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An Expanded Mixed Finite Element Method for Two-dimensional Sobolev Equations ⋆

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Abstract

In this paper we provide an expanded mixed finite element method for a class of two-dimensional Sobolev equations. The optimal error estimates for a semi-discrete scheme and a fully discrete scheme are obtained. Also numerical examples are stated to verify our theoretical results.

Keywords: Sobolev Equation; Expanded Mixed Finite Element; Error Estimate; Numerical Experiment

1 Introduction

Let $\Omega \subset R^2$ be a bounded domain with piecewise smooth boundary $\partial\Omega$. For fixed $0 < T < \infty$, we consider the following initial and boundary value problem:

$$\begin{cases} u_t - \mu\Delta u_t - \gamma\Delta u = f, & (x,y,t) \in \Omega \times (0,T], \\ u(x,y,t) = 0, & (x,y,t) \in \partial\Omega \times (0,T], \\ u(x,y,0) = \varphi_0(x,y), & (x,y) \in \Omega, \end{cases} \tag{1.1}$$

where u_t denotes the time derivative of the function u, μ and γ are two positive constants, the source term $f(x,y,t)$ and the initial value function $\varphi_0(x,y)$ are sufficiently smooth.

The above equation is characterized by the occurrence of mixed time and space derivatives appearing in the highest-order term, which we call Sobolev equation. As a class of important Sobolev equations, Eq.(1.1) possesses the important physical background. In 1960, Eq.(1.1) appeared in the theory developed by Barenblett, Zheltov and Kochia for flow through fissured

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rock [1], where u represents the average pressure of the liquid in the fissures in the neighborhood of the given point. Its one-dimensional counterpart $u_t = u_{xx} + au_{txx}$ was derived by Coleman and Noll as governing the simple shearing motion of a fluid of a second grade[2]. Subsequently, when Chen and Gurtin studied the heat conduction problems in different viscous media, they came to the conclusion that Eq.(1.1) related to be the simplified energy equation, and u denotes the conductive temperature[3]. Based on that concept, Ting performed further mathematical studies of the theory by considering a simple cooling process and thereby derived physically significant results[4]. The more and more extensive applications of this kind of equation in mathematics and physics have attracted the attention of many scholars. So far, a few different schemes have been proposed for the numerical solution of this type of equations([5, 6, 7, 8]).

The mixed finite element method, basically a finite element method [9] with constrained conditions, plays an important role in studying the numerical solution of higher order partial differential equations(PDEs) or PDEs including two (or more) unknown functions. Its general theory was proposed by Babuska [10] and Brezzi [11] in the early 1970s, and later on improved and incorporated with adaptivity by Falk and Osborn[12]. The mixed finite element method has now been widely used for solving fluid flow and transport equations[13, 14], for example, when the governing equations describing two-phase flow within a petroleum reservoir are expressed as a fractional flow formulation (i.e., in terms of a global pressure and a saturation), mixed methods can be very efficient and accurate in solving the pressure equation. However, mixed finite element methods have not yet applied to groundwater hydrology, for underground reservoirs often need to specify complex boundary conditions involving combinations of individual fluid fluxes and pressures, and it is sometimes impractical to express them in terms of the total quantifies. Consequently, two-pressure formulations are commonly used by hydrologists, since the complex individual boundary conditions can easily be handled. However, the coefficient in the two-pressure formulation may tend to zero because of low permeability, so that its reciprocal is not readily usable as in standard mixed finite element methods. Therefore, it is not feasible to apply mixed methods to a two-pressure formulation.

The expanded mixed finite element(EMFE) method[15, 16, 17], a new formulation expanding the standard mixed formulation, introduces three (or more) auxiliary variables for practical problems. This method was developed and analyzed by Arbogast, Wheeler and Yotov ([18]). Chen([19, 20]) detailed the EMFE method for linear and quasi-linear second-order elliptic problems. Woodward and Dawson([21]) proposed an analysis of EMFE method applied to Richards' equation, which simulates the flow of the water into a variably saturated porous medium. EMFE method can be applied to the above two-pressure formulation, so that individual boundary conditions can be handled. In addition, this method works for small diffusion or low permeability fluid problems. Using this method, we can get optimal order error estimates for certain nonlinear problems, while standard mixed formulation sometimes gives only suboptimal error estimates([22]). While the papers mentioned above are the introduction of two variables, in this paper we will introduce three parameter variables to get better approximation of more physical quantities.

The purpose of this paper is to present the expanded mixed finite element method for the 2D Sobolev equation. We show the existence and uniqueness of the solution of the mixed finite element method, and obtain its optimal error estimates. The outline of this paper is as follows. In Sect.2, we describe a semi-discrete formulation for Eq. (1.1) and provide the error estimates for the solution of the formulation. The optimal error estimate is obtained. In Sect.3, we present a fully discrete formulation from the semi-discrete formulation and analyze its error. In Sect.4, we provide two numerical examples to verify that numerical results are consistent with theoretical conclusions.

Throughout this paper we use C to denote a generic positive constant independent of the discretization parameters h and t unless otherwise stated, which has different values in different appearances. We also adopt the standard definitions and notations of Sobolev spaces and their norms in [9] and [23].

2 Semi-discrete Scheme Based on EMFE Method

2.1 Semi-discrete Scheme and Existence and Uniqueness of Solution

By introducing three auxiliary variables

$$\mathbf{q} = \nabla u, \quad w = u_t, \quad \mathbf{p} = \nabla w,$$

the problem (1.1) can be rewritten into the following equivalent first-order differential system with regard to the solution $\{u, \mathbf{q}, w, \mathbf{p}\}$. It reads that, $\forall (x, y, t) \in \Omega \times (0, T]$

$$u_t = w, \quad \mathbf{q}_t = \mathbf{p}, \quad w - \mu \mathrm{div}(\mathbf{p}) = f + \gamma \mathrm{div}(\mathbf{q}), \quad \mathbf{p} - \nabla w = 0, \tag{2.1}$$

with boundary value

$$u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T],$$

and initial value

$$u(x, y, 0) = \varphi_0(x, y), \qquad \mathbf{q}(x, y, 0) = \nabla\varphi_0(x, y), \quad (x, y) \in \Omega.$$

Let

$$W = H(\mathrm{div}, \Omega) = \{\tau | \tau \in (L^2(\Omega))^2, \mathrm{div}\tau \in L^2(\Omega)\}$$

normed by $\|\tau\|_W = \|\tau\| + \|\mathrm{div}\tau\|$, and let $V = L^2(\Omega)$.

Note that $u(x, y, t)|_{\partial\Omega} = 0$ implies $w(x, y, t)|_{\partial\Omega} = u_t(x, y, t)|_{\partial\Omega} = 0$ and integrating by parts, we obtain the weak form of (2.1): find a map $\{u, \mathbf{q}, w, \mathbf{p}\} : [0, T] \to V \times W \times V \times W$ such that

$$\begin{cases} (u_t, v) = (w, v), & \forall v \in V, 0 < t \le T, \\ (\mathbf{q}_t, \tau) = (\mathbf{p}, \tau), & \forall \tau \in W, 0 < t \le T, \\ (w, v) - \mu(\mathrm{div}\mathbf{p}, v) = (f, v) + \gamma(\mathrm{div}\mathbf{q}, v), & \forall v \in V, 0 < t \le T, \\ (\mathbf{p}, \tau) + (w, \mathrm{div}\tau) = 0, & \forall \tau \in W, 0 < t \le T. \end{cases} \tag{2.2}$$

where v, τ can take different values in different equations.

In order to clarify a proper finite element approximation procedure for $\{u, \mathbf{p}, w, \mathbf{q}\}$, we consider the finite-dimensional subspace $V_h \times W_h \times V_h \times W_h$ of $V \times W \times V \times W$ associated with a quasi-uniform partition τ_h of Ω into triangles, where the diameter of τ_h is not greater than $h(0 < h < 1)$, and every angle of each triangle is bounded below by a positive constant. The boundary elements of τ_h are allowed to have one curvilinear edge. We choose $V_h \times W_h \times V_h \times W_h$ as the Raviart-Thomas-Nedelec space [24, 25, 26] of index $k \ge 0$ and introduce the L^2-projection $R_h : V \to V_h$, and the Raviart-Thomas projection[26] $\pi_h : H^1(\Omega)^2 \to W_h$, which have the following useful commuting property:

$$\mathrm{div} \circ \pi_h = R_h \circ \mathrm{div} : H^1(\Omega)^2 \to V_h.$$

These projections have the following approximation properties([27, 28]):

$$\|v - R_h v\|_{-s} \le Ch^{l+s}\|v\|_l, \qquad 0 \le l, s \le k+1, \tag{2.3}$$
$$\|v - R_h v\|_{0,q} \le Ch^{l}\|v\|_{l,q}, \qquad 0 \le l \le k+1, 1 \le q \le \infty, \tag{2.4}$$
$$\|\tau - \pi_h \tau\|_{0,q} \le Ch^{l}\|\tau\|_{l,q}, \qquad \frac{1}{q} < l \le k+1, 1 \le q \le \infty, \tag{2.5}$$
$$\|\mathrm{div}(\tau - \pi_h \tau)\| \le Ch^{l}\|\mathrm{div}\tau\|_{l}, \qquad 0 \le l \le k+1. \tag{2.6}$$

Our semi-discrete scheme, that is, the continuous-in-time mixed finite element approximation to (2.2) is defined by determining $\{u_h, \mathbf{q}_h, w_h, \mathbf{p}_h\} : [0,T] \to V_h \times W_h \times V_h \times W_h$ such that

$$\begin{cases} (u_{h,t}, v_h) = (w_h, v_h), & \forall v_h \in V_h, 0 < t \le T, \\ (\mathbf{q}_{h,t}, \tau_h) = (\mathbf{p}_h, \tau_h), & \forall \tau_h \in W_h, 0 < t \le T, \\ (w_h, v_h) - \mu(\mathrm{div}\mathbf{p}_h, v_h) = (f, v_h) + \gamma(\mathrm{div}\mathbf{q}_h, v_h), & \forall v_h \in V_h, 0 < t \le T, \\ (\mathbf{p}_h, \tau_h) + (w_h, \mathrm{div}\tau_h) = 0, & \forall \tau_h \in W_h, 0 < t \le T. \\ (u_h(0), v_h) = (\varphi_0, v_h), & \forall v_h \in V_h, \\ (\mathbf{q}_h(0), \tau_h) = (\nabla\varphi_0, \tau_h), & \forall \tau_h \in W_h. \end{cases} \tag{2.7}$$

Theorem 1 *The problem (2.7) has the unique solution.*

Proof. In fact, since (2.7) is linear, it suffices to show that the associated homogeneous system

$$(u_{h,t}, v_h) = (w_h, v_h), \qquad \forall v_h \in V_h, 0 < t \le T, \tag{2.8}$$
$$(\mathbf{q}_{h,t}, \tau_h) = (\mathbf{p}_h, \tau_h), \qquad \forall \tau_h \in W_h, 0 < t \le T, \tag{2.9}$$
$$(w_h, v_h) - \mu(\mathrm{div}\mathbf{p}_h, v_h) = \gamma(\mathrm{div}\mathbf{q}_h, v_h), \qquad \forall v_h \in V_h, 0 < t \le T, \tag{2.10}$$
$$(\mathbf{p}_h, \tau_h) + (w_h, \mathrm{div}\tau_h) = 0, \qquad \forall \tau_h \in W_h, 0 < t \le T, \tag{2.11}$$
$$(u_h(0), v_h) = 0, \qquad \forall v_h \in V_h, \tag{2.12}$$
$$(\mathbf{q}_h(0), \tau_h) = 0, \qquad \forall \tau_h \in W_h, \tag{2.13}$$

has only the trivial solution.

