UNIVERSAL CONSTRUCTIONS FOR (CO)RELATIONS: CATEGORIES, MONOIDAL CATEGORIES, AND PROPS

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ABSTRACT. Calculi of string diagrams are increasingly used to present the syntax and algebraic structure of various families of circuits, including signal flow graphs, electrical circuits and quantum processes. In many such approaches, the semantic interpretation for diagrams is given in terms of relations or corelations (generalised equivalence relations) of some kind. In this paper we show how semantic categories of both relations and corelations can be characterised as colimits of simpler categories. This modular perspective is important as it simplifies the task of giving a complete axiomatisation for semantic equivalence of string diagrams. Moreover, our general result unifies various theorems that are independently found in literature and are relevant for program semantics, quantum computation and control theory.

1. INTRODUCTION

Network-style diagrammatic languages appear in diverse fields as a tool to reason about computational models of various kinds, including signal processing circuits, quantum processes, Bayesian networks and Petri nets, amongst many others. In the last few years, there have been more and more contributions towards a uniform, formal theory of these languages which borrows from the well-established methods of programming language semantics. A significant insight stemming from many such approaches is that a *compositional* analysis of network diagrams, enabling their reduction to elementary components, is more effective when system behaviour is thought of as a *relation* instead of a function.

A paradigmatic case is the one of signal flow graphs, a foundational structure in control theory: a series of recent works [3, 1, 5, 6, 14] gives this graphical language a syntax and a semantics where each signal flow diagram is interpreted as a subspace (a.k.a. linear relation) over streams. The highlight of this approach is a sound and complete axiomatisation for

semantic equivalence: what is of interest for us is how this result is achieved in [3], namely through a *modular* account of the domain of subspaces. The construction can be studied for any field k: one considers the prop[1] $\mathsf{SV_k}$ whose arrows $n \to m$ are subspaces of $\mathsf{k}^n \times \mathsf{k}^m$, composed as relations. As shown in in [7, 34], $\mathsf{SV_k}$ enjoys a universal characterisation: it is the pushout (in the category of props) of props of *spans* and of *cospans* over $\mathsf{Vect_k}$, the prop with arrows $n \to m$ the linear maps $\mathsf{k}^n \to \mathsf{k}^m$:

$$
\begin{array}{ccc}
\mathsf{Vect_k} + \mathsf{Vect_k}{}^{op} & \longrightarrow & \mathsf{Span}(\mathsf{Vect_k}) \\
\downarrow & & \downarrow \\
\mathsf{Cospan}(\mathsf{Vect_k}) & \longrightarrow & \mathsf{SV_k}.
\end{array}
\tag{1.1}
$$

In linear algebraic terms, the two factorisation properties expressed by (1.1) correspond to the representation of a subspace in terms of a basis (span) and the solution set of a system of linear equations (cospan). Most importantly, this picture provides a roadmap towards a complete axiomatisation for $\mathsf{SV_k}$: one starts from the domain $\mathsf{Vect_k}$ of linear maps, which is axiomatised by the equations of Hopf algebras, then combines it with its opposite $\mathsf{Vect_k}{}^{op}$ via two distributive laws of props [24], one yielding an axiomatisation for $\mathsf{Span}(\mathsf{Vect_k})$ and the other one for $\mathsf{Cospan}(\mathsf{Vect_k})$. Finally, merging these two axiomatisations yields a complete axiomatisation for $\mathsf{SV_k}$, called the theory of interacting Hopf algebras [7, 34].

It was soon realised that this modular construction was of independent interest, and perhaps evidence of a more general phenomenon. In [35] it is shown that a similar construction could be used to characterise the prop ER of equivalence relations, using as ingredients In, the prop of injections, and F, the prop of total functions. The same result is possible by replacing equivalence relations with partial equivalence relations and functions with partial functions, forming a prop PF. In both cases, the universal construction yields a privileged route to a complete axiomatisation, of ER and of PER respectively [35].

$$
\begin{array}{ccc}
\mathsf{In} + \mathsf{In}^{op} & \longrightarrow & \mathsf{Span}(\mathsf{In}) \\
\downarrow & & \downarrow \\
\mathsf{Cospan}(\mathsf{F}) & \longrightarrow & \mathsf{ER}
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{In} + \mathsf{In}^{op} & \longrightarrow & \mathsf{Span}(\mathsf{In}) \\
\downarrow & & \downarrow \\
\mathsf{Cospan}(\mathsf{PF}) & \longrightarrow & \mathsf{PER}.
\end{array}
\tag{1.2}
$$

Even though a pattern emerges, it is certainly non-trivial: for instance, if one naively mimics the linear case (1.1) in the attempt of characterising the prop of relations, the construction collapses to the terminal prop $\mathbf{1}$.

$$
\begin{array}{ccc}
\mathsf{F} + \mathsf{F}^{op} & \longrightarrow & \mathsf{Span}(\mathsf{F}) \\
\downarrow & & \downarrow \\
\mathsf{Cospan}(\mathsf{F}) & \longrightarrow & \mathbf{1}
\end{array}
\tag{1.3}
$$

More or less at the same time, diagrammatic languages for various families of circuits, including linear time-invariant dynamical systems [14], were analysed using so-called *core-lations*, which are generalised equivalence relations [12, 11, 13, 2]. Even though they were not originally thought of as arising from a universal construction like the examples above,

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[1] A prop is a symmetric monoidal category with objects the natural numbers [25]. It is the typical setting for studying both the syntax and the semantics of network diagrams.

corelations still follow a modular recipe, as they are expressible as a quotient of $\mathsf{Cospan}(\mathcal{C})$, for some prop $\mathcal{C}$. Thus by analogy we can think of them as yielding one half of the diagram

$$
\begin{array}{c}
\mathcal{C} + \mathcal{C}^{op} \\
\downarrow \\
\mathsf{Cospan}(\mathcal{C}) \longrightarrow \mathsf{Corel}(\mathcal{C}).
\end{array}
\tag{1.4}
$$

In this paper we clarify the situation by giving a unifying perspective for all these constructions. We prove a general result, which

- implies (1.1) and (1.2) as special cases;
- explains the failure of (1.3);
- extends (1.4) to a pushout recipe for corelations.

More precisely, our theorem individuates sufficient conditions for characterising the category $\mathsf{Rel}(\mathcal{C})$ of $\mathcal{C}$-relations as a pushout. A dual construction yields the category $\mathsf{Corel}(\mathcal{C})$ of $\mathcal{C}$-corelations as a pushout. For the case of interest when $\mathcal{C}$ is a prop, the two constructions look as follows.

$$
\begin{array}{ccc}
\mathcal{A} + \mathcal{A}^{op} \longrightarrow \mathsf{Span}(\mathcal{C}) & \qquad & \mathcal{A} + \mathcal{A}^{op} \longrightarrow \mathsf{Span}(\mathcal{A}) \\
\downarrow \qquad\quad \downarrow & & \downarrow \qquad\quad \downarrow \\
\mathsf{Cospan}(\mathcal{A}) \longrightarrow \mathsf{Rel}(\mathcal{C}). & & \mathsf{Cospan}(\mathcal{C}) \longrightarrow \mathsf{Corel}(\mathcal{C}).
\end{array}
$$

The variant ingredient $\mathcal{A}$ is a subcategory of $\mathcal{C}$. In order to make the constructions possible, $\mathcal{A}$ has to satisfy certain requirements in relation with the factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ which defines $\mathcal{C}$-relations (as jointly-in-$\mathcal{M}$ spans) and $\mathcal{C}$-corelations (as jointly-in-$\mathcal{E}$ cospans). For instance, taking $\mathcal{A}$ to be $\mathcal{C}$ itself succeeds in (1.1) (and in fact, for any abelian $\mathcal{C}$), but fails in (1.3).

Besides explaining existing constructions, our result opens the lead for new applications. In particular, we observe that under mild conditions the construction of relations lifts to the category $\mathcal{C}^T$ of T-algebras for a monad $T \colon \mathcal{C} \to \mathcal{C}$, and dually for corelations and comonads. We leave the exploration of this and other ramifications for future work.

Synopsis. Section 2 introduces factorisation systems and (co)relations, and shows the subtleties of mapping spans into corelations in a functorial way. Section 3 states our main result and some of its consequences. We first formulate the construction for categories (Theorem 3.1), and then for props (Theorem 3.6), which are our prime object of interest in applications. Section 4 is devoted to show various instances of our construction. We illustrate the case of equivalence relations, of partial equivalence relations, of subspaces, of linear corelations, and finally of (co)relations of (co)algebras over a (co)monad. Section 5 summarises our contribution and looks forward to further work. Appendix A unfolds the full proof of the main result, Theorem 3.1, and Appendix B does the same for the extension to monoidal categories.

This work is based on the conference paper [15], which has been expanded to include all the omitted proofs, the monoidal case of the main construction (Section 3.3), and new explanations and examples, including two additional case studies: the non-example of plain relations (Section 4.4) and the example of corelations for coalgebras (Section 4.10). Also, the generic construction of relations for algebras (Section 3.5) has been amended and extended.

Conventions. We write $f \,; g$ for composition of $f \colon X \to Y$ and $g \colon Y \to Z$ in a category $\mathcal{C}$. It will be sometimes convenient to indicate an arrow $f \colon X \to Y$ of $\mathcal{C}$ as $X \xrightarrow{f \in \mathcal{C}} Y$ or also $\xrightarrow{\in \mathcal{C}}$, if names are immaterial. Also, we write $X \xleftarrow{f \in \mathcal{C}} Y$ for an arrow $X \xrightarrow{f \in \mathcal{C}^{op}} Y$. We use $\oplus$ for the monoidal product in a monoidal category, with unit object I. Monoidal categories and functors will be strict when not stated otherwise.

2. (Co)relations

In this section we review the categorical approach to relations, which originates in the work of Hilton [18] and Klein [21]. This categorical approach is based on the observation that, in Set, relations are the jointly mono spans—that is, spans $X \xleftarrow{f} N \xrightarrow{g} Y$ such that the pairing $\langle f, g \rangle \colon N \to X \times Y$ is an injection. We introduce in parallel the dual notion, called corelations [12]. These are generalisations of jointly epi cospans—that is, cospans $X \xrightarrow{f} N \xleftarrow{g} Y$ such that $[f, g] \colon X + Y \to N$ is surjective—and can be seen as an abstraction of the concept of equivalence relation.

