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# HIGHER-ORDER WINKLER SOLUTIONS FOR LATERALLY-LOADED PILES

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## ABSTRACT

Novel analytical solutions are derived for the response of a flexible elastic pile in homogeneous soil to dynamic inertial and kinematic loads. The solutions are based on the Winkler model of soil reaction and encompass three soil constants (three-parameter model) instead of one in the classical formulation (one-parameter model). This extension allows for a more rational calibration of the model against reference solutions such as finite or boundary elements, by matching all three stiffness constants – in swaying, rocking and cross-swaying rocking – at the pile head. This approach leads to a more realistic representation of pile-soil interaction and a better estimation of internal forces– notably peak pile bending moments – along the pile. Both inertial and kinematic interaction is examined, induced by pile head loads and vertically propagating shear waves, respectively. Closed-form solutions are obtained for: (1) the stiffness coefficients at the pile head, (2) the maximum bending moments, (3) the kinematic response coefficients. Remarkably, the method does not lead to a significant increase in complexity of the analysis, as the order of the governing differential equation and the boundary conditions at the pile head and tip are the same as in the classical model. A novel geometric interpretation of the three elastic constants is provided.


*Keywords: Three-parameter Winkler model; Flexible piles; Stiffness constants; Kinematic coefficients*

## 1. INTRODUCTION

A new analytical approach based on the Winkler model of soil reaction for flexible elastic piles embedded in homogeneous soil, is presented. Contrary to the classical approach which uses a single soil constant (modulus of subgrade reaction), the proposed method employs three soil constants which generate shear tractions, external moments, and internal moments on the pile, in proportion to displacement, rotation and curvature, respectively. The use of three independent constants provides a level of continuity in the Winkler bed and facilitates the calibration of the model against reference solutions based on a continuum representation of the soil, say by matching all three stiffness coefficients (in swaying, rocking and cross-swaying-rocking) at the pile head. This enhanced representation of soil-pile interaction seems to capture better the bending moments along the pile.

Another advantage of the proposed model over the classical one lies in the simplicity of the analysis, since the order of the differential equation and the boundary conditions are not altered – contrary to corresponding gradient theories of elasticity which induce significant complexities in the analysis. Based on dimensional analysis, it is shown that soil constants are dependent on pile-soil stiffness contrast, Poisson's ratio and boundary conditions at the pile head.

The proposed method is a generalization of the classical models by Hetenyi (1946) and Pasternak (1954), which employ two constants for the soil reaction. The Winkler model of soil reaction (1867) represents soil-structure interaction by means of a bed of uniformly distributed soil springs. According to this theory, the static stiffness at the head of a flexible pile in homogeneous soil is given by means of the ~~simple~~ well-known equations (Hetenyi 1946, Scott 1981, Mylonakis 1995):

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$$K_{HH} = 4E_p I\lambda^3, \quad K_{HR} = 2E_p I\lambda^2, \quad K_{RR} = 2E_p I\lambda \tag{1a-c}$$

where

$$\lambda = \left[ k/\left(4E_p I\right) \right]^{1/4} \tag{2}$$

is the Winkler parameter (units of 1/Length) and $E_pI$ the flexural rigidity of the pile (units of Force x Length$^2$). Equations 1a-c provide the stiffness coefficients in swaying, rocking and cross-swaying-rocking, respectively, at the pile head. For a free-head pile, the lateral and rotational stiffnesses are given by

$$K_H = 2E_p I\lambda^3 \quad , \quad K_R = E_p I\lambda \tag{3a,b}$$

and are equal to ½ of the corresponding stiffness of a fixed-head pile.
It is known that the representation of soil by means of a spring bed characterized by a single constant $k$, compared to the representation of soil as a continuum, may induce a level of inconsistency in the results for soil-pile interaction. This is due to the inherent inability of the Winkler model to capture the coupling effects between adjacent springs. Following the early work of Wieghardt (1922), several investigators including Filonenko-Borodich (1940), Hetenyi (1946), Pasternak (1954) and Vlasov and Leontiev (1966) proposed improvements to the original model by introducing a second soil constant ($k_\varphi$). This constant may be interpreted either as a membrane under tension connecting the base of the springs (Hetenyi 1946), or as a bed of rotational springs distributed along the foundation-soil interface (Shanchez-Salinero 1982).

