



Crispin, J., Su, H., Round, K., & Mylonakis, G. (2018). Energy Formulation for Axial Pile Head Stiffness in Inhomogeneous Soil. In *Sustainability Issues for the Deep Foundations: Proceedings of the 2nd GeoMEast International Congress and Exhibition on Sustainable Civil Infrastructures, Egypt 2018 – The Official International Congress of the Soil-Structure Interaction Group in Egypt (SSIGE)* (pp. 181-190). Springer, Cham.  
[https://doi.org/10.1007/978-3-030-01902-0\\_15](https://doi.org/10.1007/978-3-030-01902-0_15)

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# Energy Formulation for Axial Pile Head Stiffness in Inhomogeneous Soil

Jamie Crispin<sup>1</sup>, Huize Su<sup>2</sup>, Katherine Round<sup>2</sup> and George Mylonakis<sup>3</sup>, PhD

<sup>1</sup>PhD Student, University of Bristol, Bristol, UK; [j.crispin@bristol.ac.uk](mailto:j.crispin@bristol.ac.uk)

<sup>2</sup>Student, University of Bristol, Bristol, UK;

<sup>3</sup>Chair in Geotechnics and Soil Structure Interaction, University of Bristol, Bristol, UK

**ABSTRACT:** Approximate closed-form solutions are derived for the settlement of axially loaded piles embedded in inhomogeneous soil by employing an energy method based on the Winkler model of soil reaction. Simple, linear shape functions for the attenuation of pile displacement with depth result in easy-to-use formulas. Two alternative methods are discussed for selecting the slope of the linear function. The proposed methodology is compared with rigorous (yet complicated) closed form solutions and an alternative approximation using a representative homogeneous soil.

## INTRODUCTION

The Winkler model for axially loaded piles is a popular and cost-effective approach as it allows the development of simple, closed form solutions for predicting pile settlement without resorting to numerical analysis (Randolph and Wroth 1978; Poulos and Davis 1980; Scott 1981; Mylonakis and Gazetas 1998; Guo 2012; Crispin *et al.* 2018). Comparison of these solutions to more rigorous continuum analyses allows the Winkler spring stiffness to be determined by matching the settlement at the pile head between the two models (Mylonakis 2001; Syngros 2004; Guo 2012; Anoyatis 2013). In addition, the model can be extended to inhomogeneous soils in a straightforward manner by employing depth-dependent springs. For soil stiffness varying according to a power-law function of depth, closed-form solutions for pile head stiffness are available in the literature (Scott 1981; Guo 2012; Crispin *et al.* 2018). However, these solutions are complex, expressed in terms of special functions of mathematical physics, and are limited to a handful of inhomogeneous soil profiles.

This paper investigates an energy method based on a corresponding solution for laterally loaded piles by Karatzia and Mylonakis (2012, 2016). The proposed method applies the theory of virtual work to the Winkler model via an approximate shape function describing the attenuation of pile displacement with depth, to provide an easier to apply yet still realistic solution. By relating the Winkler modulus to soil properties using either the concentric cylinder model (Randolph and Wroth 1978) or by matching the results to the aforementioned continuum analyses, this solution can be applied to design problems.

## METHOD

Eq. (1) shows the familiar governing equation for an axially loaded pile attached to a bed of Winkler springs with depth dependent modulus,  $k(z)$  (Scott 1981).

$$E_p A w''(z) - k(z)w(z) = 0 \quad (1)$$

where  $E_p$  and  $A$  denote the pile Young's modulus and cross-sectional area, respectively, and  $w(z)$  is the pile settlement at depth  $z$ . By considering the work done by a small virtual displacement profile,  $w^*(z)$ , over the pile length,  $L$ , Eq. (1) can be expressed in an integral form as:

$$\int_0^L E_p A w''(z) w^*(z) dz - \int_0^L k(z) w(z) w^*(z) dz = 0 \quad (2)$$

Following the related solution for laterally loaded piles by Karatzia and Mylonakis (2012, 2016), a dimensionless shape function,  $\phi(z)$ , is used to describe both the real and virtual displacements by normalising with the corresponding value at the pile head. The pile head stiffness,  $K_0$ , can then be expressed in terms of this shape function in the following virtual work equation:

$$K_0 = E_p A \int_0^L [\phi'(z)]^2 dz + \int_0^L k(z) [\phi(z)]^2 dz + \beta^2 K_b \quad (3)$$

where  $\beta = \phi(L)$  is the fraction of the pile head settlement propagating to the pile base, and  $K_b$  is the stiffness of the lumped spring used to model the pile base reaction. Note that the first term expresses the energy stored in the pile body, the second term the energy stored in the distributed Winkler springs that represent the stiffness (mainly through shearing) of the surrounding soil, and the final term the energy stored in the base spring that represents the soil below the pile. The sum of the three energies equals the work done by the force at the pile head under a unit virtual displacement.

Unlike in the lateral case where most piles of practical dimensions may be treated as infinitely long beams, the pile base conditions have a significant effect on the axial response. Therefore, the shape functions derived from the homogeneous soil case are relatively complex and do not lend themselves to the simple formulations desired. As an alternative, this paper investigates the benefits from using a simple linear shape function given in Eq. (4) for predicting pile head stiffness.

$$\phi(z) = \frac{w(z)}{w(0)} = \frac{w^*(z)}{w^*(0)} = 1 - (1 - \beta) \frac{z}{L} \quad (4)$$

### Homogeneous Soil

In the case of homogeneous soil,  $k(z)=k$ , and the normalised pile head stiffness approximated using Eqs. (3) and (4) is given by:

$$\frac{K_0}{E_p A \lambda} = \frac{1}{\lambda L} (1 - \beta)^2 + \frac{1}{3} \lambda L (\beta^2 + \beta + 1) + \beta^2 \Omega \quad (5)$$

where  $\lambda$  is a load transfer parameter (units of  $\text{Length}^{-1}$ ) controlling the attenuation of pile displacement with depth and  $\Omega$  is a dimensionless base stiffness coefficient:

$$\lambda = \sqrt{\frac{k}{E_p A}}, \quad \Omega = \frac{K_b}{E_p A \lambda} \quad (6a,b)$$

The coefficient  $\Omega$  can take any value between 0, representing a fully-floating pile, and  $\infty$ , representing an end-bearing pile. Using Eq. (6a) and  $k \approx 0.6E_s$  (Roesset and Angelides 1980),  $\lambda L$  can be approximated by (Mylonakis 1995):

$$\lambda L \approx 0.85 \left( \frac{E_p}{E_s} \right)^{\frac{1}{2}} \left( \frac{L}{d} \right) \quad (7)$$

where  $E_s$  is the Young's modulus of the soil and  $d$  is the pile diameter.

A simple method to estimate  $\beta$ , referred to here as *exact*  $\beta$ , is to use the exact solution for homogeneous soil (Crispin *et al.* 2018):

$$\beta = \frac{w(L)}{w(0)} = [\Omega \sinh(\lambda L) + \cosh(\lambda L)]^{-1} \quad (8)$$

However, in light of the approximate shape function in Eq. (4), this choice would result in equilibrium of external forces on the pile not being satisfied. Considering equilibrium of vertical forces on the pile yields the following expression for pile head stiffness:

$$K_0 = \int_0^L k(z)\varphi(z)dz + \beta K_b \quad (9)$$

For the homogeneous soil case, using the shape function in Eq. (4), Eq. (9) evaluates to:

$$\frac{K_0}{E_p A \lambda} = \frac{1}{2} (1 + \beta) \lambda L + \beta \Omega \quad (10)$$

This expression can be used with the *exact*  $\beta$  in Eq. (8). Alternatively, an improved equation for  $\beta$  that obeys the equilibrium condition, referred to here as *matched*  $\beta$ , can be obtained by equating Eq. (10) with Eq. (5) and solving for  $\beta$ :

$$\beta = \frac{6 - (\lambda L)^2}{6[1 + \Omega \lambda L] + 2(\lambda L)^2} \quad (11)$$

Figure 1a compares the proposed linear shape functions using both the *matched*  $\beta$  and *exact*  $\beta$  against the exact shape function for an example pile in homogeneous soil. It is evident that the *matched*  $\beta$  results in a slightly closer fit to the exact shape function. Figure 1b shows the variation of  $\beta$  with dimensionless length,  $\lambda L$ , when  $\Omega=0$ . Both estimates are very similar for  $\lambda L < 1.5$ , which covers a wide range of pile dimensions. However, the *matched*  $\beta$  behaves unexpectedly at larger values, predicting upward displacement of the pile base, which is clearly inadmissible.

