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On k-11-representable graphs

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Abstract

Distinct letters x and y alternate in a word w if after deleting in w all letters but the copies of x and y we either obtain a word of the form $xyxy\cdots$ (of even or odd length) or a word of the form $yxyx\cdots$ (of even or odd length). A simple graph $G=(V,E)$ is word-representable if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if xy is an edge in E. Thus, edges of G are defined by avoiding the consecutive pattern 11 in a word representing G, that is, by avoiding xx and yy.

In 2017, Jeff Remmel introduced the notion of a k-11-representable graph for a non-negative integer k, which generalizes the notion of a word-representable graph. Under this representation, edges of G are defined by containing at most k occurrences of the consecutive pattern 11 in a word representing G. Thus, word-representable graphs are precisely 0-11-representable graphs. Our key result in this paper is showing that every graph is 2-11-representable by a concatenation of permutations, which is rather surprising taking into account that concatenation of permutations has limited power in the case of 0-11-representation. Also, we show that the class of word-representable graphs, studied intensively in the literature, is contained strictly in the class of 1-11-representable graphs. Another result that we prove is the fact that the class of interval graphs is precisely the class of 1-11-representable graphs that can be represented by uniform words containing two copies of each letter. This result can be compared with the known fact that the class of circle graphs is precisely the class of 0-11-representable

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graphs that can be represented by uniform words containing two copies of each letter.

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1 Introduction

The theory of word-representable graphs is a young but very promising research area. It was introduced by the fourth author in 2004 based on the joint research with Steven Seif [12] on the celebrated *Perkins semigroup*, which has played a central role in semigroup theory since 1960, particularly as a source of examples and counterexamples. However, the first systematic study of word-representable graphs was not undertaken until the appearance in 2008 of [11], which started the development of the theory.

Up to date, about 20 papers have been written on the subject, and the core of the book [10] is devoted to the theory of word-representable graphs. It should also be mentioned that the software packages [5, 17] are often of great help in dealing with word-representation of graphs. Moreover, a recent paper [8] offers a comprehensive introduction to the theory. Some motivation points to study these graphs are given in Section 1.

A simple graph $G = (V, E)$ is *word-representable* if and only if there exists a word w over the alphabet V such that letters x and y, $x \neq y$, alternate in w if and only if $xy \in E$. In other words, $xy \in E$ if and only if the subword of w induced by x and y avoids the *consecutive pattern* 11 (which is an occurrence of xx or yy). Such a word w is called G's *word-representant*. In this paper we assume V to be $[n] = \{1, 2, \dots, n\}$ for some $n \geq 1$. For example, the cycle graph on 4 vertices labeled by 1, 2, 3 and 4 in clockwise direction can be represented by the word 14213243. Note that a complete graph K_n can be represented by any permutation of $[n]$, while an edgeless graph (i.e. empty graph) on n vertices can be represented by $1122 \cdots nn$. Not all graphs are word-representable, and the minimum non-word-representable graph is the wheel graph W_5 in Figure 1, which is the only non-word-representable graph on six vertices [10, 11].

In 2017, Jeff Remmel [15] introduced the notion of a k-11-representable graph for a non-negative integer k, which generalizes the notion of a word-representable graph. Under this representation, edges of G are defined by containing at most k occurrences of the consecutive pattern 11 in a word representing G. Thus, word-representable graphs are precisely 0-11-representable graphs. The new definition allows to

- represent *any* graph; Theorem 5.2 shows that any graph is 2-11-representable by a concatenation of permutations, which is rather surprising taking into account that concatenation of permutations has limited power in the case of 0-11-representation (see Theorem 1.4). 2-11-representation could be compared with the possibility to *u-represent any* graph, where $u \in \{1, 2\}^*$ of length at least 3 [9]. We refer the Reader to [9] for the relevant definitions just mentioning that the case of $u = 11$ corresponds to word-representable graphs.

- 1-11-represent at least some of non-word-representable graphs including W_5 and all such graphs on seven vertices (see Section 4).

- give a new characterization of interval graphs; see Theorem 3.1, which should be compared with Theorem 1.3 characterizing circle graphs.

The paper is organized as follows. In the rest of the section, we give more details about word-representable graphs. In Section 2, we introduce rigorously the notion of a k-11-representable graph and provide a number of general results on these graphs. In particular, we show that a $(k-1)$-11-representable graph is necessarily k-11-representable (see Theorem 2.2). In Section 3, we study the class of 1-11-representable graphs. These studies are extended in Section 4, where we 1-11-represent all non-word-representable graphs on at most 7 vertices. In Section 5 we prove that any graph is 2-11-representable. Finally, in Section 6, we state a number of open problems on k-11-representable graphs.

Motivation points to study word-representable graphs include the fact exposed in [10] that these graphs generalize several important classes of graphs such as *circle graphs* [3], 3-*colourable graphs* and *comparability graphs* [14]. Relevance of word-representable graphs to scheduling problems was explained in [7] and it was based on [6]. Furthermore, the study of word-representable graphs is interesting from an algorithmic point of view as explained in [10]. For example, the *Maximum Clique problem* is polynomially solvable on word-representable graphs [10] while this problem is generally NP-complete [2]. Finally, word-representable graphs are an important class among other graph classes considered in the literature that are defined using words. Examples of other such classes of graphs are *polygon-circle graphs* [13] and *word-digraphs* [1].

The following two theorems are useful tools to study word-representable graphs. For the second theorem, we need the notion of a *cyclic shift* of a word. Let a word w be the concatenation uv of two non-empty words u and v. Then, the word vu is a cyclic shift of w.

Theorem 1.1 ([11]). *A graph is word-representable if and only if it can be represented uniformly, i.e. using the same number of copies of each letter.*

Theorem 1.2 ([11]). *Any cyclic shift of a word having the same number of copies of each letter represents the same graph.*

A *circle graph* is the intersection graph of a set of chords of a circle, i.e. it is an undirected graph whose vertices can be associated with chords of a circle such that two vertices are adjacent if and only if the corresponding chords cross each other. In this paper, we get used of the following theorem.

Theorem 1.3 ([7]). *The class of circle graphs is precisely the class of word-representable graphs that can be represented by uniform words containing two copies of each letter.*

An orientation of a graph is *transitive*, if the presence of the edges $u \to v$ and $v \to z$ implies the presence of the edge $u \to z$. An oriented graph G is a *comparability graph* if G admits a transitive orientation. A graph is *permutationally representable* if it can be represented by concatenation of permutations of (all) vertices. Thus, permutationally representable graphs are a subclass of word-representable graphs. The following theorem classifies these graphs.

