The Phillips curve in a matching model*

Tai-Wei Hu[1] Neil Wallace

University of Bristol Penn State University

January 8, 2019

Abstract

Following ideas in Hume, monetary shocks are embedded in the Lagos-Wright model in a new way: there are only nominal shocks that are accomplished by way of individual transfers and there is sufficient noise in individual transfers so that realizations of those transfers do not permit the agents to deduce much about the aggregate realization. The last condition is achieved by assuming that the distribution of aggregate shocks is almost degenerate. For such rare shocks, aggregate output increases with the growth rate of the stock of money—our definition of the Phillips curve. This almost-degeneracy assumption is far from being necessary; under some mild conditions, the Phillips curve result holds for a large class of distributions.

Key words: Phillips curve, matching model, imperfect information

JEL classification number: E30

*Submission date: Feb 2nd, 2018; revision received: Sep 13th, 2018.

[1]Corresponding author. We are grateful to seminar participants at National Taiwan University, University of Bath, Penn State-Cornell Macro workshop, Richmond Fed, and UNSW. We are also grateful to the referees for helpful comments on an earlier draft of this paper.

1 Introduction

In his Nobel lecture [see "Monetary Neutrality" (Lucas (1996)], Lucas begins by describing Hume's (1970) views about the effects of changes in the money supply. Lucas emphasizes that Hume's views were dependent on how changes in the quantity of money come about. In order to get neutrality, Hume set out very special conceptual experiments which, when correct, amount to changes in monetary units. For some other kinds of changes, Hume says that there is a short-run Phillips curve:

> Accordingly we find that, in every kingdom into which money begins to flow in greater abundance than formerly, everything takes a new face: labour and industry gain life, the merchant becomes more enterprising...
>
> To account, then, for this phenomenon, we must consider, that though the high price of commodities be a necessary consequence of the encrease of gold and silver, yet it follows not immediately upon that encrease, but some time is required before the money circulates through the whole state, and makes its effect be felt on all ranks of people. At first, no alteration is perceived; by degrees the price rises, first of one commodity, then of another, till the whole at last reaches a just proportion with the new quantity of specie in the kingdom. In my opinion, it is only in this interval or intermediate situation, between the acquisition and rise of prices, that the encreasing quantity of gold and silver is favorable to industry. When any quantity of money is imported into a nation, it is not at first dispersed into many hands but is confined to the coffers of a few persons, who immediately seek to employ it to advantage. [Hume (1970), page 37.]

Hume asserts that there is a positive association between the changes in the stock of money and real economic activity, which is our definition of the Phillips curve.[2] He also offers what may

[2] To be precise, we use the term Phillips-curve to mean that total output is strictly increasing in the growth rate of the stock of money. As is true of many models of the relationship between money and economic activity, there is no unemployment in our model.

at some time have been regarded as an explanation of it. A modern economist would not treat his discussion as an explanation, but might look to it for hints about modeling ingredients that when rigorously analyzed could conceivably constitute an explanation.

The passage from which the above excerpt comes contains at least two hints about modeling ingredients. First, changes in the quantity of money come about in a way that gives rise to changes in relative money holdings among people. In particular, the changes for individuals are not uniformly proportional to initial holdings as is required for neutrality. Second, trade seems to be occurring within small groups, rather than in a centralized market. That suggests the use of some sort of search/matching model. Given those ingredients, the passage hints at two conjectures that might be studied. One is that a change in relative money holdings has Phillips-curve type effects that dissipate over time through the effects of subsequent trades on those relative holdings. The other is that the change occurs in a way that is not seen by everyone when it occurs and that the Phillips-curve type effects dissipate when people learn about it. Although these are not mutually exclusive conjectures, we pursue only the second here.

In order to study it, we embed monetary shocks in the Lagos and Wright (2005) (LW) model and assume that the aggregate monetary shocks (i) are observed with a lag and (ii) are accomplished by way of individual transfers in such a way that those transfers are imperfect signals about the aggregate shock. The information imperfection is modeled in the usual way: there is a fixed support for individual transfers and the aggregate shock determines the distribution over that fixed support. Our main contribution is to show that when these transfers are relatively uninformative about the aggregate shock, there is a Phillips curve. For general parameter values, we show this when the distribution of the aggregate shock is close to a degenerate distribution—that is, when shocks are rare. But this near-degeneracy is far from being necessary: under some mild conditions, we obtain the Phillips curve for a large class of distributions of the aggregate shock. We also provide a counterexample that illustrates how things could go wrong.

We are not the first to study the Phillips curve in a model of small-group trade. Wallace (1997) and Katzman *et. al.* (2003) do so in a model in which money holdings are limited to be in the set $\{0, 1\}$. However, Phillips curve results in both papers depend on the assumption that less than half the population has money, an assumption that is troublesome because it has no analogue when money is divisible. Nor are we the first to use some version of LW to study the Phillips curve. Faig and Li (2009) embed a version of the signal-extraction problem in Lucas (1972) into that model. That signal-extraction problem involves a delicate confounding of real and nominal shocks, and the sign of the slope of the Phillips curve depends on preferences. In contrast, our Phillips curve relies on the assumption that individual transfers are relatively uninformative about the aggregate shock. Moreover, our specification is closer to Hume, is conceptually simpler, and is strategic.[3] In it, people meet in pairs and do not see the transfers received or the trades in other meetings. Therefore, they can only use what they experience in their meeting to draw inferences about the aggregate shock.

2 The model

The background model is that in LW. Time is discrete and there are two stages at each date. In the first stage, the decentralized market (the DM), production and consumption occur in pairwise meetings that occur at random in the following way. Just prior to such meetings, each person looks forward to being a consumer (buyer) who meets a random producer (seller) with probability σ, looks forward to being a producer who meets a random consumer with probability σ, and looks forward to a no-coincidence meeting with probability $1 - 2\sigma$, where $\sigma \leq 1/2$. In the DM, $u(y)$ is the utility of consuming and $c(y)$ is the disutility of producing, where u and c are twice differentiable, $u(0) = c(0) = 0$, u is strictly concave, c is convex, and $u'(0) - c'(0)$ is infinite. We also assume that $y^* = \arg\max_y[u(y) - c(y)]$ exists. In the second stage all agents can consume and produce and meet in the centralized market (the CM), where the utility of consuming is z and where negative z is production. At each stage, production is perishable and the discount factor between dates is $\beta \in (0, 1)$.

[3]Lucas (1972) and Faig and Li (2009) use rational-expectations competitive-equilibrium as a solution concept and have agents learn from the prices they see. Such learning has not been given a strategic foundation.

In our version, the gross growth rate of the money stock follows an *iid* process with finite support S, where $S = \{s_1, s_2, ..., s_N\}$ with $1 \leq s_n < s_{n+1}$. We let π denote the distribution over S. The changes in the stock of money are accomplished by random proportional transfers to individuals. We let $T = \{\tau_1, \tau_2, ..., \tau_I\}$ be the set of possible gross proportional transfers to a person, where $1 \leq \tau_i < \tau_{i+1}$. We let $\mu_s(\tau)$ be the probability that an agent receives the transfer $\tau \in T$ conditional on the aggregate state $s \in S$. (As this suggests, conditional on s, agents receive independently drawn transfers.) We assume that μ_s satisfies the following conditions. First, the individual transfers aggregate to s. That is,

$$\sum_{i=1}^{I} \mu_s(\tau_i)\tau_i = s \tag{1}$$

for each $s \in S$. Second, except for one of the neutrality results, we assume that μ_s has full support for all $s \in S$ and that it satisfies the following strict version of first-order stochastic dominance: for any $n' > n$ and for all $i = 1, ..., I - 1$,

$$\sum_{k=1}^{i} \mu_{s_{n'}}(\tau_k) < \sum_{k=1}^{i} \mu_{s_n}(\tau_k). \tag{2}$$

The weak inequality of version of (2) is first-order stochastic dominance.[4]

There are two versions of the model; one with an information lag and one without such a lag. The sequence of actions when there is an information lag is as follows. After people leave the CM, the growth rate of the stock of money, $s \in S$, is realized, but not observed. Then agents meet at random in pairs. Then each agent receives a proportional money transfer, a draw from $\mu_s(\tau)$. Within a meeting, both the pre-transfer and post-transfer money holdings are common knowledge. In a meeting where the transfers received are τ_i and τ_j, the common posterior about s is given by Bayes' rule:

$$p_{ij}(s) = \frac{\pi(s)\mu_s(\tau_i)\mu_s(\tau_j)}{\sum_{n=1}^{N} \pi(s_n)\mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j)}. \tag{3}$$

[4]Condition (1) and the full support assumption about μ_s imply that the range of S is a strict subset of the range of T.