In Figure 2, Eq. (5) and Eq.(10) with both the *exact*  $\beta$  and *matched*  $\beta$  are compared to the closed form solution to Eq. (1) for homogeneous soil (Mylonakis and Gazetas 1998):

$$\frac{K_0}{E_p A \lambda} = \frac{\Omega + \tanh(\lambda L)}{1 + \Omega \tanh(\lambda L)} \quad (12)$$

Both approximate solutions match Eq. (12) well when  $\lambda L < 1$  (i.e.  $L/d < 40$  for  $E_p/E_s = 10^3$ ), with the *matched*  $\beta$  providing the closest results.

### Inhomogeneous Soil

For homogeneous soils, the utility of the proposed approximate solution is limited by the simplicity of the exact solution. However, in inhomogeneous soils exact solutions have only been found for a limited number of soil stiffness variations with depth. The resulting formulae use special functions, such as those of the Airy or the Bessel type, that are unfamiliar to many practicing engineers. A simpler solution to derive and implement is desired. Extension of the proposed energy solution to inhomogeneous soil profiles only requires integration of the second term in Eq. (3) using a depth varying spring stiffness,  $k(z)$ , rather than solving the full governing differential equation. This paper investigates Winkler modulus varying according to the power-law function of depth given in Eq. (13). Closed form solutions to similar stiffness variations have been provided by Scott (1981), Guo (2012) and Crispin *et al.* (2018).

$$k(z) = k_L \left[ a + (1-a) \frac{z}{L} \right]^n, \quad a = \left( \frac{k_0}{k_L} \right)^{\frac{1}{n}} \quad (13a,b)$$

where  $k_0$  and  $k_L$  are the Winkler modulus at the ground surface and pile base, respectively,  $a$  is an inhomogeneity parameter accounting for non-zero surface stiffness and  $n$  is an inhomogeneity exponent. Substituting this variation in Winkler modulus into Eq. (3) and performing the integration yields:

$$\frac{K_0}{E_p A \lambda_L} = \frac{(1-\beta)^2}{\lambda_L} + \frac{\lambda_L L}{(1-a)^3} \left[ \frac{(1-a^{n+1})(1-a\beta)^2}{(n+1)} - \frac{2(1-a^{n+2})(1-a\beta)(1-\beta)}{(n+2)} + \frac{(1-a^{n+3})(1-\beta)^2}{(n+3)} \right] + \beta^2 \Omega_L \quad (14)$$

where  $\lambda_L$  and  $\Omega_L$  are the values of  $\lambda$  and  $\Omega$  considering the soil properties at the pile base:

$$\lambda_L = \sqrt{\frac{k_L}{E_p A}}, \quad \Omega_L = \frac{K_b}{E_p A \lambda_L} \quad (15a,b)$$

An alternative approximate approach to account for soil inhomogeneity is to use the exact solution for homogeneous soil in Eq. (12) with the average Winkler modulus over the pile length,  $k_{av}$ . In this case:

$$\frac{k_{av}}{k_L} = \frac{1-a^n}{(n+1)(1-a)}, \quad \lambda = \lambda_L \sqrt{\frac{k_{av}}{k_L}}, \quad \Omega = \Omega_L \sqrt{\frac{k_L}{k_{av}}} \quad (16a,b,c)$$

This is a generalisation of the approximation for linear stiffness variations proposed by Randolph and Wroth (1978).

