Booker, A. (2019). A converse theorem without root numbers. *Mathematika*, *65*(4), 862-873. https://doi.org/10.1112/S0025579319000184

A CONVERSE THEOREM WITHOUT ROOT NUMBERS

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Abstract. We answer a challenge posed in [3, §1.3] by proving a version of Weil's converse theorem [20] that assumes a functional equation for character twists but allows their root numbers to vary arbitrarily.

1. Introduction

When Weil introduced his converse theorem [20], he had in mind what eventually became known as the Shimura–Taniyama conjecture connecting elliptic curves over $\mathbb{Q}$ with classical modular forms. Soon after, Weil's theorem was recast in representation-theoretic terms by Jacquet and Langlands [12, Theorem 11.3], for whom the motivation was Artin's conjecture, now seen as the prototypical case of Langlands' functoriality [12, §12]. At the time, much more was known about the analytic properties of Artin L-functions than of Hasse–Weil L-functions (though as fate would have it, Shimura–Taniyama is now a theorem, while some cases of Artin's conjecture for 2-dimensional representations over $\mathbb{Q}$ remain open). However, one hypothesis in the converse theorem emerged as a sticking point in the way of easily applying it to Artin L-functions, namely the behavior under twist of the root number in the functional equation, which is tantamount to proving the existence of local root numbers for Artin representations. Langlands [16] solved this problem by a direct (i.e. local) but very involved computation. Shortly after, Deligne [10] gave a simpler global proof by the method of "stability of ϵ-factors".

In this paper we show that the issue could have been circumvented[1], in the sense that knowledge of the root number is not needed in the converse theorem. We also incorporate the method of [6] to allow the non-trivial twists to have arbitrary poles. Precisely, we show the following:

Theorem 1.1. *Let ξ be a Dirichlet character modulo N, k a positive integer satisfying $\xi(-1) = (-1)^k$, and $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of complex numbers satisfying $\lambda_n = O(\sqrt{n})$ and the Hecke relations, so that*

$$\sum_{n=1}^{\infty} \lambda_n n^{-s} = \prod_p \frac{1}{1 - \lambda_p p^{-s} + \xi(p) p^{-2s}}, \qquad \text{with } \overline{\lambda_p} = \overline{\xi(p)}\lambda_p \text{ for each prime } p \nmid N. \tag{1.1}$$

For any primitive Dirichlet character χ of conductor q coprime to N, define

$$\Lambda_\chi(s) = \Gamma_{\mathbb{C}}\big(s + \tfrac{k-1}{2}\big) \sum_{n=1}^{\infty} \lambda_n \chi(n) n^{-s}$$

The author was partially supported by EPSRC Grant EP/K034383/1.

[1] That is not to say that it should have been. As Langlands makes clear in his commentary [14] and [15], the existence of local root numbers for Artin representations was an important confirmation of the nascent local theory.

for $s \in \mathbb{C}$ with $\Re(s) > \frac{3}{2}$, where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Suppose, for every such χ, that $\Lambda_\chi(s)$ continues to a meromorphic function on $\mathbb{C}$ and satisfies the functional equation

$$\Lambda_\chi(s) = \epsilon_\chi (Nq^2)^{\frac{1}{2}-s}\overline{\Lambda_\chi(1-\bar{s})}, \tag{1.2}$$

for some $\epsilon_\chi \in \mathbb{C}$ (necessarily of magnitude 1). Let $\mathbf{1}$ denote the character of modulus 1, and suppose that there is a nonzero polynomial P such that $P(s)\Lambda_{\mathbf{1}}(s)$ continues to an entire function of finite order.

Then one of the following holds:

- (i) *$k = 1$ and there are primitive characters $\xi_1 \pmod{N_1}$ and $\xi_2 \pmod{N_2}$ such that $N_1N_2 = N$, $\xi_1\xi_2 = \xi$ and $\lambda_n = \sum_{d|n} \xi_1(n/d)\xi_2(d)$ for every n.*
- (ii) *$\sum_{n=1}^\infty \lambda_n n^{\frac{k-1}{2}} e^{2\pi i n z}$ is a normalized Hecke eigenform in $S_k^{\mathrm{new}}(\Gamma_0(N), \xi)$.*

The result can also be stated in representation-theoretic terms, as follows.