## 2. PROBLEM DEFINITION

The problem considered in this study is depicted in Figure 1. An infinitely long pile embedded in homogeneous soil over rigid bedrock is subjected to a lateral load and/or a bending moment at its head. The pile is modeled as a linear elastic homogeneous cylindrical Euler-Bernoulli beam of diameter $d$, length $L$ and Young's modulus $E_p$. The pile is sufficiently long and flexible, so that it deforms only up to a certain length, known as "active" length, $L_a$ (Randolph 1981) beyond which it ceases to respond to lateral loads imposed at its head. The soil medium is assumed to be linearly elastic with Young's modulus $E_s$ and Poisson's ratio $v_s$. Moreover, the contact at the interface between pile and soil is considered to be perfectly bonded, without sliding or separation between the two materials. For simplicity and to avoid use of complex arithmetic, the results presented in this study refer to undamped conditions.
According to sub-structuring concepts, the pile-soil system can be replaced by three equivalent springs accounting for swaying, rocking and cross-swaying-rocking at the pile head. For static conditions, the main dimensional parameters of the problem are the pile diameter, $d$, the pile Young's modulus, $E_p$, and the soil Young's modulus, $E_s$. The fundamental dimensions are Force [F] and Length [L]. Accordingly, the number of main dimensional parameters is $M = 3$ and the corresponding number of fundamental dimensions is $N = 2$. By applying the Buckingham's theorem (1914), it turns out that $M-N = 1$ dimensionless parameter suffices to describe the solution. This parameter can be conveniently selected ~~to be~~ as the pile-soil stiffness contrast $E_p/E_s$. Additionally, the dimensionless quantities of the pile and soil Poisson's ratio, $v_p$ and $v_s$, have second-order influence on the solution.
For kinematic conditions, the input motion is specified at the base of the pile-soil system in the form of a harmonic horizontal displacement, $u_g = u_{go} \cos(\omega z / V_s)$, which generates vertically propagating $S$ waves. In the dynamic regime ~~and considering dimensional analysis~~, the kinematic interaction for long piles ~~depends~~ can be shown to depend solely on a dimensionless frequency parameter. Following Anoyatis et al (2013) and Di Laora and Rovithis (2015), the frequency parameter ($\omega / \lambda V_s$) is adopted in

this study instead of the conventional ~~choice~~ parameter $(\omega d / V_s)$.

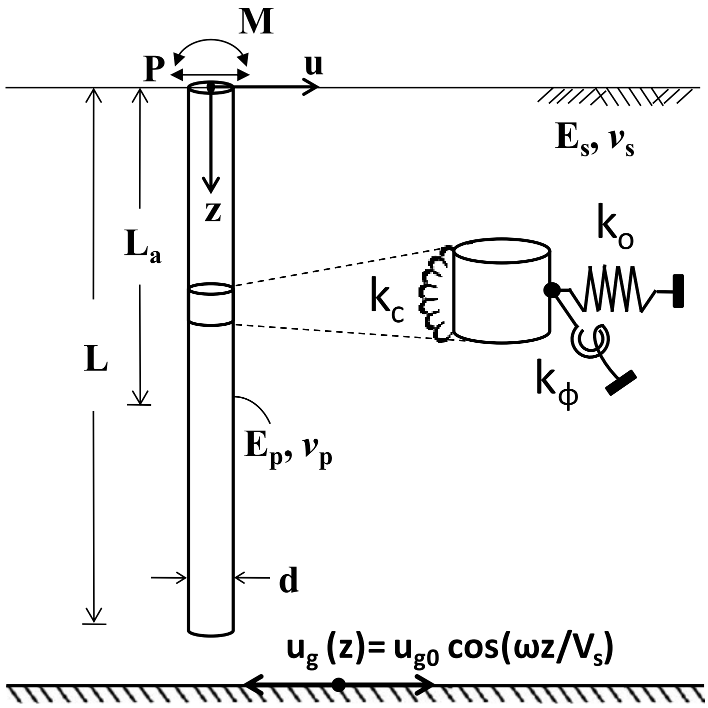


Figure 1. Problem considered and infinitesimal pile segment of three-parameter Winkler model.

## 3. INERTIAL THREE-PARAMETER WINKLER MODEL

### *3.1 Forward Analysis*

For a more accurate description of flexural pile behavior, an enhanced Winkler model equipped with three soil constants $k_o$, $k_\varphi$ and $k_c$ is proposed. The equilibrium equation of the model is (Agapaki 2014)

$$\left(E_p I - k_c\right) u^{(4)} - k_\varphi u^{(2)} + k_o u = 0 \tag{4}$$

where the superscript *( )* denotes differentiation with respect to depth, $z$. The three soil constants induce normal reactions and moments on the pile, as depicted in Figure 2.

