MAXIMALLY SMOOTH DIRICHLET INTERPOLATION FROM COMPLETE AND INCOMPLETE SAMPLE POINTS ON THE UNIT CIRCLE

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ABSTRACT

This paper introduces a cost function for the smoothness of a continuous periodic function, of which only some samples are given. This cost function is important e.g. when associating samples in frequency bins for problems such as analytic singular or eigenvalue decompositions. We demonstrate the utility of the cost function, and study some of its complexity and conditioning issues.

Index Terms— Analytic functions; Dirichlet interpolation; approximation.

1. INTRODUCTION

For some problems, one particular, desired solution amongst a manifold of others may be defined by its analyticity. This is the case e.g. for the analytic eigenvalue decomposition (EVD) $\boldsymbol{A}(\omega) = \boldsymbol{Q}(\omega)\boldsymbol{\Lambda}(\omega)\boldsymbol{Q}^{\rm H}(\omega)$ of a self-adjoint matrix $\boldsymbol{A}(\omega) : \mathbb{R} \to \mathbb{C}^{M\times M}$, $\omega \in \mathbb{R}$, s.t. $\boldsymbol{A}(\omega) = \boldsymbol{A}^{\rm H}(\omega)$, which can be accomplished with analytic factors $\boldsymbol{Q}(\omega)$ and $\boldsymbol{\Lambda}(\omega)$ [1–3]. Similarly, a general matrix $\boldsymbol{B}(\omega) : \mathbb{R} \to \mathbb{C}^{M\times N}$ admits an analytic singular value decomposition (SVD) $\boldsymbol{B}(\omega) = \boldsymbol{U}(\omega)\boldsymbol{\Sigma}(\omega)\boldsymbol{V}^{\rm H}(\omega)$, with analytic unitary $\boldsymbol{U}(\omega)$ and $\boldsymbol{V}(\omega)$, and analytic and diagonal $\boldsymbol{\Sigma}(\omega)$ [4–8].

When moving from the dependence on a real-valued continuous variable $\omega \in \mathbb{R}$ to a complex valued $z \in \mathbb{C}$, then similar interest has arisen for a parahermitian matrix $\boldsymbol{R}(z)$, with the parahermitian conjugate $\boldsymbol{R}^{\rm P}(z) = \boldsymbol{R}^{\rm H}(1/z^*) = \boldsymbol{R}(z)$ [9]. An analytic parahermitian matrix $\boldsymbol{R}(z) : \mathbb{C} \to \mathbb{C}^{M\times M}$ admits a parahermitian matrix EVD $\boldsymbol{R}(z) = \boldsymbol{U}(z)\boldsymbol{\Gamma}(z)\boldsymbol{U}^{\rm P}(z)$ in almost all cases [10, 11], with analytic paraunitary and parahermitian diagonal factors $\boldsymbol{U}(z)$ and $\boldsymbol{\Gamma}(z)$, respectively. On the unit circle, the parameterisation $z = {\rm e}^{{\rm j}\Omega}$ leads to a self-adjoint matrix similar to the analytic EVD in [1], which however differs by a cyclic dependency on $\Omega \in \mathbb{R}$.

For iterative approximations of problems such as the above factorisations, often choices other than the analytic solution are possible. Algorithms such as sequential best rotation (SBR2, [12–14]) and sequential matrix diagonalisation (SMD, [15–17]) encourage or even guarantee [18] a spectrally majorised approximation [9] $\hat{\boldsymbol{\Gamma}}(z)$ of $\boldsymbol{\Gamma}(z)$, which will violate analyticity if eigenvalues cross. An example of an analytic versus a spectrally majorised solution for the functions $F_m({\rm e}^{{\rm j}\Omega})$, $m = 1, 2$, is given in Fig. 1. Analytic factors are attractive because they can be approximated by lower order polynomials compared to spectrally majorised ones, with a direct impact on the implementation cost for applications such as broadband beamforming [19–22], angle of arrival estimation [23, 24], or source separation [25].