The error introduced by both approximate solutions, Eq. (14) with the *matched*  $\beta$  in Eq. (11) and Eqs. (12) and (16), is compared in Figure 3 for  $\lambda L = 1$  (i.e.  $L/d \approx 40$  for  $E_p/E_s = 10^3$ ) and  $\Omega = 0$  (when the error in both approximations peaks). The exact solution to Eq. (1) when  $k(z)$  is described by Eq. (13) is given by (Crispin *et al.* 2018):

$$\frac{K_0}{E_p A \lambda_L} = a^2 \frac{[I_{\nu-1}(\chi_0)I_{1-\nu}(\chi_L) - I_{1-\nu}(\chi_0)I_{\nu-1}(\chi_L)] + \Omega_L [I_{\nu-1}(\chi_0)I_{-\nu}(\chi_L) - I_{1-\nu}(\chi_0)I_{+\nu}(\chi_L)]}{[I_{-\nu}(\chi_0)I_{\nu-1}(\chi_L) - I_{+\nu}(\chi_0)I_{1-\nu}(\chi_L)] + \Omega_L [I_{-\nu}(\chi_0)I_{+\nu}(\chi_L) - I_{+\nu}(\chi_0)I_{-\nu}(\chi_L)]} \quad (17)$$

where  $I_\nu(\chi)$  is the modified Bessel functions of the first kind, of order  $\nu$  and argument  $\chi$ :

$$\nu = \frac{1}{n+2}, \quad \chi_L = \frac{2\lambda_L L}{(1-a)(n+2)}, \quad \chi_0 = \chi_L a^{\frac{n+2}{2}} \quad (18a,b,c)$$

The performance of this solution has been evaluated against a database of pile field test results in London Clay (Voyagaki *et al.* 2018; Crispin *et al.* 2018) by employing the concentric cylinder model and magical radius proposed in Randolph and Wroth (1978).

The average homogeneous solution performs better the closer the stiffness variation is to a uniform distribution ( $a=1$  or  $n=0$ ), while the accuracy of the proposed solution is almost entirely independent of the inhomogeneity function. For  $a>0.3$ , the average homogeneous solution provides a better approximation of the pile head stiffness. However, for  $a=0$  and  $n=1$ , the error in the average homogeneous solution is approximately double that for the proposed solution. Similar results have been observed for different pile lengths and base conditions with only the magnitude of the errors varying. Consequently, Eqs. (14) and (9) have been used to derive a new expression for pile head stiffness when  $a=0$  incorporating an inhomogeneous *matched*  $\beta$ . The general expression is given in Eq. (19) and simplified expressions for different soil stiffness variations and base conditions are provided in Table 1.

$$\frac{K_0}{E_p A \lambda_L} = \frac{(n+2)\lambda_L L [(n+2)(n+3) + 2\Omega_L \lambda_L L] + (\lambda_L L)^3 + (n+1)(n+2)^2 (n+3)\Omega_L}{(n+1)(n+2)^2 (n+3) [1 + \Omega_L \lambda_L L] + (n+1)(n+2)^2 (\lambda_L L)^2} \quad (19)$$

Figure 4 shows the variation of predicted pile head stiffness with length for the proposed expression, the equivalent homogeneous soil approximation using Eqs. (12) and (16) and the exact solution in Eq. (17) when  $\Omega = 0$ . The variation in error with pile length of Eq. (19) is compared to the average homogeneous solution when  $\Omega = 0$  in Figure 5. The proposed expression has approximately half the error for  $n=1$  and it is less than 12% for  $\lambda L < 1.5$ , covering most likely pile dimensions (i.e.  $L/d < 60$  for  $E_p/E_s = 10^3$ ).

## CONCLUSIONS

An approximate solution has been developed using the Winkler model and virtual work to predict the response of axially loaded piles in inhomogeneous soils. The performance of this solution has been evaluated against an exact closed-form solution and an approximation considering an equivalent homogeneous soil and the same Winkler model. The main findings of this study are:

- The proposed energy solution, in conjunction with an elementary shape function, approximates pile head stiffness well for piles of most practical

dimensions. The best results are obtained when equilibrium of the pile is considered to establish the slope of the shape function (*matched  $\beta$* ).