If the meeting is between a buyer and seller, then the buyer makes a take-it-or-leave-it offer. After meetings, agents learn s and enter the next CM.[5] If an agent leaves the pairwise trade stage, the DM, with m amount of money, then the agent enters the next CM with m/s amount of money. (This is the standard way of normalizing inflation so that the per capita quantity of money and the price of money can be constant.) This also allows us to normalize average money holding to be unity. Then, in price-taking trade, the good trades for money at the next CM.[6]

The sequence of actions when there is no information-lag is identical except that the growth rate of the stock of money, s, is observed when it is realized. In that case, $p_{ij}(s)$ in (3) is replaced by a distribution that is degenerate on the realized s.[7]

Several comments are in order about this specification. First, we model the transfers as proportional to money holdings in order to isolate the effect of heterogeneous transfers (see Corollary 2 below). Our Phillips curve result would also hold if the transfers were lump-sum. However, then, as is well-known, the transfers would have real effects even without heterogeneity. Second, the only role of assuming that the transfers are realized within meetings as opposed to before meetings is to rationalize the assumption that each person in a meeting knows the transfer received by their trading partner. Third, the assumption that the buyer makes a take-it-or-leave-it offer in meetings is not crucial. We suspect that the results also hold for many of the bargaining rules described in Gu and Wright (2016).

[5]In a somewhat different context, Araujo and Shevchenko (2006) study the complications that arise when the information lag is longer.

[6]As in Hu *et.al.* (2009), this price-taking trade can easily be modeled as a game.

[7]As may be evident, the information-lag version and the no information-lag version are special cases of a more general specification. The buyer's transfer and the seller's transfer play very different roles in the model. Both symmetrically affect the posterior over the aggregate realization in the information-lag version. In addition, the buyer's transfer affects spending directly in both versions. Therefore, the information-lag version is equivalent to a setting in which only the buyer realizes a transfer, but the pair in a meeting see the buyer's transfer and also see the buyer's transfer in one other randomly chosen meeting. But, then, what if the pair in a meeting see the buyer's transfer and that in l randomly chosen meetings? Obviously, as $l \to \infty$, there is convergence to the no-information-lag version.

3 Stationary equilibrium

We start with existence of stationary monetary equilibrium. The objects in a stationary monetary equilibrium are output in buyer-seller meetings—denoted y_{ij}, where τ_i is the transfer received by the buyer and τ_j is that received by the seller—and the price of money in the CM—denoted v.

Proposition 1. In each version of the model, either with or without an information-lag, there exists a unique valued-money stationary equilibrium. In that equilibrium $y_{ij} < y^*$ for at least some (i, j). Moreover, y_{ij} and v are continuous in π.

The proof, which follows, is familiar from other expositions of LW. Standard arguments show that the value function entering the CM is affine, and that the rate-of-return of money from one CM to next CM is equal to one. Given that result, the equilibrium condition reduces to the solution to an optimal saving decision in the CM, a decision which is common to everyone.

Proof. Assume that there is an information-lag. (As we remark at the end, the no-information-lag version is a special case.) We refer to the current centralized market as the CM. As is well-known, the value of entering the next CM with m' amount of money is $v'm' + A$, where v' is the price of money and A is a constant. In a stationary equilibrium, $v' = v$, the price of money in the current CM.

A person who enters the DM with m amount of money is a buyer with probability σ, a seller with probability σ, or is in a no-coincidence meeting with probability $1 - 2\sigma$. Given the assumed continuation value in the next CM, we start by considering the buyer's problem, the only significant one in the DM,

Consider a buyer who enters the DM with m amount of money and is in meeting (i, j), one in which the buyer receives transfer τ_i and the seller receives transfer τ_j. The problem of the buyer is to choose output, y_{ij}, and the amount of money to offer, d_{ij}, to maximize $u(y_{ij}) + v'(m\tau_i - d_{ij})E_{ij}(1/s)$ subject to $m\tau_i - d_{ij} \geq 0$ and

$$c(y_{ij}) \leq v'd_{ij}E_{ij}(1/s). \tag{4}$$

Here, $E_{ij}(1/s) = \sum_{n=1}^{N} p_{ij}(s_n)/s_n$, where $p_{ij}(s_n)$ is given by (3) and where $E_{ij}(1/s) = E_{ji}(1/s)$. Because the value function in the next CM is affine, the seller's money holding appears in (4) only by way of $E_{ij}(1/s)$. At a solution to this problem, (4) holds at equality. (If not, then increase y_{ij}.) Substituting (4) at equality into the objective, the objective becomes $u(y_{ij}) - c(y_{ij})$. Therefore, the solution has two branches depending on whether the constraint $m\tau_i - d_{ij} \geq 0$ is binding. The non-binding case has $y_{ij} = y^*$ and d_{ij} given by (4) at equality. The binding case has $d_{ij} = m\tau_i$ and y_{ij} given by (4) at equality. That is,

$$y_{ij} = \begin{cases} y^* \text{ if } c(y^*) \leq v' m\tau_i E_{ij}(1/s) \\ \\ c^{-1}(v' m\tau_i E_{ij}(1/s)) \text{ if } c(y^*) > v' m\tau_i E_{ij}(1/s) \end{cases} \tag{5}$$

and

$$d_{ij} = \begin{cases} c(y^*)/v' E_{ij}(1/s) \text{ if } c(y^*) \leq v' m\tau_i E_{ij}(1/s) \\ \\ m\tau_i \text{ if } c(y^*) > v' m\tau_i E_{ij}(1/s) \end{cases} . \tag{6}$$

Therefore, the buyer's payoff is $u(y_{ij}) + (m\tau_i - d_{ij})v' E_{ij}(1/s) + A$.

Now suppose the person with m is not a buyer but realizes transfer τ_i when the trading partner realizes transfer τ_j. This person's payoff is the same whether the person is a seller or is in a no-coincidence meeting because sellers receive no gains from trade—(4) at equality with the roles of i and j reversed. Thus, the person's payoff as a seller or in a no-coincidence meeting is $v' m\tau_i E_{ij}(1/s) + A$.

Therefore, ignoring constant terms, both money brought into the CM and the constant A, the CM problem is

$$\max_{m \geq 0} -vm + \beta \sum_{i,j} \gamma_{ij} \left\{ \sigma \left[u(y_{ij}) - v' d_{ij} E_{ij}(1/s) \right] + v' m\tau_i E_{ij}(1/s) \right\}, \tag{7}$$

where $\gamma_{ij} = \sum_n \pi(s_n)\mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j)$, which by the law of iterated expectations, is the unconditional probability that the person receives transfer τ_i and is in a meeting with a person who receives τ_j. Now, because $\sum_{i,j} \gamma_{ij}\tau_i E_{ij}(1/s) = 1$ (see Lemma 1 in the appendix), this becomes

$$\max_{m\geq 0} -vm + \beta v' m + \beta\sigma \sum_{i,j} \gamma_{ij}\left[u(y_{ij}) - v' d_{ij} E_{ij}(1/s)\right]. \tag{8}$$

Letting $x = mv$, real saving, and imposing $v = v'$, an equivalent choice problem is

$$\max_{x\geq 0} G(x) \equiv -(\beta^{-1} - 1)x + \sigma \sum_{i,j} \gamma_{ij}\{u[\tilde{y}_{ij}(x)] - v\tilde{d}_{ij}(x)E_{ij}(1/s)\}, \tag{9}$$

where, by (5) and (6),

$$\tilde{y}_{ij}(x) = \begin{cases} y^* \text{ if } c(y^*) \leq x\tau_i E_{ij}(1/s) \\ \\ c^{-1}(x\tau_i E_{ij}(1/s)) \text{ if } c(y^*) > x\tau_i E_{ij}(1/s) \end{cases} \tag{10}$$

and

$$v\tilde{d}_{ij}(x) = \begin{cases} c(y^*)/E_{ij}(1/s) \text{ if } c(y^*) \leq x\tau_i E_{ij}(1/s) \\ \\ x\tau_i \text{ if } c(y^*) > x\tau_i E_{ij}(1/s) \end{cases}. \tag{11}$$

The rest of the proof appears in the appendix. There, it is shown that G is differentiable with derivative denoted $G'(x)$. It is then shown that $G'(0) > 0$ and that $G'(x) < 0$ for $x \geq \bar{x}$, where

$$\bar{x} = \frac{\min_{i,j}[\tau_i E_{ij}(1/s)]}{c(y^*)}. \tag{12}$$

It follows that the choice of x can be limited to the compact domain $[0, \bar{x}]$, that G has a maximum, and that any maximum occurs in the interval $(0, \bar{x})$. Then it is shown that $G'(x)$ is strictly decreasing on $(0, \bar{x})$, which implies that G is strictly concave on $(0, \bar{x})$. That implies that the maximum, denoted $\hat{x}$, is unique. And, because $\bar{x}$ is continuous in π, the Theorem of the Maximum implies that $\hat{x}$ is continuous in π. Because y_{ij} is continuous in x, it also is continuous in π. Because we normalize the per capita quantity of money to be unity, $v = \hat{x}$ follows from equating real saving to the real value of money. In order to see that $y_{ij} < y^*$ for some (i, j), we consider two cases. If $\min_{i,j}[\tau_i E_{ij}(1/s)] = \tau_k E_{kl}(1/s)$ for all (k, l), then $\hat{x} < \bar{x}$ implies $y_{kl} < y^*$ for all (k, l). Otherwise, $y_{kl} < y^*$ for all (k, l) such that $\min_{i,j}[\tau_i E_{ij}(1/s)] < \tau_k E_{kl}(1/s)$. The proof for the no-information-lag version is identical except that $E_{ij}(1/s)$ is replaced by $1/s$.■

Two neutrality results follow from the above exposition.