Let $v_h = w_h$ in (2.10), we can get that

$$\|w_h\|^2 - \mu(\mathrm{div}\mathbf{p}_h, w_h) = \gamma(\mathrm{div}\mathbf{q}_h, w_h). \tag{2.14}$$

Choosing $\tau_h = \mathbf{p}_h$ and $\tau_h = \mathbf{q}_h$ in (2.11) respectively, then we have

$$\|\mathbf{p}_h\|^2 + (w_h, \mathrm{div}\mathbf{p}_h) = 0, \tag{2.15}$$

$$(\mathbf{p}_h, \mathbf{q}_h) + (w_h, \mathrm{div}\mathbf{q}_h) = 0. \tag{2.16}$$

From (2.14),(2.15) and (2.16), using the $\varepsilon-$inequality, we obtain

$$\|w_h\|^2 + \mu\|\mathbf{p}_h\|^2 = -\gamma(\mathbf{p}_h, \mathbf{q}_h) \le \gamma\|\mathbf{p}_h\|\|\mathbf{q}_h\| \le \varepsilon\|\mathbf{p}_h\|^2 + C\|\mathbf{q}_h\|^2. \tag{2.17}$$

From (2.17) and taking $\varepsilon < \mu$, we get

$$\|w_h\|^2 + \|\mathbf{p}_h\|^2 \le C\|\mathbf{q}_h\|^2. \tag{2.18}$$

From (2.9) we know that if we choose $\tau_h = \mathbf{q}_{h,t}$ and $\tau_h = \mathbf{p}_h$ respectively, then we have

$$\|\mathbf{q}_{h,t}\| = \|\mathbf{p}_h\|. \tag{2.19}$$

Combining (2.13),(2.18) with (2.19) yields

$$\|\mathbf{q}_h\| = \|\int_0^t \mathbf{q}_{h,t}\mathrm{d}t\| \leq \int_0^t \|\mathbf{q}_{h,t}\|\mathrm{d}t \leq C\int_0^t \|\mathbf{q}_h\|\mathrm{d}t. \tag{2.20}$$

Using Gronwall's lemma, we have $\|\mathbf{q}_h\| = 0$, and from (2.18), $\|w_h\| = \|\mathbf{p}_h\| = 0$. Hence $\mathbf{q}_h \equiv \mathbf{0}$, $w_h \equiv 0$ and $\mathbf{p}_h \equiv \mathbf{0}$.By taking $v_h = u_{h,t}$ and $v_h = w_h$ in (2.8), we obtain $\|u_{h,t}\| = \|w_h\|$, and then $u_{h,t} = 0$, considering (2.12), we get $u_h \equiv 0$. □

Therefore the solution $\{u_h, \mathbf{q}_h, w_h, \mathbf{p}_h\}$ of (2.7) is well defined.

2.2 Error Estimate

In study of mixed methods for parabolic problems, we usually introduce a mixed elliptic projection related to our equations. According to our Sobolev equations, we modify this idea and define a map $\{\overline{u}, \overline{\mathbf{q}}, \overline{w}, \overline{\mathbf{p}}\} : [0,T] \to V_h \times W_h \times V_h \times W_h$ such that

$$(u - \overline{u}, v_h) = 0, \qquad \forall v_h \in V_h, 0 < t \leq T, \tag{2.21}$$
$$(\mathbf{q} - \overline{\mathbf{q}}, \tau_h) = 0, \qquad \forall \tau_h \in W_h, 0 < t \leq T, \tag{2.22}$$
$$(w - \overline{w}, v_h) - \mu(\mathrm{div}(\mathbf{p} - \overline{\mathbf{p}}), v_h) = \gamma(\mathrm{div}(\mathbf{q} - \overline{\mathbf{q}}), v_h), \qquad \forall v_h \in V_h, 0 < t \leq T, \tag{2.23}$$
$$(\mathbf{p} - \overline{\mathbf{p}}, \tau_h) + (w - \overline{w}, \mathrm{div}\tau_h) = 0, \qquad \forall \tau_h \in W_h, 0 < t \leq T, \tag{2.24}$$

To begin with, let us demonstrate the existence and uniqueness of the solution of (2.21)-(2.24). Similar to the proof of Theorem 1, since (2.21)-(2.24) is linear, we only need to prove that there is only the trivial solution to the associated homogeneous system

$$(\overline{u}, v_h) = 0, \qquad \forall v_h \in V_h, 0 < t \leq T, \tag{2.25}$$
$$(\overline{\mathbf{q}}, \tau_h) = 0, \qquad \forall \tau_h \in W_h, 0 < t \leq T, \tag{2.26}$$
$$(\overline{w}, v_h) - \mu(\mathrm{div}\overline{\mathbf{p}}, v_h) = \gamma(\mathrm{div}\overline{\mathbf{q}}, v_h), \qquad \forall v_h \in V_h, 0 < t \leq T, \tag{2.27}$$
$$(\overline{\mathbf{p}}, \tau_h) + (\overline{w}, \mathrm{div}\tau_h) = 0, \qquad \forall \tau_h \in W_h, 0 < t \leq T. \tag{2.28}$$

Let $v_h = \overline{u}$ in (2.25) and $\tau_h = \overline{\mathbf{q}}$ in (2.26), we can easily get $\overline{u} = 0, \overline{\mathbf{q}} = \mathbf{0}$.

By taking $v_h = \overline{w}$ in (2.27) and $\tau_h = \overline{\mathbf{p}}$ in (2.28), we obtain

$$\|\overline{w}\|^2 + \mu\|\overline{\mathbf{p}}\|^2 = \gamma(\mathrm{div}\overline{\mathbf{q}}, \overline{w}) = 0,$$

which implies that $\overline{w} = 0$ and $\overline{\mathbf{p}} = \mathbf{0}$.

So, the existence and uniqueness of the solution of (2.21)-(2.24) have been demonstrated and $\{\overline{u}, \overline{\mathbf{q}}, \overline{w}, \overline{\mathbf{p}}\}$ in (2.21)-(2.24) is well defined. Next we give some error estimates of $\{\overline{u}, \overline{\mathbf{q}}, \overline{w}, \overline{\mathbf{p}}\}$.

Lemma 1 *Let $\{u, \mathbf{q}, w, \mathbf{p}\}$ and $\{\overline{u}, \overline{\mathbf{q}}, \overline{w}, \overline{\mathbf{p}}\}$ satisfy the relation(2.2) and (2.21)-(2.24), respectively. Assume that $\{u, \mathbf{q}, w, \mathbf{p}\}$ are sufficiently smooth and that Ω is 2-regular (for the definition of*

2-regularity see [24]). Then for all $0 < t \leq T$, there exists a constant $C > 0$ independent of h and t, such that

$$\begin{aligned}
&\| u - \overline{u} \| \leq Ch^l \| u \|_l, \\
&\| \mathbf{q} - \overline{\mathbf{q}} \| \leq Ch^l \| \mathbf{q} \|_l, \\
&\| w - \overline{w} \| \leq Ch^l (\| w \|_l + \| \mathbf{q} \|_l + \| \mathbf{p} \|_l + \| \mathrm{div}\mathbf{q} \|_l + \| \mathrm{div}\mathbf{p} \|_l), \\
&\| \mathbf{p} - \overline{\mathbf{p}} \| + \| \mathrm{div}(\mathbf{p} - \overline{\mathbf{p}}) \| + \| \mathrm{div}(\mathbf{q} - \overline{\mathbf{q}}) \| \leq Ch^l (\| \mathbf{q} \|_l + \| \mathbf{p} \|_l + \| \mathrm{div}\mathbf{q} \|_l + \| \mathrm{div}\mathbf{p} \|_l),
\end{aligned}$$

where $0 < l \leq k+1$.

Proof. Let $v_h = R_h u - \overline{u}$ in (2.21), we get

$$\|R_h u - \overline{u}\|^2 = -(u - R_h u, R_h u - \overline{u}) \leq \|u - R_h u\| \|R_h u - \overline{u}\|,$$

which implies that $\|R_h u - \overline{u}\| \leq \|u - R_h u\| \leq Ch^l \|u\|_l$. Noting (2.3), we obtain

$$\|u - \overline{u}\| \leq \|u - R_h u\| + \|R_h u - \overline{u}\| \leq Ch^l \|u\|_l.$$

Similarly, let $\tau_h = \pi_h \mathbf{q} - \overline{\mathbf{q}}$ in (2.22), we get $\|\pi_h \mathbf{q} - \overline{\mathbf{q}}\| \leq Ch^l \|\mathbf{q}\|_l$, then $\|\mathbf{q} - \overline{\mathbf{q}}\| \leq Ch^l \|\mathbf{q}\|_l$.

