^{-\infty}\right) = \frac{|\boldsymbol{\alpha}_\mathbf{n}|}{\sqrt{q}} + O\left(|\mathbf{n}|^{-\infty}\right),$$

as was announced.

Clustering. If $\ell, \ell' : S_d \to \{0,1\}$ agree on τ_1, it follows from

$$H_{a_j,\ell(j)}(x) - H_{a_j,\ell'(j)}(x) = O\left(x^{-\infty}\right)$$

that the corresponding eigenvalues satisfy

$$\sigma_{\mathbf{n},\ell} - \sigma_{\mathbf{n},\ell'} = O\left(|\mathbf{n}|^{-\infty}\right).$$

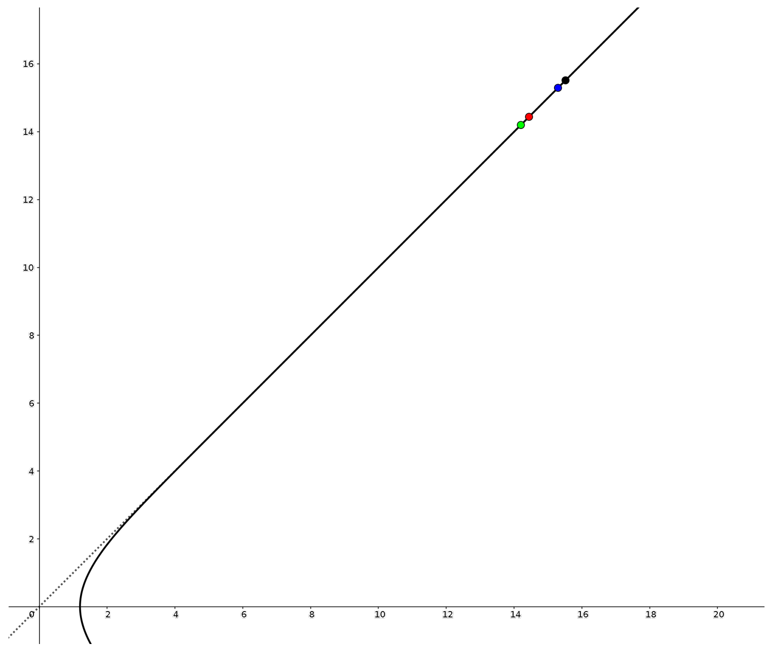

FIGURE 3. The curve C_H corresponding to $\ell(3) = 1$ and $\ell(4) = 0$: $x_3 \tanh(x_3) = x_4 \coth(x_4)$.

The various choices of the restriction $\ell : \tau_2 \to \{0,1\}$ therefore lead to 2^q eigenvalues satisfying

$$\sigma_{\mathbf{n}}^k = \frac{|\boldsymbol{\alpha}_{\mathbf{n}}|}{\sqrt{q}} + O\left(|\mathbf{n}|^{-\infty}\right) \qquad \text{for } k = 1, \ldots, 2^q.$$

Exceptional eigenvalues. For $\mathbf{n} \in \mathbb{N}_0^p \setminus \mathbb{N}^p$ we have that $n_i = 0$ for at least one $i \in \tau_1$. On the interval $\left(0, \frac{\pi}{2a_i}\right]$, the function $T_{a_i,0}$ is positive while $T_{a_i,1}$ is negative, hence an eigenvalue can only correspond to $\ell(i) = 0$. In this case, the range of $T_{a_i,0}$ is $\left[0, a_i^{-1}\right)$. A corresponding eigenvalue is therefore bounded above by a_i^{-1}. There is only a finite number of these, proving assertion (iv).

This concludes the proof of Theorem 2.6. $\square$

In the next section we will take up the task of understanding the asymptotic behavior of the counting function $N(\sigma)$.

3. EIGENVALUE ASYMPTOTICS

The goal of Section 3 is to prove Theorem 1.1. The plan is to represent the counting function $N(\sigma)$ as a sum of auxiliary counting functions corresponding to different families of eigenvalues provided by Theorem 2.6. Each of those counting functions will be then investigated using lattice counting techniques.

3.1. **A hierarchy of counting functions.** Let $p \in \{1, 2, \ldots, d-1\}$. Given $\tau = (\tau_1, \tau_2) \in \mathscr{T}_p$ and $\ell : S_d \to \{0,1\}$, define the counting function $N^{\tau,\ell} : \mathbb{R} \to \mathbb{N}$ by

$$N^{\tau,\ell}(\sigma) = \#\{j \in \mathbb{N} : \sigma_j \in S_{\tau,\ell} \text{ and } \sigma_j < \sigma\}.$$

Recall that the bipartition τ defines the location τ_1 of the trigonometric factors, and the location τ_2 of the hyperbolic factors, whereas the function ℓ distinguishes between sin and cos trigonometric factors, and sinh and cosh hyperbolic factors. We also introduce

$$N^{\tau}(\sigma) := \sum_{\ell:S_d\to\{0,1\}} N^{\tau,\ell}(\sigma) \quad \text{and} \quad N_p(\sigma) := \sum_{\tau\in\mathcal{T}_p} N^{\tau}(\sigma). \tag{3.1.1}$$

Since there is only a finite number of eigenfunctions with linear factors, one has

$$N(\sigma) = \sum_{p=1}^{d-1} N_p(\sigma) + O(1).$$

Set $q = d-p$ and let $\partial^q\Omega$ denote the union of p-dimensional facets of a cuboid Ω. Our goal is to prove the following asymptotics for $N_p(\sigma)$.

Proposition 3.1. *For each $p = 1,\dots,d-1$, we have:*

$$N_p(\sigma) = \frac{\sqrt{q^p}}{(2\pi)^p}\omega_p \operatorname{Vol}_p(\partial^q(\Omega))\sigma^p + c_p \operatorname{Vol}_{p-1}(\partial^{q+1}\Omega)\sigma^{p-1} + O\left(\sigma^{\eta_p}\right), \tag{3.1.2}$$

where c_p are some explicitly computable constants and

$$\eta_p = \max\left(p-1-\frac{1}{p},\, p-2+\frac{2}{p+1}\right) = \begin{cases} 2/3 & \text{if } p=2, \\ p-1-1/p & \text{otherwise.}\end{cases}$$

We prove Proposition 3.1 in subsection 3.5.

3.2. **Quasi-eigenvalues.** In this section, we observe that the clustering of eigenvalues in Theorem 2.6 allows us to simplify the eigenvalue counting problem. Essentially, we will count every cluster as one eigenvalue with a weight equal to the number of eigenvalues in the cluster.

Definition 3.2. Given $p \in S_d$, $q = d-p$, $\tau \in \mathcal{T}_p$, $\ell : S_d \to \{0,1\}$ and $\mathbf{n} \in \mathbb{N}^p$, the number $\frac{|\boldsymbol{\alpha}_{\mathbf{n}}|}{\sqrt{q}}$ defined in (2.3.2) is called a *quasi-eigenvalue of multiplicity* 2^q.

It is clear from Theorem 2.6 that

$$N(\sigma) = \sum_{p=1}^{d-1} 2^q \# \left\{ \mathbf{n} \in \mathbb{N}^p : \frac{|\boldsymbol{\alpha}_{\mathbf{n}}|}{\sqrt{q}} < \sigma \right\} + O(1). \tag{3.2.1}$$

The factor 2^q accounts for the clustering of eigenvalues around the corresponding quasi-eigenvalue. Note that the $O(1)$ error can be absorbed in the error term in (1.2.1). Therefore, in view of (3.2.1), for our purposes there is no need to distinguish between counting eigenvalues and quasi-eigenvalues.

3.3. **Eigenfunctions with a single trigonometric factor.** Consider first the case $p = 1$.The choice of sin or cos for the trigonometric factor and the choice of the coordinate corresponding to the trigonometric factor yields $2d$ families of eigenfunctions, each having 2^{d-1} possibilities for the choice of the hyperbolic factor. As follows from Theorem 2.6, each of the $2d$ families contributes a cluster of 2^{d-1} eigenvalues which correspond to the same quasi-eigenvalue. Therefore, as was mentioned earlier, this cluster can be counted for our purposes as a single quasi-eigenvalue of multiplicity 2^{d-1}. The compatibility equations

$$H_{a_i,\ell(i)}(\beta_i) = H_{a_j,\ell(j)}(\beta_j) \quad \forall i,j \in \tau_2 \tag{3.3.1}$$

define a connected curve in $\mathbb{R}^{d-1}$ which goes to infinity along the diagonal while its value increases to $+\infty$. Equating (3.3.1) to $T_{a_k,\ell(k)}$, $k \in \tau_1$ amounts to solving the following equations:

$$\alpha_k \cot(a_k\alpha_k) = \frac{\alpha_k}{\sqrt{d-1}} + O\left(\alpha_k^{-\infty}\right) \qquad \text{if } \ell(k) = 0,$$

and

$$-\alpha_k \tan(a_k\alpha_k) = \frac{\alpha_k}{\sqrt{d-1}} + O\left(\alpha_k^{-\infty}\right) \qquad \text{if } \ell(k) = 1.$$

This yields eigenvalues of the form

$$\sigma = \begin{cases} \frac{\pi j}{a_j\sqrt{d-1}} + \frac{1}{a_j\sqrt{d-1}} \operatorname{arccot}((d-1)^{-1/2}) + O\left(j^{-\infty}\right) & \text{if } \ell(k) = 0, \\ \frac{\pi j}{a_j\sqrt{d-1}} + \frac{1}{a_j\sqrt{d-1}} \arctan((d-1)^{-1/2}) + O\left(j^{-\infty}\right) & \text{if } \ell(k) = 1, \end{cases}$$

each with quasi-multiplicity 2^{d-1}. Given that arccot and arctan are bounded functions, and since

$$\operatorname{Vol}_1(\partial^{d-1}\Omega) = 2^d \sum_{j=1}^d a_j,$$

we have that

$$N_1(\sigma) = \frac{\omega_1\sqrt{d-1}}{2\pi} \operatorname{Vol}_1(\partial^{d-1}\Omega)\,\sigma + O(1)\,.$$

This concludes the proof of Theorem 1.1 for $d = 2$, since $p = 1$ is the only possibility in this case. Observe that for $d = 2$, this is indeed the expected first term of Weyl's law (1.1.2).