Theorem 1.2. *Let $\mathbb{A}_{\mathbb{Q}}$ denote the adèle ring of $\mathbb{Q}$, and let $\pi = \pi_\infty \otimes \bigotimes_{v<\infty} \pi_v$ be an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with automorphic central character and conductor N. Assume that each π_v is unitary and that π_∞ is a discrete series or limit of discrete series representation. For each unitary idèle class character ω of conductor q coprime to N, suppose that the complete L-functions*

$$\Lambda(s, \pi \otimes \omega) = \prod_v L(s, \pi_v \otimes \omega_v) \quad \text{and} \quad \Lambda(s, \widetilde{\pi} \otimes \omega^{-1}) = \prod_v L(s, \widetilde{\pi}_v \otimes \omega_v^{-1}),$$

defined initially for $\Re(s) > \frac{3}{2}$, continue to meromorphic functions on $\mathbb{C}$ and satisfy a functional equation of the form

$$\Lambda(s, \pi \otimes \omega) = \epsilon_\omega (Nq^2)^{\frac{1}{2}-s}\Lambda(1-s, \widetilde{\pi} \otimes \omega^{-1})$$

for some complex number ϵ_ω. Suppose also that there is a nonzero polynomial P such that $P(s)\Lambda(s, \pi)$ continues to an entire function of finite order. Then there is an automorphic representation $\Pi = \bigotimes_v \Pi_v$ which is either cuspidal or an isobaric sum of unitary idèle class characters, and satisfies $\pi_v \cong \Pi_v$ for $v = \infty$ and every finite v at which π_v is unramified.

Many extensions and variations of the hypotheses of the two theorems are possible. We mention a few:

1. The restriction to discrete series representations was made for convenience and could be removed with more work. It is also likely possible to formulate a version over number fields, starting along the lines of [4, 5].
2. The assumptions that $\lambda_n = O(\sqrt{n})$ in Theorem 1.1 and that π_v be unitary in Theorem 1.2 could be relaxed to polynomial growth of the Satake parameters, at the expense of allowing solutions corresponding to Eisenstein series of higher weight. Since we have allowed the untwisted L-function to have finitely many poles, that would include the Eisenstein series of weight 2 and level 1 (which is not modular), as in [6].
3. If we assume that the twisted L-functions are entire then, using the method of [11], it is enough to assume the functional equation for the trivial character and characters of a single well-chosen prime conductor q (depending on N). As shown in [2, Theorem 2.5], the set of suitable q has density 1 in the set of all primes.

(4) Our proof makes use of the Euler product, an idea that originates with Conrey and Farmer [9]. It is not required in Weil's original converse theorem, thanks to an abundance of twists, and one might ask whether it is possible to eliminate both the root numbers and Euler product. It is plausible that the answer is no, as the analogous question for additive twists has a negative answer, as shown in [19]. However, adopting the philosophy espoused in [2], it is likely possible to linearize the Euler product, replacing it by the functional equation under twist by Ramanujan sums, $c_q(n)$.

Finally, we note that stability of ϵ-factors for more general reductive groups and representations of their L-groups remains an active area of research (see [8] for a recent survey), motivated in part by applications involving converse theorems for GL_n. It would be interesting to understand the extent to which our result can be generalized to higher rank.

2. Lemmas

We begin with a few preparatory lemmas. We assume the notation and hypotheses of Theorem 1.1 throughout.

Lemma 2.1. *Let $q \nmid N$ be a prime number, and let*

$$c_q(n) = \sum_{\substack{a \pmod q \\ (a,q)=1}} e\left(\frac{an}{q}\right)$$

be the associated Ramanujan sum, where $e(x) = e^{2\pi i x}$. Define

$$\Lambda_{c_q}(s) = \Gamma_{\mathbb{C}}\big(s + \tfrac{k-1}{2}\big) \sum_{n=1}^{\infty} \lambda_n c_q(n) n^{-s}.$$

Then the ratio $D_q(s) = \Lambda_{c_q}(s)/\Lambda_{\mathbf{1}}(s)$ is a Dirichlet polynomial satisfying the functional equation

$$D_q(s) = \xi(q) q^{1-2s} \overline{D_q(1-\bar{s})}.$$

Proof. This holds more generally for positive integers q coprime to N, as shown in [2, Lemma 4.12]. For completeness, we prove the claim for prime values of q. Let χ_0 denote the trivial character mod q. Then a straightforward calculation shows that $c_q(n) = q-1-q\chi_0(n)$, so that