Corollary 1. If economies 1 and 2 are identical except that $S^2 = \alpha S^1$ and $T^2 = \alpha T^1$ for some $\alpha > 1$, then both economies have the same real equilibria.

Proof. It is immediate that $E^2_{ij}(1/s) = (1/\alpha)E^1_{ij}(1/s)$. From that it follows that α does not appear in (9)-(12).■

It follows that we can without loss of generality impose $\tau_1 = 1$, as we do in some examples below.

Corollary 2. If there is no heterogeneity in realized individual transfers—meaning that for each s, $\mu_s(\tau) = 1$ for some τ—then the equilibrium is the same as that of an economy with a degenerate π (no aggregate uncertainty).

Proof. Let s be the realized aggregate shock. By the hypothesis, there exists $\tau(s)$ such that $\mu_s[\tau(s)] = 1$. It follows that $E_{ij}(1/s) = 1/s$ and from (1) that $\tau(s) = s$. Therefore, using (10), and (11), $u[\tilde{y}_{ij}(x)] - v\tilde{d}_{ij}(x)E_{ij}(1/s)$ does not depend on s and (9) reduces to the special case of no randomness. ■

4 The Phillips Curve under near degeneracy

Now we turn to our main result, the existence of a Phillips curve. Proposition 1 establishes the existence of a unique stationary monetary equilibrium. For any $\pi \in \Delta(S)$, let $y^1_{ij}(\pi)$ be the corresponding equilibrium DM output for meeting type (τ_i, τ_j) with an information lag, and let $y^0_{nij}(\pi)$ be that without an information lag for meeting type (s_n, τ_i, τ_j). Then, the respective aggregate DM outputs are

$$Y^1(s_n, \pi) = \sigma \sum_{i,j} \mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j) y^1_{ij}(\pi) \tag{13}$$

and

$$Y^0(s_n, \pi) = \sigma \sum_{i,j} \mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j) y^0_{nij}(\pi) = \sigma \sum_{i} \mu_{s_n}(\tau_i) y^0_{ni}(\pi), \tag{14}$$

where the second equality in (14) holds because output does not depend on the seller's transfer when there is no information lag. We focus on output in the DM, because output in the CM does not depend on the realized aggregate shock. We say that there is a Phillips curve if $Y^1(s_n, \pi)$ is strictly increasing in s.

According to (13), aggregate DM output is a weighted average of meeting specific outputs. The meeting specific outputs, represented by the matrix $[y^1_{ij}]_{i,j}$, do not depend on s. And according to (2), the higher is s, the more weight is placed on those components of the matrix with higher transfers. Intuitively, a Phillips curve would be obtained if meetings with higher transfers are associated with higher outputs. However, that is delicate. Although high transfers to buyers tend to increase spending in the meetings, high transfers (no matter whether to buyers or to sellers) also suggest that the aggregate shock is high and, therefore, tend to offset the higher spending effect. Our main Phillips-curve result, proposition 2, limits the informational role of transfers by describing what happens in a neighborhood of a degenerate π. (In the next section on robustness, we present some results that go beyond near-degeneracy to study the two effects.)

Proposition 2 establishes two results: one is about the dependence of $Y^1(s, \pi)$ on s and the other is about the dependence of $Y^1(s, \pi) - Y^0(s, \pi)$ on s, both for a neighborhood of a degenerate π. The first is the main Phillips curve result. The second says that the information lag plays a role.

Proposition 2. Let $\tilde{\pi} \in \Delta(S)$ and $\tilde{n}$ be such that $\tilde{\pi}(s_{\tilde{n}}) = 1$. There is a neighborhood of $\tilde{\pi}$ such that for all π in that neighborhood: (i) $Y^1(s, \pi)$ is strictly increasing in s; and (ii) $Y^1(s, \pi) - Y^0(s, \pi) > 0$ for all $s > s_{\tilde{n}}$ and $Y^1(s, \pi) - Y^0(s, \pi) < 0$ for all $s < s_{\tilde{n}}$.

Proof. (i) Because Y^1 is continuous in π, it suffices to show that $Y^1(s,\tilde{\pi})$ is strictly increasing in s. For $\pi = \tilde{\pi}$, $p_{ij}(s_{\tilde{n}}) = 1$. This implies that $E_{ij}(1/s) = 1/s_{\tilde{n}}$ for all i, j. Therefore, $y^1_{ij}(\tilde{\pi})$ does not depend on j and, as implied by Proposition 1, $y^1_{1j}(\tilde{\pi}) < y^*$. Also, by (10), $y^1_{ij}(\tilde{\pi})$ is weakly increasing in i and is not constant. Hence, by our strict stochastic dominance assumption (see (2)), $Y^1(s,\tilde{\pi})$ is strictly increasing in s.

(ii) Because $Y^1(s,\pi)$ and $Y^0(s,\pi)$ are continuous in π, it is enough to establish the inequalities for $\pi = \tilde{\pi}$. For $\pi = \tilde{\pi}$, the equilibrium real balance, $\hat{x}$, does not depend on whether or not there is an information-lag. Therefore, $y^1_{ij}(\tilde{\pi}) = y^0_{\tilde{n}ij}(\tilde{\pi})$. By (10) with $E_{ij}(1/s) = 1/s$, $y^0_{nij}(\tilde{\pi}) < y^0_{n'ij}(\tilde{\pi})$ for all $n > n'$and all i, j such that $y^0_{nij}(\tilde{\pi}) < y^*$ (which necessarily holds for $i = 1$). Therefore, for $n > \tilde{n}$,

$$\begin{aligned} & [Y^1(s_n,\tilde{\pi}) - Y^0(s_n,\tilde{\pi})] \\ = \; & \sigma \sum_{i,j} \mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j)[y^1_{ij}(\tilde{\pi}) - y^0_{nij}(\tilde{\pi})] \qquad (15) \\ = \; & \sigma \sum_{i,j} \mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j)[y^0_{\tilde{n}ij}(\tilde{\pi}) - y^0_{nij}(\tilde{\pi})] > 0. \end{aligned}$$

Similarly, for $n < \tilde{n}$,

$$\begin{aligned} & [Y^1(s_n,\tilde{\pi}) - Y^0(s_n,\tilde{\pi})] \\ = \; & \sigma \sum_{i,j} \mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j)[y^1_{ij}(\tilde{\pi}) - y^0_{nij}(\tilde{\pi})] \qquad (16) \\ = \; & \sigma \sum_{i,j} \mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j)[y^0_{\tilde{n}ij}(\tilde{\pi}) - y^0_{nij}(\tilde{\pi})] < 0, \end{aligned}$$

which completes the proof. ■

Notice that part (ii) of Proposition 2 says that the output effects of (rare) shocks are larger when there is an information-lag. There is no claim about whether there is a Phillips curve when there is no information-lag. Corollary 3 and example 1 demonstrate that little can be said in general about the Phillips curve when there is no information lag.

Corollary 3. If (i) $c(y) = y$ and (ii) β is small enough so that $y^0_{ni} < y^*$ for all n and i, then

$Y^0(s_n, \pi)$ does not depend on s_n.

Proof. Let x_π be the equilibrium real balance in the CM when there is no information-lag. Under the assumptions,

$$Y^0(s_n, \pi) = \sigma \sum_i \mu_{s_n}(\tau_i) y^0_{ni}(\pi) = \sigma \sum_i \mu_{s_n}(\tau_i)\tau_i x_\pi / s_n = \sigma x_\pi \left[\sum_i \mu_{s_n}(\tau_i)\tau_i \right] / s_n = \sigma x_\pi, \quad (17)$$

where the second equality follows from the second line of (10) and $E_{ij}(1/s) = 1/s$, and where the last equality follows from (1).■

The following example shows that curvature in $c(y)$ is enough to make $Y^0(s_n, \pi)$ non-monotone in s_n.