Theorem 1.4 ([12]). *A graph is permutationally representable if and only if it is a comparability graph.*

2 Definitions and general results

A *factor* in a word $w_1w_2\dots w_n$ is a word $w_iw_{i+1}\dots w_j$ for $1 \le i \le j \le n$. For a letter or a word x, we let x^k denote $\underbrace{x\dots x}_{k \text{ times}}$. For any word w, we let $\pi(w)$ denote the *initial permutation* of w obtained by reading w from left to right and recording the leftmost occurrences of the letters in w. For example, if $w = 2535214421$ then $\pi(w) = 25314$. Similarly, the *final permutation* $\sigma(w)$ of w is obtained by reading w from right to left and recording the rightmost occurrences of w. For the w above, $\sigma(w) = 35421$. Also, for a word w, we let $r(w)$ denote the *reverse* of w, that is, w written in the reverse order. For example, if $w = 22431$ then $r(w) = 13422$. Finally, for a pair of letters x and y in a word w, we let $w|_{\{x,y\}}$ denote the word induced by the letters x and y. For example, for the word $w = 2535214421$, $w|_{\{2,5\}} = 25522$. The last definition can be extended in a straightforward way to defining $w|_S$ for a set of letters S. For example, for the same w, $w|_{\{1,2,3\}} = 232121$.

Throughout this paper, we denote by $G\backslash v$ the graph obtained from a graph G by deleting a vertex $v \in V(G)$ and all edges adjacent to it.

Let $k \ge 0$. A graph $G = (V, E)$ is k-11-*representable* if there exists a word w over the alphabet V such that the word $w|_{\{x,y\}}$ contains in total at most k occurrences of the factors in $\{xx, yy\}$ if and only if xy is an edge in E. Such a word w is called G's k-11-*representant*. A *uniform* (resp., t-*uniform*) representation of a graph G is a word, satisfying the required properties, in which each letter occurs the same (resp., t) number of times. As is stated above, in this paper we assume V to be $[n] = \{1, 2, \dots, n\}$ for some $n \ge 1$. Note that 0-11-representable graphs are precisely word-representable graphs, and that 0-11-representants are precisely word-representants. We also note that the "11" in "k-11-representable" refers to counting occurrences of the *consecutive pattern* 11 in the word induced by a pair of letters $\{x, y\}$, which is exactly the total number of occurrences of the factors in $\{xx, yy\}$. Throughout the paper, we normally omit the word "consecutive" in "consecutive pattern" for brevity. Finally, we let $\mathcal{G}^{(k)}$ denote the class of k-11-representable graphs.

Lemma 2.1. *Let $k \ge 0$ and a word w k-11-represent a graph G. Then the word $r(\pi(w))w$ $(k+1)$-11-represents G. Also, the word $wr(\sigma(w))$ $(k+1)$-11-represents G. Moreover, if $k = 0$ then the word ww 1-11-represents G.*

Proof. Suppose x and y are two vertices in G. If xy is an edge in G then $w|_{\{x,y\}}$ contains at most k occurrences of the pattern 11, so $(r(\pi(w))w)|_{\{x,y\}}$ (resp., $(wr(\sigma(w)))|_{\{x,y\}}$) contains at most $k+1$ occurrences of the pattern 11, and xy will be an edge in the new representation.

On the other hand, if xy is not an edge in G, then $w|_{\{x,y\}}$ contains at least $k+1$ occurrences of the pattern 11, so $(r(\pi(w))w)|_{\{x,y\}}$ (resp., $(wr(\sigma(w)))|_{\{x,y\}}$) contains at least $k+2$ occurrences of the pattern 11, and xy will not be an edge in the new representation.

Finally, if x and y alternate in w, then ww contains at most one occurrence of xx or yy, while non-alternation of x and y in w leads to at least two occurrence of the pattern 11 in ww, which involves x or/and y. These observations prove the last claim. □

Theorem 2.2. *We have $\mathcal{G}^{(k)} \subseteq \mathcal{G}^{(k+1)}$ for any $k \geq 0$.*

Proof. This is an immediate corollary of Lemma 2.1. □

Lemma 2.3. *Let $k \geq 0$, G be a k-11-representable graph, and i and j be vertices in G, possibly $i = j$. Then there are infinitely many words w k-representing G such that $w = iw'j$ for some words w'.*

Proof. Let u k-represent G. Then note that any word v of the form $\pi(u)\cdots\pi(u)u\sigma(u)\cdots\sigma(u)$ k-represents G. Deleting all letters to the left of the leftmost i in v, and all letters to the right of the rightmost j in v, we clearly do not change the number of occurrence of the pattern 11 for any pair of letters $\{x, y\}$. The obtained word w satisfies the required properties. □

There is a number of properties that is shared between word-representable graphs and k-11-representable graphs for any $k \geq 1$. These properties can be summarized as follows:

- The class $\mathcal{G}^{(k)}$ is hereditary. Indeed, if a word w k-11-represents a graph G, and v is a vertex in G, then clearly the word obtained from w by removing v k-11-represents the graph $G\backslash\{v\}$.
- In the study of k-11-representable graphs, we can assume that graphs in question are connected (see Theorem 2.4).
- In the study of k-11-representable graphs, we can assume that graphs in question have no vertices of degree 1 (see Theorem 2.5).
- In the study of k-11-representable graphs, we can assume that graphs in question have no two vertices having the same neighbourhoods up to removing these vertices, if they are connected (see Theorem 2.6).
- Glueing two k-11-representable graphs in a vertex gives a k-11-representable graph (see Theorem 2.7).
- Connecting two k-11-representable graphs by an edge gives a k-11-representable graph (see Theorem 2.8).

Theorem 2.4. *Let $k \geq 0$. A graph G is k-11-representable if and only if each connected component of G is k-11-representable.*

Proof. If G is k-11-representable then each of G's connected components is k-11-representable by the hereditary property of k-11-representable graphs.

Conversely, suppose that C_i's are the connected components of G for $1 \leq i \leq \ell$, and w_i k-11-represents C_i. Adjoining several copies of $\pi(w_i)$ to the left of w_i, if necessary, we can assume that each letter in any w_i occurs at least $k+2$ times. But then, the word $w = w_1 w_2 \cdots w_\ell$ k-11-represents G, since

- edges/non-edges in each C_i are represented by the w_i, and
- for $x \in C_i$ and $y \in C_j$, $i \neq j$, the word $w|_{\{x,y\}}$ contains at least $2k+2$ occurrences of the pattern 11 making x and y be disconnected in G,

we are done. □

Theorem 2.5. *Let $k \geq 0$, G be a graph with a vertex x, and G_{xy} be the graph obtained from G by adding to it a vertex y connected only to x. Then, G is k-11-representable if and only if G_{xy} is k-11-representable.*

Proof. The backward direction follows directly from the hereditary nature of k-11-representability. For the forward direction, suppose that w k-11-represents G. Adjoining several copies of $\pi(w)$ to the left of w, if necessary, we can assume that x occurs at least $2k+2$ times in w. Replacing every other occurrence of x in w, starting from the leftmost one, with yxy, we obtain a word w' that k-11-represents G_{xy}. Indeed, clearly, the letters x and y alternate in w' so xy is an edge in G_{xy} no matter what k is. On the other hand, if $z \neq x$ is a vertex in G, then $w'|_{\{z,y\}}$ has at least $k+1$ occurrences of the pattern 11 (formed by y's) ensuring that zy is not an edge in G_{xy}. Any other alternation of letters in w is the same as that in w'. □

Theorem 2.6. *Let $k \geq 0$ and G be a graph having two, possibly connected vertices, x and y, with the same neighbourhoods up to removing x and y. Then, G is k-11-representable if and only if $G \setminus x$ is k-11-representable.*