$$\begin{aligned} D_q(s) = \frac{\Lambda_{c_q}(s)}{\Lambda_{\mathbf{1}}(s)} &= q - 1 - \frac{q\sum_{n=1}^{\infty} \lambda_n \chi_0(n) n^{-s}}{\sum_{n=1}^{\infty} \lambda_n n^{-s}} = q - 1 - q(1 - \lambda_q q^{-s} + \xi(q) q^{-2s}) \\ &= -1 + \lambda_q q^{1-s} - \xi(q) q^{1-2s}, \end{aligned}$$

by (1.1). Hence

$$\begin{aligned} \xi(q) q^{1-2s} \overline{D_q(1-\bar{s})} &= \xi(q) q^{1-2s} \big(-1 + \overline{\lambda_q} q^{s} - \overline{\xi(q)} q^{2s-1}\big) = -1 + \xi(q)\overline{\lambda_q} q^{1-s} - \xi(q) q^{1-2s} \\ &= D_q(s), \end{aligned}$$

since $\overline{\lambda_q} = \overline{\xi(q)}\lambda_q$, by (1.1). ☐

Lemma 2.2.

(1) *$\Lambda_\chi(s)$ is entire of finite order for every primitive character χ of prime conductor $q \nmid N$.*

(2) $\Lambda_{\mathbf{1}}(s)$ *is entire apart from at most simple poles at* $s \in \{0,1\}$ *for* $k = 1$ *and* $s \in \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$ *for* $k = 2$.

Proof. These follow from the proof of [6, Theorem 1.1]. Although the statement of that theorem includes a formula for the root number, we verify that no use of that hypothesis is made until §3.1, where Weil's converse theorem is applied. Thus we find that (1) holds and that $\Lambda_{\mathbf{1}}(s)$ is entire apart from at most simple poles at $s \in \{\frac{1\pm k}{2}\}$ for $k \neq 2$ and $s \in \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\}$ for $k = 2$. Finally, the estimate $\lambda_n = O(\sqrt{n})$, together with the functional equation (1.2), rules out poles in the case $k > 2$. $\square$

Lemma 2.3. *For any prime* $q \nmid N$ *and any integer* a*, there is a positive integer* $n \equiv a \pmod{q}$ *such that* $\lambda_n \neq 0$.

Proof. Suppose the conclusion is false for some q and a. If $a \equiv 0 \pmod{q}$ then we must have $\lambda_q = 0$, but then the Euler product (1.1) implies that $\lambda_{q^2} = -\xi(q) \neq 0$. Hence we may assume that $(a, q) = 1$.

Letting χ_0 denote the trivial character mod q, we have $\chi_0(n) = \frac{q-1-c_q(n)}{q}$, and thus

$$\frac{1}{q} - \frac{1}{q(q-1)}c_q(n) + \frac{1}{q-1}\sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi\neq\chi_0}} \overline{\chi(a)}\chi(n)$$

is the indicator function of the residue class of a. Hence, by hypothesis we have

$$\Lambda_{\mathbf{1}}(s) = \frac{1}{q-1}\Lambda_{c_q}(s) - \frac{q}{q-1}\sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi\neq\chi_0}} \overline{\chi(a)}\Lambda_\chi(s).$$

Applying the functional equation and making use of Lemma 2.1, this implies that

$$\epsilon_{\mathbf{1}} N^{\frac{1}{2}-s}\overline{\Lambda_{\mathbf{1}}(1-\bar{s})} = \frac{\epsilon_{\mathbf{1}}\xi(q)}{q-1}(Nq^2)^{\frac{1}{2}-s}\overline{\Lambda_{c_q}(1-\bar{s})} - \frac{q}{q-1}(Nq^2)^{\frac{1}{2}-s}\sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi\neq\chi_0}} \overline{\chi(a)}\epsilon_\chi\overline{\Lambda_\chi(1-\bar{s})}.$$

Multiplying both sides by $\overline{\epsilon_{\mathbf{1}}}(Nq^2)^{s-\frac{1}{2}}$, replacing s by $1-\bar{s}$ and conjugating, we obtain

$$q^{1-2s}\Lambda_{\mathbf{1}}(s) = \frac{\overline{\xi(q)}}{q-1}\Lambda_{c_q}(s) - \frac{q}{q-1}\sum_{\substack{\chi \,(\mathrm{mod}\ q)\\ \chi\neq\chi_0}} \chi(a)\epsilon_{\mathbf{1}}\overline{\epsilon_\chi}\Lambda_\chi(s).$$

Comparing the Dirichlet coefficients of both sides at q, we see that $\lambda_q = 0$. In turn, as above, this implies that $\lambda_{q^2} = -\xi(q)$. Comparing coefficients at q^2, we thus have $q = -1$, which is absurd. This concludes the proof. $\square$

In [17, Theorem 9], Li proved that a cuspform whose L-function satisfies both the Euler product (1.1) and the functional equation (1.2) for $\chi = \mathbf{1}$ must be primitive. Our final result of this section constitutes an extension of Li's result that includes the Eisenstein series.