3.4. **Eigenfunctions with many trigonometric factors.** In this subsection, we count the number of eigenvalues associated with eigenfunctions with more than one trigonometric factor. The idea is to write the eigenvalues as the norms of points $\boldsymbol{\alpha} \in \mathbb{R}^p$ that are close to some lattice points. The main difficulty is that the compatibility equations are transcendental, making it impossible to explicitly find $\boldsymbol{\alpha}$. We will therefore approximate the eigenvalues in a controlled way, and we will show that this approximation results in a small enough error that could be absorbed in the remainder in the two-term asymptotics for the eigenvalue counting function. Finally, we will use the lattice point counting techniques going back to [8, 15], and more recently used in [11].

3.4.1. *Approximate eigenvalues.* Suppose that $d \geq 3$ and $p \in \{2, \ldots, d-1\}$. Let $\tau \in \mathcal{T}_p$ and $\ell : S_d \to \{0,1\}$ be given.

Given $\mathbf{n} \in \mathbb{N}^p$, it follows from Theorem 2.6 and the compatibility equations (2.2.3), that the corresponding solution $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\mathbf{n}} \in I_{\mathbf{n}}$ satisfies the following for each $i, j \in \tau_1$

$$\alpha_i \cot\left(\alpha_i a_i + \frac{\ell(i)\pi}{2}\right) = \alpha_j \cot\left(\alpha_j a_j + \frac{\ell(j)\pi}{2}\right) = \frac{|\boldsymbol{\alpha}_{\mathbf{n}}|}{\sqrt{q}} + O\left(|\mathbf{n}|^{-\infty}\right).$$

Hence, for each $i \in \tau_1$, we have, choosing the principal branch of arccot, a family of solutions indexed by $\mathbf{n} \in \mathbb{N}^p$

$$\alpha_i a_i = \left(n_i + \frac{\ell(i)}{2}\right)\pi + \operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1 + \sum_{j \neq i \in \tau_1}\left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^{1/2}\right) + O\left(|\mathbf{n}|^{-\infty}\right).$$

Since $\alpha_i = \frac{\left(n_i + \frac{\ell(i)}{2}\right)\pi}{a_i} + O(1)$, we can rewrite the previous equation as follows

$$\alpha_i = \frac{\left(n_i + \frac{\ell(i)}{2}\right)\pi}{a_i} + \frac{1}{a_i}\operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1 + \sum_{j\neq i}\left(\frac{\frac{\left(n_j + \frac{\ell(j)}{2}\right)\pi}{a_j} + t_{\alpha_j}(\mathbf{n})}{\frac{\left(n_i + \frac{\ell(i)}{2}\right)\pi}{a_i} + t_{\alpha_i}(\mathbf{n})}\right)^2\right]^{1/2}\right) + O\left(|\mathbf{n}|^{-\infty}\right), \tag{3.4.1}$$

where the functions t_{α_j} are bounded. Since $\ell(i)$ ranges over $\{0,1\}$, the solution set to the previous equation is the same as the one to

$$\alpha_i = \frac{n_i\pi}{2a_i} + \frac{1}{a_i}\operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1 + \sum_{j\neq i}\left(\frac{\frac{n_j\pi}{2a_j} + t_{\alpha_j}(\mathbf{n})}{\frac{n_i\pi}{2a_i} + t_{\alpha_i}(\mathbf{n})}\right)^2\right]^{1/2}\right) + O\left(|\mathbf{n}|^{-\infty}\right). \tag{3.4.2}$$

Lemma 3.3. *Define* $\widetilde{\alpha}_i$ *as*

$$\widetilde{\alpha}_i = \frac{n_i\pi}{2a_i} + \frac{1}{a_i}\operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1 + \sum_{j\neq i}\left(\frac{a_i n_j}{a_j n_i}\right)^2\right]^{1/2}\right). \tag{3.4.3}$$

Then,

$$\widetilde{\alpha}_i = \alpha_i + O\left(|\mathbf{n}|^{-1}\right) \tag{3.4.4}$$

Proof. In Lemma A.1 in the Appendix, take $x_i = \frac{n_i\pi}{a_i}$ and $\psi_i = t_{\alpha_i}$. Then, one readily sees that

$$|\mathbf{x}| \asymp |\mathbf{n}|,$$

where $f \asymp g$ means that $f = O(g)$ and $g = O(f)$. The lemma then follows. □

Note that the right hand side of equation (3.4.3) does not depend on α_i anymore, which makes it easier to analyse.

We now have eigenvalues indexed by $\mathbf{n} \in \mathbb{N}^p$ given by

$$\sigma_{\mathbf{n}} = \sqrt{\frac{1}{q}\sum_{i\in\tau_1}\widetilde{\alpha}_i^2} + O\left(|\mathbf{n}|^{-1}\right). \tag{3.4.5}$$

Definition 3.4. The numbers

$$\widetilde{\sigma}_{\mathbf{n}} = \sqrt{\frac{1}{q}\sum_{i\in\tau_1}\widetilde{\alpha}_i^2} \tag{3.4.6}$$

are called the *approximate eigenvalues.*

Remark 3.5. Up until now, eigenvalues, quasi-eigenvalues and approximate eigenvalues were indexed by $\mathbf{n} \in \mathbb{N}^p$. In the following two theorems it is convenient to use $n \in \mathbb{N}$ to index them in ascending order.

The following lemma allows us to estimate the error induced by counting approximate eigenvalues instead of eigenvalues.

Lemma 3.6. *Let $(a_n), (b_n)$ be two sequences of positive numbers which tend to infinity. Suppose there exists a number $s > -1$ such that $a_n = b_n + O\left(b_n^{-s}\right)$. Let*

$$N_a(\lambda) = \#\{n : a_n < \lambda\} \qquad \textit{and} \qquad N_b(\lambda) = \#\{n : b_n < \lambda\}.$$

Suppose that there exists a number K such that

$$N_a(\lambda) = \sum_{k=0}^{K} c_k \lambda^{p-k} + O\left(\lambda^r\right),$$

with $r < p - K$. Then,

$$N_b(\lambda) = \sum_{k=0}^{K} c_k \lambda^{p-k} + O\left(\lambda^{r'}\right) \tag{3.4.7}$$

where $r' = \max(r, p-1-s)$.

Remark 3.7. Note that if $r' \geq p - K$, some of the terms in the sum in (3.4.7) might be absorbed in the error term.

Proof. Indeed, the assumption on the sequences a_n and b_n implies that there exists $c > 0$ such that

$$N_a\left(\lambda + \frac{c}{\lambda^s}\right) \leq N_b(\lambda) \leq N_a\left(\lambda - \frac{c}{\lambda^s}\right).$$

A direct computation of $N_a(\lambda \pm c\lambda^{-s})$ completes the proof of the lemma. □

Recall now the definition of $N^\tau(\sigma)$ given by (3.1.1). We will write $\widetilde{N}^\tau$ for the counting function of the corresponding approximate eigenvalues.

Lemma 3.8. *We have:*

$$\left|\widetilde{N}^\tau(\sigma) - N^\tau(\sigma)\right| = O\left(\sigma^{p-1-1/p}\right).$$

Proof. Both the eigenvalues and the approximate eigenvalues are, up to a bounded error, the norms of the points of the lattice $\Gamma = \bigoplus_{i=1}^{p} \frac{\pi}{2a_i\sqrt{q}}\mathbb{N}$, repeated 2^q times. Denote by $l_n := \left\{|\boldsymbol{\gamma}| : \boldsymbol{\gamma} \in \Gamma\right\}_n$ the sequence of norms of the points of the lattice Γ arranged in ascending order. It is well known that there is a constant C such that

$$N_l(\sigma) = C\sigma^p + O\left(\sigma^{p-1}\right),$$

where C depends on Γ and N_l denotes the counting function of the sequence l_n as in Lemma 3.6. Applying Lemma 3.6 with $s = 0$ yields

$$N^\tau(\sigma) = 2^q C\sigma^p + O\left(\sigma^{p-1}\right).$$

Reversing this expression tells us that

$$\sigma_n = \left(\frac{n}{2^q C}\right)^{1/p} + o\left(n^{1/p}\right). \tag{3.4.8}$$

From equations (3.4.6) and (3.4.8) we have that

$$\widetilde{\sigma}_n = \sigma_n + O\left(n^{-1/p}\right).$$

Therefore, applying once again Lemma 3.6, but this time with $s = 1/p$, yields

$$N^\tau(\sigma) = \widetilde{N}^\tau(\sigma) + O\left(\sigma^{p-1-1/p}\right). \tag{3.4.9}$$

□

3.4.2. *Another representation of the counting function.* For every τ, let us now define a family of sets $E_\sigma \subset \mathbb{R}^p$ with the property that

$$\widetilde{N}^\tau(\sigma) = \sum_{\mathbf{n}\in\mathbb{N}^p} 2^q \chi\left(\frac{\mathbf{n}}{\sigma}\right) + O(1), \tag{3.4.10}$$

where $\chi := \chi_\sigma$ is the indicator function of E_σ. Let us define elliptic polar coordinates in $\mathbb{R}^p$ with the convention that $\theta_p = 0$:

$$\begin{aligned} r^2 &= \sum_{i\in\tau_1} \left(\frac{\pi x_i}{2a_i\sqrt{q}}\right)^2, \\ x_j &= r\frac{2a_j\sqrt{q}}{\pi}\cos(\theta_j)\prod_{i<j}\sin(\theta_i). \end{aligned} \tag{3.4.11}$$