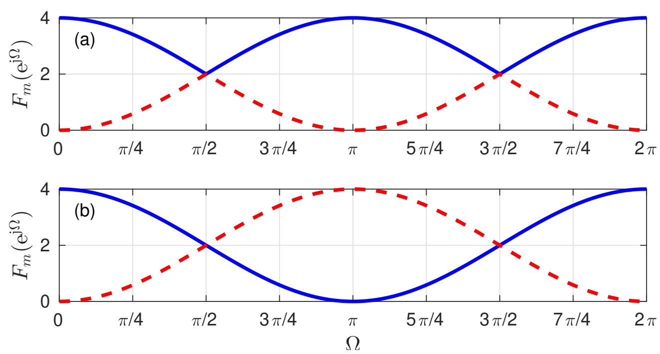

Fig. 1. Selection of (a) spectrally majorised vs (b) analytic functions.

To enforce analyticity over spectral majorisaton in the association across frequency requires a suitable cost function. An example for the arising challenge for the functions in Fig. 1 sampled uniformly at $N = 8$ points in Fig. 2(a) is shown in Figs. 2(b) and (c), where two interpolated curves are woven through a unique and complete assignment of all samples in every frequency bin. Since an analytic function is infinitely differentiable, a smoothness criterion appears to be a good approach to distinguish between the analytic solution and e.g. the spectrally majorised one.

As a metric for smoothness, in [5] the arc length of a singular value is considered, which is related to its first derivative. In [26–28], a lack of smoothness (and therefore of analyticity) of the eigenvalues translates into a discontinuity in the eigenvectors, which can form a cost function, even though this can lead to problems at or near an algebraic multiplicity of eigenvalues that is greater than one. We therefore choose to follow a different approach and directly evaluate a metric that relates to the analyticity of eigenvalues, which is used for the iterative extraction of analytic eigenvalues in [29].

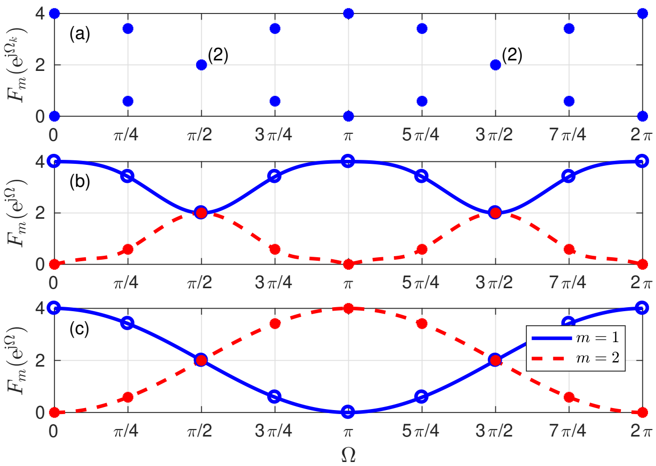

Fig. 2. (a) $N = 8$ sample points obtained from Fig. 1, with (b) spectrally majorised and (c) analytic interpolations.

Below, we investigate the formulation of a cost function based on the infinite differentiability of an analytic function by measuring the power in its derivatives, in order to drive a new set of DFT-based PhEVD algorithms [29].

2. DIRICHLET INTERPOLATION

2.1. Dirichlet Kernel

When describing a 2π-periodic function $F(e^{j\Omega})$ by N equispaced samples $F_k = F(e^{j\Omega_k})$, $\Omega_k = 2\pi k/N$ that have been obtained obeying the Nyquist theorem, the underlying interpolation function is the Dirichlet kernel or periodic sinc function, $P_N(e^{j\Omega})$. This kernel arises as the discrete-time Fourier transform $P_N(e^{j\Omega}) = \sum_n p_N[n]e^{-j\Omega n}$ (or short $P_N(e^{j\Omega}) \bullet\!\!-\!\!\circ\, p_N[n]$) of a rectangular window $p_N[n]$ of length N, s.t. $p_N[n] = 1$ for $n = -L_N \dots (N-L_N-1)$ and $p_N[n] = 0$ otherwise, whereby we define $L_N = (N-1)/2$ for N odd, and $L_N = N/2$ for N even.

