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ORIGINAL ARTICLE

Primitive Recursion and Isaacson's Thesis

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Although Peano arithmetic (PA) is necessarily incomplete, Isaacson argued that it is in a sense conceptually complete: proving a statement of the language of PA that is independent of PA will require conceptual resources beyond those needed to understand PA. This paper gives a test of Isaacon's thesis. Understanding PA requires understanding the functions of addition and multiplication. It is argued that grasping these primitive recursive functions involves grasping the double ancestral, a generalized version of the ancestral operator. Thus, we can test Isaacon's thesis by seeing whether when we phrase arithmetic in a context with the double ancestral operator, the result is conservative over PA. This is a stronger version of the test given by Smith, who argued that understanding the predicate "natural number" requires understanding the ancestral operator, but did not investigate what is required to understand the arithmetic functions.

Keywords arithmetic; ancestral logic; incompleteness; double ancestral; nominalism

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Though first-order Peano arithmetic [PA] is necessarily incomplete, Isaacson (1987, 1992) famously argues that there is a sense in which it is complete: it captures the purely arithmetical content of our concept of natural number. The idea is that to prove an arithmetical sentence which is unprovable in PA, one will have to employ further ideas, such as higher order concepts or reflections on the consistency or truth of the axioms of PA. These further ideas go beyond the purely arithmetic. The thesis that PA is complete in this sense is known as Isaacson's thesis.

Isaacson argues mainly by looking at examples of true sentences unprovable in PA, and seeing what is needed to prove them. Smith (2008) gives a different argument for the same basic thesis, arguing that understanding the predicate "natural number" amounts to understanding the ancestral operator—and thus that the truth of Isaacon's thesis rests on whether when you supplement PA with the ancestral operator in the appropriate way, the result is conservative over PA. It is not difficult to show that it is, giving positive support to Isaacson's thesis.

This is all very well as far as it goes, but it does not (in my view) go far enough. The same questions asked of the predicate "natural number" should be asked of the functions of

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addition and multiplication. In PA, these functions are assigned symbols in the language, governed by the axioms

$$x + 0 = x$$

$$x + Sy = S\left(x + y\right)$$

$$x \times 0 = 0$$

$$x \times Sy = \left(x \times y\right) + x$$

where S is the successor operation on numbers. Just as we can ask how we form the predicate "natural number" and know its axiomatization in PA is appropriate, we can ask this of $+$ and $\times$. We are not simply positing these functions, assuming that there are valid operations on numbers with these properties. We are not imposing these operations by fiat—perhaps starting with multiple candidate infinite sequences, and then narrowing our attention to those which happen to allow these operations. We feel we can see that numbers are the kind of things which can be added and multiplied in the way these axioms describe. Grasping this is part of grasping the axioms of PA. But how do we grasp this? How do we come to see that we can introduce functions like this, which are total and single and satisfy the relevant equations? As we will see, this question has a satisfying answer in the form of a double version of the ancestral operator.

Since the double ancestral provides a plausible and satisfying account of our grasp of these primitive recursive functions, we obtain a new test of Isaacson's thesis: when arithmetic is phrased in terms of the double ancestral, is the resulting theory conservative over PA? This is a stronger test than Smith's since this theory straightforwardly interprets the ancestral arithmetic used in his test.

As another incidental application, since the double ancestral is an ontologically innocent operator, it follows that a nominalist can make blameless use of primitive recursion (without any need to quantify over functions as objects).

1 The thesis

The first task is to clarify what the thesis states. Without going into the details of Isaacson's original arguments, one can extract from his writings the following central idea (here we let L_A be the language of PA):

> Proving a statement of L_A that is unprovable in PA will require employing concepts beyond those required to grasp the basic concepts of arithmetic: natural number, successor, induction, addition and multiplication.

This is what I mean by Isaacson's thesis in this paper.

How strong a thesis this is will depend on how strong a notion of "grasp" one works with—to fully understand a certain concept, how much further one's understanding should extend. In general, these questions can be difficult. Sometimes there is a consensus, as in the generally held view that one can properly grasp first-order logic without being able to understand second-order logic. Things are not always so clear though — can one fully grasp fifth-order logic without being able to grasp sixth-order logic?