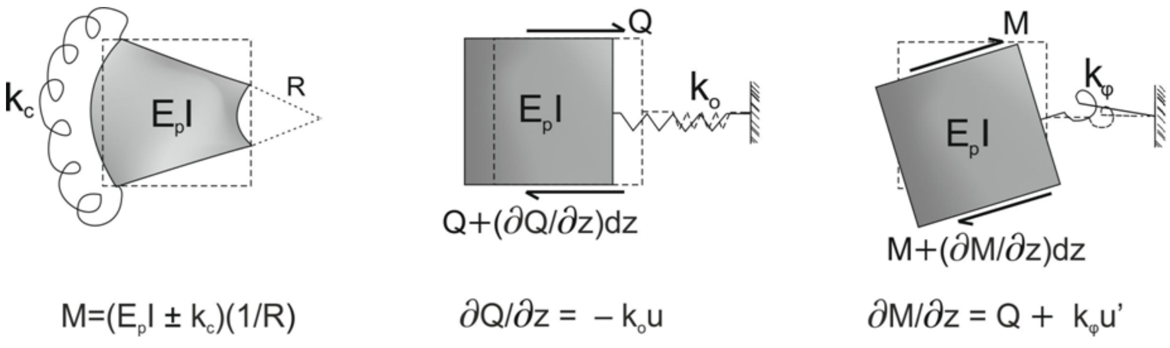


Figure 2. Interpretation of the soil constants ~~utilized~~ employed in the three-parameter Winkler model

The constants of the distributed Winkler springs, $k_o$, $k_\varphi$ and $k_c$, can be linked to soil stiffness through the following relationships

$$k_o = \delta_o E_s, \quad k_\varphi = \delta_\varphi E_s d^2, \quad k_c = \delta_c E_s d^4 \tag{5}$$

with $\delta_o$, $\delta_\varphi$ and $\delta_c$ being dimensionless Winkler constants. Their values are examined in the ensuing. Note that one may derive Equation 4 by assuming that the soil reaction, $p$, at a particular point in the soil depends on a suite of derivatives of displacement, $u$, at the same point, in the form (Scott 1981)

$$p = k_o u - k_\varphi u^{(2)} - k_c u^{(4)} \tag{6}$$

Although this assumption leads to the same governing equation as in Equation 4, the two formulations are not equivalent, because Equation 6 ignores the coupling between the internal pile bending moment and the pile rotation shown in the right graph of Figure 2.
Utilizing the three-parameter Winkler model, one may obtain the stiffness of an infinitely long pile embedded in homogeneous soil according to the following equations

$$K_{HH} = 2\left(E_p I\right)' \lambda\left(\lambda^2 + \mu^2\right) \tag{7}$$

$$K_{HR} = -\left(E_p I\right)' \left(\lambda^2 + \mu^2\right) \tag{8}$$

$$K_{RR} = 2\left(E_p I\right)' \lambda \tag{9}$$

where

$$\left(E_p I\right)' = E_p I - \delta_c E_s d^4 \tag{10}$$

with $\lambda$, $\mu$ being the Winkler parameters (units of 1/Length) and $E_pI$ the flexural stiffness of pile's cross section (units of Force x Length$^2$). For static conditions, the Winkler parameters are obtained as

$$\lambda^4 = \frac{k_o}{4\left(E_p I\right)'}\left(1 + \frac{k_\varphi}{2\sqrt{\left(E_p I\right)' k_o}}\right)^2, \quad \mu^4 = \frac{k_o}{4\left(E_p I\right)'}\left(1 - \frac{k_\varphi}{2\sqrt{\left(E_p I\right)' k_o}}\right)^2 \tag{11a,b}$$

For free-head piles, the lateral stiffness due to an imposed force at the pile head under zero moment, and the rocking stiffness due to a bending moment at the pile head under zero force are

$$K_H = \frac{\left(E_p I\right)' \left(\lambda^2 + \mu^2\right)\left(3\lambda^2 - \mu^2\right)}{2\lambda}, \quad K_R = \frac{\left(E_p I\right)' \left(3\lambda^2 - \mu^2\right)}{2\lambda} \tag{12a,b}$$

Note that, for $\lambda = \mu$ and $(E_pI)' = (E_pI)$ the above relations duly reduce to those obtained from the conventional one-parameter Winkler model.

### *3.2 Inverse Analysis*

In the realm of an inverse analysis, it is assumed that the stiffness coefficients $K_{HH}$, $K_{HR}$ and $K_{RR}$ are known, and the constants of the distributed Winkler springs, i.e. $\delta_o$, $\delta_\varphi$ and $\delta_c$ need to be determined. The flexural rigidity of the pile is computed from Equations 7 – 9 in the form

$$\left(E_p I\right)' = \frac{K_{RR}}{2\lambda} = \frac{K_{RR}\left|K_{HR}\right|}{K_{HH}} \tag{13}$$

which is valid for the one-parameter Winkler model. Substituting Equation 10 in 13 yields

$$\delta_c = \frac{E_p I}{E_s d^4}\left(1 - \frac{K_{RR}\left|K_{HR}\right|}{E_p I K_{HH}}\right) \tag{14}$$

The sign of the $K_{HR}$ term in Equation 14 may be either plus or minus depending on the selection of the reference system, hence the absolute value employed in Equations 13 and 14.