Example 1. Consider the model without an information-lag. Suppose that $c(y) = y^2$, $T = \{1, \tau\}$, and that β is small enough so that $y_{ni}(\pi) < y^*$ for all n, i. Then, $Y^0(s_n, \pi)$ is non-monotone in s_n. (The proof is in the appendix.)

5 Robustness

There are special aspects of the model and special aspects of the process for aggregate shocks in Proposition 2. Here we discuss whether the Phillips curve result will survive if we depart from some of them.

5.1 More general *iid* shocks

As mentioned, a higher transfer has two distinct effects on the quantity produced in a given meeting—the informational effect decreases output while the spending effect increases output. Proposition 2, by assuming near-degeneracy, limits the first effect. That result is valid for any distribution over T and any parameters of the model. Here, we focus on some special classes of distributions, which allow us to analytically study the relative strength of these two forces. The first class shows that the spending effect is stronger for a wide range of parameters, much broader than near-degeneracy. The second and more special class shows that the informational effect can overcome the spending effect, at least for some realizations.

Proposition 3. Assume that $T = \{1, \tau\}$ and that π is a symmetric distribution with $E_\pi(s) = (1+\tau)/2$. Also, assume that $c(y) = y$ and that β is sufficiently small so that the buyer constraint is always binding in equilibrium. Then, $Y^1(s, \pi)$ is strictly increasing in s if the support of π is contained in $[\alpha + (1-\alpha)\tau, (1-\alpha) + \alpha\tau]$ with

$$\alpha < \sqrt{\frac{1+\tau}{4(3\tau - 1)}} + \frac{1}{2}. \tag{18}$$

The proof of Proposition 3 can be found in the Appendix. A two-point support for T is convenient because condition (1) then determines $\mu_s(\tau)$ and condition (2) is implied, but we can obtain a similar result by introducing intermediate transfers in a symmetric fashion (see Proposition 4 in the Appendix for details). For such T, feasibility requires that the support of S be bounded by $[\alpha + (1-\alpha)\tau, (1-\alpha) + \alpha\tau]$ for some $\alpha \in (0, 1)$. The larger is τ, the more stringent is the inequality in (18). However, even when τ converges to infinity, the right side of (18) converges to 0.78, and, therefore, allows for a wide range of supports for s. For example, Proposition 3 implies that, when $\tau \leq 2$, the Phillips curve result holds for any (discrete) uniform distribution over $[\alpha + (1-\alpha)\tau, (1-\alpha) + \alpha\tau]$ with $\alpha \leq 0.88$. Moreover, by the continuity established in Proposition 2, both the assumed symmetry and the condition on the expected value of s need only hold approximately. Thus, Proposition 3 shows that near-degeneracy is far from being necessary for the Phillips curve.

The following example, however, shows that some restriction on the support is necessary for the Phillips curve result.

Example 2. Maintain all the assumptions in Proposition 3, except for (18). Instead, for each $N \in \mathbb{N}$, let $S_N = \{1+\varepsilon, 1+2\varepsilon, ..., 1+N\varepsilon\}$ with $\varepsilon = (\tau - 1)/(N+1)$ and let

$$\pi(1+\varepsilon) = \pi(\tau - \varepsilon) = \frac{1}{2}(1-\delta),$$

and

$$\pi(1+n\varepsilon) = \frac{\delta}{N-2} \text{ for all } 2 \leq n \leq N-1$$

for some small $\delta > 0$. Then, there exist (large) $\bar{N}$ and (small) $\bar{\delta} > 0$ such that if $N > \bar{N}$ and if $\delta < \bar{\delta}$, then $Y^1(s, \pi)$ is not monotone in s. (The proof is in the appendix.)

In this example, π assigns high probability to the end-points of S. When both the buyer and the seller receive τ, they are almost certain that the aggregate state is near τ, while if both receive 1, then they are almost certain that the aggregate state is near 1. That is, the information effect has full force in those meetings. If they were certain, then output would be the same in those two meetings. If they are almost certain, then output when both receive τ is slightly larger than when both receive 1. When one receives τ and the other receives 1, their interim belief is almost identical to the prior belief, and average output over those two meetings is higher than that for the other two kinds of meetings. Hence, aggregate output is increasing in the measure of meetings with mixed transfers, a measure which is increasing in s for small s and decreasing in s for large s.

Notice, also, that Example 2 is a counter-example only because our definition of the Phillips curves calls for $Y^1(s, \pi)$ to be monotone in s—even over parts of S that occur with very low probability. A different approach would define the Phillips curve probabilistically—for example, as a positive correlation between total output and s. Defined as a correlation, Example 2 would not be a counter-example because the endpoints of S occur with arbitrarily high probability, and output at the high endpoint is larger than at the low endpoint for N large. For example, if $u(y) = \sqrt{y}$, $\beta = 1/1.05$, $\sigma = 0.3$, $\tau = 1.2$, and π is as specified in Example 3 with $N = 200$ and $\delta = 0.05$, then $Y^1(s, \pi)$ is non-monotone in s but the correlation between $Y^1(s, \pi)$ and s is 0.99.

5.2 Markov aggregate shocks

With Markov shocks, the state entering a CM is the realized s from the previous period. Then the guess is that there is a stationary equilibrium in which the price of money in the CM depends on the realized aggregate shock from the previous period, denoted $v(s)$. If so, then people in the

CM in state s face a return distribution between that CM and the next CM, where the realized return is $v(s')/v(s)$ and s' is the realized aggregate shock in the current period. The stationary equilibrium conditions implied by optimal saving choices give rise to N simultaneous equations in $v(s)$ for $s \in S$. In that case, existence requires a fixed point argument and the well-known challenge is to choose a domain for $v(s)$ for $s \in S$ that excludes $v(s) = 0$ for all $s \in S$.

However, if we assume that the Markov process is nearly degenerate—has a transition matrix for s which has a column all of whose elements are near unity—then our result in Proposition 2 applies by way of the implicit function theorem. In addition, our technique for getting a Phillips curve should also apply for a Markov process that is nearly degenerate in a different sense—namely, has a transition matrix that is close to the identity matrix.

5.3 Wealth effects

Barro (1989, pages 2-3) expresses the following concern about the robustness of the Phillips curve in Lucas (1972). The transfer in Lucas goes entirely to potential consumers—as it does in Faig and Li (2009). If both consumers and producers receive transfers, then richer producers may want to produce less and that could offset the greater spending by consumers. We have transfers going to both consumers and producers, but the structure of LW precludes wealth effects on producers. In particular, the transfer that the seller receives matters for the trade in the DM only because it is a signal about the aggregate shock.[8]

There are generalizations of LW that have wealth effects. The LW structure imposes a periodic quasi-terminal condition: when the CM stage occurs, the economy restarts from that stage with a degenerate distribution of money holdings and, hence, with no state variable. If there are many DM stages before the CM stage in each period, then there are wealth effects before the last DM stage because the continuation value of money will be strictly concave at least over some of its domain. Indeed, with a large number of DM stages, the model at intermediate DM stages resembles the divisible-money versions of the Shi (1995) and Trejos and Wright (1995)

[8]Katzman *et. al.* (2003) have transfers going to both consumers and producers, but, as noted above, their results depend on the troublesome assumption that fewer than half the population has money.

matching models studied by Zhu (2005) and Molico (2006), versions without aggregate shocks. With aggregate shock realizations at each DM stage and even with the realized aggregate shock at a stage announced just prior to the next stage, such a model combines both the role of an information lag and the role of shocks on the evolution of the distribution of money holdings.

As might be expected, not much is known about such a version of the model. Some very preliminary investigation of a version with two DM stages suggests that the implied seller wealth effects do tend to weaken the kind of Phillips curve effects found for the one DM-stage model. However, the implications seem to depend on the details of the model.

6 Concluding remarks

We set out to explore the validity of an idea that we find in Hume (1970); namely, that higher increases in the quantity of money are accompanied by increases in total output if two conditions are met. One is that the increase comes about in a way that gives rise to relative holdings among people. The other is that trade occurs in small groups so that people cannot immediately see the relevant aggregates.