What the thesis amounts to may also depend on what one takes the right interpretation of arithmetic to be. For instance, suppose one defended a conception of arithmetic as being about strings formed from some particular symbol |. This is essentially the conception of Hilbert (1990), and is developed in more detail by Parsons (2007), though they are concerned particularly with intuitive aspects of the theory and avoid arbitrary quantification over the domain. It may be that one could describe a first-order theory of the strings, interpret PA in this theory, and argue that our grasp of the concepts of arithmetic amounted to a grasp of this theory of strings (this is not what the authors mentioned argue). Then if one could show that this theory of strings was conservative over PA, one would have evidence for one version of Isaacson's thesis. However, this would be a very limited version of Isaacson's thesis, entirely dependent on the claim that a proper grasp of arithmetic amounts to a grasp of this theory of strings and should be expected to extend no further. It would be more an argument for a particular interpretation of arithmetic than for Isaacson's thesis in general.

The best defense of Isaacson's thesis would be one which examines the axioms of PA themselves, rather than relying on any particular interpretation of them. One will also obtain a stronger version of Isaacson's thesis if one is liberal in the notion of "grasp" one uses — liberal in questions of what further concepts proper grasp of a particular concept entails.

2 The argument

Smith (2008) gives a better argument for the thesis than the hypothetical string based one just sketched. He focuses on what is required to grasp the concept of natural number. The basic thought is that

> understanding quantification over the [natural] numbers involves understanding that the numbers are zero, the next number, the one after that, *and so on*, *without limit* — and understanding too that these are the *only* numbers. Which is in effect to grasp the thought that every number stands in the ancestral of the successor relation to zero. (ibid., pp. 3 – 4, emphasis his)

Smith thus argues that grasping the concept "natural number" amounts to grasping the ancestral operator. This allows him to set a test for Isaacson's thesis. He supplements PA with the ancestral operator, to give what he calls "ancestral arithmetic." Then, if grasping the concept "natural number" amounts to grasping the ancestral operator, Isaacson's thesis requires that anything provable in this ancestral arithmetic is already provable in PA, that is, that ancestral arithmetic is conservative over PA. This Smith shows straightforwardly, giving positive support to Isaacson's thesis.

However, Smith's account misses out a crucial part of arithmetic, as discussed initially: the functions of addition and multiplication. We want to know how we grasp that numbers are the kinds of things that can be added and multiplied. One approach is to define addition and multiplication in full second-order logic, which one can do given the successor operation and a second-order induction axiom. It would be a surprise if quantifying over relations was necessary to grasp these primitive recursive functions, however. If true, that would presumably disprove Isaacson's thesis as understood here.

It would also suggest that primitive recursion will be unavailable to a nominalist: although a nominalist will not be discussing addition and multiplication of numbers, there may be other contexts where they wish to use primitive recursion, for instance involving concretely instantiated infinite sequences. If understanding primitive recursion required quantifying over relations that looks impossible, however.

The main claim of this paper is that the double ancestral gives us a satisfying analysis of how one can grasp these kinds of primitive recursive functions, in the same way the ancestral does for the concept of "natural number." This gives rise to a new, stronger test of Isaacson's thesis, in terms of what I call double ancestral arithmetic.

In the course of this argument, certain judgments about what is involved in the grasp of some concept are required. As noted above, these will be a necessary part of any discussion of Isaacson's thesis. In particular, I will take it as given that appealing to a pairing function for natural numbers would be a very bad explanation of our grasp of addition and multiplication. The ability to form a pair (m, n) of two natural numbers, either as a self-standing object or via an injection $\mathbb{N}^2 \to \mathbb{N}$, seems to be no part of our usual understanding of PA. Pedagogically, our understanding of addition and multiplication has nothing to do with a pairing function $\mathbb{N}^2 \to \mathbb{N}$—we learn addition and multiplication long before learning about a pairing function, and students are often surprised to discover that such an injection exists. Grasping abstract pairs or general tuples of natural numbers also seems to be no part of our initial conception of arithmetic.

One issue I will not address is the Neo-Fregean analysis of arithmetic. They might argue that arithmetic is properly understood in terms of cardinality, using second-order logic augmented with Hume's principle. If that were true it would present a major challenge to Isaacson's thesis as understood here. I do not find the Neo-Fregean arguments convincing, but they are not the subject of this paper, and will have to be set to one side.