Adding Equations 11a and 11b and combining Equations 7 and 9, the constant $k_o$ is obtained as

$$k_o = \left(E_p I - \delta_c E_s d^4\right)\left(\frac{K_{HH}}{K_{RR}}\right)^2 \tag{15}$$

Combining Equations 5a and 15 yields

$$\delta_o = \left(\frac{E_p I}{E_s d^4} - \delta_c\right)\left(\frac{K_{HH} d^2}{K_{RR}}\right)^2 \tag{16}$$

By dividing by parts Equations 11a and 11b, the constant $k_\varphi$ ~~is derived~~ as

$$k_\varphi = 2\left(E_p I\right)'\left(\lambda^2 - \mu^2\right) \tag{17}$$

Based on Equation 17 and considering Equations 7 – 10 and 13, the constant $\delta_\varphi$ is obtained as a function of the pile-head stiffness coefficients

$$\delta_\varphi = 2\left(\frac{E_p I}{E_s d^4} - \delta_c\right)\left[\left(\frac{E_p I}{E_s d^4} - \delta_c\right)^{-2}\left(\frac{K_{RR}}{2E_s d^3}\right)^2 + \left(\frac{K_{HH} d}{2K_{HR}}\right)^2 - \left(\frac{K_{HH} d^2}{K_{RR}}\right)\right] \tag{18}$$

Introducing the following convenient form for the pile-head stiffness coefficients

$$K_{HH} = \chi_{HH} E_s d, \quad K_{HR} = \chi_{HR} E_s d^2, \quad K_{RR} = \chi_{RR} E_s d^3 \tag{19a-c}$$

the dimensionless Winkler constants are obtained by the simpler expressions

$$\delta_o = \frac{\chi_{HR}\chi_{HH}}{\chi_{RR}} \tag{20}$$

$$\delta_\varphi = 2\left(\frac{\chi_{RR}\chi_{HR}}{\chi_{HH}}\right)\left[\frac{1}{2}\left(\frac{\chi_{HH}}{\chi_{HR}}\right)^2 - \left(\frac{\chi_{HH}}{\chi_{RR}}\right)\right] \tag{21}$$

$$\delta_c = \frac{E_p I}{E_s d^4}\left(1 - \frac{E_s d^4}{E_p I}\frac{\chi_{RR}\chi_{HR}}{\chi_{HH}}\right) \quad (22)$$

with $\chi_{HH}$, $\chi_{HR}$ and $\chi_{RR}$ being dimensionless constants available in literature. Based on detailed finite-element analyses results, Syngros (2004) proposed the following expressions for $\chi_{HH}$, $\chi_{HR}$ and $\chi_{RR}$

$$\chi_{HH} = 0.75\left(\frac{E_p}{E_s}\right)^{1/4}, \quad \chi_{HR} = 0.21\left(\frac{E_p}{E_s}\right)^{1/2}, \quad \chi_{RR} = 0.15\left(\frac{E_p}{E_s}\right)^{3/4} \quad (23a\text{-}c)$$

Substituting Equations 23 into 20–22 and considering a cylindrical pile ($I = \pi d^4/64$), leads to the following simple expressions

$$\delta_o = 1, \quad \delta_\varphi = 0.12\left(\frac{E_p}{E_s}\right)^{1/2}, \quad \delta_c = 0.007\left(\frac{E_p}{E_s}\right) \quad (24a\text{-}c)$$

The equivalent expression for constant $\delta_o$ of the one-parameter Winkler model is equal to 1.17, which is considerably greater than the equivalent coefficient in Equation 24 – in good agreement with the expressions of Syngros (2004) and Gazetas (1991).

### *3.3 Maximum bending moments*

For a free-head pile, the maximum bending moment develops at a depth $z = \pi/2\lambda$ and is derived by solving Equation 4 for the boundary conditions $Q(0) = P$ and $M(0) = 0$. It is easy to show that for the three-parameter model, the bending moment at the pile head along the pile is (Agapaki 2014)

$$M(z) = \frac{e^{-\mu z}\left(\mu^2 + \lambda^2\right)\sin(\lambda z)P}{\lambda\left(3\lambda^2 - \mu^2\right)} \quad (25)$$

The corresponding maximum bending moment can be written in the form

$$\frac{M_{max}}{Pd} = \chi_M\left(\frac{E_p}{E_s}\right)^{1/4} \quad (26)$$

in which the dimensionless coefficient $\chi_M$ is equal to approximately 0.12. For the one-parameter Winkler model, the corresponding coefficient is equal to 0.13, which indicates that the one-parameter model compared to the three-parameter one, overestimates the bending moment at the pile head by approximately 10 % compared to the three-parameter one.
With reference to a fixed-head pile, Equation 26 is still valid, yet with $\chi_M$ being 0.28 and the maximum bending moment developing at the pile head. For the one-parameter model the corresponding coefficient $\chi_M$ is 0.32, which means that this model overestimates the bending moment at the pile head about 20%.