Definition 1. A **factorisation system** $(\mathcal{E}, \mathcal{M})$ in a category $\mathcal{C}$ comprises subcategories $\mathcal{E}$, $\mathcal{M}$ of $\mathcal{C}$ such that

(1) $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms of $\mathcal{C}$.
(2) every morphism $f \in \mathcal{C}$ admits a factorisation $f = e \,; m$, $e \in \mathcal{E}$, $m \in \mathcal{M}$.
(3) given f, f', with factorisations $f = e \,; m$, $f' = e' \,; m'$ of the above sort, for every u, v such that $f \,; v = u \,; f'$ there exists a unique s making the following diagram commute.

$$
\begin{array}{ccc}
\xrightarrow{\ e\ } & \xrightarrow{\ m\ } & \\
\Big\downarrow{u} & \Big\downarrow{\exists ! s} & \Big\downarrow{v} \\
\xrightarrow[\ e'\ ]{} & \xrightarrow[\ m'\ ]{} &
\end{array}
$$

Definition 2. Given a category $\mathcal{C}$, we say that a subcategory $\mathcal{A}$ is **stable under pushout** if for every pushout square

$$
\begin{array}{ccc}
& \xrightarrow{\ a\ } & \\
\Big\downarrow{} & & \Big\downarrow{} \\
& \xrightarrow[\ f\ ]{} &
\end{array}
$$

such that $a \in \mathcal{A}$, we also have that $f \in \mathcal{A}$. Similarly, we say that $\mathcal{A}$ is **stable under pullback** if for every pullback square labelled as above $f \in \mathcal{A}$ implies $a \in \mathcal{A}$.

A factorisation system $(\mathcal{E}, \mathcal{M})$ is **stable** if $\mathcal{E}$ is stable under pullback, **costable** if $\mathcal{M}$ is stable under pushout, and **bistable** if it is both stable and costable.

Examples of bistable factorisation systems include the trivial factorisation systems $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ and $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$ in any category $\mathcal{C}$, where $\mathcal{I}_{\mathcal{C}}$ is the subcategory containing exactly the isomorphisms in $\mathcal{C}$, the epi-mono factorisation system in any topos, or the epi-mono factorisation system in any abelian category. Stable factorisation systems include the (regular epi, mono) factorisation system in any regular category, such as any category monadic over Set. Dually, costable factorisation systems include the (epi, regular mono) factorisation system in any coregular category, such as the category of topological spaces and continuous maps.

Definition 3. Given a category $\mathcal{C}$ with pushouts, the category $\mathsf{Cospan}(\mathcal{C})$ has the same objects as $\mathcal{C}$ and arrows $X \to Y$ isomorphism classes of cospans $X \xrightarrow{f}\xleftarrow{g} Y$ in $\mathcal{C}$. The composite of $X \xrightarrow{f}\xleftarrow{g} Y$ and $Y \xrightarrow{h}\xleftarrow{i} Z$ is obtained by taking the pushout of $\xleftarrow{g}\xrightarrow{h}$.

Given a category $\mathcal{C}$ with pullbacks, the category $\mathsf{Span}(\mathcal{C})$ has the same objects as $\mathcal{C}$ and arrows $X \to Y$ isomorphism classes of spans $X \xleftarrow{f}\xrightarrow{g} Y$ in $\mathcal{C}$. The composite of $X \xleftarrow{f}\xrightarrow{g} Y$ and $Y \xleftarrow{h}\xrightarrow{i} Z$ is obtained by taking the pullback of $\xrightarrow{g}\xleftarrow{h}$.

When $\mathcal{C}$ also has a (co)stable factorisation system, we may define a category of (co)relations with respect to this system.

Definition 4. Given a category $\mathcal{C}$ with pushouts and a costable factorisation system $(\mathcal{E}, \mathcal{M})$, the category $\mathsf{Corel}(\mathcal{C})$ has the same objects as $\mathcal{C}$. The arrows $X \to Y$ are equivalence classes of cospans $X \xrightarrow{f} N \xleftarrow{g} Y$ under the symmetric, transitive closure of the following relation: two cospans $X \xrightarrow{f} N \xleftarrow{g} Y$ and $X \xrightarrow{f'} N' \xleftarrow{g'} Y$ are related if there exists $N \xrightarrow{m} N'$ in $\mathcal{M}$ such that

$$\begin{array}{ccc} & N & \\ f \nearrow & \downarrow m & \searrow g \\ X & & Y \\ f' \searrow & \nearrow g' & \\ & N' & \end{array} \tag{2.1}$$

commutes. This notion of equivalence respects composition of cospans, and so $\mathsf{Corel}(\mathcal{C})$ is indeed a category. We call the morphisms in this category **corelations**.

Given a category $\mathcal{C}$ with pullbacks and a stable factorisation system, we can dualise the above to define the category $\mathsf{Rel}(\mathcal{C})$ of **relations**.

(Co)stability is needed in order to ensure that composition of (co)relations is associative, *cf.* [12, §3.3]. For proofs it is convenient to give an alternative description of (co)relations.

Proposition 2.1. *If $\mathcal{C}$ has binary coproducts, corelations are in one-to-one correspondence with isomorphism classes of cospans such that the copairing $[p, q] \colon X + Y \to N$ lies in $\mathcal{E}$.*

If $\mathcal{C}$ has binary products, relations are in one-to-one correspondence with isomorphism classes of spans such that the pairing $\langle f, g \rangle \colon N \to X \times Y$ lies in $\mathcal{M}$.

Proof. We focus on relations, the proof for corelations being dual. It suffices to show that two spans $\xleftarrow{f}\xrightarrow{g}$ and $\xleftarrow{f'}\xrightarrow{g'}$ represent the same relation if and only if the $\mathcal{M}$ parts of the factorisations of $\langle f, g \rangle$ and $\langle f', g' \rangle$ are equal.

For the backward direction, write factorisations $\langle f, g \rangle = e; m$ and $\langle f', g' \rangle = e'; m$, and note that $m = \langle m; p_1, m; p_2 \rangle$, where the p_i are the canonical projections. Thus the following diagrams commute

Therefore $\xrightarrow{e \in \mathcal{E}}$ and $\xrightarrow{e' \in \mathcal{E}}$ witness that both $\xleftarrow{f}\xrightarrow{g}$ and $\xleftarrow{f'}\xrightarrow{g'}$ are in the equivalence class of $\xleftarrow{p_1}\xleftarrow{m}\xrightarrow{m}\xrightarrow{p_2}$ and so represent the same relation.

For the forward direction, note that if $\xleftarrow{f}\xrightarrow{g}$ and $\xleftarrow{f'}\xrightarrow{g'}$ represent the same relation, then there exists a sequence of spans $\xleftarrow{f_i}\xrightarrow{g_i}$ in $\mathcal{C}$ together with morphisms $\xrightarrow{e_i \in \mathcal{E}}$, $i = 0, \ldots n$, such that $f_1 = f$, $g_1 = g$, $f_n = f'$, $g_n = g'$, and for all $i = 1, \ldots, n$ either (i) $e_i; f_i = f_{i-1}$

and $e_i; g_i = g_{i-1}$, or (ii) $f_i = e_i; f_{i-1}$ and $g_i = e_i; g_{i-1}$. This implies either (i) $e_i; \langle f_i, g_i \rangle = \langle f_{i-1}, g_{i-1} \rangle$ or (ii) $\langle f_i, g_i \rangle = e_i; \langle f_{i-1}, g_{i-1} \rangle$. In either case, by the uniqueness of factorisations, we see that the $\mathcal{M}$ parts of $\langle f_i, g_i \rangle$ are the same for all i. $\square$

We call a span $\xleftarrow{f} \xrightarrow{g}$ **jointly-in-$\mathcal{M}$** if the pairing $\langle f, g \rangle$ lies in $\mathcal{M}$, and analogously call a cospan $\xrightarrow{f} \xleftarrow{g}$ **jointly-in-$\mathcal{E}$** if the copairing $[f, g]$ lies in $\mathcal{E}$. To each relation there is thus, up to isomorphism, a canonical representation as a jointly-in-$\mathcal{M}$ span, and similarly to each corelation a jointly-in-$\mathcal{E}$ cospan.

Example 2.1. Many examples of relations and corelations are already familiar.
- The category Set is bicomplete and has a bistable epi-mono factorisation system. Relations with respect to this factorisation system are simply the usual binary relations, while corelations from $X \to Y$ in Set are surjective functions $X + Y \to N$; thus their isomorphism classes—the arrows of Corel(Set)—are partitions, or equivalence relations on $X + Y$.
- The category of vector spaces over a field k is abelian, and hence bicomplete with a bistable epi-mono factorisation system. The categories of relations and corelations are isomorphic: a morphism $X \to Y$ in these categories can be thought of as a linear relations, i.e. a subspace of $X \times Y$.
- In any category $\mathcal{C}$ the trivial morphism-isomorphism factorisation system $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$ is bistable. Relations with respect to $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$ are equivalence classes of isomorphisms $N \xrightarrow{\sim} X \times Y$, and hence there is a unique relation between any two objects. Corelations are just cospans.
- Dually, relations with respect to the isomorphism-morphism factorisation $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ are just spans, and there is a unique corelation between any two objects.

We now study the functorial interpretation of cospans and spans as corelations. This discussion is instrumental in our universal construction for corelations (Theorem 3.1).

First, given two categories with the same collections of objects, we may speak of **identity-on-objects (ioo)** functors between them, i.e. functors that are the identity map on objects. Four examples of such functors will become relevant in the next section:

$$
\begin{aligned}
&\mathcal{C} \to \mathsf{Cospan}(\mathcal{C}) \text{ maps } \xrightarrow{f} \text{ to } \xrightarrow{f} \xleftarrow{id} &\qquad &\mathcal{C} \to \mathsf{Span}(\mathcal{C}) \text{ maps } \xrightarrow{f} \text{ to } \xleftarrow{id} \xrightarrow{f} \\
&\mathcal{C}^{op} \to \mathsf{Cospan}(\mathcal{C}) \text{ maps } \xleftarrow{g} \text{ to } \xrightarrow{id} \xleftarrow{g} &\qquad &\mathcal{C}^{op} \to \mathsf{Span}(\mathcal{C}) \text{ maps } \xleftarrow{g} \text{ to } \xleftarrow{g} \xrightarrow{id}
\end{aligned}
\tag{2.2}
$$

We are now ready to discuss the canonical map from cospans to corelations. This is simple: one just interprets a cospan representative as its corelation equivalence class.

Definition 5. Let $\mathcal{C}$ be a category equipped with a costable factorisation system $(\mathcal{E}, \mathcal{M})$. We define $\Gamma \colon \mathsf{Cospan}(\mathcal{C}) \to \mathsf{Corel}(\mathcal{C})$ as the ioo functor mapping the isomorphism class of cospans represented by $X \xrightarrow{f} N \xleftarrow{g} Y$ to the corelation represented by this cospan.

It is straightforward to check that this is well-defined. Moreover,

Proposition 2.2. $\Gamma \colon \mathsf{Cospan}(\mathcal{C}) \to \mathsf{Corel}(\mathcal{C})$ *is full.*

Proof. Let a be a corelation. Then choosing some representative $X \to N \leftarrow Y$ of a gives a cospan whose Γ-image is a. $\square$

Mapping spans to corelations is more subtle. Given a span, we may obtain a cospan by taking its pushout. When $\mathcal{C}$ has pushouts and pullbacks, this defines a function on morphisms $\mathsf{Span}(\mathcal{C}) \to \mathsf{Cospan}(\mathcal{C})$. This function is rarely, however, a functor: it may fail to preserve composition. To turn it into a functor, two tweaks are needed: first, we restrict

to a subcategory $\mathsf{Span}(\mathcal{A})$ of $\mathsf{Span}(\mathcal{C})$, for some carefully chosen subcategory $\mathcal{A} \subseteq \mathcal{C}$, and second, we take the jointly-in-$\mathcal{E}$ part of the pushout. We call the resulting functor Π, as it takes the *pushout* and then *projects*.