3 The ancestral and the double ancestral

The prototypical instance of the ancestral operator is the relation "ancestor." Similarly a prototypical example of the double ancestral operator is the relation "ancestor of the same generation." Figure 1 illustrates this diagrammatically: illustrating the relation $\mathrm{ASG}(x, y)$ between ancestors of Jeff and ancestors of Sarah, of x being an ancestor of Jeff of the same generation as y is an ancestor of Sarah.

We can illustrate the general case of the double ancestral operator in the same way, seen in Figure 2. To better suit its application to primitive recursive functions, we will use the reflexive form—this corresponds to a modification of the "ancestor of the same generation" relation to include Jeff as an ancestor of Jeff of the same generation as Sarah is of Sarah. We let $\phi(x, y)$ and $\psi(w, z)$ be two place relations, and write $(\phi, \psi)^*(c, d, x, y)$

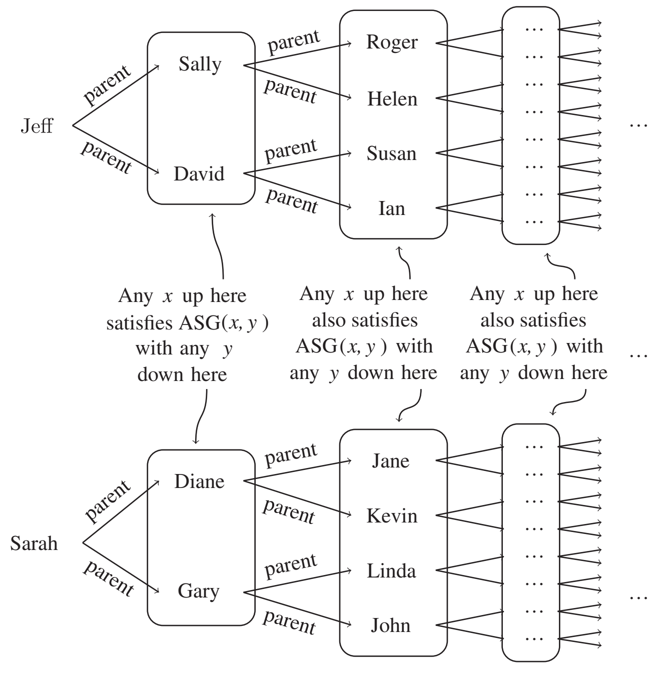

Figure 1: The "ancestor of the same generation" relation.

to indicate that the double ancestral of ϕ and ψ holds of c, d, x and y. In Figure 2, we use $a \xrightarrow{\phi} b$ to indicate that $\phi(a, b)$ holds, $a \xrightarrow{\psi} b$ similarly.

We can also informally explain $(\phi, \psi)^*$ in prose. We have that the relation $(\phi, \psi)^*(c, d, x, y)$ holds iff

- $x = c$ and $y = d$
- Or $\phi(c, x)$ and $\psi(d, y)$
- Or there are u and v such that $\phi(c, u)$ and $\phi(u, x)$, and $\psi(d, v)$ and $\psi(v, y)$
- Or there are u, u' and v, v' such that $\phi(c, u)$ and $\phi(u, u')$ and $\phi(u', x)$, and $\psi(d, v)$ and $\psi(v, v')$ and $\psi(v', y)$
- Or there are u, u', u'' and v, v', v'' such that $\phi(c, u)$ and $\phi(u, u')$ and $\phi(u', u'')$ and $\phi(u'', x)$, and $\psi(d, v)$ and $\psi(v, v')$ and $\psi(v', v'')$ and $\psi(v'', y)$
- …

and so on (and these are the only objects related by $(\phi, \psi)^*$). This is strictly analogous to how one would informally explain the ancestral, except that it is a simultaneous description involving two relations ϕ and ψ rather than just one—if you left out all mention of d, y and ψ from the above you would have a description of what is required for the ancestral $\phi^*(c, x)$ to hold.