- The homogeneous approximation performs better the closer the soil stiffness profile is to homogeneous. The accuracy of the energy solution is relatively independent of soil inhomogeneity.
- A simple expression for pile head stiffness, derived using the energy method, is provided in Eq. (19) for inhomogeneous soil with vanishing surface stiffness. This expression has half the error when compared to the homogeneous approximation for soil stiffness varying linearly with depth.

Further investigation into this method using more complex shape functions could yield accurate expressions for pile head stiffness using well known elementary functions.

## ACKNOWLEDGMENTS

The authors would like to thank Dr. Elia Voyagaki for her assistance in preparing the figures. The first author would also like to thank the Engineering and Physical Sciences Research Council for their support (grant number: EP/N509619/1).

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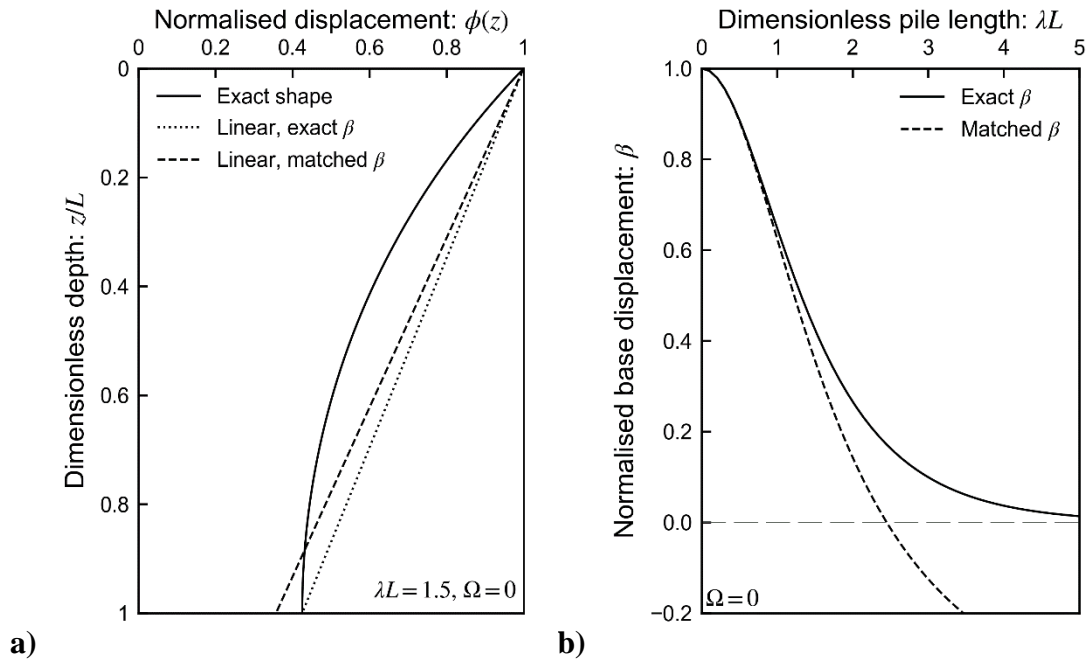
## **TABLES**

**Table 1. Summary of proposed pile head stiffness expressions (Eq. 19) for different soil stiffness variations and base conditions**