We define the family of sets

$$E_\sigma := \left\{(r,\boldsymbol{\theta}) \in \mathbb{R}^p : r^2 + \frac{2r}{\sigma}\sum_{j\in\tau_1}\frac{1}{a_j}g_j(\boldsymbol{\theta}) + \frac{H(\boldsymbol{\theta})}{\sigma^2} < 1\right\}, \tag{3.4.12}$$

with

$$g_j(\boldsymbol{\theta}) := \cos\theta_j \prod_{i<j}\sin\theta_i \operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1 + \sum_{i\neq j}\left(\frac{x_i}{x_j}\right)^2\right]^{1/2}\right), \tag{3.4.13}$$

and

$$H = H(\boldsymbol{\theta}) = \sum_{j\in\tau_1}\frac{1}{a_j^2}\operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1 + \sum_{i\neq j}\left(\frac{x_i}{x_j}\right)^2\right]^{1/2}\right)^2.$$

From equation (3.4.6), we can observe that the evaluation of χ at $\sigma^{-1}\mathbf{n}$ in coordinates (3.4.11) is 1 if and only if $\widetilde{\sigma}_{\mathbf{n}} < \sigma$. If $|\mathbf{n}| > N$ as in Theorem 2.6, there are 2^q solutions close to any order to $\widetilde{\sigma}_{\mathbf{n}}$. This achieves our stated goal of equation (3.4.10). Let us now prove a few properties of the set E_σ that will be required in the sequel.

Lemma 3.9. *There exists σ_0, such that for $\sigma > \sigma_0$, the set E_σ is strictly convex and the principal curvatures of ∂E_σ are positive and uniformly bounded away from* 0. *Furthermore, the boundary ∂E_σ is smooth for all $\sigma > \sigma_0$, and all the derivatives of the boundary defining function are uniformly bounded in $\sigma > \sigma_0$.*

Proof. From equation (3.4.12) ∂E_σ is the level set of the function

$$F(r,\boldsymbol{\theta}) = r^2 + \frac{2r}{\sigma}\sum_{j\in\tau_1}\frac{1}{a_j}g_j(\boldsymbol{\theta}) + \frac{H(\boldsymbol{\theta})}{\sigma^2},$$

which satisfies

$$\begin{aligned} F(r,\boldsymbol{\theta}) &= r^2 + O\left(\sigma^{-1}\right), \\ [\nabla F(\mathbf{x})]_i &= \frac{\pi x_i}{a_i\sqrt{q}} + O\left(\sigma^{-1}\right), \\ \operatorname{Hess} F &= \operatorname{diag}\left(\frac{\pi}{a_i\sqrt{q}}\right)_{i\in\tau_1} + O\left(\sigma^{-1}\right). \end{aligned} \tag{3.4.14}$$

One can observe that H and g_j, $j \in \tau_1$ are all smooth functions of $\boldsymbol{\theta} \in \mathbb{S}^{p-1}$. This automatically yields the claim on the smoothness and the uniform boundedness of the derivatives of the boundary defining function.

Furthermore, for σ large enough, the second fundamental form of ∂E_σ, being a positive multiple of Hess F, is positive, with its smallest eigenvalue uniformly bounded away from 0. This implies the claim on the principal curvatures, which in turn yields strict convexity of E_σ. □

This directly implies the following corollary.

Corollary 3.10. *The product of the principal curvatures of ∂E_σ is uniformly bounded away from zero for σ large enough.*

3.4.3. *Poisson Summation Formula.* In this section, we use the general scheme of the proof of [11, Theorem 1.1]. Recall that

$$
\begin{aligned}
N^\tau(\sigma) &= \sum_{\mathbf{n}\in\mathbb{N}^p} 2^q \chi\left(\frac{\mathbf{n}}{\sigma}\right) + O(1) \\
&= 2^{q-p} \sum_{\mathbf{n}\in\mathbb{Z}^p} \chi\left(\frac{\mathbf{n}}{\sigma}\right) + R_\tau(\sigma) + O(1),
\end{aligned}
\tag{3.4.15}
$$

where $R_\tau(\sigma)$ is the error term induced by the overcounting of points on hyperplanes with one vanishing coordinate.

Our goal is now to compute the terms appearing in equation (3.4.15) using the Poisson summation formula which states, under sufficient smoothness assumptions that

$$
\sum_{\mathbf{n}\in\mathbb{Z}^p} f(\mathbf{n}) = \sum_{\mathbf{m}\in\mathbb{Z}^p} \widehat{f}(\mathbf{m}) \tag{3.4.16}
$$

where the Fourier transform is given by

$$
\widehat{f}(\boldsymbol{\xi}) := \int_{\mathbb{R}^p} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}}\,\mathrm{d}\mathbf{x}.
$$

However, χ is not regular enough for us to use the Poisson summation formula, hence we need to mollify it. Let us introduce a nonnegative function $\psi \in C_c^\infty(\mathbb{R})$ supported in $[-1,1]$ and such that

$$
\int_0^\infty \psi(r) r^{p-1}\,\mathrm{d}r = \frac{1}{V_{p-1}},
$$

with V_{p-1} being the volume of the $p-1$ dimensional unit sphere in $\mathbb{R}^p$. We then define a family $\Psi_\epsilon : \mathbb{R}^p \to \mathbb{R}$ of radial bump functions of total mass 1 by

$$
\Psi_\epsilon(\mathbf{x}) = \frac{1}{\epsilon^p}\psi\left(\frac{|\mathbf{x}|}{\epsilon}\right).
$$

Set $\Psi := \Psi_1$ Consider the smooth function $\chi_\epsilon = \Psi_\epsilon * \chi$. Note that

$$
\widehat{\Psi}_\epsilon(\boldsymbol{\xi}) = \widehat{\Psi}(\epsilon\boldsymbol{\xi})
$$

We now prove the following lemma.

Lemma 3.11. *Let $\chi_\epsilon^+, \chi_\epsilon^- : \mathbb{R}^p \to \mathbb{R}$ be defined by*

$$
\begin{aligned}
\chi_\epsilon^+(\mathbf{x}) &= \chi_\epsilon\big((1-\eta_+\epsilon)\mathbf{x}\big) \\
\chi_\epsilon^-(\mathbf{x}) &= \chi_\epsilon\big((1+\eta_-\epsilon)\mathbf{x}\big)
\end{aligned}
$$

for some $\eta_-, \eta_+ > 0$. One can choose η_-, η_+ in such a way that for all σ large enough

$$
\chi_\epsilon^-(\mathbf{x}) \le \chi(\mathbf{x}) \le \chi_\epsilon^+(\mathbf{x})
$$

for all $\mathbf{x} \in \mathbb{R}^p$ and all $\epsilon > 0$ small enough.

Proof. For the first inequality, observe that

$$\begin{aligned}\chi_\epsilon((1+\eta_-\epsilon)\mathbf{x}) &= \int_{\mathbb{R}^p} \chi(\mathbf{y})\Psi_\epsilon((1+\eta_-\epsilon)\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\\ &= \int_{B_{(1+\eta_-\epsilon)\mathbf{x}}(\epsilon)} \chi(\mathbf{y})\Psi_\epsilon((1+\eta_-\epsilon)\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\\ &\le \sup_{B_{(1+\eta_-\epsilon)\mathbf{x}}(\epsilon)} \chi(\mathbf{y}).\end{aligned}$$

Hence, to show that $\chi_\epsilon((1+\eta_-\epsilon)\mathbf{x}) \le \chi(\mathbf{x})$ for all $\mathbf{x}$, by convexity of E_σ it is sufficient to show that for all $\mathbf{x} \in \partial E_\sigma$, there exists η_-, independent of σ such that the following holds for each $\epsilon > 0$ small enough

$$B_{(1+\eta_-\epsilon)\mathbf{x}}(\epsilon) \cap E_\sigma = \varnothing.$$

Note that for all $\mathbf{x} \in \partial E_\sigma$, we have that

$$\operatorname{dist}((1+t)\mathbf{x}, \partial E_\sigma) = (\mathbf{x}\cdot \mathcal{N}_{\partial E_\sigma}(\mathbf{x}))\,t + O\left(t^2\right), \tag{3.4.17}$$

where $\mathcal{N}_{\partial E_\sigma}$ is the Gauss map of the boundary. To see this, denote by $T_{\mathbf{x}}\partial E_\sigma$ the tangent hyperplane of ∂E_σ at $\mathbf{x}$, and by $P_{\mathbf{x}}$ the orthogonal projection on that hyperplane. We have by the triangle inequality that

$$|\operatorname{dist}((1+t)\mathbf{x}, \partial E_\sigma) - \operatorname{dist}((1+t)\mathbf{x}, T_{\mathbf{x}}\partial E_\sigma)| \le \operatorname{dist}(P_{\mathbf{x}}((1+t)\mathbf{x}), \partial E_\sigma).$$