One can give a precise definition of the double ancestral using finite sequences. We have that $(\phi, \psi)^*(c, d, x, y)$ holds iff for some $n \geq 0$ we have sequences $(a_0, \dots a_n)$ and

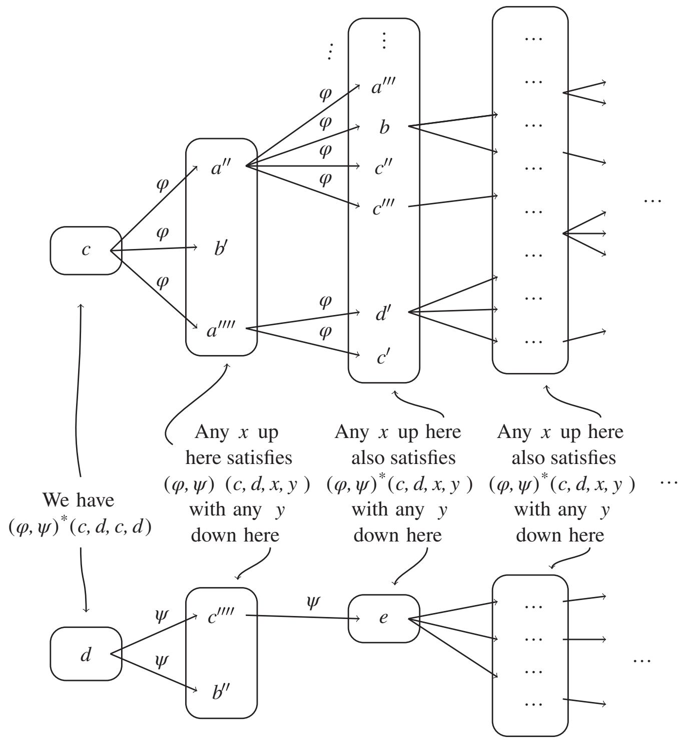

Figure 2: The double ancestral of ϕ and ψ.

$(b_0 \ldots b_n)$ such that $c = a_0$, $d = b_0$, $x = a_n$, $y = b_n$, and for all $i = 0 \ldots (n-1)$ we have $\phi(a_i, a_{i+1})$ and $\psi(b_i, b_{i+1})$. One can also give a definition in second-order logic, where the relation $\{(x, y) * (\phi, \psi)^*(c, d, x, y)\}$ is the intersection of all relations R such that $R(c, d)$ and such that if $R(x, y)$ and $\phi(x, w)$ and $\psi(y, z)$ then $R(w, z)$.

However, there seems no reason to think that understanding predicates formed from the double ancestral operator *requires* one of these definitions—any more than for the ancestral operator. Smith (2008) and Avron (2003) argue that an explicit definition of the ancestral is not necessary, and that the ancestral operator can be thought of as a conceptual primitive occupying a valid middle ground between first- and second-order logic.[1] Exactly the same arguments can be used for the double ancestral.

One attractive way to argue for the ancestral and double ancestral as primitives is to argue that we grasp them by grasping the introduction and elimination rules for them, as with other logical vocabulary (helped by informal explication, again as with other logical vocabulary). This view of ancestral style predicates is urged by Parsons (2007, Chapter 8), and exactly the same could be said of relations formed by the double ancestral.

We will see these rules for the double ancestral in a second, but first we will characterize it in stricter logical terms. Formally the double ancestral is an operator on formulae that produces relation symbols. We introduce an extra clause into the recursive definition of formulae for the language: if ϕ and ψ are formulae, x_1, x_2 are distinct variables, y_1, y_2 are distinct variables and s_1, s_2, t_2, t_2 are terms, then we obtain a formula $(\phi, \psi)^*_{x_1,x_2,y_1,y_2}\left(s_1, t_1, s_2, t_2\right)$. Free occurrences of x_1, x_2 in ϕ become bound in this formula, as do free occurrences of y_1, y_2 in ψ.

Now onto the rules for the double ancestral. We will see shortly that it can be used to play the role of the (single) ancestral, so since there is no complete effective deductive system for the ancestral operator (Shapiro 2000), there isn't one for the double ancestral operator either. However, we can still give natural deductive rules that capture the reasoning we use for it in practice. These parallel those for the ancestral described by Smith, and the rules for the predicate "natural number" described and defended by Parsons (2007, Chapter 8). It is these rules that will be used to test Isaacson's thesis, so that if one could argue that there were further inferences that a grasp of the double ancestral should license, the test of Isaacson's thesis would be undermined; there are no obvious candidates for this though. I use $\phi[t|x]$ to denote the substitution of the term t for free occurrences of variable x in ϕ.

