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Unconditionally stable modified methods for the solution of two and three dimensional telegraphic equation with Robin boundary conditions

Swarn Singh^{a*}, Suruchi Singh^b, Ping Lin^c, Rajni Arora^d

^aDepartment of Mathematics, Sri Venkateswara College, University of Delhi, New Delhi-110021, India

^bDepartment of Mathematics, Aditi Mahavidyalaya, University of Delhi, Delhi-110039, India

^cDivision of Mathematics, University of Dundee, DD1 4HN, U.K.

^dDepartment of Mathematics, University of Delhi, New Delhi-110007, India

Abstract: In this paper, we discuss modified three level implicit difference methods of order two in time and four in space for the numerical solution of two and three dimensional telegraphic equation with Robin boundary conditions. Ghost points are introduced to obtain fourth order approximations for boundary conditions. Matrix stability analysis is carried out to prove stability of the method for telegraphic equations in two and three dimensions with Neumann boundary conditions. Numerical experiments are carried out and the results are found to be better when compared with the results obtained by other existing methods.

Keywords: matrix stability, Neumann boundary conditions, Numerov type approximation, Robin boundary conditions, telegraphic equation, unconditionally stable

1. Introduction

Telegraphic equation describes an electrical signal travelling along a transmission cable. Besides, it has many applications in various other fields of sciences. It is being encountered in the field of mathematical modeling of random walk of animals [1]. Problem of solving telegraphic equation continually arises in the study and development of Magnetic Resonance Imaging (MRI) technique which is used in clinical diagnosis [2]. Hence, solving telegraphic equation is of major interest. In recent past, several techniques [3]-[16] are discovered to solve telegraphic equation in one, two and three dimensions. Dehghan and Mohebbi in [3] proposed an implicit collocation method for the solution of two-dimensional linear hyperbolic equation with Dirichlet boundary conditions. In [4], authors use variational multiscale element-free Galerkin method for solving various partial differential equations. Dehghan et al in [5]-[7], respectively discuss methods based on dual reciprocity integral equation method, Chebyshev tau method and He's variational iteration method for solving telegraph equation. Authors in [8], [12] discussed algorithms based on B-spline differential quadrature method to solve two dimensional hyperbolic telegraph equation. In [9], a combination of meshless local weak and strong (MLWS) forms is applied to solve two dimensional telegraphic equation. In [13], the author obtained unconditionally stable alternating direction implicit (ADI) methods for the solution of multi-dimensional telegraphic equations. It must be noted that major emphasis is given to the solution of problems with Dirichlet boundary conditions and hence Von Neumann stability analysis is usually carried out to verify the stability of the methods which does not bring into account the effect of boundary conditions. Very recently, Singh et al [16] proposed

* Corresponding author.

Email address: ssingh@svc.ac.in

stable schemes of $O(k^2 + k^2h^2 + h^4)$ and $O(k^4 + k^4h^2)$ for the solution of one dimensional telegraphic equation subject to Neumann boundary conditions and employed matrix stability method to test the stability of the proposed schemes. Matrix stability method analysis is more general stability analysis and can be applied to variable coefficient problem also while Von Neumann stability can usually be applied only to constant coefficient case. We, in this paper discuss methods of $O(k^2 + k^2h^2 + h^4)$ for solving telegraphic equation in two and three dimensions subject to appropriate initial and Robin boundary conditions. The proposed methods are three level implicit difference methods which are based on Numerov type approximations proposed in [14] and [15]. We discuss in detail the stability of the proposed methods by matrix stability method for solving telegraphic equation with Neumann boundary conditions. The methods are shown to be solvable by reducing each of them to two level problem. We further convert these methods into ADI methods suitable to facilitate the computation. The organization of the paper is as follows:

Section 2 is divided into three subsections. First subsection contains details of approximation for two dimensional problem and high order approximations for boundary conditions. Next two subsections carry details of stability analysis of the proposed method and corresponding ADI method. Likewise, section 3 also comprises of three subsections carrying details of approximation for three dimensional problem and high order approximations for boundary conditions; stability analysis of the proposed method; and ADI method. In section 4, we present numerical experiments to demonstrate the efficiency and accuracy of the modified methods. Finally, concluding remarks are given in section 5.

2. Two dimensional telegraphic equation

2.1. Numerical Method

In this section, we consider the following two-dimensional telegraphic equation

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + u_{yy} + f(x, y, t), 0 \leq x, y \leq 1, t > 0 \quad (2.1.1)$$

subject to the initial conditions

$$u(x, y, 0) = \phi(x, y), u_t(x, y, 0) = \psi(x, y), 0 \leq x, y \leq 1 \quad (2.1.2)$$

and the Robin boundary conditions

$$a_1 u(0, y, t) + b_1 u_x(0, y, t) = h_1(y, t), 0 \leq y \leq 1, t > 0 \quad (2.1.3)$$

$$a_2 u(1, y, t) + b_2 u_x(1, y, t) = h_2(y, t), 0 \leq y \leq 1, t > 0 \quad (2.1.4)$$

$$a_3 u(x, 0, t) + b_3 u_y(x, 0, t) = h_3(x, t), 0 \leq x \leq 1, t > 0 \quad (2.1.5)$$

$$a_4 u(x, 1, t) + b_4 u_y(x, 1, t) = h_4(x, t), 0 \leq x \leq 1, t > 0 \quad (2.1.6)$$

where $\alpha > 0, \beta \geq 0$ are constants and the functions h_1, h_2, h_3, h_4 and the forcing function $f(x, y, t)$ are assumed to be sufficiently smooth to maintain the order of accuracy of the difference method discussed. In boundary conditions (2.1.3)-(2.1.6) a_i 's and b_i 's are scalars. Consider the domain $\{(x, y, t) | 0 \leq x, y \leq 1, t > 0\}$ which is discretized uniformly into N subintervals along x and y directions with spacing of $h > 0$ such that $Nh = 1$ and J subintervals in time direction with spacing of $k > 0$, where N and J are

positive integers. Then, the discretized domain is $\Omega = \{(x_l, y_m, t_j) | 0 \leq l, m \leq N, 0 < j \leq J\}$. The grid point $(x_l, y_m, t_j) = (lh, mh, jk)$ is denoted by (l, m, j) and the approximate solution of (2.1.1) at the grid point (l, m, j) is denoted by $U_{l,m}^j$. Let $p = \frac{k}{h} > 0$ be the grid ratio parameter.

A Numerov type approximation [14] with accuracy of $O(k^2 + k^2h^2 + h^4)$ for the solution of (2.1.1) for $l, m = 1(1)N - 1, 0 < j \leq J$ may be written as

$$\begin{aligned} \delta_t^2 U_{l,m}^j + \sqrt{a}(2\mu_t \delta_t) U_{l,m}^j + \frac{\sqrt{a}}{12} (\delta_x^2 + \delta_y^2) (2\mu_t \delta_t) U_{l,m}^j + b U_{l,m}^j + \left(\frac{b}{12} - p^2\right) (\delta_x^2 + \delta_y^2) U_{l,m}^j \\ - \frac{p^2}{6} \delta_x^2 \delta_y^2 U_{l,m}^j + \frac{1}{12} (\delta_x^2 + \delta_y^2) \delta_t^2 U_{l,m}^j \\ = \frac{k^2}{12} (f_{l+1,m}^j + f_{l-1,m}^j + f_{l,m+1}^j + f_{l,m-1}^j + 8f_{l,m}^j) + O(k^4 + k^4h^2 + k^2h^4) \end{aligned} \quad (2.1.7)$$

where $a = \alpha^2 k^2$, $b = \beta^2 k^2$ and δ_t and μ_t are central and averaging operators with respect to time direction respectively. Similarly, operators with respect to spatial directions are defined.

In case, where all b_i 's = 0, we obtain Dirichlet boundary conditions and so the given problem can be easily solved by using approximation (2.1.7) whereas in case where not all b_i 's = 0, we introduce ghost points. We will discuss in detail the case when none of b_i 's are equal to zero. We extend the spatial domain by introducing two ghost points outside both x and y domains at each time level and assume that the method (2.1.7) holds on the extended domain $\Omega' = \{(x_l, y_m, t_j) | -1 \leq l, m \leq N + 1, 0 < j \leq J\}$. We derive approximations at ghost points in such a way that accuracy of the method is not degraded. For deriving an explicit approximation for $U_{-1,m}^j$ where $m = 0(1)N$, we make use of Taylor's expansion about the grid point $(0, m, j)$, which provides us

$$\frac{U_{1,m}^j - U_{-1,m}^j}{2h} = U_{x0,m}^j + \frac{h^2}{6} U_{xxx0,m}^j + O(h^4) \quad (2.1.8)$$

Making use of (2.1.1) and (2.1.3) we get

$$U_{xxx0,m}^j = \omega_{1tt0,m}^j + 2\alpha\omega_{1t0,m}^j + \beta^2\omega_{10,m}^j - \omega_{1yy0,m}^j - f_{x0,m}^j, \text{ where } \omega_{10,m}^j = \left(\frac{h_{1m}^j - a_1 U_{0,m}^j}{b_1}\right).$$

Ignoring the fourth order truncation error term, we obtain a fourth order approximation for $U_{-1,m}^j$ from (2.1.8) as

$$U_{-1,m}^j = U_{1,m}^j - 2h \left[\omega_{10,m}^j + \frac{h^2}{6} (\omega_{1tt0,m}^j + 2\alpha\omega_{1t0,m}^j + \beta^2\omega_{10,m}^j - \omega_{1yy0,m}^j - f_{x0,m}^j) \right] \quad (2.1.9)$$

Similarly, to obtain fourth order approximation at $(N + 1, m, j)$, for $m = 0(1)N$ we make use of Taylor's expansion about the grid point (N, m, j) , which provides us

$$\frac{U_{N+1,m}^j - U_{N-1,m}^j}{2h} = U_{xN,m}^j + \frac{h^2}{6} U_{xxxN,m}^j + O(h^4) \quad (2.1.10)$$

Making use of (2.1.1) and (2.1.4) we get

$$U_{xxxN,m}^j = \omega_{2ttN,m}^j + 2\alpha\omega_{2tN,m}^j + \beta^2\omega_{2N,m}^j - \omega_{2yyN,m}^j - f_{xN,m}^j, \text{ where } \omega_{2N,m}^j = \left(\frac{h_2^j - a_2 U_{N,m}^j}{b_2} \right).$$

Ignoring the fourth order truncation error term, we obtain a fourth order approximation for $U_{N+1,m}^j$ from (2.1.10) as

$$U_{N+1,m}^j = U_{N-1,m}^j + 2h \left[\omega_{2N,m}^j + \frac{h^2}{6} \left(\omega_{2ttN,m}^j + 2\alpha\omega_{2tN,m}^j + \beta^2\omega_{2N,m}^j - \omega_{2yyN,m}^j - f_{xN,m}^j \right) \right] \quad (2.1.11)$$

Similarly, approximations at points $(l, -1, j)$ and $(l, N + 1, j)$ for $l = 0(1)N$ are obtained. Next, we derive an approximation for $U_{-1,-1}^j$.