Lemma 2.4. *Let* $f \in M_k(\Gamma_0(N), \xi)$ *have Fourier expansion* $\sum_{n=0}^\infty f_n e(nz)$*, and assume that it is a normalized eigenfunction for the full Hecke algebra, so that*

$$\sum_{n=1}^\infty f_n n^{-s-\frac{k-1}{2}} = \prod_p \frac{1}{1 - f_p p^{-s-\frac{k-1}{2}} + \xi(p)p^{-2s}}.$$

Let

$$\Lambda_f(s) = \Gamma_{\mathbb{C}}\big(s + \tfrac{k-1}{2}\big) \sum_{n=1}^{\infty} f_n n^{-s-\frac{k-1}{2}}$$

be the associated complete L-function, and assume that it satisfies the functional equation

$$\Lambda_f(s) = \epsilon N^{\frac{1}{2}-s} \overline{\Lambda_f(1-\bar{s})}$$

for some $\epsilon \in \mathbb{C}$. Then one of the following holds:

(i) *f is a primitive cuspform, i.e. a normalized Hecke eigenform in $S_k^{\mathrm{new}}(\Gamma_0(N), \xi)$.*

(ii) *There are Dirichletcharacters $\xi_1 \pmod{N_1}$ and $\xi_2 \pmod{N_2}$ such that ξ_1 is primitive, $N_1 N_2 = N$, $\xi_1 \xi_2 = \xi$ and*

$$f_n = \sum_{d \mid n} \xi_1(n/d) \xi_2(d) d^{k-1}$$

for every $n > 0$. If $k \neq 2$ then ξ_2 is primitive. If $k = 2$ then ξ_2 need not be primitive (and in fact it must be imprimitive if ξ_1 and ξ_2 are both trivial), but if N_2^* denotes the conductor of ξ_2 then N_2/N_2^* is squarefree and $(N_2/N_2^*, N_2^*) = 1$.

Proof (sketch). Let X denote the set of pairs (ξ_1, ξ_2), where $\xi_1 \pmod{N_1}$ and $\xi_2 \pmod{N_2}$ are primitive Dirichlet characters such that $N_1 N_2 \mid N$, $\xi_1(-1)\xi_2(-1) = (-1)^k$ and if $k = 1$ then $\xi_1(-1) = 1$. To any pair $(\xi_1, \xi_2) \in X$ we associate the L-series

$$L_{\xi_1,\xi_2}(s) = L(s + \tfrac{k-1}{2}, \xi_1) L(s - \tfrac{k-1}{2}, \xi_2),$$

where the factors on the right-hand side are the usual Dirichlet L-functions.

Next let C denote the set of all primitive weight-k cuspforms g of conductor dividing N. To $g \in C$ with Fourier expansion $\sum_{n=1}^{\infty} g_n e(nz)$ we associate the L-series

$$L_g(s) = \sum_{n=1}^{\infty} g_n n^{-s-\frac{k-1}{2}}.$$

Let $L_f(s) = \sum_{n=1}^{\infty} f_n n^{-s-\frac{k-1}{2}}$ denote the finite L-series of f. Then by newform theory and the description of Eisenstein series in [18, §4.7], there are Dirichlet polynomials D_{ξ_1,ξ_2} and D_g such that

$$L_f(s) = \sum_{(\xi_1,\xi_2) \in X} D_{\xi_1,\xi_2}(s) L_{\xi_1,\xi_2}(s) + \sum_{g \in C} D_g(s) L_g(s). \tag{2.1}$$

Further, the coefficients of each Dirichlet polynomial are supported on divisors of N.

Following [13], we will say that Euler products $L_1(s)$ and $L_2(s)$ are *equivalent* if their Euler factors agree for all but at most finitely many primes, and *inequivalent* otherwise. It follows from the Rankin–Selberg method (see, e.g., [7, Corollary 4.4]) that the elements of $\{L_{\xi_1,\xi_2} : (\xi_1, \xi_2) \in X\} \cup \{L_g : g \in C\}$ are pairwise inequivalent. Combining this with [13, Theorem 2], we see that the right-hand side of (2.1) has exactly one nonzero term. If the nonzero term corresponds to a cuspform $g \in C$ then f is also cuspidal, and thus the conclusion follows from Li's theorem [17, Theorem 9].