Proof. The forward direction follows directly from the hereditary nature of k-11-representability. For the backward direction, let w k-11-represent $G \setminus x$. If x and y are connected in G, then replacing each y by xy in w clearly gives a k-11-representant of G because x and y will have the same properties and they will be strictly alternating. On the other hand, if x and y are not connected in G, then adjoining several copies of $\pi(w)$ to the left of w, if necessary, we can assume that y occurs at least $k+2$ times in w. We then replace every even occurrence of y in w (from left to right) by yx, and every odd occurrence by xy. This will ensure that in the subword induced by x and y, the number of occurrences of the pattern 11 is at least $k+1$ making x and y be not connected in G. On the other hand, still x and y have the same alternating properties with respect to other letters. Thus, the obtained word k-11-represents G, as desired. □

Theorem 2.7. *Let $k \geq 0$, G_1 and G_2 be k-11-representable graphs, and the graph G is obtained from G_1 and G_2 by identifying a vertex x in G_1 with a vertex y in G_2. Then, G is k-11-representable.*

Proof. Let w_1 and w_2 be k-11-representants of the graphs G_1 and G_2, respectively. Recall that if a word w k-11-represents a graph H, then the word $w' = \pi(w)w$ obtained from w by adding the initial permutation $\pi(w)$ of w in front of w also k-11-represents H. Applying this observation, we may assume that the number of occurrences of x in the word w_1 equals to that of the letter y in the word w_2. In addition, by Lemma 2.3, we may further assume that w_1 starts with the letter x, and w_2 starts with the letter y. That is, $w_1 = xg_1xg_2 \dots xg_m$, where g_i's are words over $V(G_1) \setminus \{x\}$, and $w_2 = yh_1yh_2 \dots yh_m$, where h_i's are words over $V(G_2) \setminus \{y\}$. Let π_1 (resp., π_2) be the initial permutation of the word $g_1g_2 \dots g_m$ (resp., $h_1h_2 \dots h_m$). In other words, $\pi(w_1) = x\pi_1$ and $\pi(w_2) = y\pi_2$.

Let z be the vertex in G which corresponds to the vertices x and y, i.e. $z = x = y$ in G. We claim that the word $w(G) := (z\pi_1\pi_2z\pi_2\pi_1)^{k+1}zg_1h_1zg_2h_2 \dots zg_mh_m$ k-11-represents the graph G. The induced subword of $w(G)$ on $V(G_1)$ is precisely $\pi(w_1)^{2k+2}w_1$ which k-11-represents the graph G_1. Similarly, the induced subword of $w(G)$ on $V(G_2)$ k-11-represents the graph G_2. Now, consider $v_1 \neq x$ in $V(G_1)$ and $v_2 \neq y$ in $V(G_2)$. By the definition of G, the vertices v_1 and v_2 are not adjacent in G. Thus, it remains to show that the induced subword $w(G)|_{\{v_1,v_2\}}$ has at least $k+1$ occurrences of the pattern 11, which is easy to see from $(v_1v_2v_2v_1)^{k+1}$ being a factor of $w(G)|_{\{v_1,v_2\}}$. Therefore, the word $w(G)$ indeed k-11-represents the graph G. □

Theorem 2.8. *Let $k \geq 0$, G_1 and G_2 be k-11-representable graphs, and the graph G is obtained from G_1 and G_2 by connecting a vertex x in G_1 with a vertex y in G_2 by an edge. Then G is k-11-representable.*

Proof. Let w_1 and w_2 be k-11-representants of G_1 and G_2, respectively. By the same argument as in Theorem 2.7, we can assume that the number of occurrences of the letter x in the word w_1 equals that of the letter y in the word w_2. By Lemma 2.3, we can assume that w_1 begins with x, and w_2 ends with y. In addition, we can assume that the initial permutation of w_2 ends with y. Suppose the initial permutation of w_2 does not end with y, and let AyB be the initial permutation. It is clear that the word $w_2' = BAyBw_2$ also k-11-represents G_2, so that we can consider w_2' instead of w_2, and the initial permutation of w_2' ends with y.

Now we can write $w_1 = xg_1xg_2 \dots xg_m$, where g_i's are words over $V(G_1) \setminus \{x\}$, and $w_2 = h_1yh_2y \dots h_my$, where h_i's are words over $V(G_2) \setminus \{y\}$. Let π_1 (resp., π_2) be the initial permutation of the word $g_1g_2 \dots g_m$ (resp., $h_1h_2 \dots h_m$). Observe that $\pi(w_1) = x\pi_1$ and $\pi(w_2) = \pi_2y$. We claim that the word $w(G) := (x\pi_1\pi_2y\pi_2xy\pi_1)^{k+1}xg_1h_1yxg_2h_2y \dots xg_mh_my$ is a k-11-representant of G. As in Theorem 2.7, it is clear that the word $w(G)$ k-11-represents the graphs G_1 and G_2, when restricted to $V(G_1)$ and $V(G_2)$, respectively. Also, $w(G)$ makes the vertices x and y be adjacent, because $w(G)|_{\{x,y\}} = (xy)^{2k+m+2}$. Hence, it remains to show that for every $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that $v_1 \neq x$ or $v_2 \neq y$, which must be non-adjacent in G, the induced subword $w(G)|_{\{v_1,v_2\}}$ has at least $k+1$ occurrences of the pattern 11. This is obviously the case, because $w(G)|_{\{v_1,v_2\}}$ contains $(v_1v_2v_2v_1)^{k+1}$ having at least $2k+1$ occurrences of the pattern 11. Therefore, the word $w(G)$ k-11-represents the graph G. □

Theorem 2.9. *Let G be a graph with a vertex v. If $G \setminus v$ is k-uniform word-representable for $k \geq 1$, then G is $(k-1)$-11-representable.*

Proof. Let w be a k-uniform word that represents the graph $G \setminus v$. Let $N(v) \subset V(G)$ be the set of all neighbors of v in G, and let $N^c(v)$ be the complement of $N(v)$ in $V(G) \setminus \{v\}$, i.e. $N^c(v) = V(G) \setminus (N(v) \cup \{v\})$. We will describe how to construct a $(k-1)$-11-representant $w(G)$ of G from the word w. Recall that $r(\pi(w))$ is the reverse of the initial permutation $\pi(w)$ of the word w.