For the Dirichlet kernel $P_N(e^{j\Omega})$, we have, dependent on N being odd or even:

$$P_N(e^{j\Omega}) = \begin{cases} \dfrac{\sin\left(\frac{N}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)} = \displaystyle\sum_{\ell=-L_N}^{N-L_N-1} e^{-j\Omega\ell} & N \text{ odd} \\ e^{-j\frac{\Omega}{2}} \dfrac{\sin\left(\frac{N}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)} = \displaystyle\sum_{\ell=-L_N}^{N-L_N-1} e^{-j\Omega\ell} & N \text{ even,} \end{cases} \quad (1)$$

where the latter expressions in each line are the Fourier series.

2.2. Interpolation

The kernel in (1) permits to express a 2π-periodic function $F(e^{j\Omega})$ as

$$F(e^{j\Omega}) = \frac{1}{N}\sum_{k=0}^{N-1} F_k P_N(e^{j(\Omega-\Omega_k)}) \quad (2)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} F_k \sum_{\ell=-L_N}^{N-L_N-1} e^{-j(\Omega-\Omega_k)\ell} \,. \quad (3)$$

The interpolated expression in (3) can be further simplified to

$$F(e^{j\Omega}) = \sum_{\ell=-L_N}^{N-L_N-1} \frac{1}{N}\sum_{k=0}^{N-1} F_k e^{j\Omega_k\ell} e^{-j\Omega\ell}$$

$$= \sum_{\ell=-L_N}^{N-L_N-1} a_\ell e^{-j\Omega\ell} \quad (4)$$

where a_l, $l = 0\dots(N-1)$ are the coefficients resulting from an N-point inverse discrete Fourier transform (DFT) of the sample points F_k, $k = 0\dots(N-1)$.

3. POWER OF DERIVATIVES BASED ON A COMPLETE SAMPLE SET

We first assume that a function $F(e^{j\Omega})$ is given by N equispaced sample $F(e^{j\Omega_k})$, $\Omega_k = 2\pi k/N$, $n = 0\dots(N-1)$ along the unit circle.

3.1. Power of Derivatives

As a criterion for smoothness, we are interested in the power of the pth derivative of $F(e^{j\Omega})$, which can be measured as

$$\chi_p = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \frac{d^p}{d\Omega^p} F(e^{j\Omega}) \right|^2 d\Omega \,, \quad (5)$$

and can provide a metric for the smoothness of $F(e^{j\Omega})$. Differentiating $F(e^{j\Omega})$ p times w.r.t. the frequency parameter Ω yields

$$\frac{d^p}{d\Omega^p} F(e^{j\Omega}) = \frac{1}{N}\sum_{k=0}^{N-1} F_k \frac{d^p}{d\Omega^p} P_N(e^{j(\Omega-\Omega_k)}) \quad (6)$$

$$= \sum_{\ell=-L_N}^{N-L_N1} (-j\ell)^p a_\ell e^{-j\Omega\ell} \quad (7)$$

using (4).

Note that due to orthogonality of the complex exponential terms and integration over an integer number of fundamental periods, for a Fourier series with some arbitrary coefficients b_ℓ,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \sum_l b_\ell e^{j\Omega\ell} \right|^2 d\Omega = \sum_\ell \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| b_\ell e^{j\Omega\ell} \right|^2 d\Omega = \sum_\ell |b_\ell|^2 \,.$$

Therefore, we can write

$$\chi_p = \sum_{\ell=-L_N}^{N-L_N-1} \left| (-j\ell)^p a_\ell \right|^2 = \sum_{\ell=-L_N}^{N-L_N-1} \ell^{2p} |a_\ell|^2 \,. \quad (8)$$

3.2. Matrix Formulation

We denote an N-point DFT matrix by $\mathbf{T}_N$, and assumed that it is normalised s.t. $\mathbf{T}_N\mathbf{T}_N^{\mathrm{H}} = \mathbf{I}$. With the samples points

F_k along the unit circle and their inverse DFT coefficients a_ℓ organised into vectors,

$$\mathbf{f} = [F_0,\ F_1,\ \ldots,\ F_{N-1}]^{\mathrm{T}} \tag{9}$$

$$\mathbf{a} = [a_0,\ a_1,\ \ldots,\ a_{N-1}]^{\mathrm{T}}\ , \tag{10}$$

they relate as $\mathbf{a} = \frac{1}{\sqrt{N}}\mathbf{T}_N^{\mathrm{H}}\mathbf{f}$. Further,

$$\mathbf{D} = \mathrm{diag}\{|-L_N|,\ \ldots,\ 1,\ 0,\ 1,\ \ldots\ (N-L_n-1)\}\ . \tag{11}$$