## 4. KINEMATIC THREE-PARAMETER WINKLER MODEL

The equilibrium of horizontal forces acting on the elementary pile segment of Figure 3 leads to the following equation of motion (Agapaki 2014)

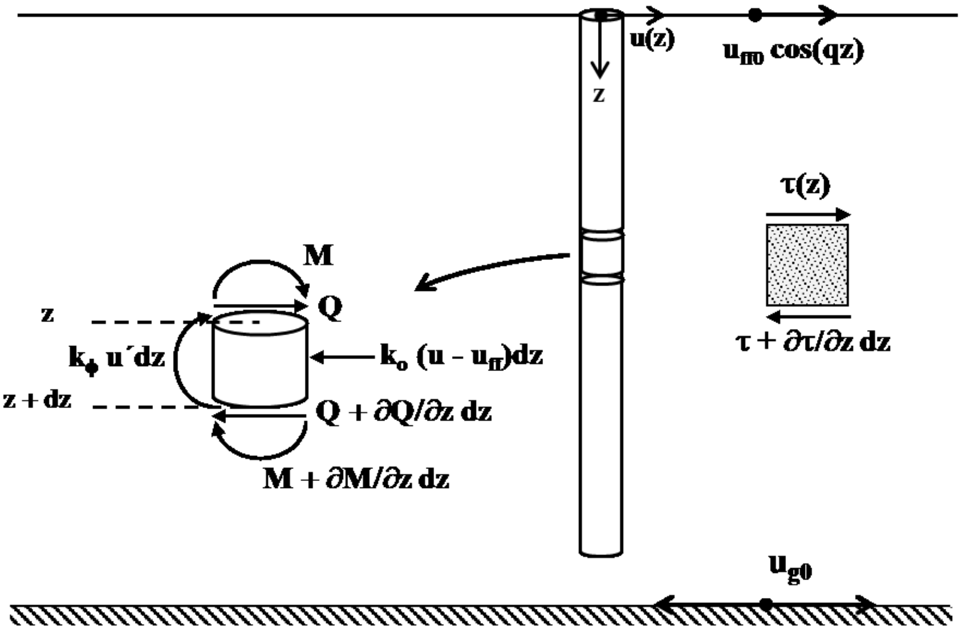


Figure 3. Kinematic three-parameter model and sign convention

$$\left(E_p I - k_c\right)u^{(4)} - k_\varphi u^{(2)} + k_o u = k_o u_{ff} \tag{27}$$

where $u$ is the pile deflection and $u_{ff}$ is the induced free-field motion.
The free-field soil motion, $u_{ff}(z)$, can be written in the form

$$u_{ff}\left(z\right) = u_{ff0}\cos qz \tag{28}$$

which corresponds to a standing wave satisfying the stress-free condition at the soil surface. $u_{ff0}$ is the vibration amplitude at the soil surface and $q$ the wavenumber of the harmonic $S$ waves

$$q = \omega / V_s \tag{29}$$

with $\omega$ being the cyclic excitation frequency and $V_s$ the propagation velocity of shear waves in the soil medium.
The general solution ~~of~~ to Equation 27 is

$$u = e^{\lambda z}\left(A\cos\mu z + B\sin\mu z\right) + e^{-\lambda z}\left(C\cos\mu z + D\sin\mu z\right) + \Gamma u_{ff0}\cos\left(\omega z / V_s\right) \tag{30}$$

In the above solution $A$, $B$, $C$ and $D$ are integration constants determined from the boundary conditions at the pile head and tip, while $\lambda$ and $\mu$ are the associated Winkler parameters in Equation 11. $\Gamma$ is a kinematic response coefficient given by the following expression

$$\Gamma = \frac{k_o}{E_p I\left(q^4 + k_\varphi q^2 + k_o\right)} = \frac{\left(\lambda^2 + \mu^2\right)^2}{q^4 + 2q^2\left(\lambda^2 - \mu^2\right) + \left(\lambda^2 + \mu^2\right)^2} \tag{31}$$