Taylor's expansion about grid point $(0,0,j)$ provides us

$$\frac{U_{1,1}^j - U_{-1,-1}^j}{2h} = (U_{x0,0}^j + U_{y0,0}^j) + \frac{h^2}{6} (U_{xxx0,0}^j + 3U_{xxy0,0}^j + 3U_{xyy0,0}^j + U_{yyy0,0}^j) + O(h^4) \quad (2.1.12)$$

Making use of (2.1.3), (2.1.5) and (2.1.1) and ignoring the fourth order truncation error term we obtain fourth order approximation for $U_{-1,-1}^j$ as

$$\begin{aligned} U_{-1,-1}^j = U_{1,1}^j - 2h & \left[\omega_{10,0}^j + \omega_{30,0}^j \right. \\ & + \frac{h^2}{6} \left(\omega_{1tt0,0}^j + 2\alpha\omega_{1t0,0}^j + \beta^2\omega_{10,0}^j + 2\omega_{1yy0,0}^j - f_{x0,0}^j + \omega_{3tt0,0}^j + 2\alpha\omega_{3t0,0}^j \right. \\ & \left. \left. + \beta^2\omega_{30,0}^j + 2\omega_{3xx0,0}^j - f_{y0,0}^j \right) \right] \end{aligned} \quad (2.1.13)$$

$$\text{where } \omega_{10,0}^j = \left(\frac{h_1^j - a_1 U_{0,0}^j}{b_1} \right), \omega_{30,0}^j = \left(\frac{h_3^j - a_3 U_{0,0}^j}{b_3} \right)$$

Similarly, approximations at other ghost points $(N + 1, N + 1, j)$, $(-1, N + 1, j)$ and $(N + 1, -1, j)$ are obtained. Finally, eliminating approximations at all the ghost points from above obtained approximations and (2.1.7) on extended domain Ω' , we obtain a method of $O(k^2 + k^2 h^2 + h^4)$ for the telegraphic equation (2.1.1) with Robin boundary conditions, which maintains the block tri-diagonal structure of the coefficient matrices.

2.2. Stability Analysis

In this section we discuss the matrix stability analysis of the proposed method subject to Neumann boundary conditions, that is, the case when all a_i 's are zero and none of b_i 's are zero. Without loss of generality, we may take each $b_i = 1$, such that

$$u_x(0, y, t) = h_1(y, t), u_x(1, y, t) = h_2(y, t) \quad 0 \leq y \leq 1, t > 0 \quad (2.2.1)$$

$$u_y(x, 0, t) = h_3(x, t), u_y(x, 1, t) = h_4(x, t) \quad 0 \leq x \leq 1, t > 0 \quad (2.2.2)$$

High order approximations for Neumann boundary conditions are obtained from the previous section. In order to obtain an unconditionally stable method, suitable for obtaining ADI method, we follow the ideas given by Chawla [17] and rewrite the method (2.1.7) in the modified form for $l, m = 0(1)N$ as

$$\begin{aligned} [1 + A_1 \delta_y^2] & \left[(A_0 + A_1 \delta_x^2) \delta_t^2 U_{l,m}^j + \sqrt{a} \left(1 + \frac{1}{12} \delta_x^2 \right) (2\mu_t \delta_t) U_{l,m}^j \right] \\ & = \left[\frac{p^2}{6} \delta_x^2 \delta_y^2 + \left(p^2 - \frac{b}{12} \right) (\delta_x^2 + \delta_y^2) - b \right] U_{l,m}^j \\ & + \frac{k^2}{12} (f_{l+1,m}^j + f_{l-1,m}^j + f_{l,m+1}^j + f_{l,m-1}^j + 8f_{l,m}^j) + O(k^4 + k^4 h^2 + k^2 h^4) \end{aligned} \quad (2.2.3)$$

$$\text{where, } A_0 = (1 + \eta b^2), A_1 = \frac{1}{12} (1 - 12\gamma p^2).$$

where η and γ are free parameters to be determined. The additional terms are of high order and do not affect the accuracy of the method. Now, when the scheme (2.2.3) is expanded for $l, m = 0(1)N$, we get ghost points which are being eliminated by using the derived high order approximations we obtained in Section 2.1. For stability, we consider the homogeneous part of the thus obtained equation, which in matrix form together with Neumann boundary conditions can be written as

$$\mathbf{Z}U^{j+1} + \mathbf{X}U^j + \mathbf{Y}U^{j-1} = \mathbf{C} \quad (2.2.4)$$

where,

$$\begin{aligned} \mathbf{Z} &= \left(\frac{1+\sqrt{a}+\eta b^2}{12} \right) \mathcal{A}_1 + (1 + \eta b^2) \gamma p^2 \mathcal{A}_2 + \frac{\sqrt{a} \gamma p^2}{12} \mathcal{A}_3 + \left(\frac{1+\sqrt{a}}{144} + \gamma^2 p^4 \right) \mathcal{A}_4 + \frac{\eta b^2}{12} \mathcal{A}_5 + \gamma p^2 \mathcal{A}_6, \\ \mathbf{X} &= \left(\frac{b-2-2\eta b^2}{12} \right) \mathcal{A}_1 - 2(1 + \eta b^2) \gamma p^2 \mathcal{A}_2 - 2 \left(\frac{1}{144} + \gamma^2 p^4 \right) \mathcal{A}_4 - \frac{2\eta b^2}{12} \mathcal{A}_5 + (p^2 - 2\gamma p^2) \mathcal{A}_6, \\ \mathbf{Y} &= \left(\frac{1-\sqrt{a}+\eta b^2}{12} \right) \mathcal{A}_1 + (1 + \eta b^2) \gamma p^2 \mathcal{A}_2 - \frac{\sqrt{a} \gamma p^2}{12} \mathcal{A}_3 + \left(\frac{1-\sqrt{a}}{144} + \gamma^2 p^4 \right) \mathcal{A}_4 + \frac{\eta b^2}{12} \mathcal{A}_5 + \gamma p^2 \mathcal{A}_6, \end{aligned}$$

$$\begin{aligned} \mathcal{A}_1 &= \begin{bmatrix} \mathbf{A}' & \mathbf{A} & & & \mathbf{0} \\ \mathbf{A} & 2\mathbf{A}' & \mathbf{A} & & \\ & \ddots & \ddots & \ddots & \\ & & \mathbf{A} & 2\mathbf{A}' & \mathbf{A} \\ \mathbf{0} & & & \mathbf{A} & \mathbf{A}' \end{bmatrix}, \mathcal{A}_2 = \begin{bmatrix} \frac{1}{2} & 0 & & & 0 \\ 0 & 1 & 0 & & \\ & & \ddots & & \\ & & 0 & 1 & 0 \\ 0 & & & 0 & \frac{1}{2} \end{bmatrix}, \mathcal{A}_3 = \begin{bmatrix} 2 & \frac{1}{2} & & & 0 \\ \frac{1}{2} & 4 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & 4 & \frac{1}{2} \\ 0 & & & \frac{1}{2} & 2 \end{bmatrix}, \\ \mathcal{A}_4 &= \begin{bmatrix} \mathbf{A} & -\mathbf{A} & & & \mathbf{0} \\ -\mathbf{A} & 2\mathbf{A} & -\mathbf{A} & & \\ & \ddots & \ddots & \ddots & \\ & & -\mathbf{A} & 2\mathbf{A} & -\mathbf{A} \\ \mathbf{0} & & & -\mathbf{A} & \mathbf{A} \end{bmatrix}, \mathcal{A}_5 = \begin{bmatrix} \mathbf{E} & -\mathbf{E} & & & \mathbf{0} \\ -\mathbf{E} & 2\mathbf{E} & -\mathbf{E} & & \\ & \ddots & \ddots & \ddots & \\ & & -\mathbf{E} & 2\mathbf{E} & -\mathbf{E} \\ \mathbf{0} & & & -\mathbf{E} & \mathbf{E} \end{bmatrix}, \end{aligned}$$

$$\mathbf{E} = \begin{bmatrix} 5 & 1 & & & 0 \\ 1 & 10 & 1 & & \\ & & \ddots & & \\ & & & 1 & 10 & 1 \\ 0 & & & & 1 & 5 \end{bmatrix}, \mathcal{A}_4 = \begin{bmatrix} -2\mathbf{B} & 2\mathbf{B} & & & 0 \\ 2\mathbf{B} & -4\mathbf{B} & 2\mathbf{B} & & \\ & \ddots & \ddots & \ddots & \\ & & 2\mathbf{B} & -4\mathbf{B} & 2\mathbf{B} \\ 0 & & & 2\mathbf{B} & -2\mathbf{B} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & & & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & & & \frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

$$\mathcal{A}_5 = \begin{bmatrix} -\mathbf{B} & \mathbf{0} & & & \mathbf{0} \\ \mathbf{0} & -2\mathbf{B} & \mathbf{0} & & \\ & \ddots & \ddots & \ddots & \\ & & \mathbf{0} & -2\mathbf{B} & \mathbf{0} \\ \mathbf{0} & & & \mathbf{0} & -\mathbf{B} \end{bmatrix}, \mathcal{A}_6 = \begin{bmatrix} -4\mathbf{B} & -2\mathbf{B} & & & \mathbf{0} \\ -2\mathbf{B} & -8\mathbf{B} & -2\mathbf{B} & & \\ & \ddots & \ddots & \ddots & \\ & & -2\mathbf{B} & -8\mathbf{B} & -2\mathbf{B} \\ \mathbf{0} & & & -4\mathbf{B} & -2\mathbf{B} \end{bmatrix},$$

\mathbf{C} is the column vector corresponding to the boundary conditions,

$\mathbf{U}^j = [U_{0,0}^j U_{1,0}^j \dots U_{N,0}^j U_{0,1}^j U_{1,1}^j \dots U_{N,1}^j \dots \dots U_{0,N}^j U_{1,N}^j \dots U_{N,N}^j]'$. The matrix \mathcal{A}_1 is strictly diagonally dominant real-symmetric matrix with all main diagonal entries positive, so all the eigen values of \mathcal{A}_1 are real and positive. Also the eigenvalues of the matrices $\mathcal{A}_i, i = 2(1) 6$ are real and non-negative. Thus it can be seen that all the eigen values of the matrix \mathbf{Z} are positive and hence the matrix \mathbf{Z} is invertible. Now, the equation (2.2.4) can be rewritten as

$$\begin{bmatrix} \mathbf{U}^{j+1} \\ \dots \\ \mathbf{U}^j \end{bmatrix} = \begin{bmatrix} -\mathbf{Z}^{-1}\mathbf{X} & \vdots & -\mathbf{Z}^{-1}\mathbf{Y} \\ \dots & \ddots & \dots \\ \mathbf{I} & \vdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U}^j \\ \dots \\ \mathbf{U}^{j-1} \end{bmatrix} + \begin{bmatrix} \mathbf{Z}^{-1}\mathbf{C} \\ \dots \\ \mathbf{0} \end{bmatrix} \quad (2.2.5)$$

Writing $\begin{bmatrix} \mathbf{U}^{j+1} \\ \dots \\ \mathbf{U}^j \end{bmatrix} = \mathbf{V}^{j+1}$, $\begin{bmatrix} \mathbf{U}^j \\ \dots \\ \mathbf{U}^{j-1} \end{bmatrix} = \mathbf{V}^j$, $\begin{bmatrix} -\mathbf{Z}^{-1}\mathbf{X} & \vdots & -\mathbf{Z}^{-1}\mathbf{Y} \\ \dots & \ddots & \dots \\ \mathbf{I} & \vdots & \mathbf{0} \end{bmatrix} = \mathbf{D}$, $\begin{bmatrix} \mathbf{Z}^{-1}\mathbf{C} \\ \dots \\ \mathbf{0} \end{bmatrix} = \mathbf{G}$, we get

$$\mathbf{V}^{j+1} = \mathbf{D}\mathbf{V}^j + \mathbf{G} \quad (2.2.6)$$

Hence, the three-time level problem has now reduced to two time level problem. Thus, the proposed method is solvable.