$$\frac{s_1 = t_1 s_2 = t_2}{(\phi, \psi)^*_{\vec{x},\vec{y}}\left(s_1, t_1, s_2, t_2\right)}$$

$$\frac{(\phi, \psi)^*_{\vec{x},\vec{y}}\left(s_1, t_1, s_2, t_2\right)\ \phi\left[s_2|x_1, s_3|x_2\right]\ \psi\left[t_2|y_1, t_3|y_2\right]}{(\phi, \psi)^*_{\vec{x},\vec{y}}\left(s_1, t_1, s_3, t_3\right)}$$

$$\frac{\forall \vec{x}\,\vec{y}\ \left(\left(\chi\left(x_1, y_1\right) \wedge \phi \wedge \psi\right) \Rightarrow \chi\left[x_2|x_1, y_2|y_1\right]\right)\ (\phi, \psi)^*_{\vec{x},\vec{y}}\left(s_1, t_1, s_2, t_2\right)}{\chi\left[s_1|x_1, t_1|y_1\right] \Rightarrow \chi\left[s_2|x_1, t_2|y_1\right]}$$

In the third rule, we require that y_1 and y_2 are not free in χ. The first two rules give ways of showing that objects lie under the generalized ancestral, the third is an induction rule: if some property χ is preserved by ϕ together with ψ, then it is preserved by $(\phi, \psi)^*_{\vec{x},\vec{y}}$.

We now quickly sketch a semantics for this. If A is a structure for the language and v a variable assignment over A, then we stipulate that $D, v \vDash (\phi, \psi)^*_{\vec{x},\vec{y}}\left(s_1, t_1, s_2, t_2\right)$ iff there exist sequences $(a_0, \ldots a_n)$ and $(b_0, \ldots b_n)$ for some $n \geq 0$ such that $a_1 = v(s_1)$, $b_1 = v(t_1)$, $a_n = v(s_2)$, $b_n = v(t_2)$, and for each $i = 0 \ldots (n-1)$ we have A, $v(x_1 \mapsto a_i, x_2 \mapsto a_{i+1}) \vDash \phi$ and $v(y_1 \mapsto b_i, y_2 \mapsto b_{i+1}) \vDash \psi$. Otherwise one can employ the generalized ancestral in the meta-language for this clause.

Next, we note that the double ancestral operator can be used to define the ancestral operator. If we have a relation $\phi(x, y)$ for which we wish to form the ancestral $(\phi)^*_{x,y}(w, z)$, we can take some variables u_1, u_2 distinct from x, y, w, z, take ψ to be "$u_1 = u_2$," and take $(\phi)^*_{x,y}(w, z)$ to be $\exists u_1\left(\,(\phi, \psi)^*_{x,y,u_1,u_2}\left(w, u_1, z, u_1\right)\,\right)$. It is an easy check that this has the right semantics and satisfies Smith's deductive rules.

The double ancestral defined here is a special case of the two place generalized ancestral, which was defined by Martin (1943). Avron (2003, pp. 157–58) also discusses the generalized ancestral, and proves that it cannot be defined in terms of the ancestral. As we will soon see, the double ancestral can be used to define primitive recursive functions, so Avron's proof also shows that the double ancestral cannot be defined in terms of the ancestral. It is possible to define the generalized ancestral (and thus the double ancestral) in terms of the ancestral if one has a pairing function on objects, as we will see in Proposition 5.1. However, as noted in Section 2, trying to explain our grasp of addition and multiplication in terms of a pairing function is a very unattractive route. One can directly form relations defined using, and see this to be valid, the double ancestral, just as one can directly form predicates defined using the ancestral.

I focus on the double ancestral rather than the generalized ancestral in this paper because it allows a simpler informal characterization, and is a closer fit for the case of primitive recursive functions. One could argue that anyone who grasps the double ancestral should be able to grasp the two place generalized ancestral; whether or not that is correct, the conservativeness argument given later would also apply to the two place generalized ancestral, so Isaacson's thesis is safe either way.

4 Primitive recursion and the double ancestral

When Smith argues that a grasp of the ancestral is used to understand the predicate "natural number," he does so by pointing out that

> understanding quantification over the [natural] numbers involves understanding that the numbers are zero, the next number, the one after that, *and so on*, *without limit*—and understanding too that these are the *only* numbers. Which is in effect to grasp the thought that every number stands in the ancestral of the successor relation to zero. (Smith 2008, pp. 3–4, emphasis his)

Fix an object a and a function f, and consider the primitive recursive function g defined by

$$g(0) = a$$

$$g(S(n)) = f\left(g(n)\right).$$

a might be any object (not necessarily a number). Exactly parallel to the above explication of what it is to understand the predicate "natural number," we can say that

> understanding the function g involves understanding that g applied to zero gives a, that g applied to the next number after zero is f of g applied to zero, that g applied to the next number after that is f of g applied to that number, that g applied to the next number after that is f of g applied to *that* number, and so on.