Therefore,

$$\chi_p = \mathbf{a}^{\mathrm{H}}\mathbf{D}^{2p}\mathbf{a} = \frac{1}{N}\mathbf{f}^{\mathrm{H}}\mathbf{T}_N\mathbf{D}^{2p}\mathbf{T}_N^{\mathrm{H}}\mathbf{f}\ . \tag{12}$$

If power is accumulated across several derivatives up to order P, then

$$\chi^{(P)} = \sum_{p=0}^{P}\chi_p = \frac{1}{N}\mathbf{f}^{\mathrm{H}}\mathbf{T}_N\sum_{p=0}^{P}\mathbf{D}^{2p}\mathbf{T}_N^{\mathrm{H}}\mathbf{f}\ . \tag{13}$$

This total power can therefore be measures as a weighted inner product of $\mathbf{f}$, $\mathbf{f}^{\mathrm{H}}\mathbf{C}\mathbf{f}$, with

$$\mathbf{C} = \frac{1}{N}\mathbf{T}_N\sum_{p=0}^{P}\mathbf{D}^{2p}\mathbf{T}_N^{\mathrm{H}}\ . \tag{14}$$

The matrix $\mathbf{D}^{2p}$ is positive semi-definite, real, and of rank $(N-1)$ for $p > 0$ by construction. The inclusion of $\mathbf{D}^0$ into (14) makes $\mathbf{C}$ full rank. With its eigenvalues $\frac{1}{N}\sum_{p=0}^{P}\mathbf{D}^{2p}$, it condition number is $\gamma = \sum_{p=0}^{P}(N-L_n-1)^2 p$, i.e. the matrix becomes ill-conditioned quickly as higher order derivatives are considered.

4. INCOMPLETE SAMPLE SET

If on a regular grid of N bins, not all sample points F_k, $k = 0\ldots(N-1)$ are available, the idea is to find a maximally smooth interpolation based on an optimum positioning of the missing sample points. Various approaches for this are suggested below.

4.1. Schur Complement

It is assumed that on a grid of N equispaced bins, only $K < N$ samples are given. W.l.o.g. these are assumed to be adjacent[1] and are contained in $\mathbf{g} \in \mathbb{C}^K$. We want to select the remaining $N-K$ coefficient such that maximum smoothness is attained for the interpolation. Packed into a vector $\mathbf{x} \in \mathbb{C}^{N-K}$, their optimum values can be found as

$$\mathbf{x}_{\mathrm{opt}} = \arg\min_{\mathbf{x}}\left[\mathbf{g}^{\mathrm{H}}\mathbf{x}^{\mathrm{H}}\right]\mathbf{C}\left[\begin{array}{c}\mathbf{g}\\ \mathbf{x}\end{array}\right]\ . \tag{15}$$

[1]Otherwise a permutation matrix can be defined, see [32].

With the partitioning of $\mathbf{C}$ into

$$\mathbf{C} = \left[\begin{array}{cc}\mathbf{C}_1 & \mathbf{C}_2^{\mathrm{H}}\\ \mathbf{C}_2 & \mathbf{C}_4\end{array}\right]\ , \tag{16}$$

where $\mathbf{C}_1 \in \mathbb{R}^{K\times K}$ and all other matrix dimensions as appropriate, we have

$$\mathbf{x}_{\mathrm{opt}} = -\mathbf{C}_4^{-1}\mathbf{C}_2\mathbf{g}\ . \tag{17}$$

The smoothness metric for the extended vector $[\mathbf{g}^{\mathrm{T}}\ \mathbf{x}_{\mathrm{opt}}^{\mathrm{T}}]^{\mathrm{T}}$ can be measured as

$$\chi = \mathbf{g}^{\mathrm{H}}\left(\mathbf{C}_1 - \mathbf{C}_2^{\mathrm{H}}\mathbf{C}_4^{-1}\mathbf{C}_2\right)\mathbf{g}\ . \tag{18}$$

4.2. Minimum Variance Distortionless Response

Based on the relation of the smoothness criterion in (12) to the Fourier coefficients in $\mathbf{a} \in \mathbb{C}^N$, we can formulate the constrained optimisation problem

$$\min_{\mathbf{a}}\mathbf{a}^{\mathrm{H}}\sum_{p=0}^{P}\mathbf{D}^{2p}\mathbf{a} \qquad \text{s.t.} \qquad \mathbf{T}_N^K\mathbf{a} = \mathbf{g}\ , \tag{19}$$

where $\mathbf{T}_N^K \in \mathbb{C}^{K\times N}$ containing the appropriate K rows of the N-point DFT $\mathbf{T}_N$. The constraint in (19) ensures that the obtained response interpolates through the sample points collected in $\mathbf{g} \in \mathbb{C}^K$.