We start with the word $\pi(w)|_{N(v)} v \pi(w)|_{N^c(v)}\ w$, where $\pi(w)|_{N(v)}$ and $\pi(w)|_{N^c(v)}$ are the induced subwords of $\pi(w)$ on $N(v)$ and $N^c(v)$, respectively. In each step, we adjoin the words $r(\pi(w))v$ and $\pi(w)v$, in turn, from the left side of the word constructed in the previous step. We stop when the current word, denoted by $w(G)$, has exactly k v's. For example, the word $w(G)$, when $k = 6$, is given by

$$w(G) = r(\pi(w))v\ \pi(w)v\ r(\pi(w))v\ \pi(w)v\ r(\pi(w))v\ \pi(w)|_{N(v)} v \pi(w)|_{N^c(v)}\ w.$$

Next, we will show that the word $w(G)$ $(k-1)$-11-represents G. First, take a vertex $x \neq v$ in G. If $x \in N(v)$, then $w(G)|_{\{x,v\}} = xv \dots xv\ w|_{\{x\}}$ has $k-1$ occurrences of the pattern 11 since $w|_{\{x\}} = x^k$. If $x \in N^c(v)$, then $w(G)|_{\{x,v\}} = xv \dots xv\ vx\ w|_{\{x\}}$ has $k+1$ occurrences of the pattern 11. Thus $w(G)$ preserves all the (non-)adjacencies of v. Now, take two distinct vertices, y, z in $V(G) \setminus \{v\}$. Without loss of generality, we can assume that $\pi(v)|_{\{y,z\}} = yz$. If y and z are adjacent in $G \setminus v$, then $w|_{\{y,z\}} = yzyz \dots yz$. Hence, the induced subword

$$w(G)|_{\{y,z\}} = \dots zy\ yz\ zy\ (\pi(w)|_{N(v)} v \pi(w)|_{N^c(v)})|_{\{y,z\}}\ yzyz \dots yz$$

has $k-1$ occurrences of the pattern 11 since the part $\dots zy\ yz\ zy$ is of length $2(k-1)$, and $(\pi(w)|_{N(v)} v \pi(w)|_{N^c(v)})|_{\{y,z\}}$ is either yz or zy. If y and z are not adjacent in $G \setminus v$, then $w|_{\{y,z\}}$ has at least one occurrence of the pattern 11 and it starts with y. Hence, $w(G)|_{\{y,z\}} = \dots zy\ yz\ zy\ (\pi(w)|_{N(v)} v \pi(w)|_{N^c(v)})|_{\{y,z\}}\ w|_{\{y,z\}}$ has at least k occurrences of the pattern 11 since the only difference from the previous case is $w|_{\{y,z\}}$, which now has at least one occurrence of the pattern 11. This proves that $w(G)$ is a $(k-1)$-11-representant of G. □

Theorem 2.10. *For any non-negative integers m and k satisfying $2m - k - 1 > 0$, the following holds. Let G be a graph with a vertex v. If $G \setminus v$ is m-uniform k-11-representable, then G is $(3m-k-1)$-uniform $(2m-2)$-11-representable.*

Proof. Let w be an m-uniform k-11-representant of $G \setminus v$, $N(v) \subset V(G)$ be the set of all neighbors of v in G, and let $N^c(v) = V(G) \setminus (N(v) \cup \{v\})$. We will describe how to construct a $(3m-k-1)$-uniform $(2m-2)$-11-representant $w(G)$ of G from the word w. Similarly to the proof of Theorem 2.9, we start with the word $\pi(w)|_{N(v)} v \pi(w)|_{N^c(v)}\ w$, and in each step, we adjoin $r(\pi(v))v$ and $\pi(w)v$, in turn, from the left side until $w(G)$ has exactly $2m-k-1$ occurrences of v. Then, we adjoin v^m from the left side. For example, when $k = 3$ and $m = 4$, the word $w(G)$ is given by

$$w(G) = vvvv\ r(\pi(w))v\ \pi(w)v\ r(\pi(v))v\ \pi(w)|_{N(v)} v \pi(w)|_{N^c(v)}\ w.$$

It is easy to see that $w(G)$ is $(3m-k-1)$-uniform. Indeed, if $x \in V(G) \setminus \{v\}$, then $w(G)$ contains $(2m-k-1) + m = 3m-k-1$ x's since w is m-uniform; also, $w(G)$ contains $m + (2m-k-1) = 3m-k-1$ v's. Next, we will show that $w(G)$ $(2m-2)$-11-represents G.

Let $x \in V(G) \setminus \{v\}$. If $x \in N(v)$, then $w(G)|_{\{x,v\}} = v^m\ xv \dots xv\ x^m$. Thus $w(G)|_{\{x,v\}}$ has $2m-2$ occurrences of the pattern 11. If $x \in N^c(v)$, then the only difference from the previous case in $w(G)|_{\{x,v\}}$ is that $\pi(w)|_{N(v)}v\pi(w)|_{N^c(v)}$ is vx, not xv. Thus, $w(G)|_{\{x,v\}}$ has $2m$ occurrences of the pattern 11. Now take two distinct vertices $x, y \in V(G) \setminus \{v\}$. Without loss of generality, we can assume that $\pi(w)|_{\{x,y\}} = xy$. If x, y are adjacent in $G \setminus v$, then $w|_{\{x,y\}}$ has at most k occurrences of the pattern 11. Hence,

$$w(G)|_{\{x,y\}} = \dots yx\ xy\ yx\ (\pi(w)|_{N(v)}v\pi(w)|_{N^c(v)})|_{\{x,y\}}\ w|_{\{x,y\}}.$$

Since the length of $\dots yx\ xy\ yx$ is $4m-2k-4$ and $(\pi(w)|_{N(v)}v\pi(w)|_{N^c(v)})|_{\{x,y\}}$ is xy or yx, $w(G)|_{\{x,y\}}$ has at most $(2m-k-3)+1+k = 2m-2$ occurrences of the pattern 11. If x, y are not adjacent in $G \setminus v$, then $w|_{\{x,y\}}$ has at least $k+1$ occurrences of the pattern 11. In this case, the only difference from the previous case in $w(G)$ is $w|_{\{x,y\}}$ and so $w(G)|_{\{x,y\}}$ has at least $(2m-k-3)+1+k+1 = 2m-1$ occurrences of the pattern 11. This proves that $w(G)$ is a $(2m-2)$-11-representant of G. □

Corollary 2.11. *For any non-negative integers n and k satisfying $2n+k-7>0$, if each graph on n vertices is $(k+n-3)$-uniformly k-11-representable, then every graph on $n+1$ vertices is $(2k+3n-10)$-uniformly $(2k+2n-8)$-11-representable.*

Proof. This is a direct consequence of Theorem 2.10. Suppose every graph on n vertices is $(k+n-3)$-uniformly k-11-representable, and G is a graph on $n+1$ vertices. Clearly, $k+n-3$ is a positive integer since we have $2n+k-7>0$. Then for any vertex v in G, the graph $G \setminus v$ obtained from G by removing a vertex v is $(k+n-3)$-uniformly k-11-representable. Since $2(k+n-3)-k-1 = 2n+k-7 > 0$, we can apply Theorem 2.10, concluding that the graph G is $(2k+3n-10)$-uniform $(2k+2n-8)$-11-representable. □

In particular, Corollary 2.11 holds for any integers $n \geq 5$ and $k \geq 0$.

3 1-11-representable graphs

An *interval graph* has one vertex for each interval in a family of intervals, and an edge between every pair of vertices corresponding to intervals that intersect. Not all interval graphs are word-representable [10]. However, all interval graphs are 1-11-representable using two copies of each letter, as shown in the following theorem. This shows that the notion of an interval graph admits a natural generalization in terms of 1-11-representable graphs (instead of 2-uniform 1-11-representants, one can deal with m-uniform 1-11-representants for $m \geq 3$).