Now, if $\lambda_{j_i} = \text{eig}(\mathcal{A}_1^{-1}\mathcal{A}_j), j = 2, 3, \dots, 6$, then

$$\frac{\left(\frac{b-2-2\eta b^2}{12}\right) - 2(\mathbf{1} + \eta b^2)\gamma p^2 \lambda_{2_i} - 2\left(\frac{1}{144} + \gamma^2 p^4\right) \lambda_{4_i} - \frac{2\eta b^2}{12} \lambda_{5_i} + (p^2 - 2\gamma p^2) \lambda_{6_i}}{\left(\frac{1 + \sqrt{a} + \eta b^2}{12}\right) + (\mathbf{1} + \eta b^2)\gamma p^2 \lambda_{2_i} + \frac{\sqrt{a}\gamma p^2}{12} \lambda_{3_i} + \left(\frac{1 + \sqrt{a}}{144} + \gamma^2 p^4\right) \lambda_{4_i} + \frac{\eta b^2}{12} \lambda_{5_i} + \gamma p^2 \lambda_{6_i}}$$

and

$$\frac{\left(\frac{1 - \sqrt{a} + \eta b^2}{12}\right) + (\mathbf{1} + \eta b^2)\gamma p^2 \lambda_{2_i} - \frac{\sqrt{a}\gamma p^2}{12} \lambda_{3_i} + \left(\frac{1 - \sqrt{a}}{144} + \gamma^2 p^4\right) \lambda_{4_i} + \frac{\eta b^2}{12} \lambda_{5_i} + \gamma p^2 \lambda_{6_i}}{\left(\frac{1 + \sqrt{a} + \eta b^2}{12}\right) + (\mathbf{1} + \eta b^2)\gamma p^2 \lambda_{2_i} + \frac{\sqrt{a}\gamma p^2}{12} \lambda_{3_i} + \left(\frac{1 + \sqrt{a}}{144} + \gamma^2 p^4\right) \lambda_{4_i} + \frac{\eta b^2}{12} \lambda_{5_i} + \gamma p^2 \lambda_{6_i}}$$

are the eigen values of $Z^{-1}X$ and $Z^{-1}Y$ respectively, having the common set of corresponding linearly independent eigen vectors. Now, the eigen values of the matrix \mathbf{D} are given by the eigen values of the matrix

$$\begin{bmatrix} -M & -N \\ 1 & 0 \end{bmatrix} \quad (2.2.7)$$

where $-M$, $-N$, 1 and 0 are the i^{th} eigen values of $-\mathbf{Z}^{-1}\mathbf{X}$, $-\mathbf{Z}^{-1}\mathbf{Y}$, \mathbf{I} and $\mathbf{0}$ respectively corresponding to i^{th} eigen vector common to all the matrices $-\mathbf{Z}^{-1}\mathbf{X}$, $-\mathbf{Z}^{-1}\mathbf{Y}$, \mathbf{I} and $\mathbf{0}$. If Λ is an eigen value of \mathbf{D} then the characteristic equation of (2.2.7) is

$$\Lambda^2 + M\Lambda + N = 0. \quad (2.2.8)$$

Using the transformation $\Lambda = \frac{1+z}{1-z}$, the characteristic equation (2.2.8) reduces to

$$(1 - M + N)z^2 + 2(1 - N)z + (1 + M + N) = 0.$$

The necessary and sufficient condition for $|\Lambda| < 1$ is that

$$(1 - M + N) > 0, (1 - N) > 0, (1 + M + N) > 0.$$

Now,

$$\begin{aligned} & (1 + M + N) \\ &= \frac{\frac{b}{12} + p^2\lambda_{6i}}{\left(\frac{1 + \sqrt{a} + \eta b^2}{12}\right) + (1 + \eta b^2)\gamma p^2\lambda_{2i} + \frac{\sqrt{a}\gamma p^2}{12}\lambda_{3i} + \left(\frac{1 + \sqrt{a}}{144} + \gamma^2 p^4\right)\lambda_{4i} + \frac{\eta b^2}{12}\lambda_{5i} + \gamma p^2\lambda_{6i}} \\ &> 0. \end{aligned}$$

Further,

$$\begin{aligned} & (1 - N) \\ &= \frac{\frac{2\sqrt{a}}{12} + \frac{2\sqrt{a}}{144}\lambda_{4i} + \frac{2\sqrt{a}\gamma p^2}{12}\lambda_{3i}}{\left(\frac{1 + \sqrt{a} + \eta b^2}{12}\right) + (1 + \eta b^2)\gamma p^2\lambda_{2i} + \frac{\sqrt{a}\gamma p^2}{12}\lambda_{3i} + \left(\frac{1 + \sqrt{a}}{144} + \gamma^2 p^4\right)\lambda_{4i} + \frac{\eta b^2}{12}\lambda_{5i} + \gamma p^2\lambda_{6i}} \\ &> 0. \end{aligned}$$

Now,

$$\begin{aligned} & (1 - M + N) \\ &= \frac{\left(\frac{-b + 4 + 4\eta b^2}{12}\right) + 4(1 + \eta b^2)\gamma p^2\lambda_{2i} + \left(\frac{4}{144} + 4\gamma^2 p^4\right)\lambda_{4i} + \frac{4\eta b^2}{12}\lambda_{5i} - p^2(1 - 4\gamma)\lambda_{6i}}{\left(\frac{1 + \sqrt{a} + \eta b^2}{12}\right) + (1 + \eta b^2)\gamma p^2\lambda_{2i} + \frac{\sqrt{a}\gamma p^2}{12}\lambda_{3i} + \left(\frac{1 + \sqrt{a}}{144} + \gamma^2 p^4\right)\lambda_{4i} + \frac{\eta b^2}{12}\lambda_{5i} + \gamma p^2\lambda_{6i}} \end{aligned}$$

$$\begin{aligned}
&= \frac{\left(\frac{(64\eta - 1)(1 - 8\eta b)^2}{192\eta}\right) - p^2(1 - 4\gamma)\lambda_{6i} + \left(\frac{4}{144} + 4\gamma^2 p^4\right)\lambda_{4i} + \frac{4\eta b^2}{12}\lambda_{5i} + 4(1 + \eta b^2)\gamma p^2\lambda_{2i}}{\left(\frac{1 + \sqrt{a} + \eta b^2}{12}\right) + (1 + \eta b^2)\gamma p^2\lambda_{2i} + \frac{\sqrt{a}\gamma p^2}{12}\lambda_{3i} + \left(\frac{1 + \sqrt{a}}{144} + \gamma^2 p^4\right)\lambda_{4i} + \frac{\eta b^2}{12}\lambda_{5i} + \gamma p^2\lambda_{6i}} \\
&> 0
\end{aligned}$$

for $\eta \geq \frac{1}{64}$, $\gamma \geq \frac{1}{4}$. Hence, $|\Lambda| < 1$ for $\eta \geq \frac{1}{64}$, $\gamma \geq \frac{1}{4}$ and we conclude that for all choices of h, k the proposed method is unconditionally stable. In the similar manner, stability can be achieved in Robin boundary case.

2.3. Alternating Direction Implicit Method

The structure of the matrices in equation (2.2.3) is block tri-diagonal type which cannot be solved directly for $U_{l,m}^j$. So, in order to facilitate the computation, we split equation (2.2.3) into two equations to obtain tri-diagonal matrices which can be easily handled. Ignoring the truncation error term, the method (2.2.3) in two-step ADI form can be written as

$$[1 + A_1 \delta_y^2] U_{l,m}^* = B_m \quad (2.3.1)$$

$$[A_0 + A_1 \delta_x^2] \delta_t^2 U_{l,m}^j + \sqrt{a} \left(1 + \frac{1}{12} \delta_x^2\right) (2\mu_t \delta_t) U_{l,m}^j = U_{l,m}^* \quad (2.3.2)$$

where $U_{l,m}^*$ is an intermediate value, $B_m = \left[\frac{p^2}{6} \delta_x^2 \delta_y^2 + \left(p^2 - \frac{b}{12}\right) (\delta_x^2 + \delta_y^2) - b\right] U_{l,m}^j + \frac{k^2}{12} (f_{l+1,m}^j + f_{l-1,m}^j + f_{l,m+1}^j + f_{l,m-1}^j + 8f_{l,m}^j)$. For each fixed l , equation (2.3.1) can be written in matrix form as

$$\begin{bmatrix} 1 - 2A_1 & A_1 & & & & & \\ A_1 & 1 - 2A_1 & A_1 & & & & \\ & & \ddots & \ddots & & & \\ & & & A_1 & 1 - 2A_1 & A_1 & \\ & & & & A_1 & 1 - 2A_1 & \\ & & & & & & 1 - 2A_1 \end{bmatrix} \begin{bmatrix} U_{l,1}^* \\ U_{l,2}^* \\ \vdots \\ U_{l,N-2}^* \\ U_{l,N-1}^* \end{bmatrix} = \begin{bmatrix} B_1 - A_1 U_{l,0}^* \\ B_2 \\ \vdots \\ B_{N-2} \\ B_{N-1} - A_1 U_{l,N}^* \end{bmatrix} \quad (2.3.3)$$