In which $k_o$ and $k_\varphi$ are given by Equation 5.
The response of a kinematically excited pile in dynamic regime is expressed through the familiar kinematic response factors $I_u$ (= $u(0)$ / $u_{ff}(0)$ ) and $I_\varphi$ (= $u'(0)\,d$ / $u_{ff}(0)$) which are defined, respectively, as the maximum pile-head displacement and rotation normalized by the corresponding maximum

displacement at the surface of the free-field soil (Blaney et al. 1976, Kaynia 1982, Fan et al. 1991). However, in In this study the pile-head rotation is described by the alternative factor $I_\theta = u'(0) \ / \ \lambda \ u_{ff}(0)$ is employed instead of the ordinary factor $I_\varphi$. For infinitely long piles and free-head conditions, the kinematic response coefficients are given by the following closed-form solutions

$$I_u = \Gamma\left(1 + \frac{q^2}{3\lambda^2 - \mu^2}\right) \tag{32}$$

and

$$I_\theta = \Gamma \frac{2q^2}{3\lambda^2 - \mu^2} \tag{33}$$

In the case of zero rotation at the pile head (fixed-head conditions), the kinematic factor $I_u$ is given by the simple expression (Flores-Berrones and Whitman 1982)

$$I_u = \Gamma \tag{34}$$

while $I_\theta = 0$.
Note that, for $\lambda = \mu$ the above relations become equal to those obtained from the one-parameter Winkler model.
By introducing the frequency parameter ($\omega \ / \ \lambda V_s$), Equations 31 – 33 are rewritten as follows

$$\Gamma = \frac{\left[1 + (\mu/\lambda)^2\right]^2}{(\omega/\lambda V_s)^4 + 2(\omega/\lambda V_s)^2\left[1 - (\mu/\lambda)^2\right] + \left[1 + (\mu/\lambda)^2\right]^2} \tag{35}$$

$$I_u = \Gamma\left[1 + \frac{(\omega/\lambda V_s)^2}{3 - (\mu/\lambda)^2}\right] \tag{36}$$

$$I_\theta = \Gamma \frac{2(\omega/\lambda V_s)^2}{3 - (\mu/\lambda)^2} \tag{37}$$

## 5. RESULTS

The three Winkler constants are depicted in Figure 4. It is observed that the values of the dimensionless parameter $\delta_o$ obtained from the three-parameter model are smaller than the one obtained from the one-parameter model. Also, $\delta_o$ is independent of $E_p/E_s$. On the other hand, constants $\delta_\varphi$ and $\delta_c$ are significantly affected by the pile-soil stiffness contrast, and increase with increasing $E_p/E_s$.

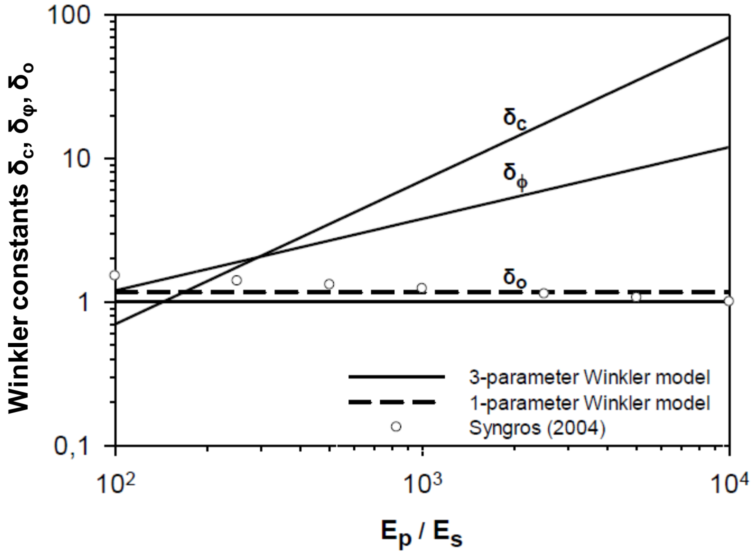


Figure 4. Winkler constants for a pile in homogeneous soil as function of $E_p/E_s$ (Equation 24).

In Figure 5a-c the stiffness coefficients are presented for a fixed-head pile (Equations 1 and 7 – 9) as a function of the stiffness ratio $E_p/E_s$. It is observed that the values of the stiffness coefficients obtained from the three-parameter model are in very good agreement with corresponding expressions in literature. Compared to the results obtained from the one-parameter model, the stiffness coefficients exhibit deviations, especially for the term $K_{HR}$. For a free-head pile, the three-parameter model seems to overestimate the swaying coefficient in comparison with other solutions, as shown in Figure 5d.