Theorem 3.1. *A graph is an interval graph if and only if it is 2-uniformly 1-11-representable.*

Proof. Let G be a 1-11-representable graph on n vertices and $w = w_1w_2\dots w_{2n}$ be a word that 2-uniformly 1-11-represents G. For any $v \in V(G) = [n]$, consider the interval $I_v = [v_1, v_2]$ on the real line such that $w_{v_1} = w_{v_2} = v$. Note that uv is an edge in G if and only if I_u and I_v overlap. But then, G is the interval graph given by the family of intervals $\{I_v : v \in [n]\}$.

To see that any interval graph G is necessarily 1-11-representable, we note a well-known easy to see fact that in the definition of an interval graph, one can assume that overlapping intervals overlap in more than one point. But then, the endpoints of an interval I_v will give the positions of the letter v in a word w constructed by recording relative positions of all the intervals. As above, one can see that such a w 1-11-represents G. □

Given a graph G with an edge xy, we let $G_{xy}^{\triangle}$ be the graph obtained from G by adding a vertex z connected only to the vertices x and y. Thus, $G_{xy}^{\triangle}$ is obtained from G by adding a triangle. If G is word-representable, that is, $G \in \mathcal{G}^{(0)}$, then $G_{xy}^{\triangle}$ is not necessarily word-representable. This can be seeing on the non-word-representable graph D_1 in Figure 2. Indeed, removing, for example, the top vertex in that graph, we obtain a word-representable graph, since the only non-word-representable graph on six vertices is the wheel W_5 [10, 11]. The following theorem establishes that adding a triangle is a safe operation in the case of 1-11-representable graphs.

Theorem 3.2. *Let $G \in \mathcal{G}^{(1)}$ and xy be an edge in G. Then $G_{xy}^{\triangle} \in \mathcal{G}^{(1)}$.*

Proof. Let w be an 1-11-representant of G. Note that, since x and y are adjacent in G, the letters x and y are either alternating in the word w, or $w|_{\{x,y\}}$ has exactly one occurrence of the pattern 11. In each case, we will construct a word $\tilde{w}$ over $V(G_{xy}^{\triangle})$, which 1-11-represents the graph $G_{xy}^{\triangle}$.

Case 1. Suppose that x and y are alternating in w. By Lemma 2.3, we can assume that w starts with x and ends with y, i.e. $w = x\ g_1\ y\ g_2 \dots x\ g_m\ y$, where g_i is a word on $V(G) \setminus \{x, y\}$. Also, we can assume that $m \geq 3$; if not, adjoin the initial permutation $\pi(w)$ to the left of w. Now, we claim that the word

$$\tilde{w} := zxz\ g_1\ y\ g_2\ x\ g_3\ zyz\ g_4\ x\ g_5\ yz\ g_6 \dots x\ g_m\ yz$$

1-11-represents the graph $G_{xy}^{\triangle}$, where $z \in V(G_{xy}^{\triangle}) \setminus V(G)$.

It is clear that $\tilde{w}$ respects the whole structure of G since the restriction of $\tilde{w}$ to $V(G)$ is w. Since $\tilde{w}|_{\{x,z\}} = zxzxzzxz \dots xz$ and $\tilde{w}|_{\{y,x\}} = zzyzyzyz \dots yz$, z is adjacent to x and y. On the other hand, for each $v \in V(G) \setminus \{x, y\}$, it is obvious that the induced subword $\tilde{w}|_{\{v,z\}}$ has at least two occurrences of the pattern 11, hence z is not adjacent to v. Therefore, $\tilde{w}$ 1-11-represents the graph $G_{xy}^{\triangle}$.

Case 2. Suppose $w|_{\{x,y\}}$ has exactly one occurrence of the pattern 11. Without loss of generality, we can assume that $w|_{\{x,y\}}$ contains the occurrence of the factor yy. By Lemma 2.3, we can also assume that w starts with x and ends with x, i.e.

$$w = x\ g_1\ y\ g_2 \dots x\ g_{m-1}\ y\ g_m\ y\ h_1\ x\ h_2 \dots y\ h_l\ x$$

for some positive integers m, l, and words g_i, h_j on $V(G) \setminus \{x, y\}$. We claim that the word

$$\tilde{w} := zxz\ g_1\ y\ g_2\ xz\ g_3\ y\ g_4 \dots xz\ g_{m-3}\ y\ g_{m-2}\ x\ g_{m-1}\ zyz\ g_m\ y\ h_1\ xz\ h_2 \dots y\ h_l\ xz$$

1-11-represents the graph $G^{\triangle}_{xy}$.

It is clear that $\tilde{w}$ respects the whole structure of G since the restriction of $\tilde{w}$ to $V(G)$ is w. Since $\tilde{w}|_{\{x,z\}} = zxzx\dots zxzzxz\dots xz$ and $\tilde{w}|_{\{y,z\}} = zzyzyz\dots yz$, z is adjacent to x and y. On the other hand, for each $v \in V(G)\backslash\{x,y\}$, the induced subword $\tilde{w}|_{\{v,z\}}$ has at least two occurrences of the pattern 11, hence z is not adjacent to v. Therefore, $\tilde{w}$ 1-11-represents the graph $G^{\triangle}_{xy}$.

□

For the next theorem, Theorem 3.3, recall the definition of a permutationally representable graph in Section 1. Note that the proof of Theorem 3.3 is similar to that of Theorem 2.9, while Theorem 3.3 deals with a stricter assumption. However, the stricter assumption is compensated by a stronger conclusion, justifying us having Theorem 2.9.

Theorem 3.3. *Let G be a graph with a vertex v. If $G \setminus v$ is permutationally representable (equivalently, by Theorem 1.4, if $G\setminus v$ is a comparability graph) then G is* 1-11*-representable.*

Proof. Let w be a 0-11-representant of $G\setminus v$. Since $G\setminus v$ is permutationally representable, we can assume that w is of the form $w = \pi_1\pi_2\dots\pi_k$ for some positive integer k and permutations π_i of $V(G\setminus v)$. Let $N(v)$ be the set of neighbours of v in G and let $N^c(v) := V(G)\setminus(N(v)\cup\{v\})$. We claim that the word

$$w(G) := r(\pi(w))\ v\ \pi(w)|_{N(v)}\ v\ \pi(w)|_{N^c(v)}\ \pi_1 v\pi_2 v\dots v\pi_k.$$

1-11-represents the graph G.