Hence we may suppose that $L_f(s) = D_{\xi_1,\xi_2}(s) L_{\xi_1,\xi_2}(s)$ for some pair $(\xi_1, \xi_2) \in X$. Since the function $\Lambda_{\xi_1,\xi_2}(s) = \Gamma_{\mathbb{C}}(s + \frac{k-1}{2}) L_{\xi_1,\xi_2}(s)$ satisfies a functional equation of level $N_1 N_2$,

$D_{\xi_1,\xi_2}(s) = \Lambda_f(s)/\Lambda_{\xi_1,\xi_2}(s)$ satisfies a functional equation of level N/N_1N_2, i.e.

$$D_{\xi_1,\xi_2}(s) = \epsilon_{\xi_1,\xi_2}\left(\frac{N}{N_1N_2}\right)^{\frac{1}{2}-s}\overline{D_{\xi_1,\xi_2}(1-\bar{s})} \tag{2.2}$$

for a suitable constant ϵ_{ξ_1,ξ_2}. On the other hand, since the coefficients of $D_{\xi_1,\xi_2}(s)$ are supported on divisors of N, from the Euler products for $L_f(s)$ and $L_{\xi_1,\xi_2}(s)$ we have

$$D_{\xi_1,\xi_2}(s) = \prod_{p\mid N}\frac{(1-\xi_1(p)p^{-s-\frac{k-1}{2}})(1-\xi_2(p)p^{-s+\frac{k-1}{2}})}{1-f_pp^{-s-\frac{k-1}{2}}}.$$

This ratio must be entire, and by the functional equation (2.2), its zeros are symmetric with respect to the line $\Re(s) = \frac{1}{2}$. By the $\mathbb{Q}$-linear independence of $\log p$ for primes p, the same is true of each individual Euler factor.

Now, if $k \neq 2$ then a straightforward case-by-case analysis shows that this is only possible if

$$\frac{(1-\xi_1(p)p^{-s-\frac{k-1}{2}})(1-\xi_2(p)p^{-s+\frac{k-1}{2}})}{1-f_pp^{-s-\frac{k-1}{2}}} = 1$$

for each p, so that $D_{\xi_1,\xi_2}(s) = 1$. Thus, $L_f(s) = L_{\xi_1,\xi_2}(s)$, and by the functional equation (2.2) we have $N = N_1N_2$. This yields the desired conclusion for $k \neq 2$.

If $k = 2$ then, since the zeros of $1-\xi_2(p)p^{-s+\frac{1}{2}}$ lie on the line $\Re(s) = \frac{1}{2}$, there are two possibilities for each p:

(i) $\dfrac{(1-\xi_1(p)p^{-s-\frac{1}{2}})(1-\xi_2(p)p^{-s+\frac{1}{2}})}{1-f_pp^{-s-\frac{1}{2}}} = 1;$

(ii) $\xi_2(p) \neq 0$ and $\dfrac{(1-\xi_1(p)p^{-s-\frac{1}{2}})(1-\xi_2(p)p^{-s+\frac{1}{2}})}{1-f_pp^{-s-\frac{1}{2}}} = 1-\xi_2(p)p^{-s+\frac{1}{2}}.$

Let S denote the set of $p \mid N$ for which case (ii) applies. Then we have

$$D_{\xi_1,\xi_2}(s) = \prod_{p\in S}(1-\xi_2(p)p^{-s+\frac{1}{2}}).$$

For each $p \in S$, note that $1-\xi_2(p)p^{-s+\frac{1}{2}}$ satisfies a functional equation of level p. Comparing with (2.2), we see that $N/N_1N_2 = \prod_{p\in S} p$. Moreover, since $\xi_2(p) \neq 0$ for each $p \in S$, we have $(N/N_1N_2, N_2) = 1$. Replacing ξ_2 by the character of modulus N/N_1 that it induces, we get the conclusion of the lemma. □

3. Proof of Theorem 1.1

We first apply Lemma 2.2 to constrain the poles of $\Lambda_{\mathbf{1}}(s)$ and $\Lambda_\chi(s)$ for primitive characters χ of prime conductor $q \nmid N$. When $k \leq 2$ we suppose for now that $\Lambda_{\mathbf{1}}(s)$ is entire, and return to the general case below. Thus, both $\Lambda_{\mathbf{1}}(s)$ and $\Lambda_\chi(s)$ are entire of finite order. By the Phragmén–Lindelöf convexity principle, they are bounded in vertical strips.