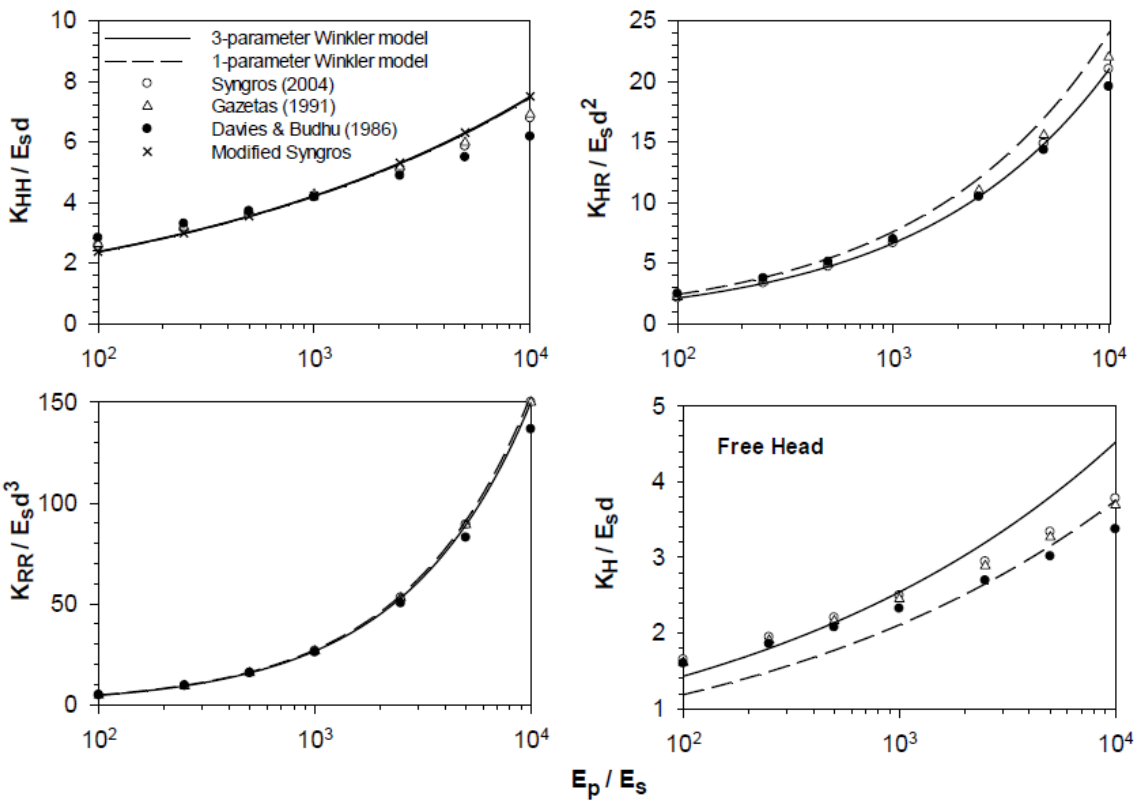


Figure 5. Pile stiffness coefficients for homogeneous soil ($v_p = 0.25$, $v_s = 0.4$).

Figure 6 shows a comparison of maximum bending moments obtained from one- and three-parameter Winkler models at the pile head and at depth $z = \pi/(2\lambda)$. Results are also compared with those obtained from Randolph (1981). The agreement between the results obtained from the three-parameter model and the numerical results from Randolph (1981) is very good, especially for small pile-soil stiffness ratios ($10^2 – 10^3$). For higher values of $E_p/E_s$ ratio, results from the numerical solution of Randolph approach those obtained from the classical solution.

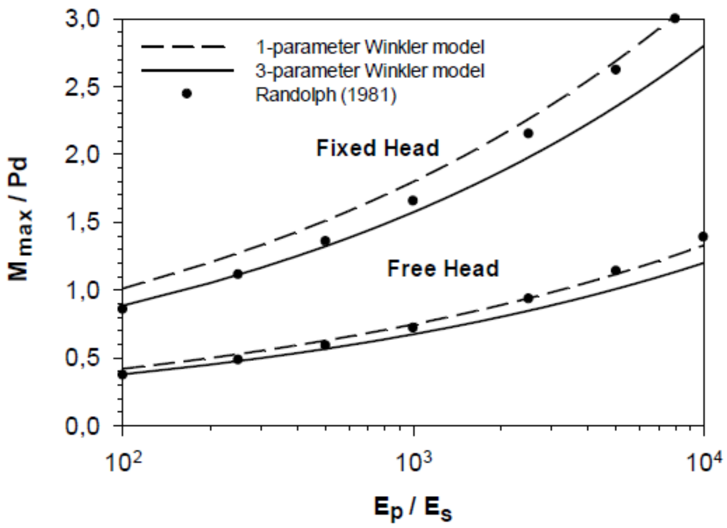


Figure 6. Maximum pile bending moments for different boundary conditions at pile head under horizontal load as a function of $E_p/E_s$ ($v_s = 0.4$).