For each $x \in V(G)\backslash\{v\}$, if $x \in N(v)$ then the induced subword $w(G)|_{\{x,v\}} = xvxv\dots xvx$ is alternating, which should be the case. If $x \in N^c(v)$, then the induced subword $w(G)|_{\{x,v\}} = xvvxxvxv\dots xvx$ has two occurrences of the pattern 11, which, again, should be the case. Thus, $w(G)$ respects all adjacencies of the vertex v. Now, take $y,z \in V(G)\setminus\{v\}$. If y and z are adjacent in $G\setminus v$, then $w|_{\{y,z\}}$ has alternating y and z. Without loss of generality, assume that $w|_{\{y,z\}} = yzyz\dots yz$. Then, the induced subword $w(G)|_{\{y,z\}} = zy\ (\pi(w)|_{N(v)}\ \pi(w)|_{N^c(v)})|_{\{y,z\}}\ yzyz\dots yz$ has at most one occurrence of the pattern 11 as $(\pi(w)|_{N(v)}\ \pi(w)|_{N^c(v)})|_{\{y,z\}}$ is either yz or zy. If y and z are not adjacent in $G\setminus v$, then $w|_{\{y,z\}}$ is not alternating, i.e. it contains either yy or zz. Without loss of generality, assume that $w|_{\{y,z\}}$ contains yy. If $\pi(w)|_{\{y,z\}} = yz$, then with the assumption on an occurrence of yy, at least one occurrence of the factor zz is not avoidable in w, so at least two occurrences of the pattern 11 in $w(G)|_{\{y,z\}}$ are guaranteed. Otherwise, $w|_{\{y,z\}} = zy\dots zy\ yz\dots$. Then, $w(G)|_{\{y,z\}} = yz\ (\pi(w)|_{N(v)}\ \pi(w)|_{N^c(v)})|_{\{y,z\}}\ zy\dots zy\ yz\dots$ has two occurrences of the pattern 11, as $(\pi(w)|_{N(v)}\ \pi(w)|_{N^c(v)})|_{\{y,z\}}$ is either yz or zy. In any case, $w(G)$ preserves the (non-)adjacency of y and z. Therefore the word $w(G)$ 1-11-represents the graph G. □

Theorem 3.4. *Let G be a word-representable graph and e be an edge in G. Let $G\setminus e$ be the graph obtained from G by removing e. Then, $G\setminus e$ is* 1-11*-representable.*

Proof. Let $e = xy$ and w be G's uniform word-representant that exists by Theorem 1.1. Without loss of generality, we can assume that $w|_{\{x,y\}} = xyxy\dots xy$. We claim that the graph G' on $V(G)$, which is 1-11-represented by the word $w' := yxwwyx$, is precisely the graph $G \setminus e$.

It is clear that x and y are not adjacent in G' since $w'|_{\{x,y\}} = yxxy\dots xyyx$. Since the word ww is a 1-11-representant of G, it remains to show that for every vertex $z \in V(G) \setminus \{x, y\}$, and a vertex $i \in \{x, y\}$, G' contains the edge iz whenever iz is an edge in G. Suppose iz is an edge in G. Then, $ww|_{\{i,z\}}$ is either $iz\dots iz$, or $zi\dots zi$. It follows that $w'|_{\{i,z\}}$ is either $iiz\dots izi$, or $izi\dots zii$. Thus, iz is an edge in G'. If iz is not an edge in G, then $ww|_{\{i,z\}}$ will contain at least two occurrences of the pattern 11, so iz is not an edge in G'. This shows that $G' = G \setminus e$. □

The following two theorems generalize Theorem 3.4. The reason that we keep Theorem 3.4 as a separate result is that it is very useful in 1-11-representing 25 non-word-representable graphs (see Section 4).

Theorem 3.5. *Let G be a word-representable graph and K be a vertex subset in G. Let G_K be the graph obtained from G by removing the edges $\{xy \in E(G) : x, y \in K\}$. Then, G_K is* 1-11-*representable.*

Proof. Let w be a uniform word-representant of G that exists by Theorem 1.1. Let p be the reverse of the initial permutation of $w|_K$, and let q be the reverse of the final permutation of $w|_K$. Note that if K is a clique in G, then $p = q$. It is straightforward to check that the word $w' := pwwq$ 1-11-represents the graph G_K. □

Theorem 3.6. *Let G be a word-representable graph, v be a vertex in G, and N be a set of some (not necessarily all) neighbors of v in G. Let G_N be the graph obtained from G by removing the edges $\{uv : u \in N\}$. Then, G_N is* 1-11-*representable.*

Proof. Let $N = \{v_1, \dots, v_k\}$ and w be a uniform word-representant of G. Since w is uniform, by Theorem 1.2, we can assume that v is the first letter in w. Without loss of generality, assume that $v_1 \dots v_k$ is the initial permutation of $w|_N$. Then, it is easy to check that the word $w' := v_k \dots v_1 vwwv_k \dots v_1 v$ 1-11-represents the graph G_N. □

4 1-11-representing non-word-representable graphs

All graphs on at most five vertices are word-representable, and there is only one non-word-representable graph, the wheel W_5, on six vertices (see Figure 1). Also, there are 25 non-word-representable graphs on 7 vertices, which are shown in Figure 2.

The following theorem shows that the notion of k-11-representability allows us to enlarge the class of word-representable graphs ($\mathcal{G}^{(0)}$), still by using alternating properties of letters in words.

Theorem 4.1. *We have $\mathcal{G}^{(0)} \subsetneq \mathcal{G}^{(1)}$.*

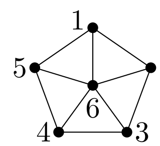

Figure 1: The wheel graph W_5

Proof. By Theorem 2.2, we have $\mathcal{G}^{(0)} \subseteq \mathcal{G}^{(1)}$. To show that the inclusion is strict, we give a word 1-11-representing the non-word-representable wheel graph W_5 in Figure 1. We start with 0-11-representing the cycle graph induced by all vertices but the vertex 6 by the 2-uniform word $w = 1521324354$. This word, and a generic approach to find it, is found on page 36 in [10]. Note that the initial permutation $\pi(w)$ is 15234, and thus, by Lemma 2.1, the word $r(\pi(w))w = 432511521324354$ 1-11-represents the cycle graph. Inserting a 6 in w to obtain $u = 4325161521324354$ gives a word 1-11-representing W_5 (which is easy to see). Note that the word $6u6$ gives a 3-uniform 1-11-representant of W_5. □

We do not know whether $\mathcal{G}^{(1)}$ coincides with the class of all graphs, but at least we can show that all 25 graphs in Figure 2 are 1-11-representable, which we do next. We will use the fact that all graphs on at most six vertices are 1-11-representable, which follows from the proof of Theorem 4.1, where we 1-11-represent the only non-word-representable graph on six vertices.

The graphs A_1 and A_5 are 1-11-representable by Theorem 2.5, since they have a vertex of degree 1. Theorem 2.6 can be applied to the graphs A_4, C_4 and C_5 since each of these graphs have a pair of vertices whose neighbourhoods are the same up to removing these vertices. Further, Theorem 3.2 gives 1-11-representability of the graphs A_6, A_7, B_5, D_1, D_4 and D_5 since each of these graphs has a triangle with a vertex of degree 2. Explicit easy-to-check 1-11-representants of the graphs A_2 and A_3 are, respectively, 437257161521324354 and 437251761521324354. For each graph G of the remaining 12 graphs in Figure 2, we provide vertices x and y connecting which by an edge results in a word-representable graph G_{xy}, so that Theorem 3.4 can be applied (removing the edge xy from G_{xy}) to see that G is 1-11-representable. The fact that G_{xy} is word-representable follows from it not being isomorphic to any of the graphs in Figure 2, where all non-word-representable graphs on seven vertices are presented. Alternatively, one can use the software packages [5, 17] to see that G_{xy} is word-representable (the software can produce an easy to check word representing G_{xy}).