Understanding some sort of informal explication along these lines is how we understand what we mean by g, and why we can introduce a function symbol with these properties—in exactly the same way as understanding the ancestral is how we know we can form the predicate "natural number." The above is doubtless less clear than the earlier explication of "natural number," but it has a parallel structure, just involving twice

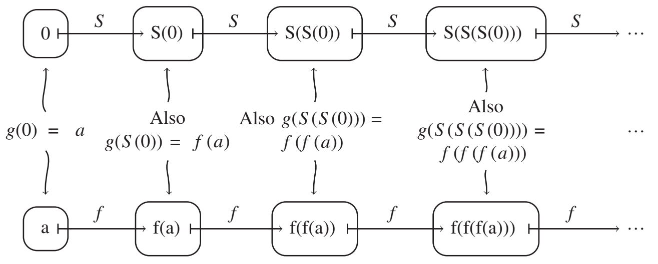

Figure 3: Primitive recursion via the double ancestral.

as many objects. It is also visibly an explication of the function g in terms of the double ancestral. This is illustrated diagrammatically in Figure 3.

Comparing this to Figure 1 and Figure 2 makes it pretty clear, I think, that definition by primitive recursive is a straightforward case of the double ancestral; and, in fact, that the double ancestral generalizes definition by primitive recursive in an exactly parallel way to how the ancestral generalizes the definition of the concept "natural number." We can see the same thing happening in prose. We can describe the above function g by saying $g(x)=y$ iff

- $x=0$ and $y=a$
- Or $x=S(0)$ and $y=f(a)$
- Or there are u and v such that $u=S(0)$ and $x=S(u)$, and $v=f(a)$ and $y=f(v)$
- Or there are u, u' and v, v' such that $u=S(0)$ and $u'=S(u)$ and $x=S(u')$, and $v=f(a)$ and $v'=f(v)$ and $y=f(v'')$
- Or there are u, u', u'' and v, v', v'' such that $u=S(0)$ and $u'=S(u)$ and $u''=S(u')$ and $x=S(u'')$, and $v=f(a)$ and $v'=f(v)$ and $v''=f(v'')$ and $y=f(v'')$
- …

and so on (and these are the only objects related by g). This is visibly an example of a definition of which the prose characterization of $(\phi, \psi)^*(c, d, x, y)$ seen in Section 3 is the general form.

Exactly as Smith argues that grasping the concept natural number means grasping it as an instance of the ancestral operator, we can argue that grasping a definition by primitive recursion means grasping it as an instance of the double ancestral operator. There seems to be no good reason why anyone who can grasp a function defined by primitive recursion should not be able to grasp other instances of the double ancestral.

There are interpretations of arithmetic on which addition and multiplication might not be seen as given by primitive recursion: for instance, the approach of Neo-Fregeanism via cardinality, and the interpretation of arithmetic in intuitive terms via strings of symbols are given by Parsons (2007). As noted in Section 1, a Neo-Fregean perspective does seem to present major problems for Isaacson's thesis, which I cannot address here. Parsons does argue that addition and multiplication in the context of strings should be seen as intuitively distinct from other primitive recursive functions (ibid., Chapter 7), but this is very much a result about his particular notion of intuition, and this string based context.

He does not argue that this string interpretation is the true interpretation of arithmetic, and does not argue that it gives an interpretation of all of PA. Thus, his views (even if correct) do not present much of a challenge to the perspective here.

5 Double ancestral arithmetic

With the logic defined, we can phrase arithmetic in it, and see how the informal characterizations of addition and multiplication earlier do correspond to simple formal definitions using the double ancestral.

We call the theory of arithmetic in double ancestral logic *double ancestral arithmetic*. It has a language with the constant 0 and the successor function S. Defining the ancestral $(\phi)^*_{x,y}(w,z)$ in terms of the double ancestral as seen in Section 3 (or taking it as an extra primitive), the axioms are:

$$\forall x \quad (v = S(u))^*_{u,v}(0,x) \tag{1}$$

$$\forall x\, S(x) \neq 0 \tag{2}$$

$$\forall xy \left(S(x) = S(y) \to x = y\right). \tag{3}$$

This is a particularly simple and natural axiomatization—all we need is that every number is a successor of 0, and that the successor function is injective without 0 in its range. It is categorical because of the standard semantics for the double ancestral, and thus for the relation $(v = S(u))^*_{u,v}$.