We observe that $\operatorname{dist}((1+t)\mathbf{x}, T_{\mathbf{x}}\partial E_\sigma) = (\mathbf{x}\cdot\mathcal{N}_{\partial E_\sigma}(\mathbf{x}))\,t$. Let F, as before, be the function in $\mathbb{R}^p$ such that the set $F \equiv 1$ coincides with ∂E_σ. Taking the Taylor expansion of F around $\mathbf{x}$, we have that

$$\operatorname{dist}(P_{\mathbf{x}}((1+t)\mathbf{x}), \partial E_\sigma) \le \|\operatorname{Hess} F(\mathbf{x})\|_\infty |P_{\mathbf{x}}((1+t)\mathbf{x})|^2 = O\left(t^2\right),$$

where we used that $\|\operatorname{Hess} F(\mathbf{x})\|_\infty$ is bounded uniformly for $\sigma > \sigma_0$ and $\mathbf{x} \in \partial E_\sigma$. Note that the strict convexity of ∂E_σ and equation (3.4.14) imply that $\mathbf{x}\cdot\mathcal{N}_{\partial E_\sigma}(\mathbf{x})$ is bounded away from zero uniformly for $\sigma > \sigma_0$. This implies that we can choose η_- large enough and independent in σ such that indeed

$$B_{(1+\eta_-\epsilon)\mathbf{x}}(\epsilon) \cap E_\sigma = \varnothing.$$

For the second inequality, we have

$$\begin{aligned}\chi_\epsilon((1-\eta_+\epsilon)\mathbf{x}) &= \int_{\mathbb{R}^p} \chi(\mathbf{y})\Psi_\epsilon((1-\eta_+\epsilon)\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\\ &= \int_{B_{(1-\eta_+\epsilon)\mathbf{x}}(\epsilon)} \chi(\mathbf{y})\Psi_\epsilon((1-\eta_+\epsilon)\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}\\ &\ge \inf_{B_{(1-\eta_+\epsilon)\mathbf{x}}(\epsilon)} \chi(\mathbf{y}).\end{aligned}$$

Hence, to show that $\chi(\mathbf{x}) \le \chi_\epsilon((1-\eta_+)\mathbf{x})$, it is sufficient to show that for all $\mathbf{x} \in \partial E_\sigma$, there exists η_+ independent of σ such that

$$B_{(1-\eta_+)\mathbf{x}}(\epsilon) \subset E_\sigma.$$

Using once again equation (3.4.17) and arguing exactly as above yields the desired number η_+. □

The following is an immediate corollary of the previous lemma:

Corollary 3.12. *We have that*

$$\sum_{\mathbf{n}\in\mathbb{Z}^p} \chi_\epsilon^-\left(\frac{\mathbf{n}}{\sigma}\right) \le \sum_{\mathbf{n}\in\mathbb{Z}^p} \chi\left(\frac{\mathbf{n}}{\sigma}\right) \le \sum_{\mathbf{n}\in\mathbb{Z}^p} \chi_\epsilon^+\left(\frac{\mathbf{n}}{\sigma}\right).$$

We will now apply the Poisson summation formula (3.4.16) to $\chi_\epsilon^\pm$, which are smooth functions. This yields, using the basic properties of the Fourier transform,

$$
\begin{aligned}
\sum_{\mathbf{n}\in\mathbb{Z}^p} \chi_\epsilon^\pm\left(\frac{\mathbf{n}}{\sigma}\right) &= \sigma^p \sum_{\mathbf{m}\in\mathbb{Z}^p} \widehat{\chi}_\epsilon^\pm(\sigma\mathbf{m}) \\
&= \sigma^p \sum_{\mathbf{m}\in\mathbb{Z}^p} (1+O(\epsilon))\,\widehat{\chi}\left(\frac{\sigma\mathbf{m}}{1\mp\eta_\pm\epsilon}\right)\widehat{\Psi}\left(\frac{\epsilon\mathbf{m}\sigma}{1\mp\eta_\pm\epsilon}\right) \\
&= \sigma^p \operatorname{Vol}(E_\sigma) + O\left(\epsilon\sigma^p\right) \\
&\quad + O\left(\sum_{\substack{\mathbf{m}\in\mathbb{Z}^p\\ \mathbf{m}\neq 0}} \sigma^p\widehat{\chi}\left(\frac{\sigma\mathbf{m}}{1\mp\eta_\pm\epsilon}\right)\widehat{\Psi}\left(\frac{\epsilon\mathbf{m}\sigma}{1\mp\eta_\pm\epsilon}\right)\right).
\end{aligned}
\tag{3.4.18}
$$

Note that for this expression to hold, we will need to later choose $\epsilon = o(1)$. Since Ψ is a Schwartz function, its Fourier transform is also Schwartz, hence to find estimates on the asymptotic behaviour of equation (3.4.18), we only need to find bounds on $\widehat{\chi}$. This is done in the following Lemma.

Lemma 3.13. *For σ large enough, the Fourier transform of χ satisfies the upper bound*

$$
\widehat{\chi}(\boldsymbol{\xi}) = O\left(|\boldsymbol{\xi}|^{-\frac{d+1}{2}}\right). \tag{3.4.19}
$$

Proof. For σ large enough, the set E_σ is strictly convex and has smooth boundary. Therefore, following [9, Theorem 2.29] we have that for any function $f \in C^\infty(\mathbb{R}^p)$ such that $f \neq 0$ on ∂E_σ,

$$
\int_{E_\sigma} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}}\,\mathrm{d}\mathbf{x} = O\left(|\boldsymbol{\xi}|^{-\frac{d+1}{2}}\right),
$$

where the implicit constants depend on the product of the principal curvatures of ∂E_σ and stay bounded as long as the principal curvatures are bounded away from 0. Hence, by equation (3.4.14), these constants will be uniformly bounded for σ large enough. Applying this result with $f(\mathbf{x}) \equiv 1$ yields the desired result. □

Remark 3.14. Note that the estimates on lower order error terms obtained in [9, Theorem 2.29] depend on the derivatives of the boundary defining function of ∂E_σ. By Lemma 3.9 those derivatives are bounded uniformly for $\sigma > \sigma_0$.

We now find the dependence on ϵ of the third summand in (3.4.18). We will choose the optimal value of ϵ such that the second and the third terms are both as small as possible. Splitting the third summand into two terms we use equation (3.4.19) and the fact that $\widehat{\Psi}$ is a Schwartz function to obtain

$$
\begin{aligned}
O\left(\sum_{\substack{\mathbf{m}\in\mathbb{Z}^p\\ m\neq 0}} \sigma^p\widehat{\chi}\left(\frac{\mathbf{m}\sigma}{1\mp\eta_\pm\epsilon}\right)\widehat{\Psi}\left(\frac{\epsilon\mathbf{m}\sigma}{1\mp\eta_\pm\epsilon}\right)\right) = O\Bigg(&\sum_{0<|\mathbf{m}|\le(\epsilon\sigma)^{-1}} \frac{\sigma^{\frac{p-1}{2}}\left(1\mp\eta_\pm\epsilon\right)^{\frac{p+1}{2}}}{|\mathbf{m}|^{\frac{p+1}{2}}} \\
&+ \sum_{|\mathbf{m}|>(\epsilon\sigma)^{-1}} \frac{\sigma^{\frac{p-1}{2}}\left(1\mp\eta_\pm\epsilon\right)^{\frac{p+1}{2}+N}}{|\mathbf{m}|^{\frac{p+1}{2}+N}(\sigma\epsilon)^N}\Bigg),
\end{aligned}
$$

for an arbitrary $N > 0$ which will be fixed below. Assuming that ϵ is small and and taking into account that the summands on the right hand side are decreasing in $|\mathbf{m}|$, we may estimate the first of those sums by

$$\sum_{0<|\mathbf{m}|\le(\epsilon\sigma)^{-1}} \frac{\sigma^{\frac{p-1}{2}}\left(1\mp\eta_\pm\epsilon\right)^{\frac{p+1}{2}}}{|\mathbf{m}|^{\frac{p+1}{2}}} \asymp \sigma^{\frac{p-1}{2}} \int_1^{(\epsilon\sigma)^{-1}} \frac{r^{p-1}}{r^{\frac{p+1}{2}}}\,\mathrm{d}r = O\left(\epsilon^{\frac{1-p}{2}}\right).$$

The second of those sums can be estimated, for N large enough that the integral converges, by

$$\sum_{|\mathbf{m}|>(\epsilon\sigma)^{-1}} \frac{\sigma^{\frac{p-1}{2}}\left(1\mp\eta_\pm\epsilon\right)^{\frac{p+1}{2}+N}}{|\mathbf{m}|^{\frac{p+1}{2}+N}(\sigma\epsilon)^N} \asymp \sigma^{\frac{p-1}{2}}(\sigma\epsilon)^{-N} \int_{(\epsilon\sigma)^{-1}}^{\infty} \frac{r^{p-1}}{r^{\frac{p+1}{2}+N}}\,\mathrm{d}r = O\left(\epsilon^{\frac{1-p}{2}}\right).$$

The optimal ϵ to make both $\sigma^p\epsilon$ and $\epsilon^{\frac{1-p}{2}}$ as small as possible is

$$\epsilon = \sigma^{\frac{-2p}{1+p}},$$

yielding that

$$\sum_{n\in\mathbb{Z}^p} \chi_\epsilon^\pm\left(\frac{n}{\sigma}\right) = \sigma^p \operatorname{Vol}(E_\sigma) + O\left(\sigma^{p-2+\frac{2}{1+p}}\right). \tag{3.4.20}$$

We now compute the volume of E_σ.

Lemma 3.15. *Let* $\Sigma = \mathbb{S}^{p-1}\cap\mathbb{R}^p_+$. *We have:*

$$\operatorname{Vol}_p(E_\sigma) = \frac{2^p\sqrt{q}^p}{\pi^p}\omega_p \prod_{j\in\tau_1} a_j - \frac{2^{2p}\sqrt{q}^p G_{p,q}}{\pi^p\sigma} \sum_{j\in\tau_1}\prod_{i\neq j} a_i + O\left(\sigma^{-2}\right), \tag{3.4.21}$$

where

$$G_{p,q} = \int_\Sigma g_j(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta}, \tag{3.4.22}$$

for any of the functions g_j *defined by equation* (3.4.13).

Remark 3.16. Note that $G_{p,q}$ does not depend on j by the symmetry of the construction of g_j.