How do we choose $\mathcal{A}$? Given a cospan $X \to A \leftarrow Y$, we may take its pullback to obtain a span $X \leftarrow P \to Y$, and then pushout this span in $\mathcal{C}$ to obtain a cospan $X \to Q \leftarrow Y$. This gives a diagram

$$
\begin{array}{ccc}
& X & \\
P & & Q \dashrightarrow^{f} A \\
& Y &
\end{array}
$$

where the map f exists and is unique by the universal property of the pushout. We want this map f to lie in $\mathcal{M}$: by Definition 4, this implies that $X \to A \leftarrow Y$ and $X \to Q \leftarrow Y$ represent the same corelation. This condition is reminiscent of that introduced by Meisen in her work on so-called categories of pullback spans [27].

Note that this pullback and pushout take place in $\mathcal{C}$. We nonetheless ask $\mathcal{A}$ to be closed under pullback, so spans $\xleftarrow{f \in \mathcal{A}} \xrightarrow{g \in \mathcal{A}}$ do indeed form a subcategory $\mathsf{Span}(\mathcal{A})$ of $\mathsf{Span}(\mathcal{C})$.[2]

Proposition 2.3. *Let $\mathcal{C}$ be a category equipped with a costable factorisation system $(\mathcal{E}, \mathcal{M})$. Let $\mathcal{A}$ be a subcategory of $\mathcal{C}$ containing all isomorphisms and stable under pullback. Further suppose that the canonical map given by the pushout of the pullback of a cospan in $\mathcal{A}$ lies in $\mathcal{M}$. Then mapping a span in $\mathcal{A}$ to the jointly-in-$\mathcal{E}$ part of its pushout cospan defines an ioo functor $\Pi \colon \mathsf{Span}(\mathcal{A}) \to \mathsf{Corel}(\mathcal{C})$.*

Proof. Recall that $\mathsf{Span}(\mathcal{A})$ is generated by morphisms of the form $\xleftarrow{id} \xrightarrow{f \in \mathcal{A}}$ and $\xleftarrow{f \in \mathcal{A}} \xrightarrow{id}$. It is thus enough to show Π preserves composition on arrows of these two types. There exist four cases: (i) $\xleftarrow{id} \xrightarrow{f} \xleftarrow{id} \xrightarrow{g}$, (ii) $\xleftarrow{f} \xrightarrow{id} \xleftarrow{g} \xrightarrow{id}$, (iii) $\xleftarrow{f} \xrightarrow{id} \xleftarrow{id} \xrightarrow{g}$, and (iv) $\xleftarrow{id} \xrightarrow{f} \xleftarrow{g} \xrightarrow{id}$. The first three cases are straightforward to prove, and in fact hold when mapping $\mathsf{Span}(\mathcal{C}) \to \mathsf{Cospan}(\mathcal{C})$. It is the case (iv) that needs our restriction to $\mathsf{Span}(\mathcal{A})$. There $\Pi(\xleftarrow{id} \xrightarrow{f})\Pi(\xleftarrow{g} \xrightarrow{id})$ is represented by the cospan $\xrightarrow{f} \xleftarrow{g}$, while $\Pi(\xleftarrow{id} \xrightarrow{f} \xleftarrow{g} \xrightarrow{id})$ is the represented by the pushout $\xrightarrow{p} \xleftarrow{q}$ of the pullback of $\xrightarrow{f} \xleftarrow{g}$. But by hypothesis, there exists a unique $\xrightarrow{m \in \mathcal{M}}$ making the following diagram commute.

$$
\begin{array}{ccc}
& \xrightarrow{f} & \\
\xrightarrow{p} & \xrightarrow{m} & \\
\xrightarrow{q} & & \\
& \xrightarrow{g} &
\end{array}
$$

This implies that $\xrightarrow{p} \xleftarrow{q}$ and $\xrightarrow{f} \xleftarrow{g}$ represent the same corelation, and so Π is functorial. $\square$

For example, if the category $\mathcal{M}$ has pullbacks and these coincide with pullbacks in $\mathcal{C}$, then we can take $\mathcal{A} = \mathcal{M}$. If $\mathcal{C}$ is abelian, we can take $\mathcal{A} = \mathcal{C}$.

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[2]Calling this subcategory $\mathsf{Span}(\mathcal{A})$ is a slight abuse of notation: it may be the case that $\mathcal{A}$ itself has pullbacks, and we have not proved that these agree with pullbacks in $\mathcal{C}$. Nonetheless, this conflict does not cause trouble in any of our examples below, and we stick to this convention for notational simplicity.

3. Main theorem: a universal property for (co)relations

This section states our main result and some consequences.

3.1. **The main theorem.** We first fix our ingredients.

Assumption 1. Let $\mathcal{C}$ be a category with
- pushouts and pullbacks;
- a costable factorisation system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{M}$ a subcategory of the monos in $\mathcal{C}$;
- a subcategory $\mathcal{A}$ of $\mathcal{C}$ containing $\mathcal{M}$, stable under pullback, and such that the canonical map given by the pushout of the pullback of a cospan in $\mathcal{A}$ lies in $\mathcal{M}$.

Building on the results of Section 2, the second requirement above allows us to form a category $\mathsf{Corel}(\mathcal{C})$ of corelations, whereas the third yields a functor $\Pi \colon \mathsf{Span}(\mathcal{A}) \to \mathsf{Corel}(\mathcal{C})$.

We shall also use the functor $\Gamma \colon \mathsf{Cospan}(\mathcal{C}) \to \mathsf{Corel}(\mathcal{C})$ (Definition 5), and a category $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$. The objects of this latter category are those of $\mathcal{A}$ and the morphisms $X \to Y$ are so-called 'zigzags' $X \xrightarrow{f} \xleftarrow{g} \xrightarrow{h} \dots \xleftarrow{k} Y$ in $\mathcal{A}$, where f, g, $\dots$, k are non-identity morphisms in $\mathcal{A}$. Composition in $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$ is given by concatenation of zigzags, where we use composition in $\mathcal{A}$ to compose any two consecutive morphisms pointing in the same direction, and remove any resulting identity morphisms in $\mathcal{A}$. The identity morphism in $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$ is the empty zigzag consisting of no morphisms. Note that there are ioo functors from $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$ to $\mathsf{Cospan}(\mathcal{C})$ and to $\mathsf{Span}(\mathcal{C})$, defined on morphisms by taking colimits and by taking limits, respectively, of zigzags—equivalently, they are defined by pointwise application of the functors in (2.2).[3]

We make all these components interact in our main theorem.

Theorem 3.1. *With $\mathcal{C}$ and $\mathcal{A}$ as in Assumption 1, the following is a pushout in* **Cat***:*

$$
\begin{array}{ccc}
\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op} & \longrightarrow & \mathsf{Span}(\mathcal{A}) \\
\downarrow & & \downarrow{\scriptstyle \Pi} \\
\mathsf{Cospan}(\mathcal{C}) & \xrightarrow[\Gamma]{} & \mathsf{Corel}(\mathcal{C})
\end{array}
\qquad (\star)
$$

We leave a complete proof of this theorem to Appendix A. In a nutshell, the key point is that, in light of (2.1), $\mathsf{Corel}(\mathcal{C})$ differs from $\mathsf{Cospan}(\mathcal{C})$ precisely because it has the extra equations $\xrightarrow{m} \xleftarrow{m} = \xrightarrow{id} \xleftarrow{id}$, with $\xrightarrow{m} \in \mathcal{M}$. But these equations arise by pullback squares in $\mathcal{A}$, and so are equations of zigzags in $\mathsf{Span}(\mathcal{A})$ (*cf.* (3.1) below). Moreover, the remaining equations of $\mathsf{Span}(\mathcal{A})$ can be generated by using these together with a subset of the equations of $\mathsf{Cospan}(\mathcal{C})$. Hence adding the equations of $\mathsf{Span}(\mathcal{A})$ to those of $\mathsf{Cospan}(\mathcal{C})$ gives precisely $\mathsf{Corel}(\mathcal{C})$, and we have a pushout square.

We now discuss some observations, consequences, and examples.

Remark 3.1. If any such $\mathcal{A}$ exists, then we may always take $\mathcal{A} = \mathcal{M}$ and the theorem holds. We record the above, more general, theorem as it explains preliminary results in this direction already in the literature; see the abelian case and examples for details.

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[3]More abstractly, one can see $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$ as the pushout of $\mathcal{A}$ and $\mathcal{A}^{op}$ over the respective inclusions of $|\mathcal{A}|$, the discrete category on the objects of $\mathcal{A}$. The functors $\mathsf{Span}(\mathcal{A}) \leftarrow \mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op} \to \mathsf{Cospan}(\mathcal{C})$ are then those given by the universal property with respect to (suitable restrictions of) the functors in (2.2).

Example 3.1. To briefly fix some examples in mind, we remark that Assumption 1 is satisfied by the (epi, mono) factorisation systems in the categories of (finite) sets and functions, vector spaces and linear transformations, and any presheaf category. Here, following Remark 3.1, we simply take the subcategory $\mathcal{A}$ to be the subcategory of monos, although other choices also suffice. It can also be shown that, given any pair of categories equipped with data satisfying Assumption 1, new examples may be obtained by taking their product and coproduct. Theorem 3.1 thus applies to all these cases. We will explore some of these examples, and many others, in detail in Section 4.

We may easily formulate the dual version of the theorem, which yields a characterisation for relations. It is based on a dual version of Assumption 1.

Assumption 2. Let $\mathcal{C}$ be a category with
- pushouts and pullbacks;
- a stable factorisation system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{E}$ a subcategory of the epis in $\mathcal{C}$;
- a subcategory $\mathcal{A}$ of $\mathcal{C}$ containing $\mathcal{E}$, stable under pushout, and such that the canonical map given by the pullback of the pushout of a span in $\mathcal{A}$ lies in $\mathcal{E}$.

Corollary 3.2 (Dual case)**.** *With $\mathcal{C}$ and $\mathcal{A}$ as in Assumption 2, the following is a pushout in* **Cat**.

$$
\begin{array}{ccc}
\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op} & \longrightarrow & \mathsf{Cospan}(\mathcal{A}) \\
\downarrow & & \downarrow \\
\mathsf{Span}(\mathcal{C}) & \longrightarrow & \mathsf{Rel}(\mathcal{C})
\end{array}
\qquad (\triangle)
$$

Proof. This corollary is obtained by noting that, given a stable factorisation system $(\mathcal{E}, \mathcal{M})$ in $\mathcal{C}$, with $\mathcal{E}$ a subcategory of the epis, we have a costable factorisation system $(\mathcal{M}^{op}, \mathcal{E}^{op})$ in $\mathcal{C}^{op}$, with $\mathcal{E}^{op}$ a subcategory of the monos. Proposition 2.3 then gives a functor $\mathsf{Cospan}(\mathcal{A}) = \mathsf{Span}(\mathcal{A}^{op}) \longrightarrow \mathsf{Rel}(\mathcal{C}) = \mathsf{Corel}(\mathcal{C}^{op})$. Noting also that $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op} = \mathcal{A}^{op} +_{|\mathcal{A}|} (\mathcal{A}^{op})^{op}$ and $\mathsf{Span}(\mathcal{C}) = \mathsf{Cospan}(\mathcal{C}^{op})$, we can hence apply Theorem 3.1. $\square$

Remark 3.2. In Theorem 3.1, the diagram $\mathsf{Cospan}(\mathcal{C}) \leftarrow \mathcal{A} +_{|\mathcal{A}|} \mathcal{A} \rightarrow \mathsf{Span}(\mathcal{A})$ 'knows' only about $\mathcal{M}$, not the factorisation system $(\mathcal{E}, \mathcal{M})$. One might then wonder how the pushout could be known to be $\mathsf{Corel}(\mathcal{C})$, since this is the category of jointly-in-$\mathcal{E}$ cospans. This is enough, however, since if $\mathcal{M}$ is part of a factorisation system, then the factorisation system is unique.