and for each fixed m , equation (2.3.2) can be written in matrix form as

$$\begin{bmatrix} A_0 - 2A_1 + \frac{5\sqrt{a}}{6} & A_1 + \frac{\sqrt{a}}{12} & & & & & \\ A_1 + \frac{\sqrt{a}}{12} & A_0 - 2A_1 + \frac{5\sqrt{a}}{6} & A_1 + \frac{\sqrt{a}}{12} & & & & \\ & & \ddots & \ddots & & & \\ & & & A_1 + \frac{\sqrt{a}}{12} & A_0 - 2A_1 + \frac{5\sqrt{a}}{6} & A_1 + \frac{\sqrt{a}}{12} & \\ & & & & A_1 + \frac{\sqrt{a}}{12} & A_0 - 2A_1 + \frac{5\sqrt{a}}{6} & \\ & & & & & A_1 + \frac{\sqrt{a}}{12} & A_0 - 2A_1 + \frac{5\sqrt{a}}{6} \end{bmatrix} \begin{bmatrix} U_{1,m}^{j+1} \\ U_{2,m}^{j+1} \\ \vdots \\ U_{N-2,m}^{j+1} \\ U_{N-1,m}^{j+1} \end{bmatrix} \\
= \begin{bmatrix} U_{1,m}^* - L_{1,m} - \left(A_1 + \frac{\sqrt{a}}{12}\right) U_{0,m}^{j+1} \\ U_{2,m}^* - L_{2,m} \\ \vdots \\ U_{N-2,m}^* - L_{N-2,m} \\ U_{N-1,m}^* - L_{N-1,m} - \left(A_1 + \frac{\sqrt{a}}{12}\right) U_{N,m}^{j+1} \end{bmatrix} \quad (2.3.4)$$

where, $L_{l,m} = -2A_1(U_{l+1,m}^j + U_{l-1,m}^j) - 2(A_0 - 2A_1)U_{l,m}^j + U_{l,m}^{j-1}(A_0 - 2A_1 - \frac{5\sqrt{a}}{6}) + (U_{l+1,m}^{j-1} + U_{l-1,m}^{j-1})(A_1 - \frac{\sqrt{a}}{12})$, $l = 1, 2, \dots, N-1$. We first solve (2.3.3) for $U_{l,m}^*$ and the intermediate approximations for boundary required for solving $U_{l,m}^*$ are obtained from (2.3.4). Then equation (2.3.4) is solved for obtaining the required solution. Clearly, both these equations carry tri-diagonal matrices and hence can be easily solved by tri-diagonal solver.

3. Three dimensional telegraphic equation

3.1. Numerical Method

In this section, we consider the following three-dimensional telegraphic equation

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + u_{yy} + u_{zz} + f(x, y, z, t), 0 \leq x, y, z \leq 1, t > 0 \quad (3.1.1)$$

subject to the initial conditions

$$u(x, y, z, 0) = \phi(x, y, z), u_t(x, y, z, 0) = \psi(x, y, z), 0 \leq x, y, z \leq 1 \quad (3.1.2)$$

and the Robin boundary conditions

$$c_1 u(0, y, z, t) + d_1 u_x(0, y, z, t) = g_1(y, z, t), 0 \leq y, z \leq 1, t > 0 \quad (3.1.3)$$

$$c_2 u(1, y, z, t) + d_2 u_x(1, y, z, t) = g_2(y, z, t), 0 \leq y, z \leq 1, t > 0 \quad (3.1.4)$$

$$c_3 u(x, 0, z, t) + d_3 u_y(x, 0, z, t) = g_3(x, z, t), 0 \leq x, z \leq 1, t > 0 \quad (3.1.5)$$

$$c_4 u(x, 1, z, t) + d_4 u_y(x, 1, z, t) = g_4(x, z, t), 0 \leq x, z \leq 1, t > 0 \quad (3.1.6)$$

$$c_5 u(x, y, 0, t) + d_5 u_z(x, y, 0, t) = g_5(x, y, t), 0 \leq x, y \leq 1, t > 0 \quad (3.1.7)$$

$$c_6 u(x, y, 1, t) + d_6 u_z(x, y, 1, t) = g_6(x, y, t), 0 \leq x, y \leq 1, t > 0 \quad (3.1.8)$$

The functions $g_i, i = 1(1)6$ and the forcing function $f(x, y, z, t)$ are assumed to be sufficiently smooth to maintain the order of accuracy of the difference method discussed. In boundary conditions (3.1.3)-(3.1.8) c_i 's and d_i 's are scalars. The domain $\{(x, y, z, t) | 0 \leq x, y, z \leq 1, t > 0\}$ is discretized uniformly into N subintervals along x, y and z directions with spacing of $h > 0$ such that $Nh = 1$ and J subintervals in time direction with spacing of $k > 0$, where N and J are positive integers such that the discretized domain is $\Phi = \{(x_l, y_m, z_n, t_j) | 0 \leq l, m, n \leq N, 0 < j \leq J\}$. For $l, m, n = 0(1)N$ and $0 < j \leq J$, the grid point $(x_l, y_m, z_n, t_j) = (lh, mh, nh, jk)$ is denoted by (l, m, n, j) . The grid spacing along x, y and z directions are chosen to be equal.

A Numerov type approximation [15] with accuracy of $O(k^2 + k^2 h^2 + h^4)$ for the solution of (3.1.1) for $l, m, n = 1(1)N-1, 0 < j \leq J$ may be written as

$$\begin{aligned}
& \delta_t^2 U_{l,m,n}^j + \sqrt{a}(2\mu_t \delta_t) U_{l,m,n}^j + \frac{\sqrt{a}}{12} (\delta_x^2 + \delta_y^2 + \delta_z^2) (2\mu_t \delta_t) U_{l,m,n}^j + b U_{l,m,n}^j \\
& + \left(\frac{b}{12} - p^2 \right) (\delta_x^2 + \delta_y^2 + \delta_z^2) U_{l,m,n}^j - \frac{p^2}{6} (\delta_x^2 \delta_y^2 + \delta_y^2 \delta_z^2 + \delta_z^2 \delta_x^2) U_{l,m,n}^j \\
& + \frac{1}{12} (\delta_x^2 + \delta_y^2 + \delta_z^2) \delta_t^2 U_{l,m,n}^j \\
& = \frac{k^2}{12} (f_{l+1,m,n}^j + f_{l-1,m,n}^j + f_{l,m+1,n}^j + f_{l,m-1,n}^j + f_{l,m,n+1}^j + f_{l,m,n-1}^j + 6f_{l,m}^j) \\
& + O(k^4 + k^4 h^2 + k^2 h^4)
\end{aligned} \tag{3.1.9}$$

As discussed in section 2.1, in this section also we will give details of the case when none of d_i 's are zero. Since, if any of the d_i 's are zero, then in that case we have Dirichlet boundary condition to deal with and no extra effort is required. So, for the derivation of approximations for boundary conditions when all $d_i \neq 0$, we extend the spatial domain by introducing two ghost points outside each of x, y and z domains at each time level and assume that the method (3.1.9) holds on the extended domain $\Phi' = \{(x_l, y_m, z_n, t_j) | -1 \leq l, m, n \leq N+1, 0 < j \leq J\}$. We derive approximations at ghost points in such a way that accuracy of the method is not degraded. For deriving an explicit approximation for $U_{-1,m,n}^j$, we make use of Taylor's expansion about the grid point $(0, m, n, j)$, which provides us

$$\frac{U_{1,m,n}^j - U_{-1,m,n}^j}{2h} = U_{x0,m,n}^j + \frac{h^2}{6} U_{xxx0,m,n}^j + O(h^4) \tag{3.1.10}$$

Using (3.1.1) and (3.1.3), we get

$$U_{xxx0,m,n}^j = \sigma_{1tt0,m,n}^j + 2\alpha \sigma_{1t0,m,n}^j + \beta^2 \sigma_{10,m,n}^j - \sigma_{1yy0,m,n}^j - \sigma_{1zz0,m,n}^j - f_{x0,m,n}^j$$

$$\text{where } \sigma_{10,m,n}^j = \left(\frac{g_{1m,n}^j - c_1 U_{0,m,n}^j}{d_1} \right).$$

Ignoring the fourth order truncation error term, we obtain a fourth order approximation for $U_{-1,m,n}^j$ as

$$\begin{aligned}
U_{-1,m,n}^j &= U_{1,m,n}^j \\
& - 2h \left[\sigma_{10,m,n}^j \right. \\
& \left. + \frac{h^2}{6} \left(\sigma_{1tt0,m,n}^j + 2\alpha \sigma_{1t0,m,n}^j + \beta^2 \sigma_{10,m,n}^j - \sigma_{1yy0,m,n}^j - \sigma_{1zz0,m,n}^j - f_{x0,m,n}^j \right) \right]
\end{aligned} \tag{3.1.11}$$

Similarly, the approximations at $(N+1, m, n, j)$, $(l, -1, n, j)$, $(l, N+1, n, j)$, $(l, m, -1, j)$, $(l, m, N+1, j)$ for $l, m, n = 0(1)N$ are obtained.

Now, we derive an approximation for $U_{-1,-1,n}^j$ with $n = 0(1)N$. Taylor's expansion about grid point $(0, 0, n, j)$ provides us

$$\begin{aligned}
& \frac{U_{1,1,n}^j - U_{-1,-1,n}^j}{2h} \\
&= (U_{x0,0,n}^j + U_{y0,0,n}^j) + \frac{h^2}{6} (U_{xxx0,0,n}^j + 3U_{xxy0,0,n}^j + 3U_{xyy0,0,n}^j + U_{yyy0,0,n}^j) \\
&+ O(h^4)
\end{aligned} \tag{3.1.12}$$

Making use of (3.1.3), (3.1.5) and (3.1.1) and ignoring the fourth order truncation error term we obtain fourth order approximation for $U_{-1,-1,n}^j$ as

$$\begin{aligned}
U_{-1,-1,n}^j &= U_{1,1,n}^j \\
&- 2h \left[\sigma_{10,0,n}^j + \sigma_{30,0,n}^j \right. \\
&+ \frac{h^2}{6} \left(\sigma_{1tt0,0,n}^j + 2\alpha\sigma_{1t0,0,n}^j + \beta^2\sigma_{10,0,n}^j + 2\sigma_{1yy0,0,n}^j - \sigma_{1zz0,0,n}^j - f_{x0,0,n}^j + \sigma_{3tt0,0,n}^j \right. \\
&\left. \left. + 2\alpha\sigma_{3t0,0,n}^j + \beta^2\sigma_{30,0,n}^j + 2\sigma_{3xx0,0,n}^j - \sigma_{3zz,0,n}^j - f_{y0,0,n}^j \right) \right]
\end{aligned} \tag{3.1.13}$$

$$\text{where } \sigma_{10,0,n}^j = \left(\frac{g_{10,n}^j - c_1 U_{0,0,n}^j}{d_1} \right), \sigma_{30,0,n}^j = \left(\frac{g_{30,n}^j - c_3 U_{0,0,n}^j}{d_3} \right).$$