5 All graphs are 2-11-representable

A simple graph $G = (V, E)$ is *permutationally k-11-representable* if there is a k-11-representant w of G which is a concatenation of permutations of V. Such a word w is called a *permutational k-11-representant* of G. In this section, we will prove that every graph is permutationally 2-11-representable, by an inductive construction of a 2-11-representant. This result

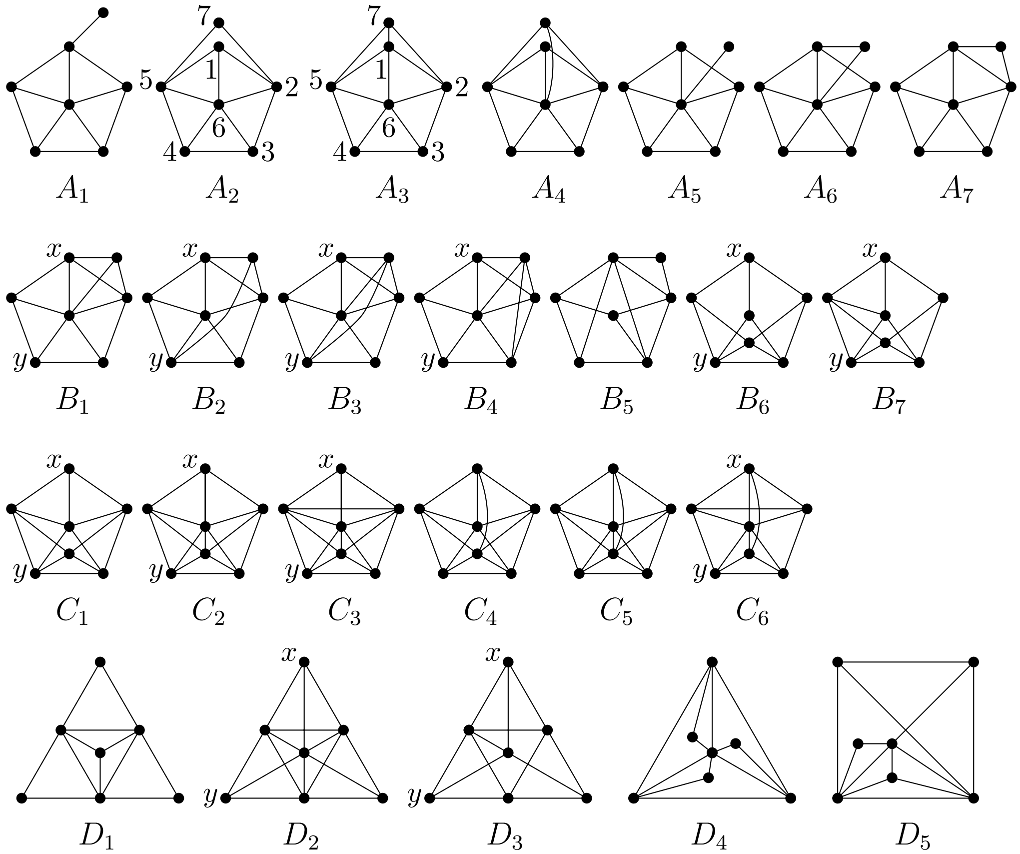

Figure 2: The 25 non-isomorphic non-word-representable graphs on 7 vertices

also implies that every graph is permutationally k-11-representable for any integer $k \geq 2$ by Lemma 2.1. We still do not know whether every graph is 1-11-representable or not.

We begin with a simple observation.

Observation 5.1. *If a word $w = w_1 P w_2$ k-11-represents a graph G where P is a permutation of $V(G)$, then the word $w' = w_1 P P w_2$ also k-11-represents the graph G.*

In the proof of the following theorem, we use the following notation. For a pair of vertices u and v in G, we write $u \sim v$ if u and v are adjacent in G, and $u \nsim v$ otherwise. Also, for convenience, we separate permutations in a permutational 2-11-representant by space.

Theorem 5.2. *Let $G = (V, E)$ be a graph on $n \geq 2$ vertices. Then there is a permutational 2-11-representant w over the alphabet V such that*

(a) w is a concatenation of at most $f(n) = n^2 - n + 2$ permutations of V, and

(b) for each $i \in V$, there exists a permutation P in w that starts with i, i.e. $P = iQ$ where Q is a permutation of $V \setminus \{i\}$.

Proof. We use induction on n. For the base case when $n = 2$, we take the 2-11-representant 12 21 12 12 of a complete graph on the vertex set $\{1, 2\}$, and we take the 2-11-representant 12 21 12 21 of two isolated vertices 1 and 2.

Suppose $n \geq 3$ and let G be a graph with the vertex set $V = [n]$. By the induction hypothesis, the induced subgraph $G \setminus n$ can be 2-11-represented by a word

$$P_1 P_2 \cdots P_{f(n-1)},$$

where each P_i is a permutation of $[n-1]$. Note that, by the condition (b), for each $i \in [n-1]$ we can choose one $k_i \in [f(n-1)]$ so that the permutation $P_{k_i} = iQ_i$ where Q_i is a permutation of $[n-1] \setminus \{i\}$.

Now we construct a 2-11-representant w of G satisfying the conditions (a) and (b). For each $i \in [f(n-1)]$, let

$$P'_i := \begin{cases} jnQ_j & \text{if } i = k_j \text{ for some } j \in [n-1] \text{ and } n \sim j \\ jnQ_j\ njQ_j\ jnQ_j & \text{if } i = k_j \text{ for some } j \in [n-1] \text{ and } n \nsim j \\ nP_i & \text{otherwise,} \end{cases}$$

and define

$$w = P'_1 \cdots P'_{f(n-1)}.$$

Note that all of nP_i, jnQ_j and njQ_j are permutations of $[n]$. Thus, the word w is a concatenation of at most $f(n-1) + 2(n-1) = n^2 - n + 2$ permutations of $[n]$ and the condition (b) obviously holds. It remains to show that w 2-11-represents the graph G.

By applying Observation 5.1 repeatedly, we observe that the subword of w induced by $[n-1]$ 2-11-represents the graph $G \setminus n$. Hence, it is sufficient to check whether the adjacency of the vertex n is preserved. For each $i \in [n-1]$, the subword of w induced by the letters i and n is given by

$$(ni)^a\ (in)\ (ni)^b \quad \text{if} \quad i \sim n,$$

having at most 2 occurrences of the consecutive pattern 11, and is given by

$$(ni)^a\ in\ ni\ in\ (ni)^b \quad \text{if} \quad i \nsim n$$

with at least 3 occurrences of the consecutive pattern 11. This completes the proof. □

Recall that, by Theorem 2.4, a graph G is 2-11-representable if and only if each connected component of G is 2-11-representable. It is obvious that every connected graph G on at least two vertices contains no isolated vertex, and that there always exists a vertex v in G such that $G \setminus v$ is again connected. Applying this observation, we can improve the function $f(n)$ in Theorem 5.2 for connected graphs as follows.

Theorem 5.3. *Let $G = (V, E)$ be a connected graph on $n \geq 2$ vertices. Then there is a 2-11-representant w over the alphabet V such that*

(a) w is a concatenation of $f(n) = n^2 - 3n + 4$ permutations of V, and

(b) for each $i \in V$, there is a permutation P in the word w which starts with i.