Indeed, suppose we have $\mathcal{E}$, $\mathcal{E}'$ such that both $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M})$ are factorisation systems. Take $e \in \mathcal{E}$. Then the factorisation system $(\mathcal{E}', \mathcal{M})$ gives a factorisation $e = e'; m_1$, while $(\mathcal{E}, \mathcal{M})$ gives a factorisation $e' = e_2; m_2$. By substitution, we have $e = e_2; m_2; m_1$. By uniqueness of factorisation, we can then assume without loss of generality that $e = e_2$ and $m_2; m_1 = id$. Next, using $e = e_2$ and substitution in $e' = e_2; m_2$, we similarly arrive at $m_1; m_2 = id$. Thus m_1 is an isomorphism, and hence lies in $\mathcal{E}'$. This implies that $e = e'; m_1 \in \mathcal{E}'$, and hence $\mathcal{E} \subseteq \mathcal{E}'$. We may similarly show that $\mathcal{E}' \subseteq \mathcal{E}$, and hence that the two categories are equal.

The next corollary is instrumental in giving categories of (co)relations a presentation by generators and equations.

Corollary 3.3. *Suppose $\mathcal{A}$ and $\mathcal{C}$ are as in Assumption 1. Then $\mathsf{Corel}(\mathcal{C})$ is freely generated by the objects of $\mathcal{C}$ and arrows $\xrightarrow{f}$, $\xleftarrow{g}$ of $\mathcal{C}$ quotiented by*

$$\xrightarrow{f \in \mathcal{A}}\xleftarrow{g \in \mathcal{A}} = \xleftarrow{p \in \mathcal{A}}\xrightarrow{q \in \mathcal{A}} \qquad\qquad \text{whenever } \xleftarrow{p}\xrightarrow{q} \text{ pulls back } \xrightarrow{f}\xleftarrow{g} \qquad (3.1)$$

$$\xleftarrow{f \in \mathcal{C}}\xrightarrow{g \in \mathcal{C}} = \xrightarrow{p \in \mathcal{C}}\xleftarrow{q \in \mathcal{C}} \qquad\qquad \text{whenever } \xrightarrow{p}\xleftarrow{q} \text{ pushes out } \xleftarrow{f}\xrightarrow{g} . \qquad (3.2)$$

Equivalently, $\mathsf{Corel}(\mathcal{C})$ is the quotient of $\mathsf{Cospan}(\mathcal{C})$ by (3.1). A dual statement holds for $\mathsf{Rel}(\mathcal{C})$.

Note that, in light of Remark 3.1, one may also replace (3.1) by the subset of axioms

$$\xrightarrow{f \in \mathcal{M}}\xleftarrow{g \in \mathcal{M}} = \xleftarrow{p \in \mathcal{M}}\xrightarrow{q \in \mathcal{M}} \qquad\qquad \text{whenever } \xleftarrow{p}\xrightarrow{q} \text{ pulls back } \xrightarrow{f}\xleftarrow{g} .$$

As $\mathcal{M} \subseteq \mathcal{A}$ by Assumption 1, this may give a smaller presentation.

Corollary 3.3 provides a modular recipe for axiomatising categories of (co)relations: a presentation for $\mathsf{Corel}(\mathcal{C})$ can be obtained by "merging" equational theories arising from $\mathcal{A}$-pullbacks and $\mathcal{C}$-pushouts, provided respectively by sets of equations (3.1) and (3.2). t is worth noticing that equations (3.1)-(3.2) enjoy an elegant description in terms of *distributive laws* of categories, see [28, 24, 34]. In practice, the reduction of Corollary 3.3 is particularly useful when $\mathcal{C}$ is "nice enough" to allow for a *finitary* axiomatisation of equations (3.1)-(3.2)— two such examples are the category of finite sets, in which pullbacks decompose according to the laws of extensive categories [24], and the category of finite-dimensional vector spaces, where pullbacks/pushouts can be computed by kernels/cokernels of matrices [7]. We shall see numerous applications of the recipe of Corollary 3.3 in the examples of Section 4 below.

In the remainder of this section, we discuss the consequences and generalisations of the main theorem when various additional structure is present, including the cases of abelian categories, monoidal categories, and props.

3.2. **The abelian case.** As a notable instance of Theorem 3.1, we can specialise to the case of abelian categories and their epi-mono factorisation system. In this case we can simply pick $\mathcal{A}$ to be $\mathcal{C}$ itself.

Corollary 3.4 (Abelian case). *Let $\mathcal{C}$ be an abelian category. Then the following is a pushout square in* **Cat***:*

$$
\begin{array}{ccc}
\mathcal{C} +_{|\mathcal{C}|} \mathcal{C}^{op} & \xrightarrow{\hspace{3cm}} & \mathsf{Span}(\mathcal{C}) \\
\downarrow & & \downarrow{\scriptstyle \Pi} \\
\mathsf{Cospan}(\mathcal{C}) & \xrightarrow[\Gamma]{\hspace{2cm}} & \mathsf{Corel}(\mathcal{C}) \cong \mathsf{Rel}(\mathcal{C})
\end{array}
$$

where we take (co)relations with respect the epi-mono factorisation system.

Proof. As $\mathcal{C}$ is abelian, it is finitely bicomplete and has a bistable factorisation system given by epis and monos. Furthermore, we need not restrict our spans to some subcategory $\mathcal{A}$: in an abelian category the pullback of a cospan $X \xrightarrow{f} A \xleftarrow{g} Y$ can be computed via the kernel of the joint map $X \oplus Y \xrightarrow{[f,-g]} A$, and similarly pushouts can be computed via cokernel, whence the canonical map from the pushout of the pullback of a given cospan to itself is always mono, being the inclusion of the image of the joint map into the apex. Similarly, the map from a span to the pullback of its pushout is simply the joint map with codomain restricted to its image, and hence always epi. Thus $\mathcal{C}$ meets both Assumptions 1 and 2 with $\mathcal{A} = \mathcal{C}$. Then

the category of corelations is the pushout of the span $\mathsf{Cospan}(\mathcal{C}) \leftarrow \mathcal{C} +_{|\mathcal{C}|} \mathcal{C}^{op} \rightarrow \mathsf{Span}(\mathcal{C})$. But by the dual theorem (Corollary 3.2), the pushout of this span is also the category of relations. Thus the two categories are isomorphic. Explicitly, the isomorphism is given by taking a corelation to the jointly mono part of its pullback span, and taking a relation to the jointly epi part of its pushout cospan. □

3.3. **The monoidal case.** Categories of (co)spans and (co)relations are often monoidal categories with, for example, the monoidal product coming from (co)products. Our theorem extends to this monoidal case.

For monoidal structure on $\mathcal{C}$ to extend to the categories of (co)spans and (co)relations, it is crucial that the monoidal product respects the ambient structure. Let $(\mathcal{C}, \oplus)$ be a symmetric monoidal category with pushouts, and let $(\mathcal{A}, \oplus)$ be a sub-symmetric monoidal category. We say that **the monoidal product preserves pushouts** in $\mathcal{A}$ if, for all spans $N \leftarrow Y \rightarrow M$ and $N' \leftarrow Y' \rightarrow M'$ in $\mathcal{A}$, we have an isomorphism

$$(N \oplus N') +_{Y \oplus Y'} (M \oplus M') \cong (N +_Y M) \oplus (N' +_{Y'} M').$$

Note that this pushout is taken in $\mathcal{C}$. This condition holds, for example, whenever $\mathcal{C}$ is monoidally closed. We say the monoidal product preserves pullbacks if the analogous condition holds for pullbacks.

Furthermore, we say that a subcategory $\mathcal{A}$ is **closed under** $\oplus$ if, given morphisms f, g in $\mathcal{A}$, the morphism $f \oplus g$ is also in $\mathcal{A}$.

Proposition 3.5. *Let $\mathcal{C}$ and $\mathcal{A}$ be symmetric monoidal categories satisfying Assumption 1. Suppose that the monoidal product of $\mathcal{C}$ preserves pushouts in $\mathcal{C}$ and pullbacks in $\mathcal{A}$, and that $\mathcal{M}$ and $\mathcal{A}$ are closed under the monoidal product. Then the following is a pushout square in the category of symmetric monoidal categories and (lax, strong, strict) monoidal functors.*

$$
\begin{array}{ccc}
\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op} & \longrightarrow & \mathsf{Span}(\mathcal{A}) \\
\downarrow & & \downarrow \Pi \\
\mathsf{Cospan}(\mathcal{C}) & \xrightarrow{\ \Gamma\ } & \mathsf{Corel}(\mathcal{C}).
\end{array}
$$

The proof of this result relies on an analysis of how monoidal structure carries through the various steps of our pushout construction. The details can be found in Appendix B.

3.4. **The case of props.** As mentioned in the introduction, the motivating examples for our construction are categories providing a semantic interpretation for circuit diagrams. These are typically props (**pro**duct and **p**ermutation categories [25]): it is thus useful to phrase our construction in this setting. This is merely a specialisation of the above result.

Recall that a prop is a symmetric monoidal category with objects the natural numbers, in which $n \oplus m = n + m$. Props form a category **Prop** with morphisms the ioo strict symmetric monoidal functors. A simplification to Theorem 3.1 is that the coproduct $\mathcal{C} + \mathcal{C}'$ in **Prop** is computed as $\mathcal{C} +_{|\mathcal{C}|} \mathcal{C}'$ in **Cat**, because the set of objects is fixed for any prop.

Theorem 3.6. *Let $\mathcal{C}$ and $\mathcal{A}$ be props satisfying Assumption 1. Suppose that the monoidal product of $\mathcal{C}$ preserves pushouts in $\mathcal{C}$ and pullbacks in $\mathcal{A}$, and that $\mathcal{M}$ is closed under the monoidal product. Then we have a pushout square in* **Prop**

$$
\begin{array}{ccc}
\mathcal{A} + \mathcal{A}^{op} & \longrightarrow & \mathsf{Span}(\mathcal{A}) \\
\downarrow & & \downarrow \Pi \\
\mathsf{Cospan}(\mathcal{C}) & \xrightarrow{\quad \Gamma \quad} & \mathsf{Corel}(\mathcal{C}).
\end{array}
\qquad (\circ)
$$

Proof. Recalling that props are certain structured strict symmetric monoidal categories and prop morphisms are structured strict monoidal functors, the theorem is an immediate consequence of Proposition 3.5.   □

We also state the prop version of the abelian case. An abelian prop is just a prop which is also an abelian category and where the monoidal product is the biproduct.