Let $v_h = \mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})$ and $v_h = \mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})$ in (2.23)respectively, we have

$$\begin{aligned}
&(R_h w - \overline{w}, \mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})) - \mu \|\mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})\|^2 - \gamma(\mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}}), \mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})) \\
&\qquad = -(w - R_h w, \mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})) + \mu(\mathrm{div}(\mathbf{p} - \pi_h \mathbf{p}), \mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})) + \\
&\qquad \gamma(\mathrm{div}(\mathbf{q} - \pi_h \mathbf{q}), \mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})),
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
&(R_h w - \overline{w}, \mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})) - \mu(\mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}}), \mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})) - \gamma \|\mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})\|^2 \\
&\qquad = -(w - R_h w, \mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})) + \mu(\mathrm{div}(\mathbf{p} - \pi_h \mathbf{p}), \mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})) + \\
&\qquad \gamma(\mathrm{div}(\mathbf{q} - \pi_h \mathbf{q}), \mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})).
\end{aligned} \tag{2.30}$$

Taking $\tau_h = \pi_h \mathbf{p} - \overline{\mathbf{p}}$ and $\tau_h = \pi_h \mathbf{p} - \overline{\mathbf{p}}$ in (2.24)respectively,

$$\|\pi_h \mathbf{p} - \overline{\mathbf{p}}\|^2 + (R_h w - \overline{w}, \mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})) = -(\mathbf{p} - \pi_h \mathbf{p}, \pi_h \mathbf{p} - \overline{\mathbf{p}}) - (w - R_h w, \mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})), \tag{2.31}$$

$$(\pi_h \mathbf{p} - \overline{\mathbf{p}}, \pi_h \mathbf{q} - \overline{\mathbf{q}}) + (R_h w - \overline{w}, \mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})) = -(\mathbf{p} - \pi_h \mathbf{p}, \pi_h \mathbf{q} - \overline{\mathbf{q}}) - (w - R_h w, \mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})). \tag{2.32}$$

Let $\tau_h = \pi_h \mathbf{p} - \overline{\mathbf{p}}$ in (2.22), we have

$$(\pi_h \mathbf{q} - \overline{\mathbf{q}}, \pi_h \mathbf{p} - \overline{\mathbf{p}}) = -(\mathbf{q} - \pi_h \mathbf{q}, \pi_h \mathbf{p} - \overline{\mathbf{p}}). \tag{2.33}$$

Combining (2.29)-(2.33), we obtain

$$\begin{aligned}
&\mu \|\pi_h \mathbf{p} - \overline{\mathbf{p}}\|^2 + \gamma^2 \|\mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})\|^2 + \mu^2 \|\mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})\|^2 \\
= &- \mu^2 (\mathrm{div}(\mathbf{p} - \pi_h \mathbf{p}), \mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})) - \gamma^2 (\mathrm{div}(\mathbf{q} - \pi_h \mathbf{q}), \mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})) - \\
&\mu(\mathbf{p} - \pi_h \mathbf{p}, \pi_h \mathbf{p} - \overline{\mathbf{p}}) - \gamma(\mathbf{p} - \pi_h \mathbf{p}, \pi_h \mathbf{q} - \overline{\mathbf{q}}) + \gamma(\mathbf{q} - \pi_h \mathbf{q}, \pi_h \mathbf{p} - \overline{\mathbf{p}}) - \\
&\mu\gamma(\mathrm{div}(\mathbf{p} - \pi_h \mathbf{p}), \mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})) - \mu\gamma(\mathrm{div}(\mathbf{q} - \pi_h \mathbf{q}), \mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})) \\
\leq &\mu^2 \|\mathrm{div}(\mathbf{p} - \pi_h \mathbf{p})\| \|\mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})\| + \gamma^2 \|\mathrm{div}(\mathbf{q} - \pi_h \mathbf{q})\| \|\mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})\| + \\
&\mu \|\mathbf{p} - \pi_h \mathbf{p}\| \|\pi_h \mathbf{p} - \overline{\mathbf{p}}\| + \gamma \|\mathbf{p} - \pi_h \mathbf{p}\| \|\pi_h \mathbf{q} - \overline{\mathbf{q}}\| + \gamma \|\mathbf{q} - \pi_h \mathbf{q}\| \|\pi_h \mathbf{p} - \overline{\mathbf{p}}\| + \\
&\mu\gamma \|\mathrm{div}(\mathbf{p} - \pi_h \mathbf{p})\| \|\mathrm{div}(\pi_h \mathbf{q} - \overline{\mathbf{q}})\| + \mu\gamma \|\mathrm{div}(\mathbf{q} - \pi_h \mathbf{q})\| \|\mathrm{div}(\pi_h \mathbf{p} - \overline{\mathbf{p}})\|.
\end{aligned}$$

Using $\varepsilon-$inequality, there is a sufficiently small ε such that

$$
\begin{aligned}
&\mu\|\pi_h\mathbf{p}-\overline{\mathbf{p}}\|^2+\gamma^2\|\mathrm{div}(\pi_h\mathbf{q}-\overline{\mathbf{q}})\|^2+\mu^2\|\mathrm{div}(\pi_h\mathbf{p}-\overline{\mathbf{p}})\|^2\\
\leq& C(\|\mathbf{q}-\pi_h\mathbf{q}\|^2+\|\mathbf{p}-\pi_h\mathbf{p}\|^2+\|\mathrm{div}(\mathbf{q}-\pi_h\mathbf{q})\|^2+\|\mathrm{div}(\mathbf{p}-\pi_h\mathbf{p})\|^2)+\\
&\varepsilon(\|\pi_h\mathbf{p}-\overline{\mathbf{p}}\|^2+\|\mathrm{div}(\pi_h\mathbf{q}-\overline{\mathbf{q}})\|^2+\|\mathrm{div}(\pi_h\mathbf{p}-\overline{\mathbf{p}})\|^2)\\
\leq& Ch^{2l}(\|\mathbf{q}\|_l^2+\|\mathbf{p}\|_l^2+\|\mathrm{div}\mathbf{q}\|_l^2+\|\mathrm{div}\mathbf{p}\|_l^2)+\varepsilon(\|\pi_h\mathbf{p}-\overline{\mathbf{p}}\|^2+\\
&\|\mathrm{div}(\pi_h\mathbf{q}-\overline{\mathbf{q}})\|^2+\|\mathrm{div}(\pi_h\mathbf{p}-\overline{\mathbf{p}})\|^2),
\end{aligned}
$$

with that,

$$
\|\pi_h\mathbf{p}-\overline{\mathbf{p}}\|^2+\|\mathrm{div}(\pi_h\mathbf{q}-\overline{\mathbf{q}})\|^2+\|\mathrm{div}(\pi_h\mathbf{p}-\overline{\mathbf{p}})\|^2\leq Ch^{2l}(\|\mathbf{q}\|_l^2+\|\mathbf{p}\|_l^2+\|\mathrm{div}\mathbf{q}\|_l^2+\|\mathrm{div}\mathbf{p}\|_l^2),
$$

then,

$$
\begin{aligned}
&\|\mathbf{p}-\overline{\mathbf{p}}\|+\|\mathrm{div}(\mathbf{q}-\overline{\mathbf{q}})\|+\|\mathrm{div}(\mathbf{p}-\overline{\mathbf{p}})\|\\
\leq&\|\mathbf{p}-\pi_h\mathbf{p}\|+\|\pi_h\mathbf{p}-\overline{\mathbf{p}}\|+\|\mathrm{div}(\mathbf{q}-\pi_h\mathbf{q})\|+\|\mathrm{div}(\pi_h\mathbf{q}-\overline{\mathbf{q}})\|+\\
&\|\mathrm{div}(\mathbf{p}-\pi_h\mathbf{p})\|+\|\mathrm{div}(\pi_h\mathbf{p}-\overline{\mathbf{p}})\|\\
\leq& Ch^l(\|\mathbf{q}\|_l+\|\mathbf{p}\|_l+\|\mathrm{div}\mathbf{q}\|_l+\|\mathrm{div}\mathbf{p}\|_l).
\end{aligned}
$$

Let $v_h=R_hw-\overline{w}$ in (2.23), we have

$$
\begin{aligned}
\|R_hw-\overline{w}\|^2=&-(w-R_hw,R_hw-\overline{w})+\mu(\mathrm{div}(\mathbf{p}-\overline{\mathbf{p}}),R_hw-\overline{w})+\gamma(\mathrm{div}(\mathbf{q}-\overline{\mathbf{q}}),R_hw-\overline{w})\\
\leq&(\|w-R_hw\|+\mu\|\mathrm{div}(\mathbf{p}-\overline{\mathbf{p}})\|+\gamma\|\mathrm{div}(\mathbf{q}-\overline{\mathbf{q}})\|)\|R_hw-\overline{w}\|,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\|R_hw-\overline{w}\|\leq&\|w-R_hw\|+\gamma\|\mathrm{div}(\mathbf{q}-\overline{\mathbf{q}})\|+\mu\|\mathrm{div}(\mathbf{p}-\overline{\mathbf{p}})\|\\
\leq& Ch^l(\|w\|_l+\|\mathbf{q}\|_l+\|\mathbf{p}\|_l+\|\mathrm{div}\mathbf{q}\|_l+\|\mathrm{div}\mathbf{p}\|_l),
\end{aligned}
$$

thus, we conclude

$$
\begin{aligned}
\|w-\overline{w}\|\leq&\|w-R_hw\|+\|R_hw-\overline{w}\|\\
\leq& Ch^l(\|w\|_l+\|\mathbf{q}\|_l+\|\mathbf{p}\|_l+\|\mathrm{div}\mathbf{q}\|_l+\|\mathrm{div}\mathbf{p}\|_l).\qquad\square
\end{aligned}
$$

Subtracting (2.7) from (2.2), we get the error equations

$$
\begin{cases}
((u-u_h)_t,v_h)=(w-w_h,v_h), & \forall v_h\in V_h,0<t\leq T,\\
((\mathbf{q}-\mathbf{q}_h)_t,\tau_h)=(\mathbf{p}-\mathbf{p}_h,\tau_h), & \forall\tau_h\in W_h,0<t\leq T,\\
(w-w_h,v_h)-\mu(\mathrm{div}(\mathbf{p}-\mathbf{p}_h),v_h)=\gamma(\mathrm{div}(\mathbf{q}-\mathbf{q}_h),v_h), & \forall v_h\in V_h,0<t\leq T,\\
(\mathbf{p}-\mathbf{p}_h,\tau_h)+(w-w_h,\mathrm{div}\tau_h)=0, & \forall\tau_h\in W_h,0<t\leq T.
\end{cases}\tag{2.34}
$$

Using (2.21)-(2.24), (2.34) can be written as

$$((\overline{u}-u_h)_t,v_h)=(w-w_h,v_h),\qquad\forall v_h\in V_h,0<t\leq T,\tag{2.35}$$

$$((\overline{\mathbf{q}}-\mathbf{q}_h)_t,\tau_h)=(\mathbf{p}-\mathbf{p}_h,\tau_h),\qquad\forall\tau_h\in W_h,0<t\leq T,\tag{2.36}$$

$$(\overline{w}-w_h,v_h)-\mu(\mathrm{div}(\overline{\mathbf{p}}-\mathbf{p}_h),v_h)=\gamma(\mathrm{div}(\overline{\mathbf{q}}-\mathbf{q}_h),v_h),\qquad\forall v_h\in V_h,0<t\leq T,\tag{2.37}$$

$$(\overline{\mathbf{p}}-\mathbf{p}_h,\tau_h)+(\overline{w}-w_h,\mathrm{div}\tau_h)=0,\qquad\forall\tau_h\in W_h,0<t\leq T.\tag{2.38}$$