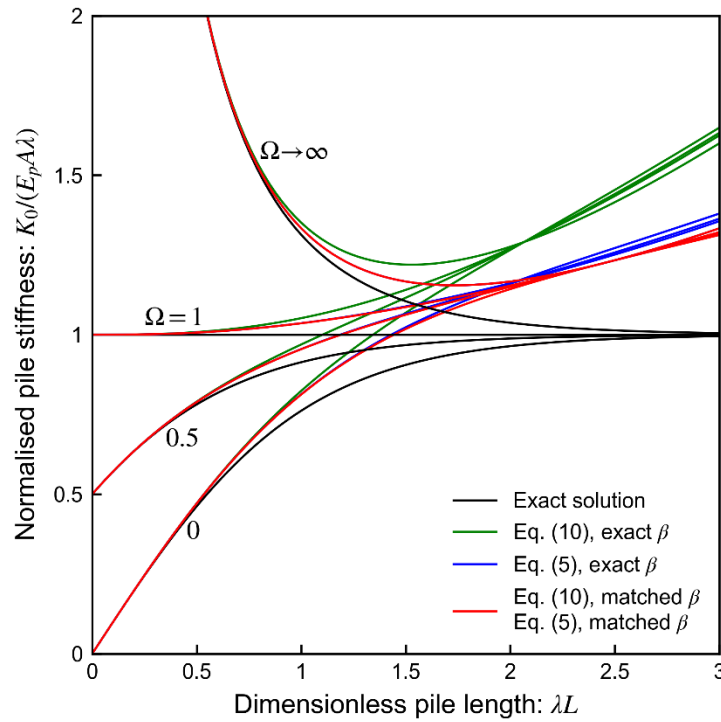
$\frac{K_0}{E_p A \lambda_L}$	General case $0 < L < \infty, 0 \leq \Omega_L < \infty$	Perfectly floating pile, $\Omega_L = 0$	Perfectly end- bearing pile, $\Omega_L \rightarrow \infty$
$n=0$	$\frac{4\lambda_L(3 + \Omega_L \lambda_L L) + (\lambda_L L)^3 + 12\Omega_L}{12(1 + \Omega_L \lambda_L L) + 4(\lambda_L L)^2}$	$\frac{12\lambda_L L + (\lambda_L L)^3}{12 + 4(\lambda_L L)^2}$	$\frac{(\lambda_L L)^2 + 3}{3\lambda_L L}$
$n = 1/2$	$\frac{10\lambda_L L(35 + 8\Omega_L \lambda_L L) + 16(\lambda_L L)^3 + 525\Omega_L}{525(1 + \Omega_L \lambda_L L) + 150(\lambda_L L)^2}$	$\frac{350\lambda_L L + 16(\lambda_L L)^3}{525 + 150(\lambda_L L)^2}$	$\frac{16(\lambda_L L)^2 + 105}{105\lambda_L L}$
$n = 1$	$\frac{6\lambda_L L(6 + \Omega_L \lambda_L L) + (\lambda_L L)^3 + 72\Omega_L}{72(1 + \Omega_L \lambda_L L) + 18(\lambda_L L)^2}$	$\frac{36\lambda_L L + (\lambda_L L)^3}{72 + 18(\lambda_L L)^2}$	$\frac{(\lambda_L L)^2 + 12}{12\lambda_L L}$