Corollary 3.7. *Suppose that $\mathcal{C}$ is an abelian prop. The following is a pushout in* **Prop**.

$$
\begin{array}{ccc}
\mathcal{C} + \mathcal{C}^{op} & \longrightarrow & \mathsf{Span}(\mathcal{C}) \\
\downarrow & & \downarrow \Pi \\
\mathsf{Cospan}(\mathcal{C}) & \xrightarrow{\quad \Gamma \quad} & \mathsf{Corel}(\mathcal{C}) \cong \mathsf{Rel}(\mathcal{C})
\end{array}
$$

Proof. Note that in an abelian prop the biproduct, being both a product and a coproduct, preserves both pushouts and pullbacks, and that the monos are closed under the biproduct. Thus we can apply Theorem 3.6.   □

The proof of Theorem 3.6 and of Corollary can be found in Appendix B.

3.5. (Co)algebras over a (co)monad.

It is of interest to identify under which conditions our universal construction of $\mathsf{Rel}(\mathcal{C})$ lifts to relations over $\mathcal{C}^T$, the category of algebras of some monad $T \colon \mathcal{C} \to \mathcal{C}$.

Proposition 3.8. *Let $\mathcal{C}$ be a complete and cocomplete regular category that obeys Assumption 2 with respect to its (regular epi, mono) factorisation system. Further assume that every regular epi splits. Next, let T be a monad on $\mathcal{C}$ such that T and T^2 preserve pushouts of regular epis. Then the category $\mathcal{C}^T$ of algebras over T is a complete and cocomplete regular category that obeys Assumption 2 with respect to its (regular epi, mono) factorisation system.*

Note that if $\mathcal{C}$ is a complete and cocomplete category, then regularity is just the statement that regular epis are stable under pullback, and Assumption 2 is just the statement that the canonical map given by the pullback of the pushout of a span of regular epis is again a regular epi.

Proof. The completeness, cocompleteness, and regularity of $\mathcal{C}^T$ under the above conditions is a standard result, see e.g. [9, Th. 4.3.5]. In brief, the completeness of $\mathcal{C}^T$ is easily inherited from the completeness of $\mathcal{C}$. The cocompleteness requires the regular epis to split and the existence of products. This allows explicit construction of coequalizers in $\mathcal{C}^T$ and, using the existence of coproducts in $\mathcal{C}$, this immediately implies the cocompleteness of $\mathcal{C}^T$ (see [9, Prop. 4.3.4]). The stability of regular epis under pullback in $\mathcal{C}^T$ follows the fact

that the forgetful functor $U \colon \mathcal{C}^T \to \mathcal{C}$ preserves and reflects both pullbacks and regular epis. Thus, since regular epis are stable under pullback in $\mathcal{C}$, the same holds true in $\mathcal{C}^T$.

It then remains to show that, in $\mathcal{C}^T$, the canonical map given by the pullback of the pushout of a span of regular epis is again a regular epi. Since T preserves pushouts of regular epis, such a pushout in $\mathcal{C}^T$ may simply be obtained by computing the pushout P of the underlying maps in $\mathcal{C}$, and using the universal property of the T-image of this pushout to get a map $c \colon TP \to P$. Since T^2 also preserves pushouts of regular epis, it can be shown that this map c is in fact an algebra, and the pushout in $\mathcal{C}^T$ (see [9, Prop. 4.3.2]). On the other hand, as limits in a category of algebras, pullbacks in $\mathcal{C}^T$ can easily be computed via pullbacks in $\mathcal{C}$. Thus the canonical map in question is simply the canonical map in $\mathcal{C}$ lifted to a map of algebras, and hence is again a regular epi. This proves the claim. $\square$

We may dualise Proposition 3.8 to the case of corelations in the category of coalgebras over a comonad; in this case we assume our category is coregular, and that it is the regular monos that split in $\mathcal{C}$ and whose pullbacks are preserved by the comonad T and by T^2.
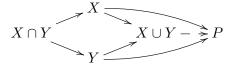
Note also that the assumptions above can be varied or weakened to still obtain that the category of (regular epi, mono) relations in a category of algebras can be constructed as a pushout. For example, it is straightforward to relax cocompleteness to, say, finite cocompleteness, or to put conditions on the properties of $\mathcal{C}^T$ instead of on $\mathcal{C}$ and T.[4]

4. EXAMPLES

4.1. **Equivalence Relations.** Our first example concerns the construction of equivalence relations starting from injective functions. For $n \in \mathbb{N}$, write $\overline{n}$ for the set $\{0, 1, \ldots, n-1\}$, and $\uplus$ for the disjoint union of sets. We fix a prop ER whose arrows $n \to m$ are the equivalence relations on $\overline{n} \uplus \overline{m}$. For composition $e_1 \,;\, e_2 \colon n \to m$ of equivalence relations $e_1 \colon n \to z$ and $e_2 \colon z \to m$, one first defines an equivalence relation on $\overline{n} \uplus \overline{z} \uplus \overline{m}$ by gluing together equivalence classes of e_1 and e_2 along common witnesses in $\overline{z}$, then obtains $e_1 \,;\, e_2$ by restricting to elements of $\overline{n} \uplus \overline{m}$.

Equivalence relations are equivalently described as corelations of functions. For this, let F be the prop whose arrows $n \to m$ are functions from $\overline{n}$ to $\overline{m}$. F has the usual factorisation system $(\mathsf{Su}, \mathsf{In})$ given by epi-mono factorisation, where Su and In are the sub-props of surjective and of injective functions respectively. Given these data, one can check that ER is isomorphic to $\mathsf{Corel}(\mathsf{F})$, the prop of corelations on F.

We are now in position to apply our construction of Theorem 3.6. First, we verify Assumption 1 with $\mathcal{C}$ instantiated as F and $\mathcal{A}$ as In. The only point requiring some work is the third, which goes as follows: given a cospan of monos $X \to P \leftarrow Y$, consider X, Y as subsets of P. Then the pullback-pushout diagram looks like

---

and $X \cup Y \to P$ is the inclusion map, hence a morphism in In. Therefore, we can construct the pushout diagram ($\circ$) as follows:

$$
\begin{array}{ccc}
\mathsf{In} + \mathsf{In}^{op} & \longrightarrow & \mathsf{Span}(\mathsf{In}) \\
\downarrow & & \downarrow \\
\mathsf{Cospan}(\mathsf{F}) & \longrightarrow & \mathsf{ER}
\end{array}
\tag{4.1}
$$

This modular reconstruction easily yields a presentation by generators and relations for (the arrows of) ER. Following the recipe of Corollary 3.3, ER is the quotient of $\mathsf{Cospan}(\mathsf{F})$ by all the equations generated by pullbacks in In, as in (3.1). Now, recall that In is presented (in string diagram notation [30]) by the generator $\boxdot \colon 0 \to 1$, and no equations. Thus, in order to present all the equations of shape (3.1) it suffices to consider one pullback square in In:

$$
, \text{ yielding the equation } \boxdot\,;\,\boxdot\ =\ \square\,;\,\square.
\tag{4.2}
$$

On the other hand, we know $\mathsf{Cospan}(\mathsf{F})$ is presented by the theory of special commutative Frobenius monoids (also termed separable Frobenius algebras), see [24]. Therefore ER is presented by the generators and equations of special commutative Frobenius monoids, with the addition of (4.2). This is known as the theory of extraspecial commutative Frobenius monoids [11]. This result also appears in [35], in both cases without the realisation that it stems from a more general construction.
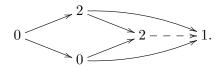
4.2. **Partial Equivalence Relations.** Partial equivalence relations (PERs) are common structures in program semantics, which date back to the seminal work of Scott [29] and recently revamped in the study of quantum computations (e.g., [20, 16]). Our approach yields a characterisation for the prop PER whose arrows $n \to m$ are PERs on $\overline{n} \uplus \overline{m}$, with composition as in ER. The ingredients of the construction generalise Example 4.1 from total to partial maps. Instead of F one starts with PF, the prop of partial functions, which has a factorisation system involving the sub-prop of partial surjections and the sub-prop of injections. The resulting prop of PF-corelations is isomorphic to PER. Theorem 3.6 yields the following pushout

$$
\begin{array}{ccc}
\mathsf{In} + \mathsf{In}^{op} & \longrightarrow & \mathsf{Span}(\mathsf{In}) \\
\downarrow & & \downarrow \\
\mathsf{Cospan}(\mathsf{PF}) & \longrightarrow & \mathsf{PER}.
\end{array}
\tag{4.3}
$$

As in Example 4.1, following Corollary 3.3, (4.3) reduces the task of axiomatising PER to the one of axiomatising $\mathsf{Cospan}(\mathsf{PF})$ and adding the single equation (4.2) from $\mathsf{Span}(\mathsf{In})$. $\mathsf{Cospan}(\mathsf{PF})$ is presented by "partial" special commutative Frobenius monoids, studied in [35].

4.3. **Non-Example: the Terminal Prop.** It is instructive to see a non-example, to show that the assumptions on $\mathcal{A}$ are not redundant. One may want consider an obvious variation of (4.1), where instead of In one takes the whole F as $\mathcal{A}$. However, with this tweak the construction collapses: the pushout is the terminal prop $\mathbf{1}$ with exactly one arrow between any two objects.

$$
\begin{array}{ccc}
\mathsf{F} + \mathsf{F}^{op} & \longrightarrow & \mathsf{Span}(\mathsf{F}) \\
\downarrow & & \downarrow \\
\mathsf{Cospan}(\mathsf{F}) & \longrightarrow & \mathbf{1}
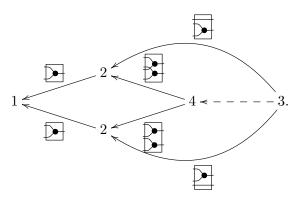\end{array}
$$

This phenomenon was noted before ([7], see also [17, Th. 5.6]), however without an understanding of its relationship with other (non-collapsing) instances of the same construction. Theorem 3.6 explains why this case fails where others succeed: the problem lies in the choice of F as the subcategory $\mathcal{A}$. Indeed, the canonical map given by the pullback of the pushout of any span in $\mathcal{A} = \mathsf{F}$ does not necessarily lie in In, i.e. it may be not injective. An example is given by the cospan $0 \to 1 \leftarrow 2$, with the canonical map from the pushout of the pullback cospan the non-injective map $2 \to 1$:

$$
\begin{array}{c}
2 \\
0 \qquad\qquad 2 \dashrightarrow 1. \\
0
\end{array}
$$

4.4. **Non-Example: Spans of Functions.** Another interesting non-example is the attempt of constructing $\mathsf{Rel}(\mathsf{F})$, the prop of relations between finite sets. Inspired by the case of corelations (Section 4.1), one may think of using again In as subcategory $\mathcal{A}$ and apply the recipe ($\triangle$) from Corollary 3.2. However, because $\mathsf{Cospan}(\mathsf{In})$ and $\mathsf{In} + \mathsf{In}^{op}$ are isomorphic props (the distributive law forming $\mathsf{Cospan}(\mathsf{In})$ does not yield any additional equations), the pushout diagrams just yields spans again.

$$
\begin{array}{ccc}
\mathsf{In} + \mathsf{In}^{op} & \longrightarrow & \mathsf{Span}(\mathsf{F}) \\
\downarrow & & \downarrow \\
\mathsf{Cospan}(\mathsf{In}) & \longrightarrow & \mathsf{Span}(\mathsf{F})
\end{array}
$$

The reason why the construction ($\triangle$) does not apply is because the third item of Assumption 2 fails. The following is a counterexample, where we start with a span $2 \leftarrow 3 \to 2$ of epis, take the pushout $2 \to 1 \leftarrow 2$, then the pullback $2 \to 4 \leftarrow 2$, and then observe that the canonical morphism $3 \to 4$ cannot be epi. As before, the functions involved are specified in string

diagrammatic notation.