We can give an informal characterization of addition similar to that of general primitive recursive functions:

> Adding 0 to x gives you x, adding 1 to x gives you Sx, adding 2 to x gives you SSx, adding 3 to x gives you $SSSx$, and so on.

It follows that the relation $x+y=z$ can be captured by the double ancestral

$$\left(u_2 = S\left(u_1\right),\ \ v_2 = S\left(v_1\right)\right)^*_{u_1,u_2,v_1,v_2}\left(0,x,y,z\right)$$

(one can visualize a diagram similar to Figure 3 to see how this works). One can prove straightforwardly in double ancestral arithmetic that this definition does indeed define a total, single-valued function with value z of its arguments x and y. Using the normal notation $x+y=z$ for it, we have that $+$ satisfies the usual equations

$$x + 0 = x$$
$$x + Sy = S\left(x + y\right).$$

Similarly, multiplication $x\times y=z$ is captured by the double ancestral

$$\left(u_2 = S\left(u_1\right),\ \ v_2 = v_1 + x\right)^*_{u_1,u_2,v_1,v_2}\left(0,,,0,y,z\right).$$

Again one can prove its usual properties in the theory.

Thus, one obtains the axioms of Smith's ancestral arithmetic and (*a fortiori*) of PA in double ancestral arithmetic—the instances of the induction axiom scheme follow from axiom (1). The double ancestral provides the general concept of which addition and multiplication are a special case. Thus, it provides a useful test case for Isaacson's thesis.

If Smith is correct that a grasp of the ancestral is what's needed to grasp the predicate "natural number," and the parallel argument here for the double ancestral and primitive recursion is also correct, then the axiomatization of arithmetic above in terms of the double ancestral appears to include everything that is needed for a full understanding of PA (as long as the deductive rules for the ancestral and double ancestral are in some sense adequate).

Thus, conservativeness of double ancestral arithmetic over PA would imply that one would have to employ ideas beyond those needed to understand PA in order to prove a statement of L_A that was unprovable in PA. On the other hand, if double ancestral arithmetic is not conservative over PA, then—on a very natural interpretation of arithmetic—we would have examples of statements of L_A provable using only the conceptual resources needed to understand PA, so Isaacson's thesis would be in trouble.

Fortunately for Isaacson's thesis, we have the following:

Proposition 5.1. *Double ancestral arithmetic is conservative over ancestral arithmetic (as defined by Smith).*

Proof. We show that ancestral arithmetic and double ancestral arithmetic are definitionally equivalent. Let T_{Anc} be the theory of ancestral arithmetic, L_{Anc} its language and let T_{DA} be the theory of double ancestral arithmetic and L_{DA} its language. We saw above how to define the relevant primitives of ancestral arithmetic—the ancestral operator, addition and multiplication—in terms of the double ancestral. Let $\phi \mapsto f(\phi)$ denote the translation by these definitions from the L_{Anc} to L_{DA}. We have that if $T_{\text{Anc}}, \Gamma \vdash$ then $T_{\text{DA}}, f(\Gamma) \vdash f(\phi)$, since double ancestral arithmetic proves the axioms for the ancestral and addition and multiplication.

Now in ancestral arithmetic we can define a bijective pairing function $\alpha : \mathbb{N}^2 \to \mathbb{N}$, where $\alpha(x, y) = \alpha(x', y')$ iff $x = x'$ and $y = y'$. We let the inverse be $z \mapsto (\beta_1(z), \beta_2(z))$. We will show how to translate statements involving the double ancestral into statements involving the (standard) ancestral using this. This translation function will be denoted by g.

The idea is simply that $(\phi, \psi)^*_{x_1,x_2,y_1,y_2}\left(s_1, t_1, s_2, t_2\right)$ is equivalent to the ancestral

$$\left(\phi\left[\beta_1(x), \beta_1(y)\right] \wedge \psi\left[\beta_2(x), \beta_2(y)\right]\right)^*_{x,y}\left(\alpha\left(s_1 t_1\right), \alpha\left(s_2 t_2\right)\right).$$

This can be easily checked to be the case semantically.