Let $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the upper half-plane. For $z \in \mathbb{H}$, set

$$f_n = \lambda_n n^{\frac{k-1}{2}}, \quad f(z) = \sum_{n=1}^{\infty} f_n e(nz) \quad\text{and}\quad \bar{f}(z) = \sum_{n=1}^{\infty}\overline{f_n}e(nz).$$

For any function $g : \mathbb{H} \to \mathbb{C}$ and any matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ of positive determinant, let $g|\gamma$ denote the function

$$(g|\gamma)(z) = (\det \gamma)^{k/2}(cz+d)^{-k} g\left(\frac{az+b}{cz+d}\right).$$

Then, by Hecke's argument [18, Theorem 4.3.5], the functional equation (1.2) for $\chi = \mathbf{1}$ implies that $f|\left(\begin{smallmatrix} & -1 \\ N & \end{smallmatrix}\right) = i^k \epsilon_{\mathbf{1}} \bar{f}$. Note that

$$\begin{pmatrix} 1 & \\ N & 1 \end{pmatrix} = \begin{pmatrix} & -1 \\ N & \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} & -1 \\ N & \end{pmatrix}^{-1}.$$

Since f and $\bar{f}$ are Fourier series, it follows that $f|\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right) = f$ and $f|\left(\begin{smallmatrix} 1 & \\ N & 1 \end{smallmatrix}\right) = f$.

If $\gamma, \gamma' \in \Gamma_0(N)$ have the same top row, then a computation shows that $\gamma'\gamma^{-1}$ is a power of $\left(\begin{smallmatrix} 1 & \\ N & 1 \end{smallmatrix}\right)$, so that $f|\gamma' = f|\gamma$. Thus, $f|\gamma$ depends only on the top row of γ. With this in mind, we will write $\gamma_{q,a}$ to denote any element of $\Gamma_0(N)$ with top row $\left(\begin{smallmatrix} q & -a \end{smallmatrix}\right)$.

Let q be a prime not dividing N, and let χ be a character modulo q, not necessarily primitive. Define

$$f_\chi = \sum_{\substack{a \pmod q \\ (a,q)=1}} \overline{\chi(a)} f \left| \begin{pmatrix} 1 & a/q \\ & 1 \end{pmatrix} \right. \quad \text{and} \quad \bar{f}_{\overline{\chi}} = \sum_{\substack{a \pmod q \\ (a,q)=1}} \chi(a) \bar{f} \left| \begin{pmatrix} 1 & a/q \\ & 1 \end{pmatrix} \right. .$$

Substituting the Fourier expansion for f, we see that f_χ has a Fourier expansion with coefficients

$$f_n \sum_{\substack{a \pmod q \\ (a,q)=1}} \overline{\chi(a)} e\left(\frac{an}{q}\right) = f_n \begin{cases} c_q(n) & \text{if } \chi \text{ is trivial,} \\ \tau(\overline{\chi})\chi(n) & \text{otherwise,} \end{cases}$$

and similarly for $\bar{f}_{\overline{\chi}}$. Set

$$C_\chi = \begin{cases} \overline{\xi(q)} & \text{if } \chi \text{ is trivial,} \\ \chi(-N)\epsilon_{\mathbf{1}} \overline{\epsilon_\chi \tau(\overline{\chi})/\tau(\chi)} & \text{otherwise.} \end{cases}$$

Then, by (1.2), Lemma 2.1 and Hecke's argument, we have $f_\chi|\left(\begin{smallmatrix} & -1 \\ Nq^2 & \end{smallmatrix}\right) = i^k \chi(-N)\epsilon_{\mathbf{1}} \overline{C_\chi} \bar{f}_{\overline{\chi}}$.

Suppose that a and m are integers satisfying $Nam \equiv -1 \pmod q$. Then

$$\gamma_{q,a} = q \begin{pmatrix} & -1 \\ N & \end{pmatrix} \begin{pmatrix} 1 & m/q \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ Nq^2 & \end{pmatrix}^{-1} \begin{pmatrix} 1 & a/q \\ & 1 \end{pmatrix}^{-1} = \begin{pmatrix} q & -a \\ -Nm & \frac{Nam+1}{q} \end{pmatrix}$$