4.5. **Isomorphisms.** In a degenerate way, our construction can actually be applied to an arbitrary category $\mathcal{C}$ with pushouts and pullbacks. Indeed, we noted in Section 2 that every category $\mathcal{C}$ has a factorisation system $(\mathcal{C}, \mathcal{I})$, where again $\mathcal{I}$ is the subcategory containing exactly the isomorphisms. Note that $\mathcal{I}$ is always a subcategory of the category of monos in $\mathcal{C}$: we may then also take $\mathcal{A} = \mathcal{I}$. Corelations with respect to this factorisation system are simply cospans. With this setup, $\mathcal{C}$ satisfies Assumption 1, and Theorem 3.1 then states that

$$
\begin{array}{ccc}
\mathcal{I} +_{|\mathcal{I}|} \mathcal{I}^{op} & \longrightarrow & \mathsf{Span}(\mathcal{I}) \\
\downarrow & & \downarrow{\scriptstyle \Pi} \\
\mathsf{Cospan}(\mathcal{C}) & \xrightarrow[\Gamma]{} & \mathsf{Cospan}(\mathcal{C})
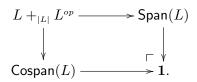\end{array}
\tag{4.4}
$$

is a pushout square. Within the axiomatic perspective of Corollary 3.3, (4.4) states that all the axioms arising from pullbacks of isomorphisms in $\mathcal{C}$ already arise from pushouts in $\mathcal{C}$. This is a manifestation of the fact that any square in $\mathcal{C}$ formed by taking the pullback of isomorphisms is also a pushout square.

4.6. **Lattices.** Another "degenerate" case is the one of lattices. Recall that a lattice is a poset with binary joins and meets. Just as for any pre-order, one can see a lattice L as a category: objects are the lattice elements and there is an arrow from $a \in L$ to $b \in L$ precisely when $a \leq b$ in the lattice.

Let us now check Assumption 1 for L. First, L has pushouts and pullbacks: they are just given by joins and meets. For the second and third requirement, we need to fix a factorisation system. Above we noticed that the factorisation system $(\mathcal{C}, \mathcal{I})$ works for any $\mathcal{C}$; in the case of a lattice, we may also consider the factorisation $(\mathcal{I}, \mathcal{C})$. Indeed, because there is at most one arrow between any two objects of L, every arrow is both an epi and a mono, and thus every subcategory of L is a subcategory of the monos in L. Also, notice that the only isomorphisms are the identities; thus the subcategory $\mathcal{I}$ of isomorphisms is just the discrete sub-category $|L|$ formed by the objects of L, with their identity arrows. Choosing the factorisation system $(|L|, L)$, and letting $\mathcal{A} = L$, then gives data that satisfies Assumption 1.

(Co)spans for a lattice turn out to be familiar order-theoretic notions. The set of arrows from $a \in L$ to $b \in L$ in $\mathsf{Cospan}(L)$ is equal to the *upper set* of $a \vee b$, and composition is given by join. Similarly, the set of arrows from $a \in L$ to $b \in L$ in $\mathsf{Span}(L)$ is equal to the *lower set*

of $a \wedge b$, and composition is given by meet. Then the category $\mathsf{Corel}(L)$ is equivalent to the terminal category $\mathbf{1}$; any two cospans in L are considered equivalent as $(|L|, L)$-corelations.

Theorem 3.1 thus yields the pushout square

$$
\begin{array}{ccc}
L +_{|L|} L^{op} & \longrightarrow & \mathsf{Span}(L) \\
\downarrow & & \downarrow \\
\mathsf{Cospan}(L) & \longrightarrow & \mathbf{1}.
\end{array}
$$

This collapse is reminiscent of Non-Example 4.3. We can understand this as follows. A morphism from a to b in $L +_{|L|} L^{op}$ is a zigzagging path from a to b in the lattice L, for example $a \leq c_1 \geq c_2 \leq c_3 \geq \cdots \leq c_n \geq b$. The category $\mathsf{Cospan}(L)$ considers two paths to be the same if they have the same least upper bound, while the category $\mathsf{Span}(L)$ considers two paths to be the same if they have the same greatest lower bound. Pushing out these two notions of equivalence, we see that in the category $\mathsf{Corel}(L)$, all paths between a and b are considered the same, and hence there is a unique morphism between any two objects. That is to say, $\mathsf{Corel}(L) \cong \mathbf{1}$.

Our example can be generalised from lattices to arbitrary posets with pushouts and pullbacks. It is straightforward to show that a poset P with all pushouts and pullbacks need not itself be a lattice, but that elements a and b have a join if and only if they have a meet, and moreover if and only if they have an upper bound or lower bound. This means that P must be a coproduct of lattices.

Suppose that P is the coproduct of n lattices. In this case, we may then use the same (identity, any morphism) factorisation system $(|P|, P)$ and apply Theorem 3.1. Here the homsets of $\mathsf{Cospan}(P)$ and $\mathsf{Span}(P)$ can still be described as upper sets of joins and lower sets of meets, with the caveat that the homset is empty when the relevant join or meet does not exist. The category of $(|P|, P)$-corelations is then equivalent to the coproduct of n copies of the terminal category, and Theorem 3.1 asserts the usual pushout square. Thus in this case the pushout of the categories of spans and cospans in P results in counting the number of lattice summands, or 'connected components', of P.

4.7. **Linear Subspaces over a Field.** We now consider an example for the abelian case: the prop SV_k whose arrows $n \to m$ are k-linear subspaces of $k^n \times k^m$, for a field k. Composition in SV_k is relational: $V \, ; W = \{(v, w) \mid \exists u.(v, u) \in V, (u, w) \in W\}$. Interest in SV_k is motivated by various recent applications. We mention the case where k is the field of Laurent series, in which SV_k constitutes a denotational semantics for signal flow graphs [3, 1, 5, 6], and the case $k = \mathbb{Z}_2$, in which SV_k is isomorphic to the phase-free ZX-calculus, an algebra for quantum observables [10, 4].

Now, in order to apply our construction, note that SV_k is isomorphic to $\mathsf{Rel}(\mathsf{Vect}_k)$, where Vect_k is the abelian prop whose arrows $n \to m$ are the linear maps of type $k^n \to k^m$ (the monoidal product is by direct sum). This follows from the observation that subspaces of $k^n \times k^m$ of dimension z correspond to mono linear maps from k^z to $k^n \times k^m$, whence to jointly mono spans $n \leftarrow z \to m$ in Vect_k.

We are then in position to use Corollary 3.7, which yields the following pushout characterisation for $\mathsf{SV_k}$.

$$
\begin{array}{ccc}
\mathsf{Vect}_k + {\mathsf{Vect}_k}^{op} & \longrightarrow & \mathsf{Span}(\mathsf{Vect}_k) \\
\downarrow & & \downarrow \\
\mathsf{Cospan}(\mathsf{Vect}_k) & \longrightarrow & \mathsf{SV_k}
\end{array}
$$

This very same pushout has been studied in [4] for the $\mathsf{k} = \mathsf{Z}_2$ case. As before, the modular reconstruction suggests a presentation by generators and relations for $\mathsf{SV_k}$, in terms of the theories for spans and cospans in Vect_k. The axiomatisation of $\mathsf{SV_k}$ is called the theory of interacting Hopf algebras [7, 34] , as it features two Hopf algebras structures and axioms expressing their combination.

On the top of existing results on $\mathsf{SV_k}$, our Corollary 3.7 suggests a novel perspective, namely that $\mathsf{SV_k}$ can be also thought as the prop of *corelations* over Vect_k. This representation can be understood by recalling the 1-1 correspondence between subspaces of $\mathsf{k}^n \times \mathsf{k}^m$ and (solution sets of) homogeneous systems of equations $Mv = 0$, where M is a $z \times (n+m)$ matrix. Writing the block decomposition $M = (M_1 \mid -M_2)$, where M_1 is a $z \times n$ matrix and M_2 a $z \times m$ matrix, this is the same as solutions to $M_1 v_1 = M_2 v_2$. These systems then yield jointly epi cospans $n \xrightarrow{M_1} z \xleftarrow{M_2} m$ in Vect_k, that is, corelations.

4.8. **Linear Corelations over a Principal Ideal Domain.** We now consider the generalisation of the linear case from fields to principal ideal domains (PIDs). In order to form a prop, we need to restrict our attention to finitely-dimensional *free* modules over a PID R. The symmetric monoidal category of such modules and module homomorphisms, with monoidal product by direct sum, is equivalent to the prop $\mathsf{FMod_R}$ whose arrows $n \to m$ are R-module homomorphisms $\mathsf{R}^n \to \mathsf{R}^m$ or, equivalently, $m \times n$-matrices in R. Because of the restriction to free modules, $\mathsf{FMod_R}$ is not abelian. However, it is still finitely bicomplete and has a costable (epi, split mono)-factorisation system.[5] Note that the fact that the ring R is a PID matters for the existence of pullbacks, as it is necessary for submodules of free R-modules to be free—pushouts exist by self-duality of $\mathsf{FMod_R}$.

Write $\mathsf{MFMod_R}$ for the prop of split monos in $\mathsf{FMod_R}$. It is a classical, although nontrivial, theorem in control theory that this category obeys the required condition on pushouts of pullbacks [14]. Hence Theorem 3.6 yields the pushout square

$$
\begin{array}{ccc}
\mathsf{FMod_R} + {\mathsf{FMod_R}}^{op} & \longrightarrow & \mathsf{Span}(\mathsf{MFMod_R}) \\
\downarrow & & \downarrow \\
\mathsf{Cospan}(\mathsf{FMod_R}) & \longrightarrow & \mathsf{Corel}(\mathsf{FMod_R})
\end{array}
\tag{4.5}
$$

in **Prop**. This modular account of $\mathsf{Corel}(\mathsf{FMod_R})$ is relevant for the semantics of dynamical systems. When $\mathsf{R} = \mathbb{R}[s, s^{-1}]$, the ring of Laurent polynomials in some formal symbol s with coefficients in the reals, the prop $\mathsf{Corel}(\mathsf{FMod}_{\mathbb{R}[s,s^{-1}]})$ models complete linear time-invariant discrete-time dynamical systems in $\mathbb{R}$; more details can be found in [14]. In that paper, it is also proven that $\mathsf{Corel}(\mathsf{FMod_R})$ is axiomatised by the presentation of $\mathsf{Cospan}(\mathsf{FMod_R})$ with the addition of the law $\boxdot \,;\, \boxdot = \square$. By Corollary 3.3, it follows that $\boxdot \,;\, \boxdot = \square$

---

[5]The factorisation given by (epi, mono) morphisms is not unique up to isomorphism, whence the restriction to split monos—see [14].

originates by a pullback in $\mathsf{Span}(\mathsf{MFMod_R})$ and in this case it is the only contribution of spans to the presentation of corelations.