Similarly, approximations at all other such points with ghost points along any two directions and interior point along third direction are obtained. Finally, approximations at points with ghost points occurring simultaneously along all three x, y and z directions are obtained. For example,

$$\begin{aligned}
U_{-1,-1,-1}^j &= U_{1,1,1}^j \\
&- 2h \left[\sigma_{10,0,0}^j + \sigma_{30,0,0}^j + \sigma_{50,0,0}^j \right. \\
&+ \frac{h^2}{6} \left(\sigma_{1tt0,0,0}^j + 2\alpha\sigma_{1t0,0,0}^j + \beta^2\sigma_{10,0,0}^j + 2\sigma_{1yy,0,0}^j + 2\sigma_{1zz,0,0}^j - f_{x0,0,0}^j + \sigma_{3tt,0,0}^j \right. \\
&+ 2\alpha\sigma_{3t0,0,0}^j + \beta^2\sigma_{30,0,0}^j + 2\sigma_{3xx0,0,0}^j + 2\sigma_{3zz0,0,0}^j - f_{y0,m,0}^j + \sigma_{5tt0,0,0}^j + 2\alpha\sigma_{5t0,0,0}^j \\
&\left. \left. + \beta^2\sigma_{50,0,0}^j + 2\sigma_{5xx0,0,0}^j + 2\sigma_{5yy0,0,0}^j - f_{z0,0,0}^j \right) \right]
\end{aligned} \tag{3.1.14}$$

$$\text{where, } \sigma_{10,0,0}^j = \left(\frac{g_{10,0}^j - c_1 U_{0,0,0}^j}{d_1} \right), \sigma_{30,0,0}^j = \left(\frac{g_{30,0}^j - c_3 U_{0,0,0}^j}{d_3} \right), \sigma_{50,0,0}^j = \left(\frac{g_{50,0}^j - c_5 U_{0,0,0}^j}{d_5} \right)$$

Finally, eliminating all the approximations at ghost points and (3.1.9) which is assumed to hold on extended domain Φ' , we obtain a method of $O(k^2 + k^2h^2 + h^4)$ for the telegraphic equation (3.1.1) with Robin boundary conditions.

3.2. Stability Analysis

In this section, we discuss the stability of the proposed method when applied to (3.1.1) subject to Neumann boundary conditions, i.e., when all a_i 's are zero and all b_i 's = 1.

In order to obtain an unconditionally stable method, we rewrite (3.1.9) in modified form for $l, m, n = 0(1)N$ as

$$\begin{aligned}
& [1 + A_1 \delta_y^2][1 + A_1 \delta_z^2] \left\{ [A_0 + A_1 \delta_x^2] \delta_t^2 U_{l,m,n}^j + \sqrt{a} \left(1 + \frac{1}{12} \delta_x^2 \right) (2\mu_t \delta_t) U_{l,m,n}^j \right\} \\
& = \left[\frac{p^2}{6} (\delta_x^2 \delta_y^2 + \delta_y^2 \delta_z^2 + \delta_z^2 \delta_x^2) + \left(p^2 - \frac{b}{12} \right) (\delta_x^2 + \delta_y^2 + \delta_z^2) - b \right] U_{l,m,n}^j \\
& + \frac{k^2}{12} (f_{l+1,m,n}^j + f_{l-1,m,n}^j + f_{l,m+1,n}^j + f_{l,m-1,n}^j + f_{l,m,n+1}^j + f_{l,m,n-1}^j + 6f_{l,m,n}^j) \\
& + O(k^4 + k^4 h^2 + k^2 h^4) \tag{3.2.1}
\end{aligned}$$

where, $A_0 = (1 + \eta b^2)$, $A_1 = \frac{1}{12}(1 - 12\gamma p^2)$, η and γ are free parameters to be determined. The additional terms are of high order and do not affect the accuracy of the method. For stability, we consider the homogeneous part of the method (3.2.1) which in matrix form together with Neumann boundary conditions can be written as

$$ZU^{j+1} + XU^j + YU^{j-1} = C \tag{3.2.2}$$

where,

$$\begin{aligned}
Z &= \frac{1}{12^3} (1 + \sqrt{a}) \mathbf{B}_{18} + \frac{\eta b^2}{144} \mathbf{B}_{17} + (\gamma^3 p^6) \mathbf{B}_{24} + \frac{\gamma p^2}{144} (\mathbf{B}_{28} + \mathbf{B}_{13} + \mathbf{B}_{19}) + \eta b^2 \gamma^2 p^4 \mathbf{B}_{22} \\
&+ \frac{\eta b^2 \gamma p^2}{12} (\mathbf{B}_{27} + \mathbf{B}_{12}) + \frac{\gamma^2 p^4}{12} (\mathbf{B}_{23} + \mathbf{B}_{29} + \mathbf{B}_{14}) + \frac{\sqrt{a} \gamma p^2}{144} (\mathbf{B}_{28} + \mathbf{B}_{13}) \\
&+ \frac{\sqrt{a} \gamma^2 p^4}{12} \mathbf{B}_{23}, \\
X &= -\frac{2}{12^3} \mathbf{B}_{18} - \frac{2\eta b^2}{144} \mathbf{B}_{17} - 2(\gamma^3 p^6) \mathbf{B}_{24} - \frac{2\gamma p^2}{144} (\mathbf{B}_{28} + \mathbf{B}_{13} + \mathbf{B}_{19}) - 2\eta b^2 \gamma^2 p^4 \mathbf{B}_{22} \\
&- \frac{2\eta b^2 \gamma p^2}{12} (\mathbf{B}_{27} + \mathbf{B}_{12}) - \frac{2\gamma^2 p^4}{12} (\mathbf{B}_{23} + \mathbf{B}_{29} + \mathbf{B}_{14}) + \frac{p^2}{6} (\mathbf{B}_{22} + \mathbf{B}_{25} + \mathbf{B}_{34}) + (p^2 \\
&- \frac{b}{12}) (\mathbf{B}_{110} + \mathbf{B}_{35} + \mathbf{B}_{32}), \\
Y &= \frac{1}{12^3} (1 - \sqrt{a}) \mathbf{B}_{18} + \frac{\eta b^2}{144} \mathbf{B}_{17} + (\gamma^3 p^6) \mathbf{B}_{24} + \frac{\gamma p^2}{144} (\mathbf{B}_{28} + \mathbf{B}_{13} + \mathbf{B}_{19}) + \eta b^2 \gamma^2 p^4 \mathbf{B}_{22} \\
&+ \frac{\eta b^2 \gamma p^2}{12} (\mathbf{B}_{27} + \mathbf{B}_{12}) + \frac{\gamma^2 p^4}{12} (\mathbf{B}_{23} + \mathbf{B}_{29} + \mathbf{B}_{14}) - \frac{\sqrt{a} \gamma p^2}{144} (\mathbf{B}_{28} + \mathbf{B}_{13}) \\
&- \frac{\sqrt{a} \gamma^2 p^4}{12} \mathbf{B}_{23}
\end{aligned}$$

where

$$\mathbf{B}_{1i} = \begin{bmatrix} 5\mathcal{A}_i & \mathcal{A}_i & & & 0 \\ \mathcal{A}_i & 10\mathcal{A}_i & \mathcal{A}_i & & \\ & \ddots & \ddots & \ddots & \\ & & \mathcal{A}_i & 10\mathcal{A}_i & \mathcal{A}_i \\ 0 & & & \mathcal{A}_i & 5\mathcal{A}_i \end{bmatrix}, i = 2,3,4,5,7,8,9,10$$

$$\mathbf{B}_{2i} = \begin{bmatrix} \mathcal{A}_i & -\mathcal{A}_i & & & 0 \\ -\mathcal{A}_i & 2\mathcal{A}_i & -\mathcal{A}_i & & \\ & & \ddots & \ddots & \\ & & & -\mathcal{A}_i & 2\mathcal{A}_i & -\mathcal{A}_i \\ 0 & & & & -\mathcal{A}_i & \mathcal{A}_i \end{bmatrix}, i = 2,3,4,7,8,9,$$

$$\mathbf{B}_{3i} = \begin{bmatrix} \frac{1}{2}\mathcal{A}_i & 0 & & & 0 \\ 0 & \mathcal{A}_i & 0 & & \\ & & \ddots & \ddots & \\ & & & 0 & \mathcal{A}_i & 0 \\ 0 & & & & 0 & \frac{1}{2}\mathcal{A}_i \end{bmatrix}, i = 2,4,5$$

$$\mathcal{A}_7 = \begin{bmatrix} 5A & A & & & 0 \\ A & 10A & A & & \\ & & \ddots & \ddots & \\ & & & A & 10A & A \\ 0 & & & & A & 5A \end{bmatrix}, \mathcal{A}_8 = \begin{bmatrix} 5E & E & & & 0 \\ E & 10E & E & & \\ & & \ddots & \ddots & \\ & & & E & 10E & E \\ 0 & & & & E & 5E \end{bmatrix},$$

$$\mathcal{A}_9 = \begin{bmatrix} -10B & -2B & & & 0 \\ -2B & -20B & -2B & & \\ & & \ddots & \ddots & \\ & & & -2B & -20B & -2B \\ 0 & & & & -2B & -10B \end{bmatrix}, \mathcal{A}_{10} = \begin{bmatrix} \frac{1}{2}A & 0 & & & 0 \\ 0 & A & 0 & & \\ & & \ddots & \ddots & \\ & & & 0 & A & 0 \\ 0 & & & & 0 & \frac{1}{2}A \end{bmatrix},$$

matrices $\mathcal{A}_i, i = 1,2, \dots, 6, \mathbf{A}, \mathbf{B}, \mathbf{E}$ are as defined in Section 2.2 and \mathbf{C} is the column vector corresponding to the boundary conditions,

$\mathbf{U}^j = [U_{0,0,0}^j, U_{1,0,0}^j, \dots, U_{N,0,0}^j, U_{0,1,0}^j, U_{1,1,0}^j, \dots, U_{N,1,0}^j, \dots, U_{0,0,N}^j, U_{1,0,N}^j, \dots, U_{N,0,N}^j, U_{0,N,N}^j, U_{1,N,N}^j, \dots, U_{N,N,N}^j]'$. Matrix \mathbf{B}_{18} is strictly diagonally dominant real symmetric matrix with all main diagonal entries positive, so all the eigen values of \mathbf{B}_{18} are real and positive [20] and hence invertible. The eigenvalues of the matrices \mathbf{B}_{1i} and \mathbf{B}_{2i} are real and non-negative. Thus all the eigen values of the matrix \mathbf{Z} are positive and hence \mathbf{Z} is invertible. Now, equation (3.2.2) can be rewritten as

$$\begin{bmatrix} \mathbf{U}^{j+1} \\ \vdots \\ \mathbf{U}^j \end{bmatrix} = \begin{bmatrix} -\mathbf{Z}^{-1}\mathbf{X} & \vdots & -\mathbf{Z}^{-1}\mathbf{Y} \\ \vdots & \ddots & \vdots \\ \mathbf{I} & \vdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U}^j \\ \vdots \\ \mathbf{U}^{j-1} \end{bmatrix} + \begin{bmatrix} \mathbf{Z}^{-1}\mathbf{C} \\ \vdots \\ \mathbf{0} \end{bmatrix} \quad (3.2.3)$$

Writing $\begin{bmatrix} \mathbf{U}^{j+1} \\ \vdots \\ \mathbf{U}^j \end{bmatrix} = \mathbf{V}^{j+1}$, $\begin{bmatrix} \mathbf{U}^j \\ \vdots \\ \mathbf{U}^{j-1} \end{bmatrix} = \mathbf{V}^j$, $\begin{bmatrix} -\mathbf{Z}^{-1}\mathbf{X} & \vdots & -\mathbf{Z}^{-1}\mathbf{Y} \\ \vdots & \ddots & \vdots \\ \mathbf{I} & \vdots & \mathbf{0} \end{bmatrix} = \mathbf{D}$, $\begin{bmatrix} \mathbf{Z}^{-1}\mathbf{C} \\ \vdots \\ \mathbf{0} \end{bmatrix} = \mathbf{G}$, we get

$$\mathbf{V}^{j+1} = \mathbf{D}\mathbf{V}^j + \mathbf{G} \quad (3.2.4)$$

Hence, similar to two dimensional case, we obtain that the difference method is solvable.