Proof. By symmetry, we have that

$$\operatorname{Vol}(E_\sigma) = \frac{2^{2p}\sqrt{q}^p}{\pi^p} \int_\Sigma \int_0^{\rho(\boldsymbol{\theta})} r^{p-1} \prod_{j\in\tau_1} a_j \,\mathrm{d}r\,\mathrm{d}\boldsymbol{\theta}$$

where $\rho(\boldsymbol{\theta})$ is the unique positive root (in r) of the equation

$$r^2 + \frac{2r}{\sigma}\sum_{j\in\tau_1}\frac{g_j(\boldsymbol{\theta})}{a_j} + \frac{H}{\sigma^2} - 1 = 0.$$

One can observe that

$$\rho(\boldsymbol{\theta}) = 1 - \frac{1}{\sigma}\sum_{j\in\tau_1}\frac{g_j(\boldsymbol{\theta})}{a_j} + O\left(\sigma^{-2}\right).$$

Thus, we get that

$$\operatorname{Vol}(E_\sigma) = \frac{2^{2p}}{\pi^p}\prod_{j\in\tau_1} a_j \int_\Sigma \frac{1}{p} - \frac{1}{\sigma}\sum_{j\in\tau_1}\frac{g_j(\boldsymbol{\theta})}{a_j} + O\left(\sigma^{-2}\right)\mathrm{d}\boldsymbol{\theta}$$

Integrating and replacing in the previous equation the definition of $G_{p,q}$ in equation (3.4.22) yields

$$\mathrm{Vol}(E_\sigma) = \frac{2^p}{\pi^p}\omega_p \prod_{j\in\tau_1} a_j - \frac{2^{2p}G_{p,q}}{\pi^p\sigma}\sum_{j\in\tau_1}\prod_{i\neq j} a_i + O\left(\sigma^{-2}\right). \tag{3.4.23}$$

□

Finally, we have to take into account the points that we have overcounted with coefficient $1/2$ on the hyperplanes $\{x_i = 0\}$. This is given in the following lemma.

Lemma 3.17. *The number of overcounted points on the hyperplanes* $\{x_i = 0\}$ *is*

$$R_\tau(\sigma) = \frac{\sqrt{q^p}\,2^p\omega_{p-1}\sigma^{p-1}}{4(2\pi)^{p-1}}\sum_{j\in\tau_1}\prod_{i\neq j} a_i + O\left(\sigma^{p-2}\right). \tag{3.4.24}$$

Proof. One can observe that R_τ is given by

$$R_\tau(\sigma) = \frac{1}{2}\sum_{i\in\tau_1}\#\left\{\sigma^{-1}\mathbb{N}^{p-1}\cap E_\sigma\cap\{x_i = 0\}\right\}$$

Since E_σ is convex, rough lattice point counting estimates due to Gauss tell us that

$$R_\tau(\sigma) = \frac{1}{2}\sigma^{p-1}\sum_{i\in\tau_1}\mathrm{Vol}_{p-1}\left(E_\sigma\cap\{x_i = 0\}\right) + O\left(\sigma^{p-2}\right).$$

Computing the volumes in the same way as in the proof of the previous lemma yields the desired result. □

3.5. **Proof of Proposition 3.1.** Recall that $\widetilde{N}_p$ is given by

$$\widetilde{N}_p(\sigma) = \sum_{\tau\in\mathcal{T}_p}\widetilde{N}^\tau(\sigma).$$

Observe that

$$\sum_{\tau\in\mathcal{T}_p} 2^{p+q}\prod_{j\in\tau_1} a_j = \mathrm{Vol}_p(\partial^q(\Omega))$$

and

$$\begin{aligned}\sum_{\tau\in\mathcal{T}_p}\sum_{j\in\tau_1}\prod_{i\neq j} 2^{p+q}a_i &= (q+1)2^{p+q}\sum_{\tau\in\mathcal{T}_{p-1}}\prod_{j\in\tau_1} a_j \\ &= (q+1)\,\mathrm{Vol}_{p-1}(\partial^{q+1}\Omega)).\end{aligned} \tag{3.5.1}$$

Combining these two formulas with equations (3.4.15), (3.4.20) and Lemmas 3.15, 3.17, yields

$$\widetilde{N}_p(\sigma) = \frac{\sqrt{q^p}}{(2\pi)^p}\omega_p\,\mathrm{Vol}_p(\partial^q(\Omega))\sigma^p + c_p\,\mathrm{Vol}_{p-1}(\partial^{q+1}\Omega)\sigma^{p-1} + O\left(\sigma^{p-2+\frac{2}{p+1}}\right).$$

Using equation (3.5.1), we have that $c_p = c_p' + c_p''$, where

$$c_p' = -\frac{(q+1)\sqrt{q^p}\,G_{p,q}}{\pi^p}$$

comes from the second term in equation (3.4.21) and

$$c_p'' = -\frac{(q+1)\sqrt{q^p}\,\omega_{p-1}}{4(2\pi)^{p-1}}$$

is obtained from the principal term in equation (3.4.24).

We then have from equation (3.4.9) that

$$N_p(\sigma) = \frac{\sqrt{q^p}}{(2\pi)^p}\omega_p \operatorname{Vol}_p(\partial^q(\Omega))\sigma^p + c_p \operatorname{Vol}_{p-1}(\partial^{q+1}\Omega)\sigma^{p-1} + O\left(\sigma^{\eta_p}\right),$$

where

$$\begin{aligned}\eta_p &= \max\left(p-1-1/p, p-2+\frac{2}{p+1}\right)\\ &= \begin{cases} 2/3 & \text{if } p=2,\\ p-1-1/p & \text{otherwise.}\end{cases}\end{aligned}$$

This completes the proof of Proposition 3.1.

Remark 3.18. Observe that for $p>2$, studying the approximate eigenvalues instead of the true eigenvalues yields an error which is larger than the one coming from the standard Poisson summation formula methods.

3.6. **Proof of Theorem 1.1.** Recall now that

$$N(\sigma) = \sum_{p=1}^{d-1} N_p(\sigma) + O(1).$$

Hence, applying the previous results we get

$$\begin{aligned} N(\sigma) &= N_{d-1}(\sigma) + N_{d-2}(\sigma) + O\left(\sigma^{\eta_{d-1}}\right)\\ &= \frac{1}{(2\pi)^{d-1}}\omega_{d-1}\operatorname{Vol}_{d-1}(\partial(\Omega))\sigma^{d-1} + c_{d-1}\operatorname{Vol}_{d-2}(\partial^2\Omega)\sigma^{d-2}\\ &\quad + \frac{2^{\frac{d-2}{2}}}{(2\pi)^{d-2}}\omega_{d-2}\operatorname{Vol}_{d-2}(\partial^2(\Omega))\sigma^{d-2} + O\left(\sigma^{\eta_{d-1}}\right)\\ &= C_1\operatorname{Vol}_{d-1}(\partial\Omega)\sigma^{d-1} + C_2\operatorname{Vol}_{d-2}(\partial^2\Omega)\sigma^{d-2} + O\left(\eta_{d-1}\right).\end{aligned}$$

We can write explicitly $C_2 = c'_{d-1} + c_{d-1''} + \frac{2^{\frac{d-2}{2}}\omega_{d-2}}{(2\pi)^{d-2}}$ to get indeed that

$$C_2 = \frac{2^{\frac{d-2}{2}}\omega_{d-2}}{(2\pi)^{d-2}} - \frac{2G_{d-1,1}}{\pi^{d-1}} - \frac{\omega_{d-2}}{2(2\pi)^{d-2}}$$

when $d\geq 3$ and that

$$N(\sigma) = \frac{\omega_1}{2\pi}\operatorname{Vol}_1(\partial\Omega)\sigma + O(1)$$

when $d=2$.

We can now give explicit expressions for the constants $G_{p,q}$:

$$\begin{aligned} G_{p,q} &= \int_0^{\pi/2}\dots\int_0^{\pi/2}\operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1+\sum_{j=1}^{p-1}\cot^2\theta_j\prod_{i>j}\csc^2\theta_i\right]^{1/2}\right)\prod_{k=1}^{p-1}\sin^k(\theta_k)\,\mathrm{d}\theta_1\dots\mathrm{d}\theta_{p-1}\\ &= \int_0^{\pi/2}\dots\int_0^{\pi/2}\operatorname{arccot}\left(\frac{1}{\sqrt{q}}\prod_{j=1}^{p-1}\csc\theta_j\right)\prod_{k=1}^{p-1}\sin^k(\theta_k)\,\mathrm{d}\theta_1\dots\mathrm{d}\theta_{p-1}.\end{aligned}$$

In particular, calculating the integrals for $q=1, p=2$ and $q=1, p=3$, we get:

$$\begin{aligned} G_{2,1} &= \frac{1}{2}\left(-1+\sqrt{2}\right)\pi\\ G_{3,1} &= \frac{1}{8}(-2+\pi)\pi\end{aligned}$$

This concludes the proof of Theorem 1.1.

4. Further results

4.1. **Concentration of eigenfunctions.** In this section, we discuss the behaviour of the eigenfunctions, more precisely how they scar on the lower-dimensional facets of a cuboid. This is made precise in the following theorem, where we will slightly abuse notation and denote by u_k both a Steklov eigenfunction and its boundary trace.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^d$ be the cuboid with parameters $a_1, \dots, a_d > 0$. Let $p \in \{1, \dots, d-1\}$ and let $\tau \in \mathcal{T}_p$. Consider the set*

$$X_\tau = \{x = (\mathbf{x}_{\tau_1}, \mathbf{x}_{\tau_2}) \in \partial\Omega : x_j = \pm a_j \text{ for } j \in \tau_2\}.$$

Then, there exists a sequence of $L^2(\partial\Omega)$-normalised eigenfunctions $\{u_k\}$ concentrating on X_τ and getting equidistributed around X_τ in the following sense: for each measurable $U \subset X_\tau$ and every $\epsilon > 0$, consider the set

$$U_\epsilon = \{\mathbf{x} = (\mathbf{x}_{\tau_1}, \mathbf{x}_{\tau_2}) \in \partial\Omega : \mathbf{x}_{\tau_1} \in U \text{ and } \operatorname{dist}(\mathbf{x}, U) < \epsilon\}.$$

Then, for every $\epsilon > 0$,

$$\lim_{k\to\infty} \int_{U_\epsilon} |u_k(\mathbf{x})|^2 \, dx = \frac{\operatorname{Vol}_p(U)}{\operatorname{Vol}_p(X_\tau)}.$$

For example, on a cuboid of dimension 3, the set X_τ is a union of four parallel edges in case $p = 1$, while for $p = 2$ it is a union of two opposite faces.