6 Open problems on k-11-representable graphs

The most intriguing open question in the theory of k-11-representable graphs is the following.

Problem 1. Is it true that any graph is 1-11-representable? If not, then which classes of graphs are 1-11-representable? In particular, are all planar graphs or all 4-chromatic graphs 1-11-representable?

Note that W_5 shows that not all planar or 4-chromatic graphs are word-representable, justifying our specific interest to these classes of graphs.

By Theorem 1.1, any word-representable graph can be represented by a uniform word. It is known that the class of permutationally word-representable graphs coincides with the class of comparability graphs [11]. Also, by Theorem 5.2 and Lemma 2.1, for every $k \geq 2$ any graph is permutationally k-11-representable. Thus, the following questions are natural.

Problem 2. Is it true that any 1-11-representable graph can be represented by a concatenation of permutations? Or, at least, by a uniform word?

It is known [7] that if a graph G with n vertices is word-representable, then it can be represented by a uniform word of length at most $2n(n-\kappa)$ where κ is the size of a maximum clique in G. An upper bound for the length of k-11-representants for $k \geq 2$ can be derived from Theorems 5.2 and 5.3 and Lemma 2.1. In particular, 2-11-representants are of length $O(n^3)$. However, we have no upper bounds for the length of words 1-11-representing graphs.

Problem 3. Provide an upper bound for the length of words 1-11-representing graphs.

Remind that Theorem 3.1 shows that the class of interval graphs is precisely the class of 1-11-representable graphs that can be represented 2-uniformly.

Problem 4. Does the class of m-uniformly 1-11-representable graphs, for $m \geq 3$, have any interesting/useful properties? In particular, is there a description of such graphs in terms of forbidden subgraphs? A good starting point to answer the last question should be the case of $m = 3$.

As the first step in the direction of Problem 4, we discuss a geometric realization of r-uniformly k-11-representable graphs, which might give new results on characterization problems for word-representable graphs. This is motivated from the fact that a graph is 2-uniformly 0-11-representable if and only if it is a circle graph.

Take a convex curve $\gamma = \gamma(t)$, $t \in [0,1]$, in the plane (not necessarily closed) and consider a set of $n \times r$ distinct real numbers $S = \{x_1, x_2, \dots, x_{nr}\}$ such that

$$0 \leq x_1 < x_2 < \cdots < x_{nr} \leq 1.$$

Color each element in S by $[n]$, i.e. we choose an injection $\phi : S \to [n]$, such that $|S_i| = r$ where $S_i := \{x \in S : \phi(x) = i\}$. Say $S_i := \{x_{i_1} x_{i_2}, \dots, x_{i_r}\}$ where $x_{i_j} < x_{i_k}$ whenever $j < k$. Now we draw n piecewise linear convex curves

$$C_i : \ \gamma(x_{i_1}) - \gamma(x_{i_2}) - \cdots - \gamma(x_{i_r}), \ \ i \in [n]$$

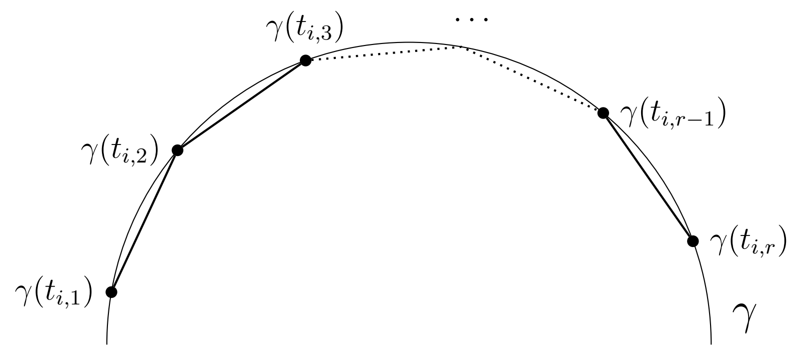

Figure 3: The piecewise linear convex curve C_i

connecting the points $\gamma(x_{i_1}), \ldots, \gamma(x_{i_r})$. See Figure 3 for an illustration.

Note that for every two distinct C_i and C_j, we have $|C_i \cap C_j| \leq 2r - 3$. For each $m \in [2r-3]$, we define *m-intersection graph* $I_m(\mathcal{C}\phi)$ of $\mathcal{C}_\phi = \{C_1, \ldots, C_n\}$ as the graph on $[n]$ such that two distinct vertices $i, j \in [n]$ are adjacent if and only if $|C_i \cap C_j| \geq m$. In particular, when $m = 1$, the graph $I_1(\mathcal{C}\phi)$ is just the intersection graph of $\mathcal{C}$.

On the other hand, regarding $[n]$ as an alphabet, we construct a word over $[n]$:

$$w = \phi(x_1)\phi(x_2)\ldots\phi(x_{nr}).$$

Then the subword of w induced by two distinct letters i and j has at most $2r - 3 - m$ occurences of the consecutive pattern 11 if and only if $i \sim j$ (i.e. ij is an edge) in the graph $I_m(\mathcal{C})$. As an immediate consequence of this relation, we observe the following.

Proposition 6.1. *For every positive integer m and $r \geq 2$ such that $1 \leq m \leq 2r-3$, a graph G is r-uniform $(2r-3-m)$-11-representable if and only if there exists a coloring $\phi : S \to [n]$ so that $G = I_m(\mathcal{C}_\phi)$.*

Proof. By the above argument, it is sufficent to prove that every r-uniform $(2r-3-m)$-11-representable graph G assigns a coloring $\phi : S \to [n]$ so that $G = I_m(\mathcal{C}_\phi)$. This is obvious since an r-uniform $(2r-3-m)$-11-representant of G naturally gives a coloring $\phi : S \to [n]$ with $|S_i| = r$, and the corresponding family $\mathcal{C}_\phi$ of n piecewise linear convex curves satisfies that $I_m(\mathcal{C}_\phi) = G$. □

Note that for every $k > 2r-3$, the only r-uniform k-11-representable graphs are complete graphs. When $m = 0$, the r-unifrom $(2r-3)$-11-representable graphs can be specified as a well-known graph class.

Proposition 6.2. *A graph is r-uniform $(2r-3)$-11-representable if and only if it is an interval graph.*

Proof. Given an r-unifrom $(2r-3)$-11-representable graph G on $[n]$, take any r-unifrom $(2r-3)$-11-representant w. Clearly two vertices i and j are not adjacent in G if and only if the subword of w induced by i and j consists of r consecutive i's and r consecutive

j's, i.e. either $i \dots i\ j \dots j$ or $j \dots j\ i \dots i$. Embed the word w on a line, and consider an interval J_i defined by the leftmost i and the rightmost i. Then G is the intersection graph of $\{J_1, \dots, J_n\}$.

For the other direction, let G be the intersection graph of intervals $\{J_1, \dots, J_n\}$. Since n is finite, we may assume that each J_i is bounded and no two intervals share an end-point. We label the end-points of J_i by i, and construct a 2-uniform word by reading the labels from left to right. Then we insert $r - 2$ i's in arbitrary positions between two original i's in the word. This gives us a r-unifrom $(2r - 3)$-11-representant of G. □

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