Numerical results for the kinematic response factors of an infinitely long pile based on the three-parameter Winkler model are presented in Figures 7 and 8. Evidently, at low frequencies $I_u$ is approximately equal to one, which implies that the pile follows the free-field soil motion. For $\mu \neq \lambda$, $I_u$ decreases monotonically with increasing frequency and tends to zero as $\omega/\lambda V_s$ approaches infinity. For $\mu = \lambda$ (one-parameter Winkler model), $I_u$ is constant (for fixed-head piles) or increases (for free-head piles) with increasing frequency up to a certain value and then starts to decrease. This can be interpreted by considering the wavelengths developed in the soil for different frequencies. As the excitation frequency increases, the wavelength decreases inducing greater rotations along the pile, which yields greater displacement at the pile head. On the contrary, when $\mu \neq \lambda$, the rotational springs along the soil-pile interface resist the rotational deformation of the pile and the displacement at the pile head is reduced even at low frequencies. Evidently, as the ratio $\mu / \lambda$ decreases, $I_u$ decreases.
With reference to the kinematic factor $I_\theta$, it is seen that the curves for $\mu \neq \lambda$ have the same shape with that of the one-parameter model, however, $I_\theta$ decreases considerably with decreasing $\mu / \lambda$.

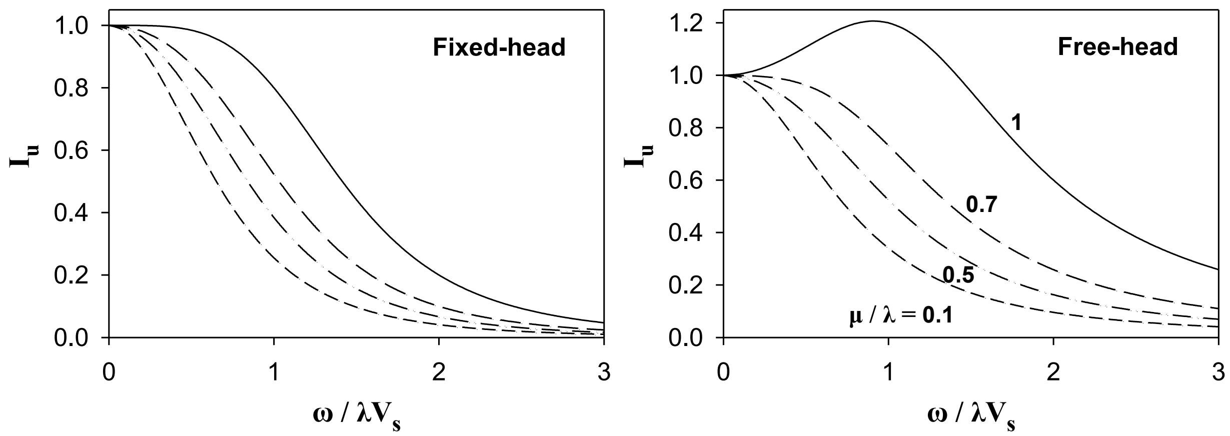


Figure 7. Translational kinematic factor for fixed- and free-head piles as a function of the dimensionless frequency $\omega/\lambda V_s$.

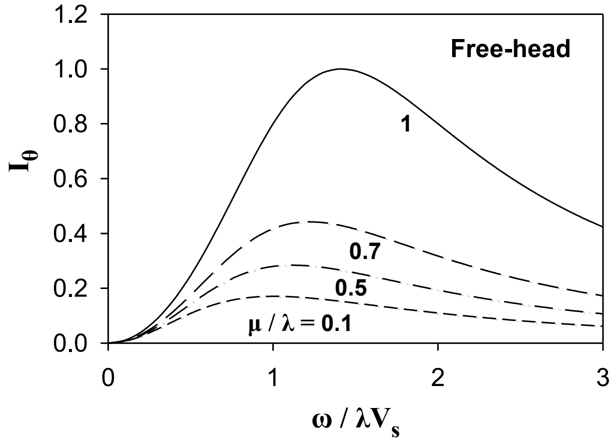


Figure 8. Rotational kinematic factor as a function of the dimensionless frequency $\omega/\lambda V_s$.

## 6. CONCLUSIONS

The main conclusions of this study are:

- The proposed enhanced Winkler model involves three soil constants and, therefore, can be configured to match three target results simultaneously, such as the three stiffness coefficients at the pile head. This is in contrast to the one-parameter Winkler model which can reproduce only a single target result, notably the horizontal pile stiffness.
- The proposed method improves the estimation of maximum bending moments for laterally loaded piles in comparison to the classical one-parameter model, especially for low pile-soil stiffness contrast ratios.
- The proposed analytical method provides reasonable results for the kinematic response factors.
- All the above are achieved without a significant increase in the complexity of the analysis, since the order of the differential equation and the boundary conditions at the pile head and tip do not change relative to the conventional Winkler model.

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