Proof. Without loss of generality, we will suppose that U is a subset of one of the connected components of X_τ, say the one where $x_j = a_j$ for all $j \in \tau_2$. For $k \in \mathbb{N}$, let $\mathbf{k} = (k, \dots, k) \in \mathbb{R}^p$ and consider the pair $(\boldsymbol{\alpha}^{(\mathbf{k})}, \boldsymbol{\beta}^{(\mathbf{k})})$ satisfying the compatibility and harmonicity conditions

$$\begin{aligned} \alpha_i^{(k)} \cot\left(\alpha_i^{(k)} a_i\right) &= \beta_j^{(k)} \tanh\left(\beta_j^{(k)} a_j\right) \qquad \forall\, i \in \tau_1, j \in \tau_2 \\ \sum_{i\in\tau_1} \left(\alpha_i^{(k)}\right)^2 &= \sum_{j\in\tau_2} \left(\beta_j^{(k)}\right)^2 \end{aligned}$$

with $\boldsymbol{\alpha}^{(\mathbf{k})} \in I_{2\mathbf{k}}$. Note that this corresponds to choosing $\ell(i) = 0$ for all $i \in \tau_1$ and $\ell(j) = 1$ for all $j \in \tau_2$. Since

$$\left(\sum_{i\in\tau_1} \left(\alpha_i^{(k)}\right)^2\right)^{1/2} = k \overbrace{\left(\sum_{i\in\tau_1} \left(\frac{\pi}{2a_i}\right)^2\right)^{1/2}}^{A} + O(1) = Ak + O(1)$$

we have that for all $j \in \tau_2$,

$$\beta_j^{(k)} = \frac{A}{\sqrt{q}} k + O(1).$$

Let $v_k(\mathbf{x})$ be the associated eigenfunction, and observe that

$$\begin{aligned} v_k(\mathbf{x})^2 &= \prod_{i\in\tau_1} \sin^2\left(\alpha_i^{(k)} x_i\right) \prod_{j\in\tau_2} \cosh^2\left(\beta_j^{(k)} x_j\right) \\ &= \frac{1}{2^p} \prod_{i\in\tau_1} \left(1 - \cos\left(2\alpha_i^{(k)} x_i\right)\right) \prod_{j\in\tau_2} \cosh^2\left(\beta_j^{(k)} x_j\right), \\ &= \frac{1}{2^p} \prod_{i\in\tau_1} \left(1 - \cos\left(\left(\frac{\pi k}{a_i} + O(1)\right) x_i\right)\right) \prod_{j\in\tau_2} \cosh^2\left(\left(\frac{A}{\sqrt{q}} k + O(1)\right) x_j\right). \end{aligned}$$

Defining the normalised eigenfunction

$$u_k = \frac{v_k}{\|v_k\|_{L^2(\partial\Omega)}},$$

we estimate both $\|v_k\|^2 := \|v_k\|^2_{L^2(\partial\Omega)}$ and $\int_{U_\epsilon} v_k(x)^2\,\mathrm{d}x$. For $\|v_k\|^2$, we have that

$$\begin{aligned}\|v_k\|^2 &= \frac{1}{2^p}\prod_{i\in\tau_1}\int_{-a_i}^{a_i} 1-\cos\left(\left(\frac{\pi k}{a_i}+O(1)\right)x_i\right)\mathrm{d}x_i \prod_{j\in\tau_2}\int_{-a_j}^{a_j}\cosh^2\left(\beta_j x_j\right)\mathrm{d}x_j\\ &= \frac{1}{2^d}\left(\mathrm{Vol}_p(X_\tau)+o(1)\right)\prod_{j\in\tau_2}\int_{-a_j}^{a_j}\cosh^2\left(\beta_j x_j\right)\mathrm{d}x_j\end{aligned} \tag{4.1.1}$$

from the Riemann-Lebesgue lemma and the fact that

$$\mathrm{Vol}(X_\tau) = 2^q\prod_{i\in\tau_1}\int_{-a_i}^{a_i}\mathrm{d}x_i.$$

Furthermore, for all $j\in\tau_2$ we have that

$$\begin{aligned}\int_{-a_j}^{a_j}\cosh^2\left(\beta_j x_j\right)\mathrm{d}x_j &= \frac{1}{4}\int_{-a_j}^{a_j} e^{2\left(\frac{A}{\sqrt{q}}k+O(1)\right)x_j}+e^{-2\left(\frac{A}{\sqrt{q}}k+O(1)\right)x_j}+2\,\mathrm{d}x_j\\ &= \frac{\sqrt{q}}{4Ak}e^{2\frac{A}{\sqrt{q}}ka_j}\left(1+o(1)\right).\end{aligned} \tag{4.1.2}$$

Setting $C=\frac{\sqrt{q}}{4A}$, equations (4.1.1) and (4.1.2) yield together that

$$\|v_k\|^2 = \frac{C^q}{2^d k^q}\mathrm{Vol}_p(X_\tau)\left(\prod_{j\in\tau_2} e^{2\frac{A}{\sqrt{q}}ka_j}\right)(1+o(1)) \tag{4.1.3}$$

We now also compute the integral of v_k^2 on U_ϵ where we get, in a similar fashion to (4.1.1) that

$$\int_{U_\epsilon} v_k(x)^2\,\mathrm{d}x = \frac{1}{2^p}\left(\mathrm{Vol}_p(U)+o(1)\right)\prod_{j\in\tau_2}\int_{a_j-\epsilon}^{a_j}\cosh^2\left(\beta_j x_j\right)\mathrm{d}x_j \tag{4.1.4}$$

We also have that

$$\int_{a_j-\epsilon}^{a_j}\cosh^2\left(\beta_j x_j\right)\mathrm{d}x_j = \frac{C}{2}e^{2\frac{A}{\sqrt{q}}ka_j}\left(1+o(1)\right), \tag{4.1.5}$$

where once again $C=\frac{\sqrt{q}}{4A}$. Together, equations (4.1.4) and (4.1.5) yield

$$\int_{U_\epsilon} v_k(\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{C^q}{2^d k^q}\mathrm{Vol}_p(U)\left(\prod_{j\in\tau_2} e^{2\frac{A}{\sqrt{q}}ka_j}\right)(1+o(1)). \tag{4.1.6}$$

Finally, putting equations (4.1.3) and (4.1.6) together yields indeed that

$$\lim_{k\to\infty}\int_{U_\epsilon} u_k(x)^2\,\mathrm{d}x = \lim_{k\to\infty}\int_{U_\epsilon}\frac{v_k(x)^2}{\|v_k\|^2}\,\mathrm{d}x = \frac{\mathrm{Vol}_p(U)}{\mathrm{Vol}_p(X_\tau)},$$

concluding the proof.

□

4.2. **The first eigenfunction.** In this section, we investigate the lowest nonzero eigenvalue σ_1 on the cuboid. Let us first find the form of an eigenfunction u associated with σ_1. By Courant's nodal theorem u has exactly 2 nodal domains. Thus, one of the factors u_j will have 2 nodal domains on the interval $[-a_j, a_j]$ and all the other factors only one nodal domain. In other words there is one odd factor, and all the others are positive even functions. We show the following proposition.

Proposition 4.2. *Suppose that $a_1 \leq \ldots \leq a_d$. Then there is $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{d-1})$ and $\alpha_d = |\boldsymbol{\beta}| < \frac{\pi}{2a_d}$ such that*

$$u(x_1, \ldots, x_d) = \sin(\alpha_d x_d) \prod_{k=1}^{d-1} \cosh(\beta_k x_k)$$

is an eigenfunction with eigenvalue σ_1.

Proof. We will first show that u is a product of one sine factor and $d-1$ hyperbolic cosines factors. Suppose that one of the trigonometric factors was a cosine. Let us study the number of nodal domains of $\cos(\alpha x_j)$ on the interval $[-a_j, a_j]$. By the Steklov boundary condition we have that

$$\cos(\alpha a_j) = -\sigma\alpha \sin(\alpha a_j),$$

There are three possible cases, whether $\sin(\alpha a_j)$ is equal to, greater than or smaller than 0. Since the eigenvalue $\sigma_0 = 0$ is simple, if $\sin(\alpha a_j) = 0$ it would imply that $\cos(\alpha a_j) = 0$, which is impossible.

If $\sin(\alpha a_j) > 0$, we have that $\cos(\alpha a_j)$ is negative. This would imply that the function $\cos(\alpha x)$ has changed sign on $[0, a_j]$ and since it is even it will have at least two zeroes on $[-a_j, a_j]$, that is at least three nodal domains, in contradiction with Courant's nodal theorem.

Finally, if $\sin(\alpha a_j) < 0$, this implies that $\alpha a_j > \pi$, meaning that $\cos(\alpha x_j)$ has changed sign at least once on $[0, a_j]$. This implies once again that there are at least three nodal domains, completing the proof that none of the factors are cosines.