Theorem 2 *Let $\{u, \mathbf{q}, w, \mathbf{p}\}$ be the solution of (2.2) and $\{u_h, \mathbf{q}_h, w_h, \mathbf{p}_h\}$ be that of (2.7). Assume $\{u, \mathbf{q}, w, \mathbf{p}\}$ are sufficiently smooth and that Ω is 2-regular. Then there exists a constant $C > 0$ independent of h and t, such that*

$$\begin{aligned}\|(u-u_h)(t)\|^2 + \|(\mathbf{q}-\mathbf{q}_h)(t)\|^2 \le Ch^{2l}[\|u\|_l^2 + \|\mathbf{q}\|_l^2 + \int_0^t (\|w\|_l^2 + \|\mathbf{q}\|_l^2 + \|\mathbf{p}\|_l^2 + \|\mathrm{div}\mathbf{q}\|_l^2 \\ + \|\mathrm{div}\mathbf{p}\|_l^2)\mathrm{d}t],\\ \|(w-w_h)(t)\|^2 + \|(\mathbf{p}-\mathbf{p}_h)(t)\|^2 \le Ch^{2l}[\|u\|_l^2 + \|w\|_l^2 + \|\mathbf{q}\|_l^2 + \|\mathbf{p}\|_l^2 + \|\mathrm{div}\mathbf{q}\|_l^2 + \|\mathrm{div}\mathbf{p}\|_l^2 \\ + \int_0^t (\|w\|_l^2 + \|\mathbf{q}\|_l^2 + \|\mathbf{p}\|_l^2 + \|\mathrm{div}\mathbf{q}\|_l^2 + \|\mathrm{div}\mathbf{p}\|_l^2)\mathrm{d}t],\end{aligned}$$

where $0 < t \le T, 0 < l \le k+1$.

Proof. Let $v_h = \overline{w} - w_h$ in (2.37), we get

$$\|\overline{w} - w_h\|^2 - \mu(\mathrm{div}(\overline{\mathbf{p}} - \mathbf{p}_h), \overline{w} - w_h) = \gamma(\mathrm{div}(\overline{\mathbf{q}} - \mathbf{q}_h), \overline{w} - w_h). \tag{2.39}$$

Taking $\tau_h = \overline{\mathbf{p}} - \mathbf{p}_h$ and $\tau_h = \overline{\mathbf{q}} - \mathbf{q}_h$ in (2.38) respectively, we obtain

$$\|\overline{\mathbf{p}} - \mathbf{p}_h\|^2 + (\overline{w} - w_h, \mathrm{div}(\overline{\mathbf{p}} - \mathbf{p}_h)) = 0, \tag{2.40}$$
$$(\overline{\mathbf{p}} - \mathbf{p}_h, \overline{\mathbf{q}} - \mathbf{q}_h) + (\overline{w} - w_h, \mathrm{div}(\overline{\mathbf{q}} - \mathbf{q}_h)) = 0. \tag{2.41}$$

From (2.39), (2.40) and (2.41), using $\varepsilon-$ inequality, for a sufficiently small ε that satisfy

$$\|\overline{w} - w_h\|^2 + \mu\|\overline{\mathbf{p}} - \mathbf{p}_h\|^2 = -\gamma(\overline{\mathbf{p}} - \mathbf{p}_h, \overline{\mathbf{q}} - \mathbf{q}_h) \le \varepsilon\|\overline{\mathbf{p}} - \mathbf{p}_h\|^2 + C\|\overline{\mathbf{q}} - \mathbf{q}_h\|^2,$$

obviously,

$$\|\overline{w} - w_h\|^2 + \|\overline{\mathbf{p}} - \mathbf{p}_h\|^2 \le C\|\overline{\mathbf{q}} - \mathbf{q}_h\|^2. \tag{2.42}$$

Let $v_h = \overline{u} - u_h$ in (2.35), and $\tau_h = \overline{\mathbf{q}} - \mathbf{q}_h$ in (2.36), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{u} - u_h\|^2 =& 2(w - w_h, \overline{u} - u_h)\\ \le& 2(\|w - \overline{w}\|^2 + \|\overline{w} - w_h\|^2 + \|\overline{u} - u_h\|^2)\\ \le& C[h^{2l}(\|w\|_l^2 + \|\mathbf{q}\|_l^2 + \|\mathbf{p}\|_l^2 + \|\mathrm{div}\mathbf{q}\|_l^2 + \|\mathrm{div}\mathbf{p}\|_l^2) + \|\overline{\mathbf{q}} - \mathbf{q}_h\|^2 + \|\overline{u} - u_h\|^2],\\ \frac{\mathrm{d}}{\mathrm{d}t}\|\overline{\mathbf{q}} - \mathbf{q}_h\|^2 =& 2(\mathbf{p} - \mathbf{p}_h, \overline{\mathbf{q}} - \mathbf{q}_h)\\ \le& 2(\|\mathbf{p} - \overline{\mathbf{p}}\|^2 + \|\overline{\mathbf{p}} - \mathbf{p}_h\|^2 + \|\overline{\mathbf{q}} - \mathbf{q}_h\|^2)\\ \le& C[h^{2l}(\|\mathbf{q}\|_l^2 + \|\mathbf{p}\|_l^2 + \|\mathrm{div}\mathbf{q}\|_l^2 + \|\mathrm{div}\mathbf{p}\|_l^2) + \|\overline{\mathbf{q}} - \mathbf{q}_h\|^2],\end{aligned}$$

obviously,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|\overline{u} - u_h\|^2 + \|\overline{\mathbf{q}} - \mathbf{q}_h\|^2) \le& C[h^{2l}(\|w\|_l^2 + \|\mathbf{q}\|_l^2 + \|\mathbf{p}\|_l^2 + \|\mathrm{div}\mathbf{q}\|_l^2 + \|\mathrm{div}\mathbf{p}\|_l^2) +\\ & \|\overline{u} - u_h\|^2 + \|\overline{\mathbf{q}} - \mathbf{q}_h\|^2],\end{aligned}$$

with that, for $0 < t \leq T$, we have

$$\begin{aligned}\|(\overline{u}-u_h)(t)\|^2+\|(\overline{\mathbf{q}}-\mathbf{q}_h)(t)\|^2 \leq & C[h^{2l}\int_0^t(\|w\|_l^2+\|\mathbf{p}\|_l^2+\|\mathbf{q}\|_l^2+\|\mathrm{div}\mathbf{p}\|_l^2+\|\mathrm{div}\mathbf{q}\|_l^2)\mathrm{d}t \\ & +\int_0^t(\|\overline{u}-u_h\|^2+\|\overline{\mathbf{q}}-\mathbf{q}_h\|^2)\mathrm{d}t].\end{aligned}$$

Using Gronwall's lemma, we have

$$\|(\overline{u}-u_h)(t)\|^2+\|(\overline{\mathbf{q}}-\mathbf{q}_h)(t)\|^2 \leq Ch^{2l}\int_0^t(\|w\|_l^2+\|\mathbf{q}\|_l^2+\|\mathbf{p}\|_l^2+\|\mathrm{div}\mathbf{q}\|_l^2+\|\mathrm{div}\mathbf{p}\|_l^2)\mathrm{d}t,$$

noticing (2.42),

$$\|(\overline{w}-w_h)(t)\|^2+\|(\overline{\mathbf{p}}-\mathbf{p}_h)(t)\|^2 \leq Ch^{2l}\int_0^t(\|w\|_l^2+\|\mathbf{q}\|_l^2+\|\mathbf{p}\|_l^2+\|\mathrm{div}\mathbf{q}\|_l^2+\|\mathrm{div}\mathbf{p}\|_l^2)\mathrm{d}t.$$

Finally, we get

$$\begin{aligned}\|(u-u_h)(t)\|^2+\|(\mathbf{q}-\mathbf{q}_h)(t)\|^2 \leq & \; Ch^{2l}[\|u\|_l^2+\|\mathbf{q}\|_l^2+\int_0^t(\|w\|_l^2+\|\mathbf{q}\|_l^2+\|\mathbf{p}\|_l^2+\|\mathrm{div}\mathbf{q}\|_l^2 \\ & +\|\mathrm{div}\mathbf{p}\|_l^2)\mathrm{d}t], \\ \|(w-w_h)(t)\|^2+\|(\mathbf{p}-\mathbf{p}_h)(t)\|^2 \leq & \; Ch^{2l}[\|u\|_l^2+\|w\|_l^2+\|\mathbf{q}\|_l^2+\|\mathbf{p}\|_l^2+\|\mathrm{div}\mathbf{q}\|_l^2+\|\mathrm{div}\mathbf{p}\|_l^2 \\ & +\int_0^t(\|w\|_l^2+\|\mathbf{q}\|_l^2+\|\mathbf{p}\|_l^2+\|\mathrm{div}\mathbf{q}\|_l^2+\|\mathrm{div}\mathbf{p}\|_l^2)\mathrm{d}t]. \qquad \square\end{aligned}$$

3 Fully Discrete Scheme Based on EMFE Method

Let $0 = t_0 < t_1 < ... < t_N = T$ be a partition of the domain $[0, T]$, $\Delta t_n = t_n - t_{n-1} (n = 1, 2, ..., N)$, $(u_h^n, \mathbf{q}_h^n, w_h^n, \mathbf{p}_h^n)$ is the approximation of $(u, \mathbf{q}, w, \mathbf{p})$ at t_n. In order to make the scheme more stable, the Euler backward difference scheme with first order accuracy is adopted to discretize the time variables t. If the two order accuracy is to be reached, the Crank-Nicolson format can be considered.