In order to do that we embedded random aggregate and individual monetary transfers in a simple and well-known model, the LW model, in a way that we hoped would not prejudice the results. Part of our attempt to be non-prejudicial is our assumption that both buyers and sellers receive transfers. Our results suggest that the Hume claim is delicate in two respects. First, the wealth effect of the shocks on buyers has to be strong enough to offset the informational updating implied by the two shocks that the buyer and seller see in a meeting. We provided a sufficient condition for that to be true—near-degeneracy of aggregate shocks—and also suggested that near-degeneracy is far from being necessary. The other respect is the potential role of wealth effects on sellers. We could not study it because we used a model, the LW model, that precludes wealth effects for sellers. Hence, that remains to be explored.

Although Hume wrote in the first half of the 18th century, economists were not able until recently to work with his ideas about the circumstances in which increases in the growth rate of money stock would be accompanied by increases in real economic activity. One can only wonder how different the history of macroeconomics would have been if those ideas could have been analyzed much earlier.

7 Appendix

Here we give the missing proofs. We begin with a lemma that is used in the proof of Proposition 1.

Lemma 1. $\sum_{i,j} \gamma_{ij}\tau_i E_{ij}(1/s) = 1$.

Proof. We have

$$\begin{aligned}
& \sum_{i,j} \gamma_{ij}\tau_i E_{ij}(1/s) \\
= & \sum_{i,j} \gamma_{ij}\tau_i \left[\sum_{s'} p_{ij}(s')\frac{1}{s'} \right] \\
= & \sum_{s,i,j} \gamma_{ij}\tau_i \left[\pi(s)\mu_s(\tau_i)\mu_s(\tau_j)/\gamma_{ij}\right] \frac{1}{s} \\
= & \sum_{s,i,j} \gamma_{ij}\pi(s)\mu_s(\tau_i)\mu_s(\tau_j)\frac{1}{s} \\
= & \sum_{s,j} \pi(s)\mu_s(\tau_j)\frac{1}{s}\left[\sum_i \mu_s(\tau_i)\tau_i\right] \\
= & \sum_{s,j} \pi(s)\mu_s(\tau_j) = 1. \blacksquare
\end{aligned}$$

Completion of Proof of Proposition 1.

In order to complete the proof, we need to establish the two claims about G set out below. We start by repeating the definition of G.

$$\max_{x \geq 0} G(x) \equiv -(\beta^{-1} - 1)x + \sigma \sum_{i,j} \gamma_{ij}\{u[\tilde{y}_{ij}(x)] - v\tilde{d}_{ij}(x)E_{ij}(1/s)\}, \tag{19}$$

where, by (5) and (6),

$$\tilde{y}_{ij}(x) = \begin{cases} y^* \text{ if } c(y^*) \le x\tau_i E_{ij}(1/s) \\ \\ c^{-1}(x\tau_i E_{ij}(1/s)) \text{ if } c(y^*) > x\tau_i E_{ij}(1/s) \end{cases} \tag{20}$$

and

$$v\tilde{d}_{ij}(x) = \begin{cases} c(y^*)/E_{ij}(1/s) \text{ if } c(y^*) \le x\tau_i E_{ij}(1/s) \\ \\ x\tau_i \text{ if } c(y^*) > x\tau_i E_{ij}(1/s) \end{cases}. \tag{21}$$

Claim 1. G is differentiable with derivative denoted G'. Moreover, $G'(0) > 0$ and $G'(x) < 0$ for all $x \ge \bar{x}$, where

$$\bar{x} = \frac{\min_{i,j}[\tau_i E_{ij}(1/s)]}{c(y^*)}. \tag{22}$$

Proof. For existence of G', it suffices to show that $u[\tilde{y}_{ij}(x)] - v\tilde{d}_{ij}(x)E_{ij}(1/s)$ is differentiable for each (i, j). There are two relevant cases. If $c(y^*) > x\tau_i E_{ij}(1/s)$, then $\tilde{y}_{ij}(x) < y^*$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[u[\tilde{y}_{ij}(x)] - v\tilde{d}_{ij}(x)E_{ij}(1/s)\right] = \left[\frac{u'(\tilde{y}_{ij}(x))}{c'(\tilde{y}_{ij}(x))} - 1\right]\tau_i E_{ij}(1/s). \tag{23}$$

If $c(y^*) > x\tau_i E_{ij}(1/s)$,then, $\tilde{y}_{ij}(x) = y^*$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[u[\tilde{y}_{ij}(x)] - v\tilde{d}_{ij}(x)E_{ij}(1/s)\right] = 0. \tag{24}$$

Because both derivatives coincide at $c(y^*) = x\tau_i E_{ij}(1/s)$, G is differentiable. Also, by (23), $G'(0) > 0$. And by (24), $G'(x) < 0$ for all $x \ge \bar{x}$. □

Claim 2. $G'(x)$ is strictly decreasing on $(0, \bar{x})$.

Proof. For any $x \in (0, \bar{x})$, $G'(x)$ is a sum of terms, some of which are given by (23) and others of which are constant. Obviously, those given by (23) are strictly decreasing because $\tilde{y}_{ij}(x) < y^*$. Hence, the result follows.□

Proof of Example 1.

Let $x(\pi)$ be the equilibrium price for money in the CM. In this case, we have

$$y_{ni}(\pi) = \sqrt{\tau_i x(\pi)/s_n}.$$

Since $T = \{1, \tau\}$, we have $\mu_{s_n}(1) = (\tau - s_n)/(\tau - 1)$, and hence

$$\begin{aligned}
Y^0(s_n, \pi) &= \mu_{s_n}(1) y_{n1}(\pi) + \mu_{s_n}(\tau) y_{n2}(\pi) \\
&= \frac{\tau - s_n}{\tau - 1}\sqrt{x(\pi)/s_n} + \frac{s_n - 1}{\tau - 1}\sqrt{\tau x(\pi)/s_n} \\
&= \frac{\sqrt{x(\pi)}}{\sqrt{s_n}(\tau - 1)}\left[(s_n - 1)\sqrt{\tau} + \tau - s_n\right] \\
&= \frac{\sqrt{x(\pi)}}{\sqrt{s_n}(\tau - 1)}\left[(s_n + \sqrt{\tau})(\sqrt{\tau} - 1)\right] \\
&= \frac{\sqrt{x(\pi)}}{\sqrt{s_n}(\sqrt{\tau} + 1)}(s_n + \sqrt{\tau}) \\
&= \frac{\sqrt{x(\pi)}}{\tau - 1}\left(\sqrt{s_n} + \sqrt{\frac{\tau}{s_n}}\right).
\end{aligned}$$

Now, let $f_n = \sqrt{s_n} + \sqrt{\frac{\tau}{s_n}}$. Then

$$f_n^2 = s_n + \frac{\tau}{s_n} + 2,$$

which is increasing in s_n if and only if $s_n \geq \sqrt{\tau}$.□

Proof of Proposition 3.

To keep track of the transfers, we use τ_h to denote τ and τ_ℓ to denote 1. Because $T = \{\tau_\ell < \tau_h\}$, we have

$$E_{\tau_h,\tau_h}(1/s) = \frac{\tau_\ell^2 E_\pi(1/s) - 2\tau_\ell + E_\pi(s)}{\tau_\ell^2 - 2\tau_\ell E_\pi(s) + E_\pi(s^2)}, \tag{25}$$

$$E_{\tau_h,\tau_\ell}(1/s) = E_{\tau_\ell,\tau_h}(1/s) = \frac{(\tau_h + \tau_\ell) - E_\pi(s) - \tau_h\tau_\ell E_\pi(1/s)}{(\tau_h + \tau_\ell)E_\pi(s) - E_\pi(s^2) - \tau_h\tau_\ell}, \tag{26}$$

$$E_{\tau_\ell,\tau_\ell}(1/s) = \frac{\tau_h^2 E_\pi(1/s) - 2\tau_h + E_\pi(s)}{\tau_h^2 - 2\tau_h E_\pi(s) + E_\pi(s^2)}. \tag{27}$$

Because the buyer constraint is always binding and $c(y) = y$, we have

$$Y^1(s_n,\pi) \propto \frac{(s_n-\tau_\ell)^2}{(\tau_h-\tau_\ell)^2}E_{\tau_h,\tau_h}(1/s)\tau_h + \frac{(\tau_h-s_n)^2}{(\tau_h-\tau_\ell)^2}E_{\tau_\ell,\tau_\ell}(1/s)\tau_\ell + \frac{(s_n-\tau_\ell)(\tau_h-s_n)}{(\tau_h-\tau_\ell)^2}E_{\tau_h,\tau_\ell}(1/s)(\tau_\ell+\tau_h). \tag{28}$$

Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}Y^1(s,\pi) \propto \frac{2}{(\tau_h-\tau_\ell)^2}\left[A + B\left(s - \frac{\tau_h+\tau_\ell}{2}\right)\right],$$

where

$$\begin{aligned}
A &= \frac{\tau_h-\tau_\ell}{2}\left\{\frac{E_\pi(s)+\tau_\ell E_\pi(1/s)-2\tau_\ell}{E_\pi(s^2)-2\tau_\ell E_\pi(s)+\tau_\ell^2}\tau_h - \frac{\tau_h^2 E_\pi(1/s)-2\tau_h+E_\pi(s)}{\tau_h^2-2\tau_h E_\pi(s)+E_\pi(s^2)}\tau_\ell\right\} \\
&= \frac{(\tau_h-\tau_\ell)^2}{2E_\pi(s)}\frac{[\tau_h-E_\pi(s)]^2[E_\pi(s)-\tau_\ell]^2+V_\pi(s)[E_\pi(s)^2-\tau_h\tau_\ell]}{\left\{[\tau_h-E_\pi(s)]^2+V_\pi(s)\right\}\left\{[E_\pi(s)-\tau_\ell]^2+V_\pi(s)\right\}} \\
&\quad - \frac{(\tau_h-\tau_\ell)^2}{2E_\pi(s)}\frac{\tau_h\tau_\ell[E_\pi(s)E_\pi(1/s)-1][E_\pi(s)^2-\tau_h\tau_\ell+V_\pi(s)]}{\left\{[\tau_h-E_\pi(s)]^2+V_\pi(s)\right\}\left\{[E_\pi(s)-\tau_\ell]^2+V_\pi(s)\right\}}, \\
B &= \frac{E_\pi(s)+\tau_\ell E_\pi(1/s)-2\tau_\ell}{E_\pi(s^2)-2\tau_\ell E_\pi(s)+\tau_\ell^2}\tau_h + \frac{\tau_h^2E_\pi(1/s)-2\tau_h+E_\pi(s)}{\tau_h^2-2\tau_hE_\pi(s)+E_\pi(s^2)}\tau_\ell \\
&\quad - \frac{(\tau_h+\tau_\ell)-E_\pi(s)-\tau_h\tau_\ell E_\pi(1/s)}{(\tau_h+\tau_\ell)E_\pi(s)-E_\pi(s^2)-\tau_h\tau_\ell}(\tau_h+\tau_\ell) \\
&= -\frac{(\tau_h-\tau_\ell)^2V_\pi(s)\left\{[\tau_h-E_\pi(s)][E_\pi(s)-\tau_\ell][\tau_h+\tau_\ell-E_\pi(s)]+V_\pi(s)E_\pi(s)\right\}}{E_\pi(s)\left\{[\tau_h-E_\pi(s)]^2+V_\pi(s)\right\}\left\{[E_\pi(s)-\tau_\ell]^2+V_\pi(s)\right\}\left\{[\tau_h-E_\pi(s)][E_\pi(s)-\tau_\ell]-V_\pi(s)\right\}} \\
&\quad + \frac{(\tau_h-\tau_\ell)^2\tau_h\tau_\ell[E_\pi(s)E_\pi(1/s)-1]\left\{E_\pi(s)[\tau_h-E_\pi(s)][E_\pi(s)-\tau_\ell]+V_\pi(s)[\tau_h+\tau_\ell-E_\pi(s)]\right\}}{E_\pi(s)\left\{[\tau_h-E_\pi(s)]^2+V_\pi(s)\right\}\left\{[E_\pi(s)-\tau_\ell]^2+V_\pi(s)\right\}\left\{[\tau_h-E_\pi(s)][E_\pi(s)-\tau_\ell]-V_\pi(s)\right\}},
\end{aligned}$$

where $V_\pi(s)$ is the variance under π. Note that at degeneracy, $V_\pi(s) = 0$ and $E_\pi(s)E_\pi(1/s) - 1 = 0$. Therefore,

$$A = \frac{(\tau_h-\tau_\ell)^2}{2E_\pi(s)} > 0 \text{ and } B = 0,$$

and the function $Y^1(s,\pi)$ is linear with a strictly positive slope. Now, since $E_\pi(s) = (1+\tau)/2$, we can further simplify the expressions. It is convenient to introduce

$$E = E_\pi(s) = (1+\tau)/2 = (\tau_h+\tau_\ell)/2,\ V = V_\pi(s),\ b = [E_\pi(s)E_\pi(1/s)-1],\ e = (\tau_h-\tau_\ell)/2.$$

Then $\tau_h = E + e$, and $\tau_\ell = E - e$. With this notation, we have

$$
\begin{aligned}
A &= \frac{2e^2}{E}\frac{\left[e^2 - b(E^2 - e^2)\right]}{(V + e^2)}, \\
B &= -\frac{4e^2\left[V - b(E^2 - e^2)\right]}{(e^2 - V)\,(V + e^2)}.
\end{aligned}
$$

The main difficulty is drawing conclusions about $b = [E_\pi(s)E_\pi(1/s) - 1]$. Now, with $S = \{s_1, ..., s_N\}$, we consider the case where N is even (the other case is similar). Because π is symmetric, for some $\varepsilon_1 > ... > \varepsilon_{N/2} > 0$, $s_n = E - \varepsilon_n$ and $s_{N-n} = E + \varepsilon_n$ with $\pi(s_{N-n}) = \pi(s_n)$ for all $n = 1, ..., N/2$. Moreover, $V = \sum_{n=1}^{N/2} 2\pi(s_n)\varepsilon_n^2$. We also use π to denote the distribution over the ε_n's.

Then,

$$
b = E\sum_{n=1}^{N/2}\pi(s_n)\left[\frac{1}{E - \varepsilon_n} + \frac{1}{E + \varepsilon_n}\right] - 1 = E_\pi\left(\frac{E^2}{E^2 - \varepsilon^2} - 1\right) = E_\pi\left(\frac{\varepsilon^2}{E^2 - \varepsilon^2}\right).
$$

Thus,

$$
V - b(E^2 - e^2) = E_\pi\left(\varepsilon^2\right) - E_\pi\left(\frac{\varepsilon^2(E^2 - e^2)}{E^2 - \varepsilon^2}\right) = E_\pi\left(\frac{\varepsilon^2(e^2 - \varepsilon^2)}{E^2 - \varepsilon^2}\right),
$$

and, hence,

$$
B = -\frac{4e^2}{(e^2 - V)\,(V + e^2)}E_\pi\left(\frac{\varepsilon^2(e^2 - \varepsilon^2)}{E^2 - \varepsilon^2}\right) < 0.
$$

Similarly,

$$
\begin{aligned}
A &= \frac{2e^2}{E\,(V + e^2)}\left[e^2 - (E^2 - e^2)E_\pi\left(\frac{\varepsilon^2}{E^2 - \varepsilon^2}\right)\right] \\
&= \frac{2e^2}{E\,(V + e^2)}\left[E_\pi\left(\frac{(E^2 - \varepsilon^2)e^2 - (E^2 - e^2)\varepsilon^2}{E^2 - \varepsilon^2}\right)\right] \\
&= \frac{2e^2E}{(V + e^2)}\left[E_\pi\left(\frac{e^2 - \varepsilon^2}{E^2 - \varepsilon^2}\right)\right] > 0.
\end{aligned}
$$

Recall that we assume that $s \in [\alpha + (1 - \alpha)\tau, (1 - \alpha) + \alpha\tau]$, so that $s \in [E - \kappa e, E + \kappa e]$, where $\kappa = 2\alpha - 1$. Because $A > 0$ and $B < 0$, in order to conclude that

$$\frac{\mathrm{d}}{\mathrm{d}s} Y^1(s, \pi) \propto A + B\left(s - \frac{\tau_h + \tau_\ell}{2}\right) = A + B\,(s - E) > 0,$$

it suffices to show that

$$A + B\kappa e > 0.$$

To do so, first note that since α satisfies (18) and $\kappa = 2\alpha - 1$,

$$\kappa < \sqrt{\frac{1+\tau}{3\tau - 1}}, \tag{29}$$

which in turn implies that

$$1 + \tau > \kappa^2\left[(1+\tau) + 2(\tau - 1)\right], \text{ that is, } E - \kappa^2(E + 2e) > 0.$$

Because $\tau > 1$, (29) implies that $\kappa < 1$, and the last ineqaulity then implies that

$$E - E\kappa^2 - 2\kappa^3 e > 0. \tag{30}$$

Now,

$$\begin{aligned} A + B\kappa e &= \frac{2e^2}{(V + e^2)(e^2 - V)}\left\{E\left[E_\pi\left(\frac{e^2 - \varepsilon^2}{E^2 - \varepsilon^2} E_\pi\left(e^2 - \varepsilon^2\right)\right)\right] - 2\kappa e E_\pi\left(\frac{\varepsilon^2(e^2 - \varepsilon^2)}{E^2 - \varepsilon^2}\right)\right\} \\ &= \frac{2e^2}{(V + e^2)(e^2 - V)}\left[E_\pi\left(\frac{(e^2 - \varepsilon^2)\left\{E\left[E_\pi(e^2 - \varepsilon^2)\right] - 2\kappa e\varepsilon^2\right\}}{E^2 - \varepsilon^2}\right)\right]. \end{aligned}$$

It then suffices to show that $E\left[E_\pi(e^2 - \varepsilon^2)\right] - 2\kappa e\varepsilon^2 > 0$ for all $\varepsilon \le \kappa e$. Since the support of ε lies in $[-\kappa e, \kappa e]$, $E_\pi(e^2 - \varepsilon^2) \ge e^2 - (\kappa e)^2$, and since $2\kappa e\varepsilon^2$ is increasing in ε, it suffices to show

$$E\left[e^2 - (\kappa e)^2\right] - 2\kappa e(\kappa e)^2 > 0, \text{ that is, } e^2\left\{E - E\kappa^2 - 2\kappa^3 e\right\} > 0,$$

which holds by (30).□

Proof of Example 2.