Since there can only be one odd factor, if one is linear all the other factors are a combination of cosines and hyperbolic cosines. We just proved that none of the factors are cosines, and it is impossible for a product of linear functions with only hyperbolic cosines to respect the harmonicity condition (2.1.2). We therefore deduce that the only odd factor of u is a sine, and by the above discussion all the other factors are hyperbolic cosines. This implies that there exists some $1 \leq j \leq d$, α_j and β_k, $k \neq j$ such that

$$u(x_1, \ldots, x_d) = \sin(\alpha_j x_j) \prod_{k \neq j} \cosh(\beta_k x_k),$$

and $\alpha_j a_j < \pi/2$. The compatibility equations (2.2.3) hence become

$$\alpha_j \cot(\alpha_j a_j) = \beta_k \tanh(\beta_k a_k)$$
$$\alpha_j^2 = |\boldsymbol{\beta}|^2 = \sum_{k \neq j} \beta_k^2,$$

and σ_1 is any member of the first equality. We show that σ_1 is smallest when a_j is the largest side, i.e. $a_j = a_d$. Suppose not. Then, there is $1 \leq k \leq d-1$ such that an eigenvalue associated with

$$v(x_1, \ldots, x_d) = \sin(|\boldsymbol{\gamma}| x_j) \prod_{k \neq j} \cosh(\gamma_j x_j).$$

is smaller than the one associated with

$$u(x_1,\dots,x_d) = \sin(|\boldsymbol{\beta}|x_d)\prod_{k\neq d}\cosh(\beta_k x_k).$$

The compatibility equations imply that for all $k \neq j$ and $k \neq d$,

$$\gamma_k \tanh(\gamma_k a_k) < \beta_k \tanh(\beta_k a_k).$$

Since $x\tanh(ax)$ is an increasing function, we deduce that $\gamma_k \leq \beta_k$ for all such k. However, we also have that

$$|\boldsymbol{\gamma}|\cot(|\boldsymbol{\gamma}|a_k) < |\boldsymbol{\beta}|\cot(|\boldsymbol{\beta}|a_d)$$

and since $x\cot(ax)$ is decreasing on its first period and $a_k \leq a_d$, this implies that $|\boldsymbol{\gamma}| > |\boldsymbol{\beta}|$. From this, we therefore have that

$$\beta_j^2 + \sum_{k\neq j,d}\beta_k^2 < \gamma_d^2 + \sum_{k\neq j,d}\gamma_k^2.$$

Since for all $k \neq j,d$ we have that $\gamma_k < \beta_k$, we therefore deduce that $\beta_j < \gamma_d$. However, once again using the compatibility conditions, we have that

$$\gamma_d \tanh(\gamma_d a_d) < \beta_j \tanh(\beta_j a_j).$$

Since $a_d > a_j$, by monotonicity of $x\tanh(ax)$ we deduce that $\gamma_d < \beta_j$, a contradiction. Hence, we have that the first eigenfunction is, taking into account that $\alpha_d = |\boldsymbol{\beta}|$,

$$u(x_1,\dots,x_d) = \sin(|\boldsymbol{\beta}|x_d)\prod_{j=1}^{d-1}\cosh(\beta_j x_j),$$

□

concluding the proof of the proposition.

4.3. **Proof of Theorem 1.6.** The first eigenvalue is given by the following min-max principle :

$$\sigma_1(\Omega) = \inf_{\substack{u\in C^\infty(\Omega)\\ \int_{\partial\Omega} u=0}} R_\Omega[u] = \inf_{\substack{u\in C^\infty(\Omega)\\ \int_{\partial\Omega} u=0}} \frac{\int_\Omega |\nabla u|^2}{\int_{\partial\Omega} u^2}.$$

Denote by Ω_0 the cube $[-1,1]^d$. Then, for any cuboid $\Omega = [-a_1,a_1]\times\ldots\times[-a_d,a_d]$ we have that

$$\int_\Omega f(x)\,\mathrm{d}x = \int_{\Omega_0} f(a_1x_1,\dots,a_dx_d)\prod_{i=1}^d a_i\,\mathrm{d}x$$

and

$$\begin{aligned}\int_{\partial\Omega} f(x)\,\mathrm{d}x &= \sum_{j=1}^d \int_{\partial\Omega\cap\{x_j=\pm a_j\}} f(x)\,\mathrm{d}x \\ &= \sum_{j=1}^d \int_{\partial\Omega_0\cap\{x_j=\pm 1\}} f(a_1x_1,\dots,a_dx_d)\prod_{i\neq j} a_i\,\mathrm{d}x.\end{aligned} \tag{4.3.1}$$

This allows us to consider integration only on Ω_0 for R_Ω. Observe that the eigenspace of $\sigma_1(\Omega_0)$ has dimension d, and that a basis for it is given by

$$u_j(x_1,\dots,x_d) = \sin(|\beta|x_d)\prod_{i\neq j}\cosh(\beta_i x_i).$$

The eigenfunctions u_j are orthogonal to constants in the scalar product given by the rescaled integral (4.3.1). Indeed, on all faces where the sin factor is not constant, the integral vanishes since it is an odd function. On the pair of faces where the sin factor is constant, we have that $u_j(x_1,\dots,a_j,\dots,x_d) = -u_j(x_1,\dots,-a_j,\dots,x_d)$ hence the integrals cancel out on these two faces.

Consider the eigenfunction

$$u = \sum_{j=1}^d u_j.$$

It is easy to see that the integral of u^2 on any face of Ω_0 is identical, and we have that $R_{\Omega_0}[u] = \sigma_1(\Omega_0)$. We now compute

$$\begin{aligned}\frac{1}{R_\Omega[u]} &= \frac{\sum_{j=1}^d \prod_{i\neq j} a_i \int_{\partial\Omega_0\cap\{x_j=\pm1\}} u^2\,\mathrm{d}x}{\prod_{j=1}^d a_j \int_{\Omega_0} |\nabla u|^2\,\mathrm{d}x}\\ &= \frac{1}{R_{\Omega_0}[u]}\frac{\frac{1}{d}\sum_{j=1}^d \prod_{i\neq j} a_i}{\prod_{j=1}^d a_j}.\end{aligned}$$

Fix the volume $\mathrm{Vol}_d(\Omega) = \mathrm{Vol}_d(\Omega_0)$, hence $\prod_j a_j = 1$. Then, from the inequality of arithmetic and geometric means,

$$\frac{R_{\Omega_0}[u]}{R_\Omega[u]} = \frac{1}{d}\sum_{j=1}^d \prod_{i\neq j} a_i \geq \left(\prod_{j=1}^d a_j^{d-1}\right)^{1/d} = 1,$$

with equality if and only if for all j,k, $\prod_{i\neq j} a_i = \prod_{i\neq k} a_i$, which is true if and only if $a_j = a_k$ for all j,k, which implies in turn that $\sigma_1(\Omega) \leq \sigma_1(\Omega_0)$, with equality if and only if Ω is a cube.

On the other hand, fix the area, $\mathrm{Vol}_{d-1}(\Omega) = \mathrm{Vol}_{d-1}(\Omega_0)$, hence $\sum_j \prod_{i\neq j} a_i = d$. Then,

$$\frac{R_{\Omega_0}[u]}{R_\Omega[u]} = \left(\prod_j a_j\right)^{-1} = \left(\prod_{j=1}^d \prod_{i\neq j} a_i\right)^{\frac{d(1-d)}{d}} \geq \left(\frac{1}{d}\sum_{j=1}^d \prod_{i\neq j} a_i\right)^{\frac{1-d}{d}} = 1,$$

with equality in the same case as before. Once again, this implies that $\sigma_1(\Omega) \leq \sigma_1(\Omega_0)$, with equality if and only if Ω is a cube.

4.4. **Proof of Corollary 1.8.** We want to show that among all rectangles, the Steklov spectrum determines the lengths a_1, a_2 of its sides. From spectral asymptotics, the perimeter of the rectangle is obtained, giving $L = a_1 + a_2$, supposing without loss of generality that $a_1 \leq a_2$. On the other hand, we have σ_1, and we know that it is the smallest root of

$$\sigma_1 = \alpha\cot(\alpha a_1) = \alpha\tanh(\alpha a_2).$$

Rewriting these to yield a_2 as a function of α, L and σ_1 gives

$$a_2 = f(\alpha) = \frac{1}{\alpha}\operatorname{arctanh}\left(\frac{\sigma_1}{\alpha}\right) \tag{4.4.1}$$

and

$$a_2 = g(\alpha) = L - \frac{1}{\alpha}\operatorname{arccot}\left(\frac{\sigma_1}{\alpha}\right). \tag{4.4.2}$$

Given σ_1 and L, the intersection of these curves yield possible values a_2 for α. We now show that they intersect at only one point. Equation (4.4.1) is defined for $\alpha > \sigma_1$ and taking the derivative yields

$$f'(\alpha) = -\frac{\operatorname{arctanh}\left(\frac{\sigma_1}{\alpha}\right)}{\alpha^2} - \frac{\sigma_1}{\alpha^3\left(1-\frac{\sigma_1^2}{\alpha^2}\right)}, \tag{4.4.3}$$

which is always negative for $\alpha > \sigma_1$, hence f is decreasing. We now show that g is increasing on $[\sigma_1, \infty)$. We have that

$$g'(\alpha) = \frac{\operatorname{arccot}\left(\frac{\sigma_1}{\alpha}\right)}{\alpha^2} - \frac{\sigma_1}{\alpha^3\left(1+\frac{\sigma_1^2}{\alpha^2}\right)}.$$

Thus, g' is positive if

$$\alpha \operatorname{arccot}\left(\frac{\sigma_1}{\alpha}\right)\left(1+\frac{\sigma_1^2}{\alpha^2}\right) - \sigma_1 \geq 0.$$

However, we have that

$$\alpha \operatorname{arccot}\left(\frac{\sigma_1}{\alpha}\right)\left(1+\frac{\sigma_1^2}{\alpha^2}\right) - \sigma_1 \geq \frac{\pi}{4}\alpha + \frac{\pi}{4}\frac{\sigma_1^2}{\alpha^2} - \sigma_1$$

hence we need to have that $\alpha^2 - \frac{4\sigma_1\alpha}{\pi} + \sigma_1^2 \geq 0$. This quantity is positive at $\alpha = \sigma_1$ since $2 \geq 4/\pi$ and it is increasing since

$$2\alpha > \frac{4\sigma_1}{\pi}$$

for $\alpha \geq \sigma_1$. We conclude that g is increasing. This implies that f and g have exactly one intersection point, say at α_0. We have that $a_2 = f(\alpha_0) = g(\alpha_0)$ and $a_1 = L - a_2$. Note that since the square maximises σ_1 and since the eigenvalues are continuous functions of the side lengths of a rectangle this means that among all rectangles with given area or perimeter, σ_1 is a decreasing function of a_2.