The formulation (19) is similar to a minimum variance distortionless response problem, for which Lagrange optimisation leads to

$$\mathbf{a}_{\mathrm{opt}} = \mathbf{D}^{(P),\dagger}\mathbf{T}_N^{K,\mathrm{H}}\left(\mathbf{T}_N^K\mathbf{D}^{(P),\dagger}\mathbf{T}_N^{K,\mathrm{H}}\right)^{-1}\mathbf{g}\ . \tag{20}$$

Since we are not interested in the coefficients $\mathbf{a}_{\mathrm{opt}}$ but in the smoothness criterion χ, inserting (20) into (19) leads to

$$\chi = \mathbf{g}^{\mathrm{H}}\left(\mathbf{T}_N^K\mathbf{D}^{(P),\dagger}\mathbf{T}_N^{K,\mathrm{H}}\right)^{-1}\mathbf{g}\ . \tag{21}$$

5. IMPLEMENTATION AND RESULTS

5.1. Implementation Complexity

To calculate the metric χ for a full set of sample points in $\mathbf{f} \in \mathbb{C}^N$, it is possible to evaluate the matrix $C = \mathbf{T}_N\sum_{p=0}^{P}\mathbf{D}^{2p}\mathbf{T}_N^{\mathrm{H}}$. The smoothness metric is then given as a weighted inner product $\chi = \mathbf{f}^{\mathrm{H}}\mathbf{C}\mathbf{f}$. A computationally less expense alternative will evaluate $\mathbf{T}_N^{\mathrm{H}}\mathbf{f}$ as in inverse fast Fourier transform (IFFT) applied to $\mathbf{f}$ followed by a vector norm, which results in an overall complexity of

$$\mathcal{C}_{\text{full}} = N(2 + \log_2 N) \tag{22}$$

multiply accumulates (MACs).

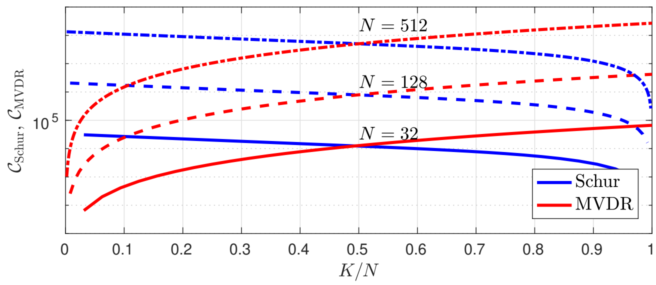

Fig. 3. Complexity of Schur vs MVDR approach.

For a reduced set of K out of possible N equispaced sample points, the Schur approach in (18) requires

$$\mathcal{C}_{\rm Schur} = (N-K)^3 + (N-K)^2 K + (N-K)K^2 \quad (23)$$

MAC operations to calculate the $K \times K$ matrix that weighs the inner product in (18). This assumes that the diagonal matrix $\sum_p \mathbf{D}^{2p}$ is precalculated or available via a look-up table. The major component in (23) is the inversion of $\mathbf{C}_4 \in \mathbb{C}^{(N-K)\times(N-K)}$, which is assumed to cost $(N-K)^3$ MACs.

For the MVDR method in (21), the cost for constructing the matrix that weighs the inner product is

$$\mathcal{C}_{\rm MVDR} = K^3 + K^2 N + KN \ , \quad (24)$$

where it is assumed that the inverse of the diagonal $\sum_p \mathbf{D}^{2p}$ is available, and that the inversion of the $K \times K$ matrix is accounted by K^3 MACs.