Introduce the mark: $\partial_t u^n = \frac{u^n - u^{n-1}}{\Delta t}$, $\partial_t \mathbf{q}^n = \frac{\mathbf{q}^n - \mathbf{q}^{n-1}}{\Delta t}$. Then, we get the fully discrete scheme: find $(u_h^n, \mathbf{q}_h^n, w_h^n, \mathbf{p}_h^n)$ such that

$$\begin{cases}(\partial_t u_h^n, v_h) = (w_h^n, v_h), & \forall v_h \in V_h, \\ (\partial_t \mathbf{q}_h^n, \tau_h) = (\mathbf{p}_h^n, \tau_h), & \forall \tau_h \in W_h, \\ (w_h^n, v_h) - \mu(\mathrm{div}\mathbf{p}_h^n, v_h) = (f^n, v_h) + \gamma(\mathrm{div}\mathbf{q}_h^n, v_h), & \forall v_h \in V_h, \\ (\mathbf{p}_h^n, \tau_h) + (w_h^n, \mathrm{div}\tau_h) = 0, & \forall \tau_h \in W_h, \\ u_h^0 = \varphi_0(x, y), \quad \mathbf{q}_h^0 = \nabla\varphi_0(x, y). \end{cases} \tag{3.1}$$

Theorem 3 *The solution of (3.1) exists uniquely.*

Proof. Since (3.1) is linear, it suffices to show that the associated homogeneous system has only the trivial solution, that is, let $u_h^{n-1} = 0, \mathbf{q}_h^{n-1} = \mathbf{0}, f^n = 0$, we get the associated homogeneous system:

$$
\begin{aligned}
&(u_h^n, v_h) = \Delta t(w_h^n, v_h), && \forall v_h \in V_h, && (3.2)\\
&(\mathbf{q}_h^n, \tau_h) = \Delta t(\mathbf{p}_h^n, \tau_h), && \forall \tau_h \in W_h, && (3.3)\\
&(w_h^n, v_h) - \mu(\mathrm{div}(\mathbf{p}_h^n), v_h) = \gamma(\mathrm{div}(\mathbf{q}_h^n), v_h), && \forall v_h \in V_h, && (3.4)\\
&(\mathbf{p}_h^n, \tau_h) + (w_h^n, \mathrm{div}\tau_h) = 0, && \forall \tau_h \in W_h. && (3.5)
\end{aligned}
$$

Taking $v_h = w_h^n$ in (3.4) and $\tau_h = \mathbf{p}_h^n$, $\tau_h = \mathbf{q}_h^n$ in (3.5) respectively, we have

$$\|w_h^n\|^2 - \mu(\mathrm{div}\mathbf{p}_h^n, w_h^n) = \gamma(\mathrm{div}\mathbf{q}_h^n, w_h), \tag{3.6}$$

$$\|\mathbf{p}_h^n\|^2 + (w_h^n, \mathrm{div}\mathbf{p}_h^n) = 0, \tag{3.7}$$

$$(\mathbf{p}_h^n, \mathbf{q}_h^n) + (w_h^n, \mathrm{div}\mathbf{q}_h^n) = 0. \tag{3.8}$$

Combine (3.6), (3.7) with (3.8), using $\varepsilon-$inequality, we obtain

$$\|w_h^n\|^2 + \mu\|\mathbf{p}_h^n\|^2 = \gamma(\mathrm{div}\mathbf{q}_h^n, w_h) = -\gamma(\mathbf{p}_h^n, \mathbf{q}_h^n) \le \varepsilon\|\mathbf{p}_h^n\|^2 + C\|\mathbf{q}_h^n\|^2,$$

with that

$$\|w_h^n\|^2 + \|\mathbf{p}_h^n\|^2 \le C\|\mathbf{q}_h^n\|^2. \tag{3.9}$$

Let $v_h = u_h^n$ and $v_h = w_h^n$ in (3.2) respectively, we have

$$\|u_h^n\|^2 = \Delta t(w_h^n, u_h^n) = (\Delta t)^2\|w_h^n\|^2. \tag{3.10}$$

Similarly, take $\tau_h = \mathbf{q}_h^n$ and $\tau_h = \mathbf{p}_h^n$ in (3.3) respectively, we have

$$\|\mathbf{q}_h^n\|^2 = (\Delta t)^2\|\mathbf{p}_h^n\|^2. \tag{3.11}$$

It thus follows that

$$\|u_h^n\|^2 + \|\mathbf{q}_h^n\|^2 \le (\Delta t)^2(\|w_h^n\|^2 + \|\mathbf{p}_h^n\|^2) \le C(\Delta t)^2\|\mathbf{q}_h^n\|^2. \tag{3.12}$$

If Δt is sufficiently small such that $C(\Delta t)^2 < 1$, we then get $\|u_h^n\| = 0, \|\mathbf{q}_h^n\| = 0$, noting (3.10) and (3.11) yields $\|w_h^n\| = 0, \|\mathbf{p}_h^n\| = 0$. Hence we can obtain

$$u_h^n = 0, \mathbf{q}_h^n = \mathbf{0}, w_h^n = 0, \mathbf{p}_h^n = \mathbf{0}. \qquad \square$$

Next, we analyze the error estimate of the format (3.1). Subtracting (3.1) from (2.2), we get the error equations

$$
\begin{cases}
(u_t^n - \partial_t u_h^n, v_h) = (w^n - w_h^n, v_h), & \forall v_h \in V_h,\\
(\mathbf{q}_t^n - \partial_t \mathbf{q}_h^n, \tau_h) = (\mathbf{p}^n - \mathbf{p}_h^n, \tau_h), & \forall \tau_h \in W_h,\\
(w^n - w_h^n, v_h) - \mu(\mathrm{div}(\mathbf{p}^n - \mathbf{p}_h^n), v_h) = \gamma(\mathrm{div}(\mathbf{q}^n - \mathbf{q}_h^n), v_h), & \forall v_h \in V_h,\\
(\mathbf{p}^n - \mathbf{p}_h^n, \tau_h) + (w^n - w_h^n, \mathrm{div}\tau_h) = 0, & \forall \tau_h \in W_h.
\end{cases} \tag{3.13}
$$

Firstly, we introduce the marks:

$$\begin{aligned} u^n - u_h^n &= u^n - \overline{u}^n + \overline{u}^n - u_h^n = \rho_u^n + \theta_u^n, \\ \mathbf{q}^n - \mathbf{q}_h^n &= \mathbf{q}^n - \overline{\mathbf{q}}^n + \overline{\mathbf{q}}^n - \mathbf{q}_h^n = \rho_{\mathbf{q}}^n + \theta_{\mathbf{q}}^n, \\ w^n - w_h^n &= w^n - \overline{w}^n + \overline{w}^n - w_h^n = \rho_w^n + \theta_w^n, \\ \mathbf{p}^n - \mathbf{p}_h^n &= \mathbf{p}^n - \overline{\mathbf{p}}^n + \overline{\mathbf{p}}^n - \mathbf{p}_h^n = \rho_{\mathbf{p}}^n + \theta_{\mathbf{p}}^n. \end{aligned}$$

using (2.21)-(2.24), the error equations can be rewritten as

$$(\theta_u^n - \theta_u^{n-1}, v_h) = \Delta t(\rho_w^n, v_h) + \Delta t(\theta_w^n, v_h) - \Delta t(u_t^n - \partial_t u^n, v_h), \quad \forall v_h \in V_h, \tag{3.14}$$

$$(\theta_{\mathbf{q}}^n - \theta_{\mathbf{q}}^{n-1}, \tau_h) = \Delta t(\rho_{\mathbf{p}}^n, \tau_h) + \Delta t(\theta_{\mathbf{p}}^n, \tau_h) - \Delta t(\mathbf{q}_t^n - \partial_t \mathbf{q}^n, \tau_h), \quad \forall \tau_h \in W_h, \tag{3.15}$$

$$(\theta_w^n, v_h) - \mu(\mathrm{div}\theta_{\mathbf{p}}^n, v_h) = \gamma(\mathrm{div}\theta_{\mathbf{q}}^n, v_h), \quad \forall v_h \in V_h, \tag{3.16}$$

$$(\theta_{\mathbf{p}}^n, \tau_h) + (\theta_w^n, \mathrm{div}\tau_h) = 0, \quad \forall \tau_h \in W_h., \tag{3.17}$$

$$\theta_u^0 = 0, \quad \theta_{\mathbf{q}}^0 = \mathbf{0}. \tag{3.18}$$

By taking $v_h = \theta_w^n$ in (3.16) and $\tau_h = \theta_{\mathbf{p}}^n$, $\tau_h = \theta_{\mathbf{q}}^n$ in (3.17) respectively, we have

$$\begin{aligned} &\|\theta_w^n\|^2 - \mu(\mathrm{div}\theta_{\mathbf{p}}^n, \theta_w^n) = \gamma(\mathrm{div}\theta_{\mathbf{q}}^n, \theta_w^n), \\ &\|\theta_{\mathbf{p}}^n\|^2 + (\theta_w^n, \mathrm{div}\theta_{\mathbf{p}}^n) = 0, \\ &(\theta_{\mathbf{p}}^n, \theta_{\mathbf{q}}^n) + (\theta_w^n, \mathrm{div}\theta_{\mathbf{q}}^n) = 0. \end{aligned}$$

Applying $\varepsilon-$inequality, it thus follows that

$$\|\theta_w^n\|^2 + \mu\|\theta_{\mathbf{p}}^n\|^2 = \gamma(\mathrm{div}\theta_{\mathbf{q}}^n, \theta_w^n) = -\gamma(\theta_{\mathbf{p}}^n, \theta_{\mathbf{q}}^n) \leq \varepsilon\|\theta_{\mathbf{p}}^n\|^2 + C\|\theta_{\mathbf{q}}^n\|^2,$$

and then

$$\|\theta_w^n\|^2 + \|\theta_{\mathbf{p}}^n\|^2 \leq C\|\theta_{\mathbf{q}}^n\|^2. \tag{3.19}$$

Considering the following inequality:

$$\begin{aligned} \|u_t^n - \partial_t u^n\| &\leq C \int_{t_{n-1}}^{t_n} \|u_{tt}\| \mathrm{d}t, \\ \|\mathbf{q}_t^n - \partial_t \mathbf{q}^n\| &\leq C \int_{t_{n-1}}^{t_n} \|\mathbf{q}_{tt}\| \mathrm{d}t, \end{aligned}$$

taking $v_h = \theta_u^n$ in (3.14) yields

$$\begin{aligned} \|\theta_u^n\| &\leq \|\theta_u^{n-1}\| + \Delta t(\|\rho_w^n\| + \|\theta_w^n\|) + \Delta t\|u_t^n - \partial_t u^n\| \\ &\leq \|\theta_u^{n-1}\| + \Delta t \cdot h^l(\|w\|_l + \|\mathbf{p}\|_l + \|\mathbf{q}\|_l + \|\mathrm{div}\mathbf{p}\|_l + \|\mathrm{div}\mathbf{q}\|_l) + \\ &\quad C\Delta t\|\theta_{\mathbf{q}}^n\| + C\Delta t \int_{t_{n-1}}^{t_n} \|u_{tt}\| \mathrm{d}t. \end{aligned}$$

then, take $\tau_h = \theta_{\mathbf{q}}^n$ in (3.15), we get

$$\begin{aligned} \|\theta_{\mathbf{q}}^n\| &\leq \|\theta_{\mathbf{q}}^{n-1}\| + \Delta t(\|\rho_{\mathbf{p}}^n\| + \|\theta_{\mathbf{p}}^n\|) + \Delta t\|\mathbf{q}_t^n - \partial_t \mathbf{q}^n\| \\ &\leq \|\theta_{\mathbf{q}}^{n-1}\| + \Delta t \cdot h^l(\|\mathbf{p}\|_l + \|\mathbf{q}\|_l + \|\mathrm{div}\mathbf{p}\|_l + \|\mathrm{div}\mathbf{q}\|_l) + C\Delta t\|\theta_{\mathbf{q}}^n\| + C\Delta t \int_{t_{n-1}}^{t_n} \|\mathbf{q}_{tt}\| \mathrm{d}t. \end{aligned}$$