First we begin with the limit case where $\delta = 0$. In this case, it is easy to verify the following.

$$\begin{aligned}
E_{\tau\tau}(1/s) &= \frac{E_\pi(s) + E_\pi(1/s) - 2}{E_\pi(s^2) - 2E_\pi(s) + 1}, \\
E_{\tau 1}(1/s) = E_{1\tau}(1/s) &= \frac{(1+\tau) - E_\pi(s) - \tau E_\pi(1/s)}{(1+\tau)E_\pi(s) - E_\pi(s^2) - \tau}, \\
E_{11}(1/s) &= \frac{\tau^2 E_\pi(1/s) - 2\tau + E_\pi(s)}{\tau^2 - 2\tau E_\pi(s) + E_\pi(s^2)}.
\end{aligned}$$

Now, assuming that the buyer constraint is always binding, we have

$$Y(s_n, \pi) \propto \frac{(s_n - 1)^2}{(\tau - 1)^2} E_{\tau\tau}(1/s)\tau + \frac{(s_n - 1)(\tau - s_n)}{(\tau - 1)^2} E_{\tau 1}(1/s)(1+\tau) + E_{11}(1/s)\frac{(\tau - s_n)^2}{(\tau - 1)^2}.$$

Under $\pi(1+\varepsilon) = \pi(\tau - \varepsilon) = 1/2$, we have

$$\begin{aligned}
E_\pi(s) &= \frac{1}{2}(1+\tau), \\
E_\pi(s^2) &= \frac{1}{2}(\tau^2 - 2(\tau - 1)\varepsilon + 2\varepsilon^2 + 1), \\
E_\pi(1/s) &= \frac{1}{2}\frac{\tau + 1}{(\tau - \varepsilon)(1+\varepsilon)}.
\end{aligned}$$

Taking $N \to \infty$, or, equivalently, $\varepsilon \to 0$, we have

$$\begin{aligned}
\lim_{\varepsilon \to 0} E_{\tau\tau}(1/s) &= \lim_{\varepsilon \to 0} \frac{\frac{1}{2}(1+\tau) + \frac{1}{2}\frac{\tau+1}{(\tau-\varepsilon)(1+\varepsilon)} - 2}{\frac{1}{2}(\tau^2 - 2(\tau-1)\varepsilon + 2\varepsilon^2 + 1) - 2\frac{1}{2}(1+\tau) + 1} = \frac{1}{\tau}, \\
\lim_{\varepsilon \to 0} E_{\tau 1}(1/s) &= \lim_{\varepsilon \to 0} \frac{(1+\tau) - \frac{1}{2}(1+\tau) - \tau\frac{1}{2}\frac{\tau+1}{(\tau-\varepsilon)(1+\varepsilon)}}{(1+\tau)\frac{1}{2}(1+\tau) - \frac{1}{2}(\tau^2 - 2(\tau-1)\varepsilon + 2\varepsilon^2 + 1) - \tau} = \frac{1}{2}\frac{\tau+1}{\tau}, \\
\lim_{\varepsilon \to 0} E_{11}(1/s) &= \lim_{\varepsilon \to 0} \frac{\tau^2\frac{1}{2}\frac{\tau+1}{(\tau-\varepsilon)(1+\varepsilon)} - 2\tau + \frac{1}{2}(1+\tau)}{\tau^2 - 2\tau\frac{1}{2}(1+\tau) + \frac{1}{2}(\tau^2 - 2(\tau-1)\varepsilon + 2\varepsilon^2 + 1)} = 1.
\end{aligned}$$

Thus, we have

$$\lim_{\varepsilon\to 0} Y(s,\pi) \propto \frac{(s-1)^2}{(\tau-1)^2} + \frac{(s-1)(\tau-s)}{(\tau-1)^2}\frac{(1+\tau)^2}{2\tau} + \frac{(\tau-s)^2}{(\tau-1)^2}$$
$$= \frac{s+\tau+s\tau-s^2}{2\tau}.$$

Therefore, $Y(s,\pi)$ is increasing iff $s \leq \frac{1+\tau}{2}$. For ε sufficiently small, the Phillips curve increases first (between $s = 1+\varepsilon$ and $s = \frac{1+\tau}{2}$), and then decreases at the tail (between $s = \frac{1+\tau}{2}$ and $s = \tau - \varepsilon$). Since the function $Y(s,\pi)$ is continuous in ε and in δ (note that π implicitly depends on δ), the non-monotonicity holds for small ε and small δ as well. □

Extension of Proposition 3.

In Proposition 3 we assumed a two-point support for T. Here we generalize the result to a more general support, with some additional assumptions on μ_s. Suppose that $T = \{\tau_{-I} < \tau_{-(I-1)} < ... < \tau_{I-1} < \tau_I\}$, and that T is symmetric around the mid-point, $M = (\tau_I + \tau_{-I})/2$, in the sense that

$$\tau_{I-1} - M = M - \tau_{-(I-1)} > \tau_{I-2} - M = M - \tau_{-(I-2)} > ... > \tau_1 - M = M - \tau_{-1} > 0. \tag{31}$$

Also, assume that

$$\mu_s(\tau_i) = \frac{1}{2I} + (s-M)p_i \text{ and } \mu_s(\tau_{-i}) = \frac{1}{2I} - (s-M)p_i \tag{32}$$

for some $(p_1, p_2, ..., p_I)$ that satisfies

$$0 < p_1 \leq p_2 \leq ... \leq p_I \text{ and } \sum_{n=1}^{I} p_i(\tau_i - M) = 1/2. \tag{33}$$

These assumptions ensure that the two conditions, (1) and (2), about μ_s are satisfied. Moreover, $\mu_s(\tau_I) \in (0,1)$ and (33) imply that $S \subset (\tau_\ell, \tau_h)$ and that $p_I \in \left[\frac{1}{2I(\tau_h - M)}, \frac{1}{2(\tau_I - M)}\right]$, where

$$\tau_h = \frac{\sum_{i=1}^{I} \tau_i}{I} \text{ and } \tau_\ell = \frac{\sum_{i=1}^{I} \tau_{-i}}{I}.$$

(One example that satisfies (33) is $p_i = 1/[2I(\tau_h - M)]$ for all i.)

Proposition 4. Suppose that T and μ_s satisfy (31)-(33), that $c(y) = y$, and that β is sufficiently small so that the buyer constraint is always binding in equilibrium. For each π symmetric around M and with support $[\alpha\tau_h + (1-\alpha)\tau_\ell, (1-\alpha)\tau_h + \alpha\tau_\ell]$ with

$$\alpha < \sqrt{\frac{\tau_h + \tau_\ell}{4(3\tau_h - \tau_\ell)}} + \frac{1}{2}, \tag{34}$$

there is a bound $\bar{p}_{\pi,\alpha} > 1/[2I(\tau_h - M)]$ such that if $p_I < \bar{p}_{\pi,\alpha}$ then $Y^1(s,\pi)$ is strictly increasing in s.

Proof. Let T satisfy (31) be given and suppose that μ_s takes the form given by (32). We first prove the result under $p_i = 1/[2I(\tau_h - M)]$ for all i, and then use continuity to establish the existence of the upper bound $\bar{p}_{\pi,\alpha}$. Note that the continuity result in Proposition 1 can be easily extended and Y^1, as a function of $(p_1, p_1, ..., p_I)$ and π, is continuous.