Appendix A. Proof of Lemma A.1

Lemma A.1. *Let*

$$f_i(\mathbf{x}) = \operatorname{arccot}\left(c\left[1+\sum_{j\neq i}\left(\frac{x_j}{x_i}\right)^2\right]^{1/2}\right).$$

for some $c > 0$ *and where by convention* $\operatorname{arccot}(\infty) = 0$*, and let* $\psi : \mathbb{R}^p \to \mathbb{R}^p$ *be a bounded function. Then,*

$$\left|f_i(\mathbf{x}+\psi(\mathbf{x})) - f_i(\mathbf{x})\right| = O\left(|\mathbf{x}|^{-1}\right). \tag{A.1}$$

Proof. We have that

$$\left|f_i(\mathbf{x}) - f_i(\mathbf{x}_0)\right| = O\left(|\mathbf{x}-\mathbf{x}_0||\nabla f(\mathbf{x}_0)|\right).$$

Consider spherical coordinates $(r, \theta_1, \ldots, \theta_{p-1})$

$$r = |\mathbf{x}|,$$
$$x_j = r\cos(\theta_j)\prod_{i<j}\sin(\theta_i),$$

where by convention $\theta_p = 0$.

Denote $\mathbf{x} = (r, \boldsymbol{\theta})$ and $\mathbf{x} + \psi(\mathbf{x}) = (r_\psi, \boldsymbol{\theta}_\psi)$. It is clear that since ψ is bounded we have that

$$|\boldsymbol{\theta} - \boldsymbol{\theta}_\psi| = O\left(r^{-1}\right).$$

Indeed, from planar geometry we get that

$$\tan(|\boldsymbol{\theta} - \boldsymbol{\theta}_\psi|) \leq \frac{\sup_{\mathbf{x}\in\mathbb{R}^p}\psi(\mathbf{x})}{r}.$$

One can observe that the functions in Equation (A.1) depend only on $\boldsymbol{\theta}_\psi$ and $\boldsymbol{\theta}$. Hence, showing that the gradient is bounded in $\boldsymbol{\theta}$ implies that $|f_i(\mathbf{x}+\psi(\mathbf{x})) - f_i(\mathbf{x})| = O\left(r^{-1}\right)$.

By symmetry, we can suppose without loss of generality that $i = p$ in Equation (A.1). Then, using repeatedly the identity $1 + \cot^2\theta = \csc^2\theta$ we have that

$$f_p(\mathbf{x}) = \operatorname{arccot}\left(c\prod_{j=1}^{p-1}\csc\theta_j\right).$$

Now, we have that

$$\partial_{\theta_j} f_p(x) = c\frac{\cot\theta_j \prod_{k=1}^{p-1}\csc\theta_k}{1 + c^2\prod_{k=1}^{p-1}\csc^2\theta_k}.$$

This is bounded since when $\theta_j \to n\pi$, the singularities are of the same order on the numerator and denominator while when it is any other $\theta_i \to n\pi$, the singularities are of order 1 in the numerator and 2 in the denominator. This concludes the proof. □

Appendix B. Positivity of the constant C_2

We can rewrite C_2 as

$$C_2 = \frac{(2^{\frac{d+2}{2}} - 2)\pi\omega_{d-2} - 2^{d+1}G_{d-1,1}}{2(2\pi)^{d-1}},$$

and we need to show that $C_2 > 0$ for $d \geq 3$. This will be done by showing that

$$\frac{(2^{\frac{d+2}{2}} - 2)\pi\omega_{d-2}}{2^{d+1}G_{d-1,1}} > 1. \tag{B.1}$$

Let us first observe that the integrand in $G_{d-1,1}$ is positive and that for any $\boldsymbol{\theta} \in [0,\pi/2]^{d-2}$, we have that

$$\operatorname{arccot}\left(\prod_{j=1}^{d-2}\csc\theta_j\right) \leq \operatorname{arccot}(1) < 1.$$

Hence,

$$\begin{aligned} G_{d-1,1} &= \int_0^{\pi/2}\cdots\int_0^{\pi/2}\operatorname{arccot}\left(\prod_{j=1}^{d-2}\csc\theta_j\right)\prod_{k=1}^{d-2}\sin^k(\theta_k)\,\mathrm{d}\theta_1\ldots\mathrm{d}\theta_{d-2} \\ &\leq \prod_{k=1}^{d-2}\int_0^{\pi/2}\sin^k(\theta_k)\,\mathrm{d}\theta_k \\ &= \frac{2^{2-d}\pi^{\frac{2-d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \end{aligned}$$

The last equality is true for $d = 3$, and is seen to be true for all $d \geq 3$ by induction using the identity [7, 3.621 (1)]

$$\int_0^{\pi/2}\sin^k(\theta)\,\mathrm{d}\theta = 2^{k-1}B\left(\frac{k+1}{2},\frac{k+1}{2}\right).$$

and the Gamma function duplication identity

$$\Gamma(\mu)\Gamma\left(\mu+1/2\right) = 2^{1-2\mu}\sqrt{\pi}\,\Gamma(2\mu).$$

Using the fact that

$$\omega_{d-2} = \frac{\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

and replacing in Equation (B.1) we have that

$$\frac{(2^{\frac{d+2}{2}}-2)\pi\omega_{d-2}}{2^{d+1}G_{d-1,1}} \geq \frac{(2^{\frac{d+2}{2}}-2)\pi}{8} > 1$$

for all $d \geq 3$, concluding the proof that $C_2 > 0$.

References

[1] M. S. Agranovich. On a mixed Poincaré-Steklov type spectral problem in a Lipschitz domain. *Russ. J. Math. Phys.*, 13(3):239–244, 2006.

[2] G. Auchmuty and M. Cho. Boundary integrals and approximations of harmonic functions. *Numer. Funct. Anal. Optim.*, 36(6):687–703, 2015.

[3] F. Brock. An isoperimetric inequality for eigenvalues of the Stekloff problem. *Z. Angew. Math. Mech.*, 81(1):69–71, 2001.

[4] D. Bucur, V. Ferone, C. Nitsch, and C. Trombetti. Weinstock inequality in higher dimensions. *ArXiv e-print: 1710.04587*, October 2017.

[5] A. Girouard and I. Polterovich. Spectral geometry of the Steklov problem (Survey article). *J. Spectr. Theory*, 7(2):321–359, 2017.

[6] Katie Gittins and Simon Larson. Asymptotic behaviour of cuboids optimising laplacian eigenvalues. *Integral Equations and Operator Theory*, 89(4):607–629, 2017.

[7] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series and products*, volume 7. Academic Press, Burlington, 2007.

[8] E. Hlawka. über Integrale auf konvexen Körpern. I. *Monatsh. Math.*, 54:1–36, 1950.

[9] A. Iosevich and E. Liflyand. *Decay of the Fourier transform, analytic and geometric aspects*. Birkhäuser/Springer, Basel, 2014.

[10] V. Ivrii. Spectral asymptotics for Dirichlet to Neumann operator in the domains with edges. *ArXiv e-print: 1802.07524*, February 2018.

[11] J. Lagacé and L. Parnovski. A generalised Gauss circle problem and integrated density of states. *J. Spectr. Theory*, 6(4):859–879, 2016.

[12] M. Levitin, L. Parnovski, I. Polterovich, and D. A. Sher. Sloshing, Steklov and corners I: Asymptotics of sloshing eigenvalues. *ArXiv e-print: 1709.01891*, September 2017.

[13] J. P. Pinasco and J. D. Rossi. Asymptotics of the spectral function for the Steklov problem in a family of sets with fractal boundaries. *Appl. Math. E-Notes*, 5:138–146, 2005.

[14] I. Polterovich and D. A. Sher. Heat invariants of the Steklov problem. *J. Geom. Anal.*, 25(2):924–950, 2015.

[15] B. Randol. A lattice-point problem. *Trans. Amer. Math. Soc.*, 121:257–268, 1966.

[16] W. A. Strauss. *Partial differential equations, an introduction*. John Wiley & Sons, Inc., New York, 1992.

[17] A. Tan. The Steklov Problem on Rectangles and Cuboids. *ArXiv e-print: 1711.00819*, November 2017.

[18] M. van den Berg and K. Gittins. Minimizing Dirichlet eigenvalues on cuboids of unit measure. *Mathematika*, 63(2):469–482, 2017.

[19] R. Weinstock. Inequalities for a classical eigenvalue problem. *J. Rational Mech. Anal.*, 3:745–753, 1954.

Département de mathématiques et de statistique, Pavillon Alexeandre-Vachon, Université Laval, Québec, QC, G1V 0A6, Canada

Email address: alexandre.girouard@mat.ulaval.ca

Département de mathématiques et de statistique, Université de Montréal, CP 6128 succ Centre-Ville, Montréal, QC H3C 3J7, Canada

Current address : Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, United Kingdom

Email address: j.lagace@ucl.ac.uk

Département de mathématiques et de statistique, Université de Montréal, CP 6128 succ Centre-Ville, Montréal, QC H3C 3J7, Canada

Email address: iossif@dms.umontreal.ca

Dipartimento SBAI, Sezione di Matematica Sapienza Università di Roma, Via Antonio Scarpa 16 00161 Roma, Italy

Email address: alessandro.savo@sbai.uniroma1.it