Using the two inequality above, we get

$$\begin{aligned}\|\theta_u^n\| + \|\theta_{\mathbf{q}}^n\| &\leq \|\theta_u^{n-1}\| + \|\theta_{\mathbf{q}}^{n-1}\| + 2\Delta t \cdot h^l(\|w\|_l + \|\mathbf{p}\|_l + \|\mathbf{q}\|_l + \|\mathrm{div}\mathbf{p}\|_l + \|\mathrm{div}\mathbf{q}\|_l) \\ &\quad + C\Delta t\|\theta_{\mathbf{q}}^n\| + C\Delta t(\int_{t_{n-1}}^{t_n} \|u_{tt}\|\mathrm{d}t + \int_{t_{n-1}}^{t_n} \|\mathbf{q}_{tt}\|\mathrm{d}t).\end{aligned}$$

If Δt is sufficiently small such that $C\Delta t < 1$, we obtain

$$\begin{aligned}\|\theta_u^n\| + \|\theta_{\mathbf{q}}^n\| &\leq C[\|\theta_u^{n-1}\| + \|\theta_{\mathbf{q}}^{n-1}\| + \Delta t \cdot h^l(\|w\|_l + \|\mathbf{p}\|_l + \|\mathbf{q}\|_l + \|\mathrm{div}\mathbf{p}\|_l + \|\mathrm{div}\mathbf{q}\|_l) + \\ &\quad \Delta t(\int_{t_{n-1}}^{t_n} \|u_{tt}\|\mathrm{d}t + \int_{t_{n-1}}^{t_n} \|\mathbf{q}_{tt}\|\mathrm{d}t)] \\ &\leq C[\|\theta_u^0\| + \|\theta_{\mathbf{q}}^0\| + n\Delta t \cdot h^l(\|w\|_l + \|\mathbf{p}\|_l + \|\mathbf{q}\|_l + \|\mathrm{div}\mathbf{p}\|_l + \|\mathrm{div}\mathbf{q}\|_l) + \\ &\quad \Delta t(\int_{t_0}^{t_n} \|u_{tt}\|\mathrm{d}t + \int_{t_0}^{t_n} \|\mathbf{q}_{tt}\|\mathrm{d}t)] \\ &\leq C[h^l(\|w\|_l + \|\mathbf{p}\|_l + \|\mathbf{q}\|_l + \|\mathrm{div}\mathbf{p}\|_l + \|\mathrm{div}\mathbf{q}\|_l) + \Delta t(\int_{t_0}^{t_n} \|u_{tt}\|\mathrm{d}t + \int_{t_0}^{t_n} \|\mathbf{q}_{tt}\|\mathrm{d}t)],\end{aligned}$$

using Theorem 2,

$$\begin{aligned}&\|u^n - u_h^n\| + \|\mathbf{q}^n - \mathbf{q}_h^n\| \\ \leq\ & \|\rho_u^n\| + \|\theta_u^n\| + \|\rho_{\mathbf{q}}^n\| + \|\theta_{\mathbf{q}}^n\| \\ \leq\ & C[h^l(\|w\|_l + \|\mathbf{p}\|_l + \|\mathbf{q}\|_l + \|\mathrm{div}\mathbf{p}\|_l + \|\mathrm{div}\mathbf{q}\|_l) + \Delta t(\int_{t_0}^{t_n} \|u_{tt}\|\mathrm{d}t + \int_{t_0}^{t_n} \|\mathbf{q}_{tt}\|\mathrm{d}t)].\end{aligned}$$

Due to (3.19), we obtain

$$\begin{aligned}\|\theta_w^n\| + \|\theta_{\mathbf{p}}^n\| &\leq C[h^l(\|w\|_l + \|\mathbf{p}\|_l + \|\mathbf{q}\|_l + \|\mathrm{div}\mathbf{p}\|_l + \|\mathrm{div}\mathbf{q}\|_l) + \\ &\quad \Delta t(\int_{t_0}^{t_n} \|u_{tt}\|\mathrm{d}t + \int_{t_0}^{t_n} \|\mathbf{q}_{tt}\|\mathrm{d}t)],\end{aligned}$$

and

$$\begin{aligned}\|w^n - w_h^n\| + \|\mathbf{p}^n - \mathbf{p}_h^n\| &\leq C[h^l(\|w\|_l + \|\mathbf{p}\|_l + \|\mathbf{q}\|_l + \|\mathrm{div}\mathbf{p}\|_l + \|\mathrm{div}\mathbf{q}\|_l) + \\ &\quad \Delta t(\int_{t_0}^{t_n} \|u_{tt}\|\mathrm{d}t + \int_{t_0}^{t_n} \|\mathbf{q}_{tt}\|\mathrm{d}t)].\end{aligned}$$

We summarize the results of the analysis into the following theorem.

Theorem 4 *Under the assumptions of Theorem 2, and $\Delta t \leq \delta$ (δ sufficiently small and independent of h) the fully discrete scheme has prior error estimate as following*

$$\begin{aligned}&\max_{1\leq n\leq N} \|u^n - u_h^n\| + \max_{1\leq n\leq N} \|\mathbf{q}^n - \mathbf{q}_h^n\| + \max_{1\leq n\leq N} \|w^n - w_h^n\| + \max_{1\leq n\leq N} \|\mathbf{p}^n - \mathbf{p}_h^n\| \\ \leq\ & C[h^l(\|w\|_{L^\infty(0,T;H^l(\Omega))} + \|\mathbf{q}\|_{L^\infty(0,T;H^l(\Omega))} + \|\mathbf{p}\|_{L^\infty(0,T;H^l(\Omega))} + \|\mathrm{div}\mathbf{q}\|_{L^\infty(0,T;H^l(\Omega))} + \\ & \|\mathrm{div}\mathbf{p}\|_{L^\infty(0,T;H^l(\Omega))}) + \Delta t(\|u_{tt}\|_{L^1(0,T;H^0(\Omega))} + \|\mathbf{q}_{tt}\|_{L^1(0,T;H^0(\Omega))})],\end{aligned}$$

where $0 < t \leq T, 0 < l \leq k+1$, C is a positive constant independent of h and Δt.

4 Numerical Experiment

In this section, two numerical examples are solved by the EMFE method in two cases ($k = 0$ and $k = 1$). All computations are performed under the P_k (piecewise polynomial of degree k) finite element space for u and w, and RT_k (Raviart-Thomas-Nedelec of index k) finite element space for $\mathbf{q}$ and $\mathbf{p}$. FreeFem++ platform and Matlab are used to calculate and show the results.

Example 1. We choose the computational domain as $\overline{\Omega} = [0,1] \times [0,1]$, $a = 1$, $b = 1$, $c = 1$, the initial value function $u_0(x,y) = \sin(\pi x)\sin(\pi y)$, and the source term $f(x,y,t) = (1 + 4\pi^2)\mathrm{e}^t \sin(\pi x)\sin(\pi y)$. It is easy to verify that the exact solution is $u = \mathrm{e}^t \sin(\pi x)\sin(\pi y)$, we calculate the equation till the final time $T = 1$.

- Let $k = 0$, that is to say, $V_h = P_0$, $W_h = RT_0$. Table 1 lists the L_2 norms of errors between u_h, $\mathbf{q}_h$, w_h, $\mathbf{p}_h$ and u, $\mathbf{q}$, w $\mathbf{p}$ with different subsections at $t = 0.2, 0.4, 0.6, 0.8, 1$, and Table 2 lists the corresponding convergence orders, from which we can see that the errors are quite small and the convergence orders are all about 1, the approximation effect is very good.

- When $k = 1$, i.e. $V_h = P_1$, $W_h = RT_1$. We plot the images of the analytical solution u and the numerical solution u_h at $t = 1$ as shown in Fig.1, and we can intuitively realize that the error between u_h, $\mathbf{q}_h$, w_h, $\mathbf{p}_h$ and u, $\mathbf{q}$, w, $\mathbf{p}$ are quite small, and the thinner the split, the smaller the error, until an almost complete coincidence. Similarly, we list L_2 error norms between u_h, $\mathbf{q}_h$, w_h, $\mathbf{p}_h$ and u, $\mathbf{q}$, w, $\mathbf{p}$ in Table 3, it can be seen that the approximation speed in this case is faster than that of $k = 0$. Table 4 lists the corresponding convergence orders, and the results show that the accuracy is in accordance with the theoretical analysis.

Example 2. We compute the same example given in [29]. The computational domain is $\overline{\Omega} = [0,1] \times [0,1]$, $a = 1$, $b = 1$, $c = 1$, the initial value function $u_0(x,y) = xy(1-x)(1-y)$, and the source term $f(x,y,t) = \mathrm{e}^t[xy(1-x)(1-y) + 4x(1-x) + 4y(1-y)]$. The analytical solution is $u = \mathrm{e}^t xy(1-x)(1-y)$.

In this example, we also show the long time stability of the method, thus we choose $T = 10$. To further show the good stability, we use a large time step. We notice that the analytical solution u contains the factor e^t, its value increases rapidly with time, we thus calculate the relative errors. It needs to be noted that the conclusions obtained in our theoretical analysis are also true for the relative errors.

- For $k = 0$, Table 5 lists the L_2 norms of relative errors between u_h, $\mathbf{q}_h$, w_h, $\mathbf{p}_h$ and u, $\mathbf{q}$, w $\mathbf{p}$ at $t = 2, 4, 6, 8, 10$. The errors show similar rule with time and subdivision as in Example 1. The corresponding convergence orders are listed in Table 6, which is in accordance with the theoretical analysis.

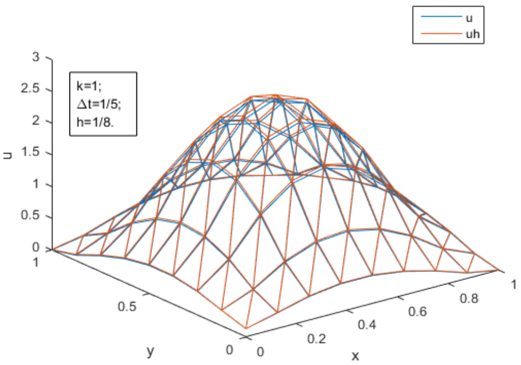
- For $k = 1$, we also list the L_2 norms of relative errors of four unknown variables and corresponding convergence orders in Table 7 and Table 8, respectively, both of them illustrate the effectiveness of the method. Compared with [29], in which different mixed finite element methods are used to solve this problem, we get not only more variables, but also smaller errors under the same partition.