Hence, if $\lambda_{mj_i} = \text{eig}(\mathbf{B}_{18}^{-1} \mathbf{B}_{mj})$, $m = 1, 2, 3, j = 2, 3, \dots, 10, i = 1(1)(N + 1)^3$ and

$$P = \frac{1}{12^3} (1 + \sqrt{a}) + \frac{\eta b^2}{144} \lambda_{17i} + (\gamma^3 p^6) \lambda_{24i} + \frac{\gamma p^2}{144} (\lambda_{28i} + \lambda_{13i} + \lambda_{19i}) + \eta b^2 \gamma^2 p^4 \lambda_{22i} + \frac{\eta b^2 \gamma p^2}{12} (\lambda_{27i} + \lambda_{12i}) + \frac{\gamma^2 p^4}{12} (\lambda_{23i} + \lambda_{29i} + \lambda_{14i}) + \frac{\sqrt{a} \gamma p^2}{144} (\lambda_{28i} + \lambda_{13i}) + \frac{\sqrt{a} \gamma^2 p^4}{12} \lambda_{23i},$$

$$Q = \frac{1}{12^3} (1 - \sqrt{a}) + \frac{\eta b^2}{144} \lambda_{17i} + (\gamma^3 p^6) \lambda_{24i} + \frac{\gamma p^2}{144} (\lambda_{28i} + \lambda_{13i} + \lambda_{19i}) + \eta b^2 \gamma^2 p^4 \lambda_{22i} + \frac{\eta b^2 \gamma p^2}{12} (\lambda_{27i} + \lambda_{12i}) + \frac{\gamma^2 p^4}{12} (\lambda_{23i} + \lambda_{29i} + \lambda_{14i}) - \frac{\sqrt{a} \gamma p^2}{144} (\lambda_{28i} + \lambda_{13i}) - \frac{\sqrt{a} \gamma^2 p^4}{12} \lambda_{23i}$$

$$R = \frac{-2}{12^3} - \frac{2\eta b^2}{144} \lambda_{17i} - 2(\gamma^3 p^6) \lambda_{24i} - \frac{2\gamma p^2}{144} (\lambda_{28i} + \lambda_{13i} + \lambda_{19i}) - 2\eta b^2 \gamma^2 p^4 \lambda_{22i} - \frac{2\eta b^2 \gamma p^2}{12} (\lambda_{27i} + \lambda_{12i}) - \frac{2\gamma^2 p^4}{12} (\lambda_{23i} + \lambda_{29i} + \lambda_{14i}),$$

Then $\frac{R}{P}$ and $\frac{Q}{P}$ are the eigen values of $\mathbf{Z}^{-1} \mathbf{X}$ and $\mathbf{Z}^{-1} \mathbf{Y}$ respectively, having the common set of corresponding linearly independent eigen vectors. Now, the eigen values of the matrix \mathbf{D} are given by the eigen values of the matrix

$$\begin{bmatrix} -M & -N \\ 1 & 0 \end{bmatrix} \quad (3.2.5)$$

where $-M$, $-N$, 1 and 0 are the i^{th} eigenvalues of $-\mathbf{Z}^{-1} \mathbf{X}$, $-\mathbf{Z}^{-1} \mathbf{Y}$, \mathbf{I} and $\mathbf{0}$ respectively corresponding to i^{th} eigenvector common to all these matrices. If Λ is an eigenvalue of \mathbf{D} then the characteristic equation of matrix (3.2.5) is

$$\Lambda^2 + M\Lambda + N = 0 \quad (3.2.6)$$

Using the transformation $\Lambda = \frac{1+z}{1-z}$, the characteristic equation (3.2.6) reduces to

$$(1 - M + N)z^2 + 2(1 - N)z + (1 + M + N) = 0$$

The necessary and sufficient condition for $|\Lambda| < 1$ is that

$$(1 - M + N) > 0, (1 - N) > 0, (1 + M + N) > 0$$

Now, as in two dimensional case by rearranging the terms it can be shown that

$$(1 + M + N) > 0, (1 - N) > 0 \text{ and } (1 - M + N) > 0$$

for $\eta \geq \frac{1}{64}$, $\gamma \geq \frac{1}{4}$ and we conclude that for all choices of h, k the proposed method is unconditionally stable. In the similar manner, stability can be achieved in Robin boundary case.

3.3. Alternating Direction Implicit Method

The structure of the matrices in equation (3.2.1) is block tri-diagonal type which cannot be solved directly for $U_{l,m,n}^j$. So, in order to facilitate the computation, we split equation (3.2.1) into three equations to obtain tri-diagonal matrices which can be easily handled. Ignoring the truncation error term, the method (3.2.1) in three-step ADI form can be written as

$$[1 + A_1 \delta_z^2] U_{l,m,n}^{**} = B_n \quad (3.3.1)$$

$$[1 + A_1 \delta_y^2] U_{l,m,n}^* = U_{l,m,n}^{**} \quad (3.3.2)$$

$$(A_0 + A_1 \delta_x^2) \delta_t^2 U_{l,m,n}^j + \sqrt{a} \left(1 + \frac{1}{12} \delta_x^2\right) (2\mu_t \delta_t) U_{l,m,n}^j = U_{l,m,n}^* \quad (3.3.3)$$

where, $B_n = \left[\frac{p^2}{6} (\delta_x^2 \delta_y^2 + \delta_y^2 \delta_z^2 + \delta_z^2 \delta_x^2) + (p^2 - \frac{b}{12}) (\delta_x^2 + \delta_y^2 + \delta_z^2) - b \right] U_{l,m,n}^j + \frac{k^2}{12} (f_{l+1,m,n}^j + f_{l-1,m,n}^j + f_{l,m+1,n}^j + f_{l,m-1,n}^j + f_{l,m,n+1}^j + f_{l,m,n-1}^j + 6f_{l,m,n}^j)$ and

$U_{l,m,n}^*$ and $U_{l,m,n}^{**}$ are intermediate values.

For fixed l, m , equation (3.3.1) can be written in matrix form as

$$\begin{bmatrix} 1-2A_1 & A_1 & & & & & \\ A_1 & 1-2A_1 & A_1 & & & & \\ & & \ddots & \ddots & & & \\ & & & A_1 & 1-2A_1 & A_1 & \\ & & & & A_1 & 1-2A_1 & \\ & & & & & & 1-2A_1 \end{bmatrix} \begin{bmatrix} U_{l,m,1}^{**} \\ U_{l,m,2}^{**} \\ \vdots \\ U_{l,m,N-2}^{**} \\ U_{l,m,N-1}^{**} \end{bmatrix} = \begin{bmatrix} B_1 - A_1 U_{l,m,0}^{**} \\ B_2 \\ \vdots \\ B_{N-2} \\ B_{N-1} - A_1 U_{l,m,N}^{**} \end{bmatrix} \quad (3.3.4)$$

For fixed l, n , equation (3.3.2) can be written in matrix form as

$$\begin{bmatrix} 1-2A_1 & A_1 & & & & & \\ A_1 & 1-2A_1 & A_1 & & & & \\ & & \ddots & \ddots & & & \\ & & & A_1 & 1-2A_1 & A_1 & \\ & & & & A_1 & 1-2A_1 & \\ & & & & & & 1-2A_1 \end{bmatrix} \begin{bmatrix} U_{l,1,n}^* \\ U_{l,2,n}^* \\ \vdots \\ U_{l,N-2,n}^* \\ U_{l,N-1,n}^* \end{bmatrix} = \begin{bmatrix} U_{l,1,n}^{**} - A_1 U_{l,0,n}^* \\ U_{l,2,n}^{**} \\ \vdots \\ U_{l,N-2,n}^{**} \\ U_{l,N-1,n}^{**} - A_1 U_{l,N,n}^* \end{bmatrix} \quad (3.3.5)$$

and for fixed m, n , equation (3.3.3) can be written in matrix form as

$$\begin{bmatrix} A_0 - 2A_1 + \frac{5\sqrt{a}}{6} & A_1 + \frac{\sqrt{a}}{12} & & & & & \\ A_1 + \frac{\sqrt{a}}{12} & A_0 - 2A_1 + \frac{5\sqrt{a}}{6} & A_1 + \frac{\sqrt{a}}{12} & & & & \\ & & \ddots & \ddots & & & \\ & & & A_1 + \frac{\sqrt{a}}{12} & A_0 - 2A_1 + \frac{5\sqrt{a}}{6} & A_1 + \frac{\sqrt{a}}{12} & \\ & & & & A_1 + \frac{\sqrt{a}}{12} & A_0 - 2A_1 + \frac{5\sqrt{a}}{6} & \\ & & & & & & A_1 + \frac{\sqrt{a}}{12} \\ & & & & & & A_0 - 2A_1 + \frac{5\sqrt{a}}{6} \end{bmatrix} \begin{bmatrix} U_{1,m,n}^{j+1} \\ U_{2,m,n}^{j+1} \\ \vdots \\ U_{N-2,m,n}^{j+1} \\ U_{N-1,m,n}^{j+1} \end{bmatrix} = \begin{bmatrix} U_{1,m,n}^* - L_{1,m,n} - \left(A_1 + \frac{\sqrt{a}}{12}\right) U_{0,m,n}^{j+1} \\ U_{2,m,n}^* - L_{2,m,n} \\ \vdots \\ U_{N-2,m,n}^* - L_{N-2,m,n} \\ U_{N-1,m,n}^* - L_{N-1,m,n} - \left(A_1 + \frac{\sqrt{a}}{12}\right) U_{N,m,n}^{j+1} \end{bmatrix} \quad (3.3.6)$$

where, $L_{l,m,n} = -2A_1(U_{l+1,m,n}^j + U_{l-1,m,n}^j) - 2(A_0 - 2A_1)U_{l,m,n}^j + U_{l,m,n}^{j-1} \left(A_0 - 2A_1 - \frac{5\sqrt{a}}{6} \right) + (U_{l+1,m,n}^{j-1} + U_{l-1,m,n}^{j-1}) \left(A_1 - \frac{\sqrt{a}}{12} \right)$, $l = 1, 2, \dots, N - 1$. We first solve (3.3.4) for $U_{l,m,n}^{**}$ and the intermediate approximations for boundary required for solving (3.3.4) are obtained from (3.3.5), then equation (3.3.5) is solved and the required approximations for boundary are obtained from (3.3.6). Finally, equation (3.3.6) is solved for obtaining the required solution. Each of the system involved in these equations has tri-diagonal structure and hence can be easily solved using tri-diagonal solver.