Now, suppose that $p_i = 1/[2I(\tau_h - M)]$ for all i. Then, for all $i, j = 1, ..., I$,

$$\begin{aligned} E_{\tau_i,\tau_j}(1/s) &= \frac{\sum_{n=1}^{N} \pi(s_n)\mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j)\frac{1}{s_n}}{\sum_{n=1}^{N} \pi(s_n)\mu_{s_n}(\tau_i)\mu_{s_n}(\tau_j)} = \frac{E_\pi\left\{\left[\frac{1}{2I} + (s-M)p_i\right]\left[\frac{1}{2I} + (s-M)p_j\right]\frac{1}{s}\right\}}{E_\pi\left\{\left[\frac{1}{2I} + (s-M)p_i\right]\left[\frac{1}{2I} + (s-M)p_j\right]\right\}} \\ &= \frac{\tau_\ell^2 E_\pi(1/s) - 2\tau_\ell + E_\pi(s)}{\tau_\ell^2 - 2\tau_\ell E_\pi(s) + E_\pi(s^2)}, \end{aligned}$$

$$\begin{aligned} E_{\tau_i,\tau_{-j}}(1/s) &= \frac{\sum_{n=1}^{N} \pi(s_n)\mu_{s_n}(\tau_i)\mu_{s_n}(\tau_{-j})\frac{1}{s_n}}{\sum_{n=1}^{N} \pi(s_n)\mu_{s_n}(\tau_i)\mu_{s_n}(\tau_{-j})} = \frac{E_\pi\left\{\left[\frac{1}{2I} + (s-M)p_i\right]\left[\frac{1}{2I} - (s-M)p_j\right]\frac{1}{s}\right\}}{E_\pi\left\{\left[\frac{1}{2I} + (s-M)p_i\right]\left[\frac{1}{2I} - (s-M)p_j\right]\right\}} \\ &= \frac{(\tau_h + \tau_\ell) - E_\pi(s) - \tau_h\tau_\ell E_\pi(1/s)}{(\tau_h + \tau_\ell)E_\pi(s) - E_\pi(s^2) - \tau_h\tau_\ell}, \end{aligned}$$

$$E_{\tau_{-i},\tau_{-j}}(1/s) = \frac{\sum_{n=1}^{N}\pi(s_n)\mu_{s_n}(\tau_{-i})\mu_{s_n}(\tau_{-j})\frac{1}{s_n}}{\sum_{n=1}^{N}\pi(s_n)\mu_{s_n}(\tau_{-i})\mu_{s_n}(\tau_{-j})} = \frac{E_\pi\left\{\left[\frac{1}{2I}-(s-M)p_i\right]\left[\frac{1}{2I}-(s-M)p_j\right]\frac{1}{s}\right\}}{E_\pi\left\{\left[\frac{1}{2I}-(s-M)p_i\right]\left[\frac{1}{2I}-(s-M)p_j\right]\right\}}$$
$$= \frac{\tau_h^2 E_\pi(1/s) - 2\tau_h + E_\pi(s)}{\tau_h^2 - 2\tau_h E_\pi(s) + E_\pi(s^2)}.$$

Note that the expression for $E_{\tau_i,\tau_j}(1/s)$ coincides with that for (25) for all i, j, and hence we may denote it by $E_{\tau_h,\tau_h}(1/s)$; similarly, we may use $E_{\tau_h,\tau_\ell}(1/s)$ given by (26)to denote $E_{\tau_i,\tau_{-j}}(1/s)$ for all i, j and use $E_{\tau_\ell,\tau_\ell}(1/s)$ given by (27) to denote $E_{\tau_{-i},\tau_{-j}}(1/s)$ for all i, j. Moreover, because the buyer constraint is always binding and $c(y) = y$, we have

$$\begin{aligned}
Y^1(s_n,\pi) \propto\ & \sum_{i,j=1}^{I}\left[\frac{1}{2I}+(s_n-M)p_i\right]\left[\frac{1}{2I}-(s_n-M)p_j\right]E_{\tau_i,\tau_j}(1/s)\tau_i \\
&+\sum_{i,j=1}^{I}\left[\frac{1}{2I}-(s_n-M)p_i\right]\left[\frac{1}{2I}-(s_n-M)p_j\right]E_{\tau_{-i},\tau_{-j}}(1/s)\tau_{-i} \\
&+\sum_{i,j=1}^{I}\left[\frac{1}{2I}+(s_n-M)p_i\right]\left[\frac{1}{2I}-(s_n-M)p_j\right]E_{\tau_i,\tau_{-j}}(1/s)\tau_i \\
&+\sum_{i,j=1}^{I}\left[\frac{1}{2I}-(s_n-M)p_i\right]\left[\frac{1}{2I}+(s_n-M)p_j\right]E_{\tau_{-i},\tau_j}(1/s)\tau_{-i} \\
=\ & \frac{(s_n-\tau_\ell)^2}{I^2(\tau_h-\tau_\ell)^2}E_{\tau_h,\tau_h}(1/s)\sum_{i,j=1}^{I}\tau_i + \frac{(\tau_h-s_n)^2}{I^2(\tau_h-\tau_\ell)^2}E_{\tau_\ell,\tau_\ell}(1/s)\sum_{i,j=1}^{I}\tau_{-i} \\
&+\frac{(s_n-\tau_\ell)(\tau_h-s_n)}{I^2(\tau_h-\tau_\ell)^2}E_{\tau_h,\tau_\ell}(1/s)\left[\sum_{i,j=1}^{I}\tau_i+\sum_{i,j=1}^{I}\tau_{-i}\right] \\
=\ & \frac{(s_n-\tau_\ell)^2}{(\tau_h-\tau_\ell)^2}E_{\tau_h,\tau_h}(1/s)\tau_h + \frac{(\tau_h-s_n)^2}{(\tau_h-\tau_\ell)^2}E_{\tau_\ell,\tau_\ell}(1/s)\tau_\ell \\
&+\frac{(s_n-\tau_\ell)(\tau_h-s_n)}{(\tau_h-\tau_\ell)^2}E_{\tau_h,\tau_\ell}(1/s)(\tau_\ell+\tau_h).
\end{aligned}$$

Again, this expression coincides with (28), and note that (34) coincides with (18) by taking $\tau = \tau_h/\tau_\ell$. Hence, we can use results in Proposition 3 and this implies that $Y^1(s_n, \pi)$ is strictly increasing in s_n. Finally, since the result holds for $p_I = 1/[2I(\tau_h - M)]$, it holds for a neighborhood around it as well.

References

[1] Araujo, L. and A. Shevchenko, Price dispersion, information and learning. *Journal of Monetary Economics*, 53, (2006), 1197-1223.

[2] Barro, R.J. (Ed.), *Modern Business Cycle Theory*, Harvard University Press, Cambridge, 1989.

[3] Faig, M. and Z. Li, The welfare costs of expected and unexpected inflation. *Journal of Monetary Economics* 56, (2009), 1004–1013.

[4] Gu, C. and R. Wright, Monetary mechanisms. *Journal of Economic Theory,* 163 (2016), 644-657.

[5] Hu, T.-W., J. Kennan, and N. Wallace, Coalition-proof trade and the Friedman rule in the Lagos-Wright model. *Journal of Political Economy*, 117, (2009), 116-137.

[6] Hume, D., On money. In *Writings on Economics*, edited by E. Rotwein, Madison , (1970), Univ of Wisconsin Press.

[7] Katzman, B., J. Kennan, and N. Wallace, Output and price level effects of monetary uncertainty in a matching model. *Journal of Economic Theory*, 108, (2003), 217-255.

[8] Lagos, R. and R. Wright, A unified framework for monetary theory and policy analysis. *Journal of Political Economy*, 113, (2005), 463-484.

[9] Lucas, R., Expectations and the neutrality of money. *Journal of Economic Theory,* 4, (1972), 103-124.

[10] Lucas, R., Nobel lecture: Monetary neutrality. *Journal of Political Economy*, 104, (1996), 661-82.

[11] Molico, M., The distribution of money and prices in search equilibrium. *International Economic Review*, 47, (2006), 701-722.

[12] Shi, S., Money and prices: a model of search and bargaining. *Journal of Economic Theory,* 67, (1995), 467-496.

[13] Trejos, A. and R. Wright, Search, bargaining, money and prices. *Journal of Political Economy,* 103, (1995), 118-141.

[14] Wallace, N., Short-run and long-run effects of changes in money in a random-matching model. *Journal of Political Economy*, 105, (1997), 1293-1307.

[15] Zhu, T., Existence of a monetary steady state in a matching model: divisible money. *Journal of Economic Theory*, 123, (2005), 135–160.