Table 1: L_2 norms of errors for u, $\mathbf{q}$, w, and $\mathbf{p}$

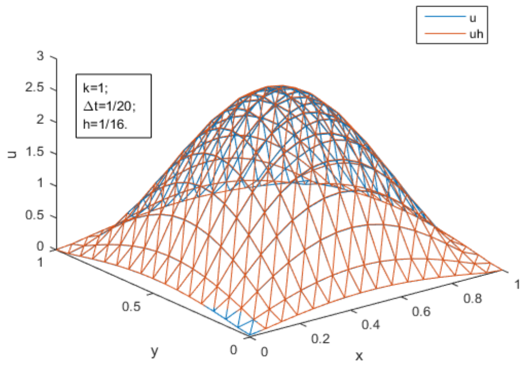
		$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1.0$
$\|u-u_h\|$	$h_1=\frac{1}{8}$, $\Delta t_1=\frac{1}{5}$	9.63793e-3	1.97476e-2	3.06969e-2	4.28932e-2	5.67993e-2
	$h_2=\frac{1}{16}$, $\Delta t_2=\frac{1}{10}$	4.94052e-3	1.01264e-2	1.57514e-2	2.20292e-2	2.92016e-2
	$h_3=\frac{1}{32}$, $\Delta t_3=\frac{1}{20}$	2.50091e-3	5.12698e-3	7.97771e-3	1.11626e-2	1.48055e-2
	$h_4=\frac{1}{64}$, $\Delta t_4=\frac{1}{40}$	1.25809e-3	2.57936e-3	4.01427e-3	5.61828e-3	7.45391e-3
$\|\mathbf{q}-\mathbf{q}_h\|$	$h_1=\frac{1}{8}$, $\Delta t_1=\frac{1}{5}$	4.57217e-2	9.42955e-2	1.47518e-1	2.07395e-1	2.76223e-1
	$h_2=\frac{1}{16}$, $\Delta t_2=\frac{1}{10}$	2.26422e-2	4.65397e-2	7.25957e-2	1.01809e-2	1.35313e-1
	$h_3=\frac{1}{32}$, $\Delta t_3=\frac{1}{20}$	1.12733e-2	2.31304e-2	3.60253e-2	5.04574e-2	6.69906e-2
	$h_4=\frac{1}{64}$, $\Delta t_4=\frac{1}{40}$	5.62575e-3	1.15323e-2	1.79473e-2	2.51207e-2	3.33338e-2
$\|w-w_h\|$	$h_1=\frac{1}{8}$, $\Delta t_1=\frac{1}{5}$	9.63163e-3	1.98354e-2	3.11424e-2	4.38879e-2	5.85412e-2
	$h_2=\frac{1}{16}$, $\Delta t_2=\frac{1}{10}$	4.73150e-3	9.80166e-3	1.53485e-3	2.15724e-2	2.87115e-2
	$h_4=\frac{1}{32}$, $\Delta t_3=\frac{1}{20}$	2.36306e-3	4.88055e-3	7.62302e-3	1.06946e-2	1.42146e-2
	$h_3=\frac{1}{64}$, $\Delta t_4=\frac{1}{40}$	1.18348e-3	2.43667e-3	3.62159e-3	5.12362e-3	7.07249e-3
$\|\mathbf{p}-\mathbf{p}_h\|$	$h_1=\frac{1}{8}$, $\Delta t_1=\frac{1}{5}$	2.78376e-2	6.48858e-2	1.08826e-1	1.57067e-1	2.12917e-1
	$h_2=\frac{1}{16}$, $\Delta t_2=\frac{1}{10}$	1.64798e-2	3.75976e-2	6.07238e-2	8.65551e-2	1.16061e-1
	$h_3=\frac{1}{32}$, $\Delta t_3=\frac{1}{20}$	9.37106e-3	2.03038e-2	3.21754e-2	4.54276e-2	6.05747e-2
	$h_4=\frac{1}{64}$, $\Delta t_4=\frac{1}{40}$	5.00782e-3	1.05482e-2	1.65567e-2	2.32655e-2	3.09375e-2

Table 2: Convergence orders of u, $\mathbf{q}$, w, and $\mathbf{p}$

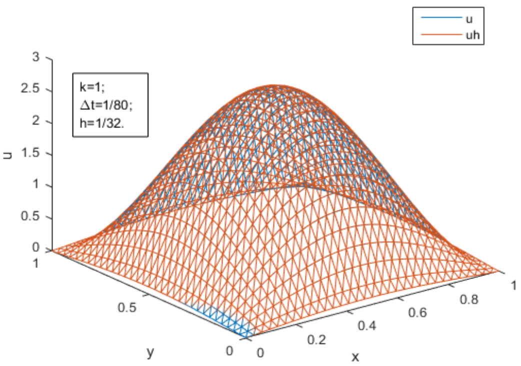
		$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1.0$
$\|u-u_h\|$	h_1/h_2	0.964060	0.963556	0.962613	0.961332	0.959826
	h_2/h_3	0.982210	0.981940	0.981433	0.980744	0.979914
	h_3/h_4	0.991218	0.991096	0.990837	0.990480	0.990064
$\|\mathbf{q}-\mathbf{q}_h\|$	h_1/h_2	1.013865	1.018727	1.022935	1.026516	1.029533
	h_2/h_3	1.006104	1.008672	1.010874	1.012727	1.014270
	h_3/h_4	1.002793	1.004110	1.005244	1.006189	1.006973
$\|w-w_h\|$	h_1/h_2	1.025482	1.016979	1.020782	1.024637	1.027824
	h_2/h_3	1.001641	1.005982	1.009663	1.012304	1.014255
	h_3/h_4	0.997621	1.002133	1.073739	1.061647	1.007083
$\|\mathbf{p}-\mathbf{p}_h\|$	h_1/h_2	0.756336	0.787262	0.841689	0.859689	0.875408
	h_2/h_3	0.814415	0.888891	0.916304	0.930050	0.938096
	h_3/h_4	0.904030	0.944753	0.958543	0.965377	0.969359



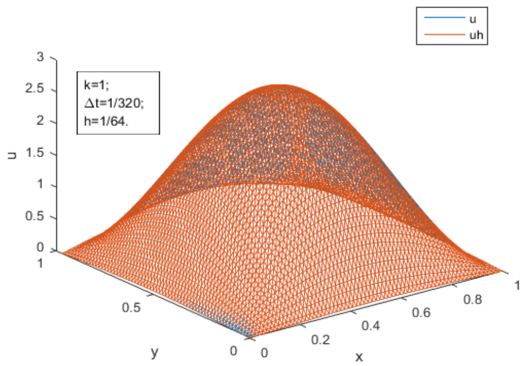
(a) u, u_h at $h = \frac{1}{8}$ and $\Delta t = \frac{1}{5}$



(b) u, u_h at $h = \frac{1}{16}$ and $\Delta t = \frac{1}{20}$



(c) u, u_h at $h = \frac{1}{32}$ and $\Delta t = \frac{1}{80}$



(d) u, u_h at $h = \frac{1}{64}$ and $\Delta t = \frac{1}{320}$

Fig. 1: Comparison of numerical solution u_h with exact solution u at $t = 1$

Table 3: L_2 norms of errors for u, $\mathbf{q}$, w, and $\mathbf{p}$

		$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1.0$
$\|u-u_h\|$	$h_1=\frac{1}{8}$, $\Delta t_1=\frac{1}{5}$	9.67088e-3	1.99375e-2	3.11787e-2	4.38175e-2	5.83378e-2
	$h_2=\frac{1}{16}$, $\Delta t_2=\frac{1}{20}$	2.50417e-3	5.13787e-3	8.00183e-3	1.12069e-2	1.48783e-2
	$h_3=\frac{1}{32}$, $\Delta t_3=\frac{1}{80}$	6.31323e-4	1.29356e-3	2.01232e-3	2.81565e-3	3.73515e-3
	$h_4=\frac{1}{64}$, $\Delta t_4=\frac{1}{320}$	1.58101e-4	3.23839e-4	5.03615e-4	7.04489e-4	9.34362e-4
$\|\mathbf{q}-\mathbf{q}_h\|$	$h_1=\frac{1}{8}$, $\Delta t_1=\frac{1}{5}$	4.25240e-2	8.76765e-2	1.37125e-1	1.92731e-1	2.56625e-1
	$h_2=\frac{1}{16}$, $\Delta t_2=\frac{1}{20}$	1.10826e-2	2.27408e-2	3.54210e-2	4.96141e-2	6.58747e-2
	$h_3=\frac{1}{32}$, $\Delta t_3=\frac{1}{80}$	2.80302e-3	5.74391e-3	8.93647e-3	1.25053e-2	1.63174e-2
	$h_4=\frac{1}{64}$, $\Delta t_4=\frac{1}{320}$	7.03103e-4	1.44028e-3	2.24014e-3	3.13397e-3	4.15698e-3
$\|w-w_h\|$	$h_1=\frac{1}{8}$, $\Delta t_1=\frac{1}{5}$	9.12603e-3	1.86851e-2	2.92441e-2	4.11360e-2	5.48032e-2
	$h_2=\frac{1}{16}$, $\Delta t_2=\frac{1}{20}$	2.38605e-3	4.85550e-3	7.55967e-3	1.05903e-2	1.40633e-2
	$h_3=\frac{1}{32}$, $\Delta t_3=\frac{1}{80}$	6.05522e-4	1.22851e-3	1.90900e-3	2.67063e-3	3.54280e-3
	$h_4=\frac{1}{64}$, $\Delta t_4=\frac{1}{320}$	1.52210e-4	3.08387e-4	4.78838e-4	6.69553e-4	8.87910e-4
$\|\mathbf{p}-\mathbf{p}_h\|$	$h_1=\frac{1}{8}$, $\Delta t_1=\frac{1}{5}$	4.53065e-2	8.62345e-2	1.32585e-1	1.85173e-1	2.45813e-1
	$h_2=\frac{1}{16}$, $\Delta t_2=\frac{1}{20}$	1.15993e-2	2.22267e-2	3.41311e-2	4.75588e-2	6.29896e-2
	$h_3=\frac{1}{32}$, $\Delta t_3=\frac{1}{80}$	2.91513e-3	5.60208e-3	8.60146e-3	1.19788e-2	1.58561e-2
	$h_4=\frac{1}{64}$, $\Delta t_4=\frac{1}{320}$	7.29291e-4	1.40352e-3	2.15526e-3	3.00126e-3	3.97223e-3