4. Numerical Experiments

In this section we will apply the proposed methods to various test problems. In each of the example, we compute root mean square (*RMS*) error, L_∞ error or Relative (Rel.) error by using the formulae

$$RMS \text{ error} = \sqrt{\frac{\sum_{i=0}^N |u_i - U_i|^2}{N+1}}, \quad L_\infty \text{ error} = \max_i |u_i - U_i|, \quad \text{Rel. error} = \sqrt{\frac{\sum_{i=0}^N |u_i - U_i|^2}{\sum_{i=0}^N |u_i|^2}}$$

where u_i and U_i denote analytical and numerical solutions respectively. Order of convergence of the method is calculated by using the formula

$$\frac{\log\left(\frac{e_{h_1}}{e_{h_2}}\right)}{\log\left(\frac{h_1}{h_2}\right)}$$

where e_{h_1} and e_{h_2} are L_∞ errors for grid sizes h_1 and h_2 respectively.

Example 1. Consider the following two dimensional telegraphic equation

$$u_{tt} + 2u_t + u = u_{xx} + u_{yy} - 2e^{x+y-t}$$

subject to initial conditions

$$u(x, y, 0) = e^{x+y}, \quad u_t(x, y, 0) = -e^{x+y}, \quad 0 \leq x, y \leq 1$$

and the boundary conditions

$$\begin{aligned} u(0, y, t) &= e^{y-t}, \quad u(1, y, t) = e^{1+y-t}, \quad 0 \leq y \leq 1, \quad t > 0 \\ u_y(x, 0, t) &= e^{x-t}, \quad u(x, 1, t) = e^{1+x-t}, \quad 0 \leq x \leq 1, \quad t > 0 \end{aligned}$$

The analytical solution of this example is $u(x, y, t) = e^{x+y-t}$. *RMS* and L_∞ errors are computed with $k = 0.01, h = 0.1$ in Table 1. Comparison is done with the L_∞ errors obtained in [8]. Clearly, results obtained with the proposed method are much better than the results obtained in [8]. We also compute errors with and $k = 0.001, h = 0.05$ and register our results in Table 2. L_∞ and relative errors are compared with the errors obtained in [8] and [9]. It can be seen that the proposed method produces much better results.

Table 1: Errors calculated for Example 1 with $k = 0.01, h = 0.1, \eta = .5, \gamma = 1/3$

$t(\text{sec})$	Proposed Method			Mittal and Bhatia[8]
	RMS error	L_∞ error	CPU Time (in sec)	L_∞ error
1	1.12016e-05	2.4904e-05	1.22	2.9996e-02
2	6.8015e-07	2.4001e-06	2.71	3.9711e-03
3	1.3001e-06	2.6090e-06	4.16	2.2178e-03
5	9.5245e-08	2.4211e-07	6.97	2.0618e-05
7	6.2980e-09	1.8679e-08	10.11	3.0052e-05
10	1.4571e-09	3.0008e-09	14.45	2.5354e-06

Table 2: Errors calculated for Example 1 with $k = 0.001, h = 0.05, \eta = .5, \gamma = 1/3$

$t(\text{sec})$	Proposed Method			Mittal and Bhatia[8]		Dehghan and Ghesmati[9]
	RMS error	L_∞ error	Rel. error	L_∞ error	Rel. error	Rel. error
.5	6.1700e-07	1.9132e-06	3.2256e-07	9.5129e-03	8.4225e-05	8.014e-05
1	5.4281e-07	1.2010e-06	4.6904e-07	7.4749e-03	1.2906e-04	2.020e-04
2	4.0111e-08	1.7234e-07	9.4998e-08	1.0361e-03	3.0957e-05	9.791e-05
3	5.4191e-08	1.3982e-07	3.5020e-07	5.7859e-04	9.1555e-05	7.029e-04

Example 2. Consider the following telegraphic equation

$$u_{tt} + 2u_t + u = u_{xx} + u_{yy} + 2\pi^2 e^{-t} \sin(\pi x) \sin(\pi y)$$

subject to initial conditions

$$u(x, y, 0) = \sin(\pi x) \sin(\pi y), u_t(x, y, 0) = -\sin(\pi x) \sin(\pi y), 0 \leq x, y \leq 1$$

and the boundary conditions

$$u_x(0, y, t) = \pi e^{-t} \sin(\pi y), u(1, y, t) = 0, 0 \leq y \leq 1, t > 0$$

$$u(x, 0, t) = 0, u_y(x, 1, t) = -\pi e^{-t} \sin(\pi x), 0 \leq x \leq 1, t > 0$$

The analytical solution of this example is $u(x, y, t) = e^{-t} \sin(\pi x) \sin(\pi y)$. We compute RMS and L_∞ errors with $k = 0.01, h = 0.1, \eta = .5, \gamma = 1$ at various time levels in Table 3. Our results are found to be better when compared with the results obtained by Mittal and Bhatia in [8]. Graphs of analytical and numerical solution with $k = 0.01, h = 0.05, \eta = .5, \gamma = 1$ at $t = 5$ are given in Figure 1. Moreover, we compute RMS and L_∞ errors at $t = 1$ for $\frac{k}{h^2} = 3.2$ and $\eta = 1, \gamma = 0.5$ in Table 4 and show that the proposed method is fourth order convergent.

Table 3: Errors calculated for Example 2 with $k = 0.01, h = 0.1, \eta = .5, \gamma = 1$

$t(\text{sec})$	Proposed Method			Mittal and Bhatia[8]
	RMS error	L_∞ error	CPU Time (in sec)	L_∞ error
1	1.1923e-03	2.6000e-03	1.28	3.6006e-03
2	2.5043e-04	5.2501e-04	2.99	5.7068e-03
3	1.4985e-04	2.7854e-04	4.24	1.2479e-03
5	2.4200e-05	6.1759e-05	7.14	2.1003e-04
10	1.4729e-08	4.5999e-08	14.63	1.4083e-06

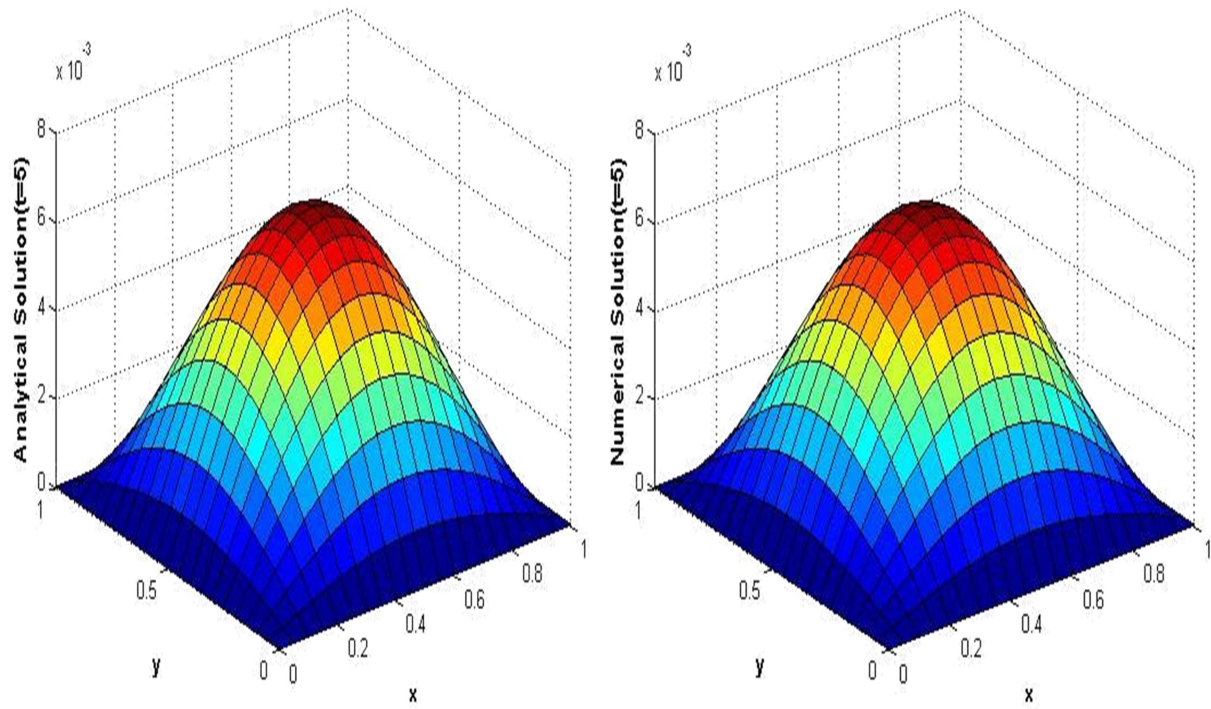


Figure 1: Graphs of analytical and numerical solution of Example 2 at $t = 5$

Table 4: Errors calculated for Example 2 at $t = 1$ for $\frac{k}{h^2} = 3.2, \eta = 1, \gamma = 0.5$

h	RMS error	L_∞ error	CPU Time (in sec)	Order
1/8	2.0001e-04	3.1638e-04	0.04	-
1/16	1.1289e-05	2.1534e-05	0.09	3.90
1/32	1.0712e-05	1.3046e-06	0.41	4.02
1/64	9.7456e-07	1.0011e-06	1.01	3.74

Example 3. Consider the following telegraphic equation for $\alpha = 1, \beta = 1$

$$\begin{aligned}
 u_{tt} + 2\alpha u_t + \beta^2 u & \\
 &= u_{xx} + u_{yy} + (1 + x + y + t)^{-2}(1 + 2\alpha(1 + x + y + t)) \\
 &+ \beta^2(1 + x + y + t)^2 \log(1 + x + y + t)
 \end{aligned}$$

subject to initial conditions

$$u(x, y, 0) = \log(1 + x + y), u_t(x, y, 0) = \frac{1}{1 + x + y}, 0 \leq x, y \leq 1$$

and the boundary conditions

$$u(0, y, t) = \log(1 + y + t), u_x(1, y, t) = \frac{1}{2 + y + t}, 0 \leq y \leq 1, t > 0$$

$$u_y(x, 0, t) = \frac{1}{1 + x + t}, u(x, 1, t) = \log(2 + x + t), 0 \leq x \leq 1, t > 0$$

The analytical solution of this example is $u(x, y, t) = \log(1 + x + y + t)$. We tabulate errors for this example in Table 5 for $k = 0.001, h = 0.05, \eta = .1, \gamma = 1/3$ at various time levels and compare our results with results obtained in [8] and [9]. The results obtained by the proposed method are found to be much better. Moreover, we compute *RMS* and L_∞ errors at $t = 1$ for $\frac{k}{h^2} = 3.2$ and $\eta = .1, \gamma = 1/3$ in Table 6 and show that the proposed method is fourth order convergent. It is evident from the tables that whether the chosen grid sizes are big or small, the proposed methods produce accurate results.