is an element of $\Gamma_0(N)$ with top row $\left(\begin{smallmatrix} q & -a \end{smallmatrix}\right)$. Thus, we have
(3.1)
$$\begin{aligned}\sum_{\substack{a \ (\mathrm{mod}\ q)\\ (a,q)=1}} C_\chi \overline{\chi(a)} f \bigg| \begin{pmatrix} 1 & a/q \\ & 1 \end{pmatrix} &= C_\chi f_\chi = i^k \chi(-N)\epsilon_{\mathbf{1}} \bar{f}_{\overline{\chi}} \bigg| \begin{pmatrix} & -1 \\ Nq^2 & \end{pmatrix}^{-1} \\ &= i^k \epsilon_{\mathbf{1}} \sum_{\substack{m \ (\mathrm{mod}\ q)\\ (m,q)=1}} \chi(-Nm) \bar{f} \bigg| \begin{pmatrix} 1 & m/q \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ Nq^2 & \end{pmatrix}^{-1} \\ &= \sum_{\substack{m \ (\mathrm{mod}\ q)\\ (m,q)=1}} \chi(-Nm) f \bigg| \begin{pmatrix} & -1 \\ N & \end{pmatrix} \begin{pmatrix} 1 & m/q \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ Nq^2 & \end{pmatrix}^{-1} \\ &= \sum_{\substack{a \ (\mathrm{mod}\ q)\\ (a,q)=1}} \overline{\chi(a)} f \bigg| \gamma_{q,a} \begin{pmatrix} 1 & a/q \\ & 1 \end{pmatrix}. \end{aligned}$$

Fix a residue b coprime to q. Multiplying both sides by $\chi(b)$ and averaging over χ, we obtain

$$f \bigg| \gamma_{q,b} \begin{pmatrix} 1 & b/q \\ & 1 \end{pmatrix} = \frac{1}{\varphi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} \chi(b) \sum_{\substack{a \ (\mathrm{mod}\ q)\\ (a,q)=1}} C_\chi \overline{\chi(a)} f \bigg| \begin{pmatrix} 1 & a/q \\ & 1 \end{pmatrix}.$$

Replacing a by ab on the right-hand side, writing

$$\widehat{C}_q(a) = \frac{1}{\varphi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} C_\chi \overline{\chi(a)}$$

and applying $\left(\begin{smallmatrix} 1 & -b/q \\ & 1 \end{smallmatrix}\right)$ on the right, we obtain

$$f|\gamma_{q,b} = \sum_{\substack{a \ (\mathrm{mod}\ q)\\ (a,q)=1}} \widehat{C}_q(a) f \bigg| \begin{pmatrix} 1 & (a-1)b/q \\ & 1 \end{pmatrix}.$$

From this we see that $f|\gamma_{q,b}$ has a Fourier expansion, with Fourier coefficients $f_n S_q(bn)$, where

$$S_q(x) = \sum_{\substack{a \ (\mathrm{mod}\ q)\\ (a,q)=1}} \widehat{C}_q(a) e\left(\frac{(a-1)x}{q}\right).$$

Now, let $\gamma = \left(\begin{smallmatrix} q & -b \\ -Nm & r \end{smallmatrix}\right)$ be an arbitrary element of $\Gamma_1(N)$. If $m = 0$ then γ is (up to sign, if $N \le 2$) a power of $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$, so that $f|\gamma = f$. Otherwise, multiplying γ on the left by $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)^{-j}$ leaves $f|\gamma$ unchanged and replaces q by $q + jmN$. By Dirichlet's theorem, we may assume without loss of generality that q is prime. Since $q \equiv 1 \pmod{N}$, we have

$$\begin{pmatrix} q & -1 \\ 1-q & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \\ N & 1 \end{pmatrix}^{\frac{1-q}{N}},$$

so that $f|\gamma_{q,1} = f$. Given any residue $x \pmod q$, by Lemma 2.1 we may choose n such that $n \equiv x \pmod q$ and $f_n \neq 0$. Equating Fourier coefficients of $f|\gamma_{q,1}$ and f, it follows that $S_q(x) = 1$. In turn, this implies that $f|\gamma_{q,b} = f$, and thus $f|\gamma = f$ for all $\gamma \in \Gamma_1(N)$.