Table 4: Convergence orders of u, $\mathbf{q}$, w, and $\mathbf{p}$

		$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1.0$
$\|u-u_h\|$	h_1/h_2	1.949315	1.956242	1.962159	1.967120	1.971221
	h_2/h_3	1.987882	1.989823	1.991470	1.992847	1.993972
	h_3/h_4	1.997532	1.998038	1.998467	1.998819	1.999113
$\|\mathbf{q}-\mathbf{q}_h\|$	h_1/h_2	1.939981	1.946907	1.952815	1.957767	1.961865
	h_2/h_3	1.983242	1.990210	1.986828	1.988211	2.013313
	h_3/h_4	1.995174	1.995684	1.996116	1.996476	1.972803
$\|w-w_h\|$	h_1/h_2	1.935363	1.944197	1.951750	1.957658	1.962325
	h_2/h_3	1.978373	1.982710	1.985506	1.987491	1.988973
	h_3/h_4	1.992116	1.994096	1.995208	1.995910	1.996405
$\|\mathbf{p}-\mathbf{p}_h\|$	h_1/h_2	1.965680	1.955971	1.957759	1.961090	1.964376
	h_2/h_3	1.992406	1.988259	1.988433	1.989229	1.990076
	h_3/h_4	1.998994	1.996913	1.996720	1.996843	1.997017

Table 5: L_2 norms of relative errors for u, $\mathbf{q}$, w, and $\mathbf{p}$

		$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1.0$
$\|u-u_h\|$	$h_1=\frac{1}{8}$, $\Delta t_1=\frac{1}{2}$	1.18648e-1	1.21844e-1	1.21902e-1	1.21899e-1	1.21898e-1
	$h_2=\frac{1}{16}$, $\Delta t_2=\frac{1}{4}$	6.13049e-2	6.27734e-2	6.28006e-2	6.27999e-2	6.27997e-2
	$h_3=\frac{1}{32}$, $\Delta t_3=\frac{1}{8}$	3.11322e-2	3.18297e-2	3.18431e-2	3.18431e-2	3.18431e-2
	$h_4=\frac{1}{64}$, $\Delta t_4=\frac{1}{16}$	1.56832e-2	1.60221e-2	1.60289e-2	1.60289e-2	1.60289e-2
$\|\mathbf{q}-\mathbf{q}_h\|$	$h_1=\frac{1}{8}$, $\Delta t_1=\frac{1}{2}$	1.18571e-1	1.21997e-1	1.22100e-1	1.22104e-1	1.22104e-1
	$h_2=\frac{1}{16}$, $\Delta t_2=\frac{1}{4}$	6.09759e-2	6.24793e-2	6.25171e-2	6.25181e-2	6.25181e-2
	$h_3=\frac{1}{32}$, $\Delta t_3=\frac{1}{8}$	3.09221e-2	3.16161e-2	3.16318e-2	3.16322e-2	3.16322e-2
	$h_4=\frac{1}{64}$, $\Delta t_4=\frac{1}{16}$	1.55709e-2	1.59028e-2	1.59099e-2	1.59101e-2	1.59101e-2
$\|w-w_h\|$	$h_1=\frac{1}{8}$, $\Delta t_1=\frac{1}{2}$	1.14025e-1	1.17325e-1	1.17421e-1	1.17424e-1	1.17424e-1
	$h_2=\frac{1}{16}$, $\Delta t_2=\frac{1}{4}$	5.81838e-2	5.96423e-2	5.96790e-2	5.96799e-2	5.96800e-2
	$h_4=\frac{1}{32}$, $\Delta t_3=\frac{1}{8}$	2.93591e-2	3.00335e-2	3.00490e-2	3.00494e-2	3.00494e-2
	$h_3=\frac{1}{64}$, $\Delta t_4=\frac{1}{16}$	1.47430e-2	1.50656e-2	1.50727e-2	1.50729e-2	1.50729e-2
$\|\mathbf{p}-\mathbf{p}_h\|$	$h_1=\frac{1}{8}$, $\Delta t_1=\frac{1}{2}$	1.08438e-1	1.11635e-1	1.11726e-1	1.11729e-1	1.11729e-1
	$h_2=\frac{1}{16}$, $\Delta t_2=\frac{1}{4}$	5.69473e-2	5.83682e-2	5.84031e-2	5.84040e-2	5.84040e-2
	$h_3=\frac{1}{32}$, $\Delta t_3=\frac{1}{8}$	2.91739e-2	2.98324e-2	2.98471e-2	2.98475e-2	2.98475e-2
	$h_4=\frac{1}{64}$, $\Delta t_4=\frac{1}{16}$	1.47639e-2	1.50791e-2	1.50859e-2	1.50860e-2	1.50860e-2

Table 6: Convergence orders of u, $\mathbf{q}$, w, and $\mathbf{p}$

		$t=0.2$	$t=0.4$	$t=0.6$	$t=0.8$	$t=1.0$
$\|u-u_h\|$	h_1/h_2	0.952613	0.956810	0.956872	0.956852	0.956845
	h_2/h_3	0.977595	0.979780	0.979798	0.979781	0.979777
	h_3/h_4	0.989188	0.990310	0.990305	0.990305	0.990305
$\|\mathbf{q}-\mathbf{q}_h\|$	h_1/h_2	0.959440	0.965395	0.965740	0.965765	0.965765
	h_2/h_3	0.979601	0.982719	0.982875	0.982880	0.982880
	h_3/h_4	0.989786	0.991379	0.991451	0.991451	0.991451
$\|w-w_h\|$	h_1/h_2	0.970661	0.976103	0.976395	0.976410	0.976408
	h_2/h_3	0.986810	0.989763	0.989906	0.989909	0.989911
	h_3/h_4	0.993778	0.995314	0.995379	0.995379	0.995379
$\|\mathbf{p}-\mathbf{p}_h\|$	h_1/h_2	0.929171	0.935535	0.935848	0.935865	0.935865
	h_2/h_3	0.994949	0.968303	0.968454	0.968457	0.968457
	h_3/h_4	0.982604	0.984330	0.984390	0.984400	0.984400

Table 7: L_2 norms of relative errors for u, $\mathbf{q}$, w, and $\mathbf{p}$

		$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1.0$
$\|u - u_h\|$	$h_1 = \frac{1}{8}$, $\Delta t_1 = \frac{1}{2}$	1.19091e-1	1.22462e-1	1.22553e-1	1.22555e-1	1.22555e-1
	$h_2 = \frac{1}{16}$, $\Delta t_2 = \frac{1}{4}$	3.09925e-2	3.16823e-2	3.16971e-2	3.16973e-2	3.16973e-2
	$h_3 = \frac{1}{32}$, $\Delta t_3 = \frac{1}{8}$	7.82004e-3	7.98155e-3	7.98479e-3	7.98485e-3	7.98485e-3
	$h_4 = \frac{1}{64}$, $\Delta t_4 = \frac{1}{16}$	1.95945e-3	1.99916e-3	1.99962e-3	1.99963e-3	1.99963e-3
$\|\mathbf{q} - \mathbf{q}_h\|$	$h_1 = \frac{1}{8}$, $\Delta t_1 = \frac{1}{2}$	1.18045e-1	1.21453e-1	1.21556e-1	1.21559e-1	1.21560e-1
	$h_2 = \frac{1}{16}$, $\Delta t_2 = \frac{1}{4}$	3.08857e-2	3.15891e-2	3.16065e-2	3.16071e-2	3.16072e-2
	$h_3 = \frac{1}{32}$, $\Delta t_3 = \frac{1}{8}$	7.81346e-3	7.97852e-3	7.98240e-3	7.98254e-3	7.98255e-3
	$h_4 = \frac{1}{64}$, $\Delta t_4 = \frac{1}{16}$	1.96033e-3	2.00093e-3	2.00167e-3	2.00169e-3	2.00169e-3
$\|w - w_h\|$	$h_1 = \frac{1}{8}$, $\Delta t_1 = \frac{1}{2}$	1.12330e-1	1.15519e-1	1.15610e-1	1.15612e-1	1.15612e-1
	$h_2 = \frac{1}{16}$, $\Delta t_2 = \frac{1}{4}$	2.95040e-2	3.01662e-2	3.01811e-2	3.01814e-2	3.01814e-2
	$h_4 = \frac{1}{32}$, $\Delta t_3 = \frac{1}{8}$	7.44492e-3	7.59944e-3	7.60264e-3	7.60271e-3	7.60271e-3
	$h_3 = \frac{1}{64}$, $\Delta t_4 = \frac{1}{16}$	1.86538e-3	1.90328e-3	1.90394e-3	1.90395e-3	1.90395e-3
$\|\mathbf{p} - \mathbf{p}_h\|$	$h_1 = \frac{1}{8}$, $\Delta t_1 = \frac{1}{2}$	1.13293e-1	1.16536e-1	1.16629e-1	1.16631e-1	1.16631e-1
	$h_2 = \frac{1}{16}$, $\Delta t_2 = \frac{1}{4}$	2.94193e-2	3.00745e-2	3.00892e-2	3.00895e-2	3.00895e-2
	$h_3 = \frac{1}{32}$, $\Delta t_3 = \frac{1}{8}$	7.44548e-3	7.59886e-3	7.60204e-3	7.60210e-3	7.60210e-3
	$h_4 = \frac{1}{64}$, $\Delta t_4 = \frac{1}{16}$	1.86834e-3	1.90603e-3	1.90671e-3	1.90672e-3	1.90672e-3

Table 8: Convergence orders of u, $\mathbf{q}$, w, and $\mathbf{p}$

		$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1.0$
$\|u - u_h\|$	h_1/h_2	1.942073	1.950585	1.950983	1.950997	1.950997
	h_2/h_3	1.986670	1.988936	1.989024	1.989023	1.989023
	h_3/h_4	1.996727	1.997275	1.997529	1.997532	1.997532
$\|\mathbf{q} - \mathbf{q}_h\|$	h_1/h_2	1.934326	1.942899	1.943328	1.943336	1.943343
	h_2/h_3	1.982906	1.985234	1.985327	1.985329	1.985332
	h_3/h_4	1.994865	1.995450	1.995618	1.995629	1.995631
$\|w - w_h\|$	h_1/h_2	1.928761	1.937125	1.937549	1.937560	1.937560
	h_2/h_3	1.986582	1.988968	1.989073	1.989074	1.989074
	h_3/h_4	1.996787	1.997405	1.997512	1.997518	1.997518
$\|\mathbf{p} - \mathbf{p}_h\|$	h_1/h_2	1.945224	1.954163	1.954609	1.954619	1.954619
	h_2/h_3	1.982326	1.984686	1.984787	1.984790	1.984790
	h_3/h_4	1.994608	1.995212	1.995301	1.995305	1.995305

5 Conclusion and Perspective

In this article, we propose a EMFE method to solve a class of 2D Sobolev equation(1.1) by introducing three auxiliary variables. Optimal error estimates are obtained for both the semi-discrete and fully discrete schemes. Finally, two numerical examples are given to verify the optimal order of the proposed scheme. In future work, we will discuss how to extend this method to the analysis of high-dimensional nonlinear Sobolev equations.

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