Due to the tabling of $\sum_p \mathbf{D}^{2p}$ and its inverse, the complexities in (23) and (24) are independent of the order p of the derivatives, or of the accumulation up to order P. For $N = \{32, 128, 512\}$, the computational costs $\mathcal{C}_{\rm Schur}$ and $\mathcal{C}_{\rm MVDR}$ are displayed for $K = 1 \ldots N$ in Fig. 3. While the MVDR approach inverts a small matrix for small K, the opposite is true for the Schur approach, where the dimension of the inverse decreases as $K \rightarrow N$. Therefore, the MVDR methods offers generally advantages for $K < N/2$, while for $K > N/2$ the Schur approach becomes preferable.

5.2. Conditioning

Sec. 3.2 stated the full-set matrix $\mathbf{C}$ as generally ill-conditioned for large N and P, even though no matrix inversion is required for the evaluation of the smoothness metric. For the evaluation of χ for an incomplete sample set however, both Schur and MVDR approaches involve matrix inversions. For the case of $P = 5$ and $N = 32$, Fig. 4 shows the condition numbers for the required matrix inversions. Generally, the larger the matrix to be inverted — in the Schur case for small K, in the MVDR case for large K — the worse the conditioning, thus generally necessitating regularisation [30].

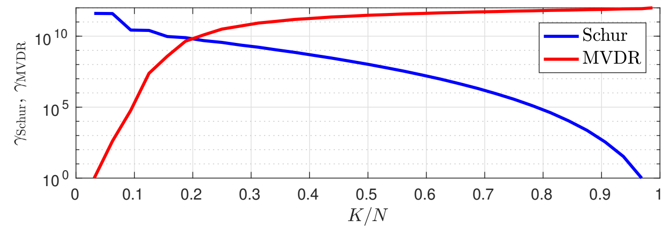

Fig. 4. Conditioning of Schur vs MVDR approach for $N = 32$, $P = 5$, with variable K.

p	$K=2$	$K=4$	$K=7$	$K=8$	σ_p^2
1	0.0270	0.3773	0.4886	0.5000	0.5000
2	0.0564	0.4606	0.4989	0.5000	0.5000
3	0.0689	0.4900	0.4999	0.5000	0.5000
4	0.0722	0.4977	0.5000	0.5000	0.5000
5	0.0730	0.4995	0.5000	0.5000	0.5000

Table 1. Results for χ_p for various derivatives p and different number of sample points K. The power of the pth derivative of $F(\mathrm{e}^{\mathrm{j}\Omega})$, σ_p^2 is shown for comparison.

5.3. Accuracy

A raised cosine $F(\mathrm{e}^{\mathrm{j}\Omega}) = 1 + \cos\Omega$, with every derivative having a power of $\sigma_p^2 = \frac{1}{2}$, is sampled at $N = 8$ equispaced bins. The first K of these sample points are used to evaluate the proposed smoothness metric, with results summarised in Tab. 1. For a sufficiently large K and p, the correct powers are attained. For lower values, an algorithm can find an interpolation with a smaller metric, as demonstrated for $\{N = 2, p = 4\}$ and $\{N = 4, p = 2\}$ in Fig. 5.

6. CONCLUSIONS

To measure the smoothness of a function associated with a full or limited set of sample points on the unit circle, this paper has suggested the power in the derivatives of the Dirichlet interpolation through the given points. For incomplete sets, also known as the 'missing samples problem' [31], a Schur complement and an MVDR approach have been suggested, whereby the latter works best for very sparse sets, while the former operates best on nearly complete sets. Cost, conditioning and the accuracy of the metric have been explored, with further evaluation and benchmarking left to [32].

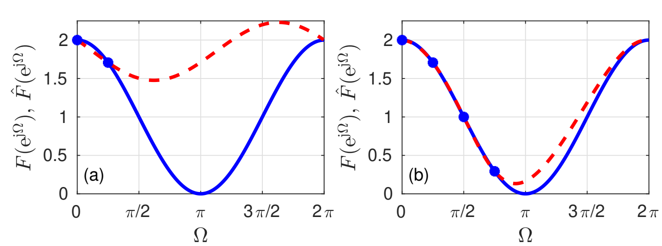

Fig. 5. Approximation of raised cosine for (a) $N = 2$, $p = 4$ and $N = 4$, $p = 2$.

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