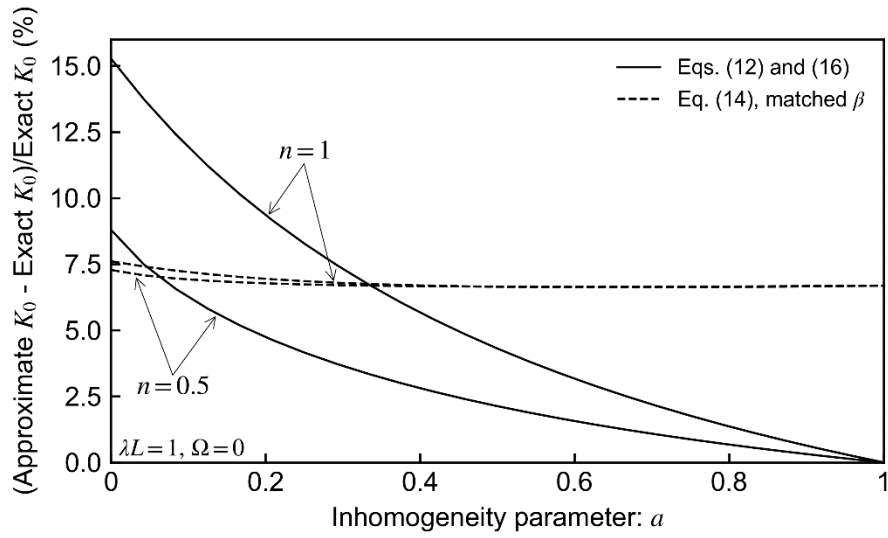
**FIGURES**



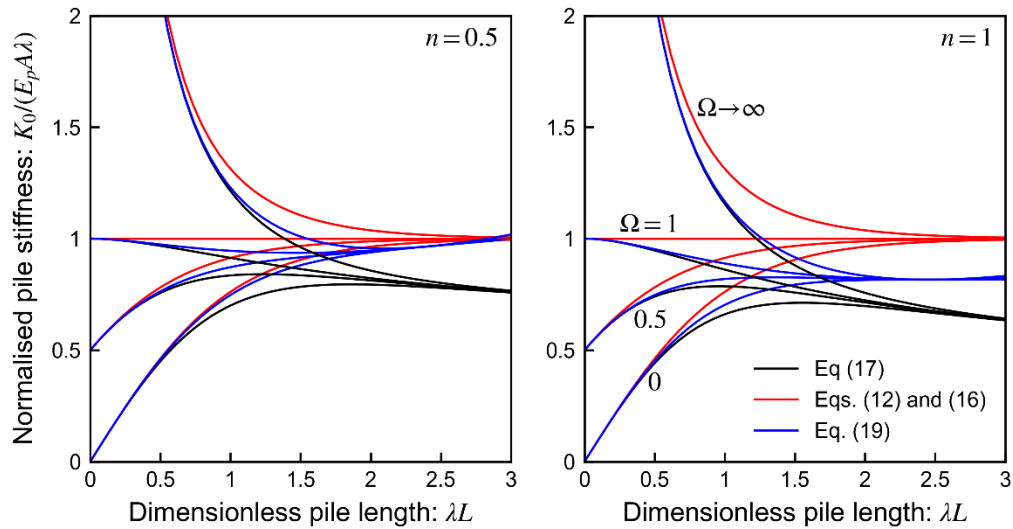
**FIG. 1. a) Exact and approximate shape functions for example pile. b) Variation of exact  $\beta$  and matched  $\beta$  with pile length,  $\Omega=0$ .**



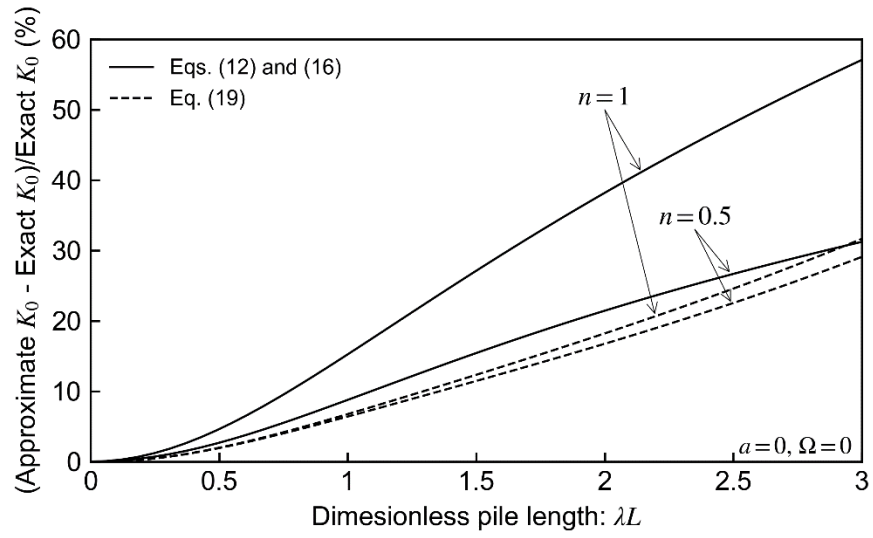
**FIG. 2. Variation of exact and approximate pile head stiffness in homogeneous soil with pile length.**



**FIG. 3. Error in pile head stiffness estimation for different soil stiffness profiles.**



**FIG. 4. Variation of exact and approximate pile head stiffness in inhomogeneous soil (described by Eq. 13) with pile length,  $a=0$ .**



**FIG. 5. Error in head stiffness estimation with pile length,  $a=0$ .**