It is worth noticing that, even though $\mathsf{FMod_R}$ is not abelian, the pushout of spans and cospans over $\mathsf{FMod_R}$ does not have a trivial outcome as for the prop F of functions (Example 4.3). Instead, in [7, 34] it is proven that we have the pushout square

$$
\begin{array}{ccc}
\mathsf{FMod_R} \oplus \mathsf{FMod_R}^{op} & \longrightarrow & \mathsf{Span}(\mathsf{FMod_R}) \\
\downarrow & & \downarrow \\
\mathsf{Cospan}(\mathsf{FMod_R}) & \longrightarrow & \mathsf{SV_k}
\end{array}
\tag{4.6}
$$

in **Prop**, where k is the *field of fractions* of R.

The pushout (4.6) is relevant for the categorical semantics for signal flow graphs pursued in [3, 5, 6]. Even though it is not an instance of Theorem 3.6 or Corollary 3.7, our developments shed light on (4.6) through the comparison with (4.5). First, note that any element $r \in \mathsf{R}$ yields a module homomorphism $x \mapsto rx$ in $\mathsf{FMod_R}$ of type $1 \to 1$, represented as a string diagram ⊡. The key observation is that, in (4.6), $\mathsf{Span}(\mathsf{FMod_R})$ is contributing to the axiomatisation of $\mathsf{SV_k}$ (*cf.* Corollary 3.3) by adding, for each r, an equation

⊡ ; ⊡ = ⊟ ; ⊟, corresponding to a pullback

in $\mathsf{FMod_R}$. Back to (4.5), the only equations of this kind that $\mathsf{Span}(\mathsf{MFMod_R})$ is contributing with are those in which ⊡ is a *split mono*, that means, when r is *invertible* in R. Therefore, the difference between (4.5) and (4.6) is that in the latter one is adding formal inverses ⊡ also for elements ⊡ which are not originally invertible in R. This explains the need of the field of fractions k of R in expanding the pushout object from $\mathsf{Corel}(\mathsf{FMod_R})$ (in (4.5)) to $\mathsf{Corel}(\mathsf{Vect_k}) \cong \mathsf{SV_k}$ (in (4.6)).

4.9. **Relations of Algebras over a Field.** We give a simple example of Proposition 3.5. The category $\mathsf{VECT_k}$ of vector spaces and linear maps over some field k, is a complete, cocomplete, regular category. Moreover, every epi splits: every surjective linear map has a section.

Consider the monad $T \colon \mathsf{VECT_k} \to \mathsf{VECT_k}$ for algebras over k (that is, for vector spaces equipped with a bilinear multiplication). This maps a vector space V to the direct sum $\bigoplus_{\{n \in \mathbb{N}\}} V^{\otimes n}$. Since each map $V \mapsto V \otimes \cdots \otimes V$ preserves both epis and pushouts of epis, so does their coproduct T. Note also that all epis in $\mathsf{VECT_k}$ are regular. This means both T and T^2 preserve pushouts of regular epis, and hence Proposition 3.5 applies. Thus we can construct the category of relations between algebras over k as a pushout of spans and cospans.

4.10. **Corelations of Coalgebras.** As mentioned, Proposition 3.8 dualises to yield a universal construction for corelations of coalgebras for a comonad T. The dualised assumptions require that T and T^2 preserve pullbacks of monos. This is true for instance when T is a polynomial endofunctor, as it is the case for comonads encountered in the setting of functional programming like the environment comonad $A \mapsto A \times E$, see [32]. Another

example is when T is the comonad defined by a geometric morphism to a coregular topos, such as Set.

An interesting class of case studies for this construction stems from the analysis of state-based systems, which are typically formalised as coalgebras $X \to FX$ for an endofunctor $F \colon \mathcal{C} \to \mathcal{C}$ [19]. When F is accessible and $\mathcal{C}$ is locally finitely presentable, one can construct the cofree comonad $T \colon \mathcal{C} \to \mathcal{C}$ over F through a so-called terminal sequence [33]. The comonad T yields an operational semantics for F-systems: given $X \to FX$, it maps X to the carrier Ω of the cofree F-coalgebra $\Omega \to F\Omega$ on X.

Now, because T is cofree, the condition of preserving pullbacks of monos required in Proposition 3.8 can actually be checked on F. Indeed, the image under T of a pullback diagram can be computed pointwise on its terminal sequence, which essentially involves repeated applications of the functor F. The condition is met for instance for coalgebras for the finite double-powerset functor $\mathcal{PP} \colon \mathsf{Set} \to \mathsf{Set}$: this functor has been used to model ground logic programs [22, 23, 8], with operational semantics given by the comonad mapping a formula to its execution tree in the logic program.

Given that bisimulation of coalgebras can be expressed both in terms of spans and cospans [31], it is an interesting question — to be explored in future work — whether corelations also model a sensible notion of behavioural equivalence, for well-chosen examples.

5. Concluding remarks

In summary, we have shown that categories of (co)relations may, under certain general conditions, be constructed as pushouts of categories of spans and cospans. In particular, especially since categories of spans and cospans can frequently be axiomatised using distributive laws, this offers a method of constructing axiomatisations of categories of (co)relations. Our results extend to the setting of props, and more generally symmetric monoidal categories. Moreover, these results are readily illustrated, unifying a diverse series of examples drawn from algebraic theories, program semantics, quantum computation, and control theory.

Looking forward, note that in the monoidal case the resulting (co)relation category is a so-named hypergraph category: each object is equipped with a special commutative Frobenius structure. Hypergraph categories are of increasing interest for modelling network-style diagrammatic languages, and recent work, such as that of decorated corelations [12] or the generalized relations of Marsden and Genovese [26], gives precise methods for tailoring constructions of these categories towards chosen applications. Our example on relations in categories of algebras for a monad (Subsection 4.9) hints at general methods for showing the present universal construction applies to these novel examples. We leave this as an avenue for future work.

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Appendix A. Proof of Theorem 3.1

We devote this section to give a step-by-step argument for Theorem 3.1.

Proposition A.1. *The square* $(\star)$ *commutes.*

Proof. As $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$ is a pushout, it is enough to show $(\star)$ commutes on the two injections of $\mathcal{A}$, $\mathcal{A}^{op}$ into $\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$. This means that we have to show, for any $f \colon a \to b$ in $\mathcal{A}$, that

$$\Pi(\xleftarrow{id}\xrightarrow{f}) = \Gamma(\xrightarrow{f}\xleftarrow{id}) \quad \text{and} \quad \Pi(\xleftarrow{f}\xrightarrow{id}) = \Gamma(\xrightarrow{id}\xleftarrow{f}).$$

These are symmetric, so it suffices to check one. This follows immediately from the fact that the pushout of $\xleftarrow{id}\xrightarrow{f}$ is $\xrightarrow{f}\xleftarrow{id}$. $\qquad\square$

Suppose we have a cocone over $\mathsf{Cospan}(\mathcal{C}) \longleftarrow \mathcal{A} +_{|\mathcal{A}|} \mathcal{A} \longrightarrow \mathsf{Span}(\mathcal{A})$. That is, suppose we have the commutative square:

$$
\begin{array}{ccc}
\mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op} & \longrightarrow & \mathsf{Span}(\mathcal{A}) \\
\downarrow & & \downarrow{\scriptstyle\Psi} \\
\mathsf{Cospan}(\mathcal{C}) & \xrightarrow[\Phi]{} & \mathcal{X}.
\end{array}
\qquad (\dagger)
$$

We prove two lemmas, from which the main theorem follows easily.

Lemma A.2. *If* $\xrightarrow{p}\xleftarrow{q}$ *is a cospan in* $\mathcal{A}$ *with pullback* $\xleftarrow{f}\xrightarrow{g}$ *(in* $\mathcal{C}$*), then* $\Psi(\xleftarrow{f}\xrightarrow{g}) = \Phi(\xrightarrow{p}\xleftarrow{q})$. *Similarly, if* $\xleftarrow{f}\xrightarrow{g}$ *is a span in* $\mathcal{A}$ *with pushout* $\xrightarrow{p}\xleftarrow{q}$ *(in* $\mathcal{C}$*), then* $\Psi(\xleftarrow{f}\xrightarrow{g}) = \Phi(\xrightarrow{p}\xleftarrow{q})$.

Proof. Consider $\xrightarrow{p}\xleftarrow{q} \in \mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$. Its image in $\mathcal{X}$ via the lower left corner of the commutative square $(\dagger)$ is $\Phi(\xrightarrow{p}\xleftarrow{q})$ while, recalling that $\xrightarrow{p}\xleftarrow{q} \in \mathcal{A} +_{|\mathcal{A}|} \mathcal{A}^{op}$ is mapped to $\xleftarrow{f}\xrightarrow{g}$ in $\mathsf{Span}(\mathcal{A})$, its image via the upper right corner is $\Psi(\xleftarrow{f}\xrightarrow{g})$. Thus $\Phi(\xrightarrow{p}\xleftarrow{q}) = \Psi(\xleftarrow{f}\xrightarrow{g})$.

The second claim is analogous, beginning instead with the span $\xleftarrow{f}\xrightarrow{g}$. $\qquad\square$

Lemma A.3. *If* $\xrightarrow{p_1}\xleftarrow{q_1}$ *and* $\xrightarrow{p_2}\xleftarrow{q_2}$ *are cospans in* $\mathcal{C}$ *such that* $\Gamma(\xrightarrow{p_1}\xleftarrow{q_1}) = \Gamma(\xrightarrow{p_2}\xleftarrow{q_2})$, *then* $\Phi(\xrightarrow{p_1}\xleftarrow{q_1}) = \Phi(\xrightarrow{p_2}\xleftarrow{q_2})$.