We define g by induction on the number of occurrences of the double ancestral in a formula. For statements θ of L_{DA} which do not involve RTC^2, $g(\theta)$ is just θ. For a statement of the form $\theta = (\phi, \psi)^*_{x_1,x_2,y_1,y_2}\left(s_1, t_1, s_2, t_2\right)$, we let ϕ' be $g(\phi)[\beta_1(x)|\, x_1, \beta_1(y)|\, x_2]$ and ψ' be $g(\psi)[\beta_2(x)|\, y_1, \beta_2(y)|\, y_2]$, and then define $g(\theta)$ to be $(\phi' \wedge \psi')^*_{x,y}\left(\alpha\left(s_1 t_1\right), \alpha\left(s_2 t_2\right)\right)$. g acts on statements built from propositional connectives or quantifiers in the obvious way, for example, $g(\theta_1 \wedge \theta_2) = g(\theta_1) \wedge g(\theta_2)$. It is an easy check that this is an adequate definition of the double ancestral, in as much as the deductive rules for the double ancestral hold: we have that if $T_{\mathrm{DA}}, \Delta \vdash \theta$ then $T_{\mathrm{Anc}}, g(\Delta) \vdash g(\theta)$.

Then, it is not difficult to show f and g give a definitional equivalence (Corcoran 1980), that is, that for any $\chi \in L_{\mathrm{Anc}}$, we have $T_{\mathrm{Anc}} \vdash \chi \Longleftrightarrow g(f(\chi))$, and for any $\theta \in L_{\mathrm{DA}}$, we have $T_{\mathrm{DA}} \vdash \theta \Longleftrightarrow f(g(\theta))$.

Thus, if χ is a formula of L_{Anc} and $T_{\mathrm{DA}} \vdash f(\chi)$ then $T_{\mathrm{Anc}} \vdash g(f(\chi))$ so $T_{\mathrm{Anc}} \vdash \chi$. In this sense, T_{DA} is conservative over T_{Anc}, as claimed. □

Putting this together with Smith's result that ancestral arithmetic is conservative over PA, we can conclude that double ancestral arithmetic is conservative over PA. Thus, Isaacson's thesis passes the test, and looks secure; even more secure than it did after passing Smith's test, since we have now taken the functions of addition and multiplication into account.

Incidentally this argument makes it clear that understanding primitive recursion does not require quantifying over relations. Since the double ancestral is an ontologically innocent operator, primitive recursion is available to a nominalist.

Note

1 Others have also used ancestral logic as a middle ground between first- and second-order logic, such as Heck (2011), pp. 274–79], though Heck does not give sustained arguments for this status.

References

Avron, Arnon. "Transitive Closure and the Mechanization of Mathematics," in *Thirty Five Years of Automating Mathematics. Applied Logic Series*. Dordrecht: Springer, 2003, 149–171.

Corcoran, J. "On Definitional Equivalence and Related Topics." *History and Philosophy of Logic* **1**.1 (1980): 231.

Heck, Richard Kimberly. *Frege's Theorem*. Oxford: OUP Oxford, 2011, September 29 322 pp.

Hilbert, David. "On the Infinite," in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, edited by Jean van Heijenoort. Cambridge, Mass.: Harvard University Press, 1990, July 1, 367–392.

Isaacson, Daniel. "Arithmetical Truth and Hidden Higher Order Concepts," in *Logic Colloquium '85*. Elsevier: The Paris Logic Group, 1987, January 1, 147–169.

Isaacson, Daniel. "Some Considerations on Arithmetical Truth and the -Rule," in *Proof, Logic and Formalization*, edited by Michael Detlefsen. London; New York: Routledge, 1992, January 9, 94–138.

Martin, R. M. "A Homogeneous System for Formal Logic." *The Journal of Symbolic Logic* **8** (1943): 1–23.

Parsons, Charles. Mathematical *Thought* and its Objects. Cambridge: Cambridge University Press, 2007, December 24 400 pp.

Shapiro, Stewart. *Foundations without Foundationalism: A Case for Second-Order Logic*. Oxford: Oxford University Press, 2000, May 18 300 pp.

Smith, Peter. "Ancestral Arithmetic and Isaacson's Thesis." *Analysis* **68**.297 (2008): 1–10.