Next consider an arbitrary $\gamma = \gamma_{q,b} \in \Gamma_0(N)$. As above, we may assume that q is prime. Moreover, for any a coprime to q, we have $\gamma_{q,a}\gamma^{-1} \in \Gamma_1(N)$, so that $f|\gamma_{q,a} = f|\gamma$. Taking χ equal to the trivial character mod q in (3.1), we thus find that

$$\sum_{\substack{a \pmod q \\ (a,q)=1}} \left(f|\gamma - \overline{\xi(q)}f\right) \Bigg| \begin{pmatrix} 1 & a/q \\ & 1 \end{pmatrix} = 0. \tag{3.2}$$

We showed above that $f|\gamma$ has a Fourier expansion. Writing a_n for the Fourier coefficients, (3.2) implies that $(a_n - \overline{\xi(q)}f_n)c_q(n) = 0$ for every n. Since $c_q(n)$ never vanishes, we have $a_n = \overline{\xi(q)}f_n$, so that $f|\gamma = \overline{\xi(q)}f$. Thus, we have shown that $f \in M_k(\Gamma_0(N), \xi)$.

Next, by Lemma 2.4, either f is a primitive cuspform or there are Dirichlet characters $\xi_1 \pmod{N_1}$ and $\xi_2 \pmod{N_2}$ such that $N_1N_2 = N$, $\xi_1\xi_2 = \xi$ and

$$f_n = \sum_{d|n} \xi_1(n/d)\xi_2(d)d^{k-1} \tag{3.3}$$

for every n. For $k \geq 2$, we consider (3.3) at $n = q_1 \cdots q_m$, where $q_1, \ldots, q_m$ are the m smallest primes $\equiv 1 \pmod N$. For this n we see that

$$\lambda_n = f_n n^{-\frac{k-1}{2}} = \prod_{i=1}^{m} \left(q_i^{\frac{k-1}{2}} + q_i^{-\frac{k-1}{2}}\right) \geq \prod_{i=1}^{m} \left(q_i^{\frac{1}{2}} + q_i^{-\frac{1}{2}}\right),$$

so that

$$\frac{\lambda_n}{\sqrt{n}} \geq \prod_{i=1}^{m} (1 + q_i^{-1}).$$

By Dirichlet's theorem, the right-hand side grows without bound as $m \to \infty$. This contradicts the hypothesis that $\lambda_n = O(\sqrt{n})$, so f must be a primitive cusp form. When $k = 1$, f need not be cuspidal, but in this case Lemma 2.4 asserts that ξ_1 and ξ_2 are primitive. Thus, we have verified the conclusion of Theorem 1.1.

It remains only to handle the possibility that $\Lambda_{\mathbf{1}}(s)$ has poles when $k \leq 2$. In this case we fix an odd prime $q \nmid N$ and a primitive character $\chi \pmod q$, and consider the sequence $\lambda_n' = \lambda_n\chi(n)$ in place of λ_n, $\xi\chi^2$ in place of ξ and Nq^2 in place of N. Then all of the hypotheses of Theorem 1.1 are satisfied for these data, and the associated L-function $\Lambda_{\mathbf{1}}(s)$ is entire. Thus, by what we have already shown, either there is a primitive cuspform $f' \in S_k^{\mathrm{new}}(\Gamma_0(Nq^2), \xi\chi^2)$ with Fourier coefficients $\lambda_n' n^{\frac{k-1}{2}}$, or $k = 1$ and there are primitive characters ξ_1' and ξ_2' such that $\lambda_n' = \sum_{d|n} \xi_1'(n/d)\xi_2'(d)$.

Consider the cuspidal case first. By newform theory [1, Theorem 3.2], we can twist f' by $\overline{\chi}$, i.e. there is a primitive cuspform f of conductor Nq^j for some j, with Fourier coefficients $\lambda_n'\overline{\chi(n)}n^{\frac{k-1}{2}} = \lambda_n n^{\frac{k-1}{2}}$ for every n coprime to q. Since q was arbitrary, we can apply this argument to two different choices of q. Then strong multiplicity one implies that f has conductor N and Fourier coefficients $\lambda_n n^{\frac{k-1}{2}}$ for every n, as desired.

In the non-cuspidal case, let $\xi_i \pmod{N_i}$ $(i = 1, 2)$ be the primitive character inducing $\xi_i'\overline{\chi}$. Then as above we find that $\lambda_n = \sum_{d|n} \xi_1(n/d)\xi_2(d)$ for all n coprime to q. In particular,

$$\lambda_p = \xi_1(p) + \xi_2(p) \quad \text{for all sufficiently large primes } p. \tag{3.4}$$

Note that ξ_1 and ξ_2 have opposite parity. If we normalize ξ_1 to be even then, since ξ_1 and ξ_2 are primitive, Dirichlet's theorem implies that they are uniquely determined by (3.4). Hence, using two choices for q, we see that $N_1N_2 = N$ and $\lambda_n = \sum_{d|n} \xi_1(n/d)\xi_2(d)$ for all n. This completes the proof.

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