Proof. Suppose $\Gamma(\xrightarrow{p_1}\xleftarrow{q_1}) = \Gamma(\xrightarrow{p_2}\xleftarrow{q_2})$ as per hypothesis. Then by Proposition 2.1 there exists $\xrightarrow{m_1}, \xrightarrow{m_2} \in \mathcal{M}$ and $\xrightarrow{p}\xleftarrow{q} \in \mathsf{Cospan}(\mathcal{C})$ such that

$$\xrightarrow{p_1}\xleftarrow{q_1} = \xrightarrow{p}\xrightarrow{m_1}\xleftarrow{m_1}\xleftarrow{q} \quad \text{and} \quad \xrightarrow{p_2}\xleftarrow{q_2} = \xrightarrow{p}\xrightarrow{m_2}\xleftarrow{m_2}\xleftarrow{q}.$$

Then

$$
\begin{aligned}
\Phi(\xrightarrow{p_1}\xleftarrow{q_1}) \quad &= \quad \Phi(\xrightarrow{p}\xrightarrow{m_1}\xleftarrow{m_1}\xleftarrow{q}) \\
&= \quad \Phi(\xrightarrow{p}\xleftarrow{id}) \,;\, \Phi(\xrightarrow{m_1}\xleftarrow{m_1}) \,;\, \Phi(\xrightarrow{id}\xleftarrow{q}) \\
&\overset{(\clubsuit)}{=} \quad \Phi(\xrightarrow{p}\xleftarrow{id}) \,;\, \Psi(\xleftarrow{id}\xrightarrow{id}) \,;\, \Phi(\xrightarrow{id}\xleftarrow{q}) \\
&= \quad \Phi(\xrightarrow{p}\xleftarrow{id}) \,;\, \Phi(\xrightarrow{id}\xleftarrow{q}) \\
&= \quad \Phi(\xrightarrow{p}\xleftarrow{q}),
\end{aligned}
$$

and similarly for $\xrightarrow{p_2}\xleftarrow{q_2}$. The equality $(\clubsuit)$ holds because, by Assumption 1, $\mathcal{M} \subseteq \mathcal{A}$ and $\xrightarrow{m_1 \in \mathcal{M}}$ is mono, thus the pullback of $\xrightarrow{m_1}\xleftarrow{m_1}$ is $\xleftarrow{id}\xrightarrow{id}$ and via Lemma A.2 $\Phi(\xrightarrow{m_1}\xleftarrow{m_1}) = \Psi(\xleftarrow{id}\xrightarrow{id})$. $\qquad\square$

Proof of Theorem 3.1. Suppose we have a commutative diagram (†). It suffices to show that there exists a functor $\theta \colon \mathsf{Corel}(\mathcal{C}) \to \mathcal{X}$ with $\theta\Gamma = \Phi$ and $\theta\Pi = \Psi$. Uniqueness is automatic by fullness (Proposition 2.2) and bijectivity on objects of Γ.

Given a corelation a, fullness yields a cospan $\xrightarrow{f}\xleftarrow{g}$ such that $\Gamma(\xrightarrow{f}\xleftarrow{g}) = a$. We then define $\theta(a) = \Phi(\xrightarrow{f}\xleftarrow{g})$. This is well-defined by Lemma A.3.

For commutativity, clearly $\theta\Gamma = \Phi$. Moreover, $\theta\Pi = \Psi$: given a span $\xleftarrow{f}\xrightarrow{g}$ in $\mathcal{M}$, let $\xrightarrow{p}\xleftarrow{q}$ be its pushout span in $\mathcal{C}$. Thus by Lemma A.2,

$$\Psi(\xleftarrow{f}\xrightarrow{g}) = \Phi(\xrightarrow{p}\xleftarrow{q}) = \theta\Gamma(\xrightarrow{p}\xleftarrow{q}) = \theta\Pi(\xleftarrow{f}\xrightarrow{g}). \qquad \square$$

Appendix B. Proof of Proposition 3.5

To begin, we discuss how to put monoidal structures on $\mathsf{Cospan}(\mathcal{C})$, $\mathsf{Corel}(\mathcal{C})$, and $\mathsf{Span}(\mathcal{A})$, and show that Γ and Π are strict symmetric monoidal functors in this case.

Proposition B.1. *Let $(\mathcal{C}, \oplus)$ be a symmetric monoidal category with pullbacks, and let $\mathcal{A}$ be a sub-symmetric monoidal category of $\mathcal{C}$ containing all isomorphisms and stable under pullback. If $\oplus$ preserves pullbacks in $\mathcal{A}$, then $(\mathsf{Span}(\mathcal{A}), \oplus)$ is a symmetric monoidal category.*

Proof. Define $\oplus \colon \mathsf{Span}(\mathcal{A}) \times \mathsf{Span}(\mathcal{A}) \longrightarrow \mathsf{Span}(\mathcal{A})$ to take a pair of objects (X, X') in $\mathsf{Span}(\mathcal{A})$ to $X \oplus X'$, and similarly map a pair of spans to the component-wise monoidal product. This is well defined since $\mathcal{A}$ is closed under $\oplus$. We first need to show that this map is functorial. That is, given two pairs $(X \leftarrow N \to Y, X' \leftarrow N' \to Y')$ and $(Y \leftarrow M \to Z, Y' \leftarrow M' \to Z')$ of spans in $\mathcal{A}$, we need to show that the composite of their images under $\oplus$:

$$X \oplus X' \longleftarrow (N \oplus N') \times_{Y \oplus Y'} (M \oplus M') \longrightarrow Z \oplus Z'$$

is isomorphic to the image under $\oplus$ of their composite:

$$X \oplus X' \longleftarrow (N \times_Y M) \oplus (N' \times_{Y'} M') \longrightarrow Z \oplus Z'.$$

This is precisely the hypothesis that $\oplus$ preserves pullbacks in $\mathcal{A}$.

It then remains to show that we have coherence maps, and these obey the requisite equations. Note that $\mathcal{A}$ contains all isomorphisms in $\mathcal{C}$, and hence the coherence maps for $(\mathcal{C}, \oplus)$ can be considered as morphisms in $\mathsf{Span}(\mathcal{A})$. It is easy to check these give the coherence of $\mathsf{Span}(\mathcal{A})$. $\square$

Note that dualising the above argument with $\mathcal{A} = \mathcal{C}$ yields the fact that $(\mathsf{Cospan}(\mathcal{C}), \oplus)$ is a symmetric monoidal category whenever $\oplus$ preseves pushouts. Also note that the inclusions $\mathcal{A} \to \mathsf{Span}(\mathcal{A})$ and $\mathcal{A}^{op} \to \mathsf{Span}(\mathcal{A})$ are strict monoidal functors.

Proposition B.2. *If $\mathcal{C}$ is a symmetric monoidal category with a costable factorisation system, and $\mathcal{M}$ is closed under $\oplus$, then $\mathsf{Corel}(\mathcal{C})$ is a symmetric monoidal category. Moreover, the quotient functor $\Gamma \colon \mathsf{Cospan}(\mathcal{C}) \to \mathsf{Corel}(\mathcal{C})$ is a strict monoidal functor.*

Proof. The first task is to show that $\mathsf{Corel}(\mathcal{C})$ is indeed a symmetric monoidal category. We show that $\oplus$ induces a monoidal product, which we shall also write $\oplus$, on $\mathsf{Corel}(\mathcal{C})$. Given two corelations a and b, with representatives $\xrightarrow{f}\xleftarrow{g}$ and $\xrightarrow{h}\xleftarrow{k}$ we define their monoidal product $a \oplus b$ to be the corelation represented by the cospan $\xrightarrow{f \oplus h}\xleftarrow{g \oplus k}$. This is well defined: given

$\xrightarrow{f'} \xleftarrow{g'}, \xrightarrow{h'} \xleftarrow{k'}$ and m_1, m_2 in $\mathcal{M}$ such that $f' = f; m_1$, $g' = g; m_1$, $h' = h; m_2$, $k' = k; m_2$, the monoidality of $\oplus$ in $\mathcal{C}$ implies $f' \oplus g' = (f \oplus g); (m_1 \oplus m_2)$ and $h' \oplus k' = (h \oplus k); (m_1 \oplus m_2)$. Since $\mathcal{M}$ is closed under $\oplus$, $m_1 \oplus m_2$ again lies in $\oplus$, and the product corelation is independent of choice of representatives.

To show that Γ is a strict monoidal functor we just need to check $\Gamma(a \oplus b) = \Gamma a \oplus \Gamma b$, where a and b are cospans. This follows immediately from the definition: the monoidal product of the corelations that two cospans represent is by definition the corelation represented by the monoidal product of the two cospans. $\square$

Proposition B.3. $\Pi \colon \mathsf{Span}(\mathcal{A}) \to \mathsf{Corel}(\mathcal{C})$ *is a strict monoidal functor.*

Proof. Again, we just need to check that $\Pi(a \oplus b) = \Pi a \oplus \Pi b$. For this we need that the monoidal product preserves pushouts. Indeed, given spans $a = X \xleftarrow{f} N \xrightarrow{g} Y$ and $b = X' \xleftarrow{f'} N' \xrightarrow{g'} Y'$, we have $\Pi(a \oplus b)$ represented by the cospan

$$X \oplus X' \longrightarrow (X \oplus X') +_{(N \oplus N')} (Y \oplus Y') \longleftarrow Y \oplus Y',$$

and $\Pi a \oplus \Pi b$ represented by the cospan

$$X \oplus X' \longrightarrow (X +_N Y) \oplus (X' +_{N'} Y') \longleftarrow Y \oplus Y'.$$

These cospans are isomorphic by the fact $\oplus$ preserves pushouts, and hence represent the same corelation. $\square$

Now having described how to interpret ($\star$) in a category of symmetric monoidal categories, it remains to show that it is a pushout.

Proof of Proposition 3.5. From Theorem 3.1, we know that ($\star$) commutes, and that given some other cocone

$$
\begin{array}{ccc}
\mathcal{A} +_{|\mathcal{C}|} \mathcal{A}^{op} & \longrightarrow & \mathsf{Span}(\mathcal{A}) \\
\downarrow & & \downarrow{\scriptstyle \Psi} \\
\mathsf{Cospan}(\mathcal{C}) & \xrightarrow{\Phi} & \mathcal{X},
\end{array}
$$

there exists a unique functor θ from $\mathsf{Corel}(\mathcal{C})$ to $\mathcal{X}$. All we need do here is check that θ is a (lax, strong, strict) monoidal functor whenever Φ and Ψ are (lax, strong, strict) monoidal functors. In fact, whether θ is lax, strong, or strict depends only on whether Φ is lax, strong, or strict.

Indeed, suppose we have corelations $a \colon X \to Y$ and $b \colon X' \to Y'$, and write $\tilde{a}$ and $\tilde{b}$ for cospans that represent them. Recall that by definition $\theta a = \Phi \tilde{a}$. Let the coherence maps $\phi_{X,X'} \colon \Phi X \oplus \Phi X' \to \Phi(X \oplus X')$ of Φ also be the coherence maps of θ. Then the coherence of Φ states that

$$
\begin{array}{ccc}
\Phi X \oplus \Phi X' & \xrightarrow{\phi_{X,X'}} & \Phi(X \oplus X') \\
{\scriptstyle \theta a \oplus \theta b = \Phi \tilde{a} \oplus \Phi \tilde{b}} \downarrow & & \downarrow {\scriptstyle \theta(a \oplus b) = \Phi(\tilde{a} \oplus \tilde{b})} \\
\Phi Y \oplus \Phi Y' & \xrightarrow{\phi_{Y,Y'}} & \Phi(Y \oplus Y')
\end{array}
$$

commutes. This shows that θ is a monoidal functor of the same type as Φ, and hence proves the theorem. $\square$