Table 5: Errors calculated for Example 3 with $k = 0.001, h = 0.05, \eta = .1, \gamma = 1/3$

t	Proposed Method			Mittal and Bhatia [8]		Dehghan and Ghesmati[9]
	<i>RMS</i> error	L_∞ error	Rel. error	L_∞ error	Rel. error	Rel. error
.5	1.7004e-08	3.3510e-08	1.5689e-08	2.4738e-03	1.1088e-03	7.939e-05
1	1.2752e-08	2.1329e-08	1.0012e-08	3.3082e-03	1.3266e-03	9.098e-05
2	4.4320e-09	7.1962e-09	3.1980e-09	1.1380e-03	3.1954e-04	8.705e-04
3	8.2202e-10	1.6545e-09	5.0203e-10	4.3577e-04	1.3024e-04	9.931e-04
4	1.0062e-09	1.7215e-09	6.1867e-10	5.4141e-04	1.4435e-05	4.703e-03
5	2.3210e-10	3.7920e-10	1.0587e-10	3.4812e-04	8.4225e-05	7.302e-03

Table 6: Errors calculated for Example 3 at $t = 1$ for $\frac{k}{h^2} = 3.2$

h	RMS error	L_∞ error	CPU Time (in sec)	Order
1/4	5.1103e-04	8.4595e-04	0.06	-
1/8	4.1699e-05	6.3657e-05	0.08	3.65
1/16	2.3321e-06	3.3209e-06	0.32	4.07
1/32	1.4978e-07	2.1032e-07	1.19	3.98

Example 4. Consider the following telegraphic equation for $\alpha = 10, \beta = 5$

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + u_{yy} + (2 - 2\alpha + \beta^2)e^{-2t} \cosh(x) \sinh(y)$$

subject to initial conditions

$$u(x, y, 0) = \cosh(x) \sinh(y), u_t(x, y, 0) = -2 \cosh(x) \sinh(y), 0 \leq x, y \leq 1$$

and the boundary conditions

$$\begin{aligned} 3u(0, y, t) + 2u_x(0, y, t) &= 3e^{-2t} \sinh(y), 0 \leq y \leq 1, t > 0 \\ u_x(1, y, t) &= e^{-2t} \sinh(1) \sinh(y), 0 \leq y \leq 1, t > 0 \\ u(x, 0, t) = 0, u(x, 1, t) &= e^{-2t} \cosh(x) \sinh(1), 0 \leq x \leq 1, t > 0 \end{aligned}$$

The analytical solution of this example is $u(x, y, t) = e^{-2t} \cosh(x) \sinh(y)$. In Table 7, RMS and L_∞ errors are obtained at $t = 1$ for $\eta = 1, \gamma = 0.5$. It can be seen that the method behaves as fourth order method for a fixed ratio $\frac{k}{h^2} = 3.2$. Graphs of analytical as well as numerical solution at $t = 1$ for $h = 1/16$ are shown in Figure 2. It is evident from the graphs that the numerical solution agrees with the analytical solution.

Table 7: Errors calculated for Example 4 at $t = 1$ for $\frac{k}{h^2} = 3.2, \eta = 1, \gamma = 0.5$

h	RMS error	L_∞ error	CPU Time (in sec)	Order
1/8	6.1001e-05	1.1080e-04	0.06	-
1/16	5.1892e-06	5.6001e-06	0.08	4.44
1/32	1.4627e-07	3.0200e-07	0.33	4.02
1/64	5.0912e-09	1.1002e-08	1.19	4.96

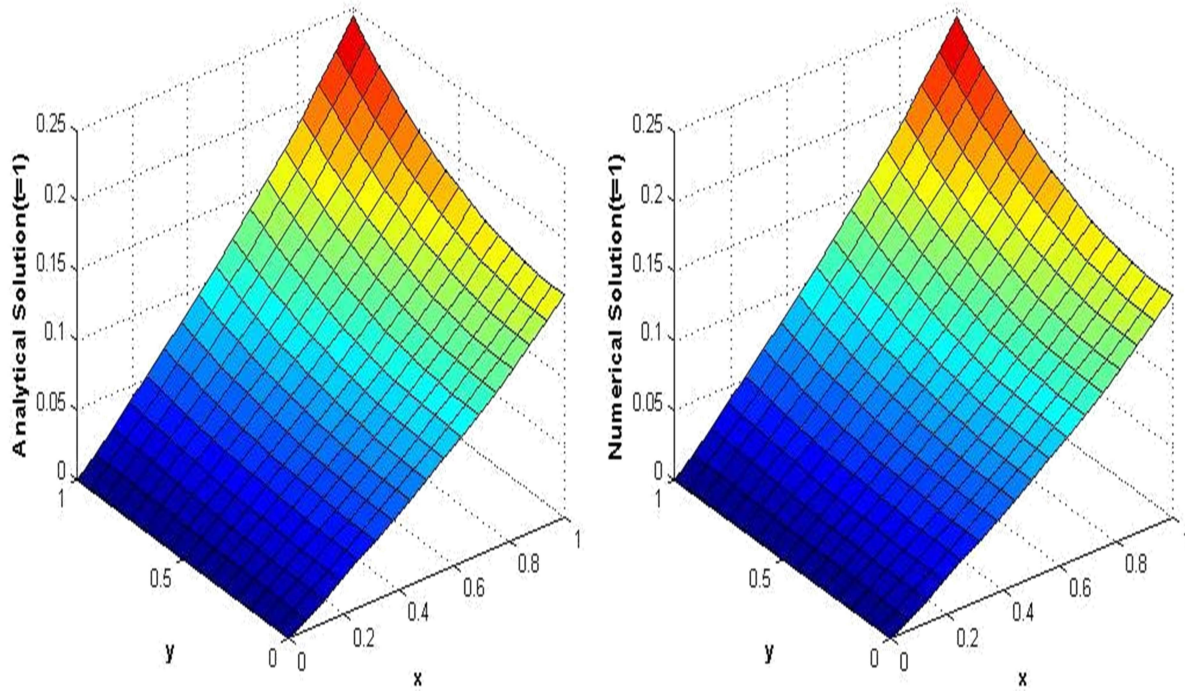


Figure 2: Graphs of analytical and numerical solution of Example 4 at $t = 1$

Example 5. Consider the following three dimensional telegraphic equation for $\alpha = 10, \beta = 5$

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + u_{yy} + u_{zz} + (1 - 4\alpha + \beta^2)e^{-2t} \cosh(x) \sinh(y) \cosh(z), 0 \leq x, y, z \leq 1, t > 0$$

subject to initial conditions

$$u(x, y, z, 0) = \cosh(x) \sinh(y) \cosh(z), u_t(x, y, z, 0) = -2 \cosh(x) \sinh(y) \cosh(z), 0 \leq x, y, z \leq 1$$

and the boundary conditions

$$\begin{aligned}
3u(0, y, z, t) + 2u_x(0, y, z, t) &= 3e^{-2t} \sinh(y) \cosh(z), 0 \leq y, z \leq 1, t > 0 \\
u(1, y, z, t) &= e^{-2t} \cosh(1) \sinh(y) \cosh(z), 0 \leq y, z \leq 1, t > 0 \\
u(x, 0, z, t) &= 0, 0 \leq x, z \leq 1, t > 0 \\
u(x, 1, z, t) &= e^{-2t} \cosh(x) \sinh(1) \cosh(z), 0 \leq x, z \leq 1, t > 0 \\
u(x, y, 0, t) + u_z(x, y, 0, t) &= e^{-2t} \cosh(x) \sinh(y), 0 \leq x, y \leq 1, t > 0 \\
u(x, y, 1, t) &= e^{-2t} \cosh(x) \sinh(y) \cosh(1), 0 \leq x, y \leq 1, t > 0
\end{aligned}$$

The analytical solution of this example is $u(x, y, z, t) = e^{-2t} \cosh(x) \sinh(y) \cosh(z)$. *RMS*, L_∞ errors and CPU time are shown in Table 8 for $k \propto h^2$, $\eta = 1/3$, $\gamma = 1/3$ at $t = 1$. Order of convergence has also been calculated in the table. It is observed from the table that the order of convergence of the method is four.

Table 8: Errors calculated for Example 5 at $t = 1$ for $\frac{k}{h^2} = 3.2$, $\eta = 1/3$, $\gamma = 1/3$

h	<i>RMS</i> error	L_∞ error	CPU Time (in sec)	Order
1/4	2.2502e-03	3.4900e-03	1.50	-
1/8	1.1725e-04	2.2555e-04	2.02	3.84
1/16	8.4909e-06	1.5097e-05	75.18	3.73
1/32	5.0472e-07	1.0074e-06	2452	3.99

5. Concluding Remarks

In this paper, we started with methods of $O(k^2 + k^2h^2 + h^4)$ proposed in [14] and [15] for the solution of telegraphic equation with Dirichlet boundary conditions. We modified the methods appropriately for the solution of telegraphic equation subject to Robin boundary conditions by obtaining fourth order approximations at the Robin boundaries and obtained respective ADI methods. For both two and three dimensional problems subject to Neumann boundary conditions, the proposed ADI methods are shown to be unconditionally stable by using matrix stability method. The parameters introduced in the process of proving the stability are obtained to be independent of the grid sizes whereas parameters introduced in methods discussed in [13], [14] and [15] depend upon the grid sizes. Various numerical experiments are carried out in order to show the efficiency and accuracy of the proposed methods. It is seen that the proposed methods behave well with both small and large grid sizes. Moreover, the methods behave as fourth order methods for $k \propto h^2$.

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