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ON r-GAPS BETWEEN ZEROS OF THE RIEMANN ZETA-FUNCTION

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ABSTRACT. Under the Riemann Hypothesis, we prove for any natural number r there exist infinitely many natural numbers n such that $(\gamma_{n+r}-\gamma_n)/(2\pi r/\log\gamma_n) > 1+\Theta/\sqrt{r}$ and $(\gamma_{n+r}-\gamma_n)/(2\pi r/\log\gamma_n) < 1-\vartheta/\sqrt{r}$ for explicit absolute positive constants Θ and ϑ, where γ denotes an ordinate of a zero of the Riemann zeta-function on the critical line. Selberg published announcements of this result several times without proof.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function, and let $\rho = \beta+i\gamma$ denote a nontrivial zero of $\zeta(s)$. Consider the sequence of ordinates of zeros in the upper half-plane

$$0 < \gamma_1 \le \gamma_2 \le \ldots \le \gamma_n \le \gamma_{n+1} \le \ldots.$$

It is well known that

$$N(T) := \sum_{0<\gamma\le T} 1 \sim \frac{T}{2\pi}\log T,$$

from which it follows that the average gap between consecutive zeros is $2\pi/\log\gamma_n$. Assuming the Riemann Hypothesis, $\beta = 1/2$ and $\gamma\in\mathbb{R}$. The result of this note is a proof of the following theorem.

Theorem. *Assuming the Riemann Hypothesis, for any natural number r there exist infinitely many n such that*

$$\frac{\gamma_{n+r}-\gamma_n}{2\pi r/\log\gamma_n} > 1+\frac{\Theta}{\sqrt{r}} \qquad \text{and} \qquad \frac{\gamma_{n+r}-\gamma_n}{2\pi r/\log\gamma_n} < 1-\frac{\vartheta}{\sqrt{r}}$$

for the absolute positive constants $\Theta = 0.574271$ and $\vartheta = 0.299856$. Moreover, for r sufficiently large, we may take $\Theta = \vartheta = 0.9065$.

There are discrepancies in the literature regarding the correct statement of this result, which we hope to now clarify. In [11, p. 199], Selberg announced, without proof, that there exists an absolute positive constant θ such that for all positive integers r

$$\limsup_{n\to\infty}\frac{\gamma_{n+r}-\gamma_n}{2\pi r/\log\gamma_n} > 1+\theta \qquad \text{and} \qquad \liminf_{n\to\infty}\frac{\gamma_{n+r}-\gamma_n}{2\pi r/\log\gamma_n} < 1-\theta.$$

This statement was later updated in the Acknowledgements section of [9], with the θ appearing above replaced with $\theta/\sqrt{r}$. Finally, in the errata of Volume 1 of his collected papers [12, p. 355], Selberg clarified the correct statement of his result.

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Selberg's Announced Result. *There exist an absolute positive constant θ such that for all positive integers r*

$$\limsup_{n\to\infty}\frac{\gamma_{n+r}-\gamma_n}{2\pi r/\log\gamma_n}>1+\theta r^{-\alpha}\qquad\text{and}\qquad\liminf_{n\to\infty}\frac{\gamma_{n+r}-\gamma_n}{2\pi r/\log\gamma_n}<1-\theta r^{-\alpha},$$

where α may be taken as 2/3, and if one assumes the Riemann Hypothesis as 1/2.

Selberg did not give an indication of a proof for either statement, however Heath-Brown in [13, p. 246-249] provides an unconditional proof of Selberg's result in the case $r=1$ using the work of Fujii [5] concerning the mean value of $S(t)$ in short intervals. (Note that $\pi S(t)$ is the argument of $\zeta(s)$ at the point $s=1/2+it$.) We remark that Heath-Brown's proof for $r=1$ shows that the result holds for a positive proportion of integers n.

The goal of this note is to give a proof of Selberg's conditional result for all $r\geq 1$ with explicit constants. To prove our theorem, we adapt a method developed by Conrey, Ghosh, and Gonek [3] on gaps between consecutive nontrivial zeros of $\zeta(s)$ in the interval $[0,T]$ for T large. The method is conditional on the Riemann Hypothesis. To our knowledge, our proof is the first to appear in the literature for $r>1$.

For a fixed, positive integer r, let

$$\lambda_r:=\limsup_{n\to\infty}\frac{\gamma_{n+r}-\gamma_n}{2\pi/\log\gamma_n}\qquad\text{and}\qquad\mu_r:=\liminf_{n\to\infty}\frac{\gamma_{n+r}-\gamma_n}{2\pi/\log\gamma_n}. \tag{1}$$

By definition $\lambda_r\geq r$ and similarly $\mu_r\leq r$, however random matrix theory predicts that $\lambda_r=\infty$ and $\mu_r=0$. Following [3], we compare averages of a well-chosen polynomial of the form

$$A(t):=\sum_{n\leq X}\frac{a^{\pm}(n)}{n^{it}}, \tag{2}$$

where $X=T^{1-\delta}$ for some small $\delta>0$. To adapt for r-gaps, we set

$$M_1:=\int_T^{2T}|A(t)|^2\,dt$$

and

$$M_2(c_r):=\int_{-\pi c_r/\log T}^{\pi c_r/\log T}\sum_{T\leq\gamma\leq 2T}|A(\gamma+\alpha)|^2\,d\alpha,$$

where c_r is some nonzero real number. We see that $M_2(c_r)$ is monotonically increasing and

$$M_2(\mu_r)\leq rM_1\leq M_2(\lambda_r).$$

Therefore, if $M_2(c_r)<rM_1$ for some choice of $a^+(n)$ and c_r then $\lambda_r>c_r$. Similarly, if $M_2(c_r)>rM_1$ for some choice of $a^-(n)$ and c_r then $\mu_r<c_r$.

Connecting their work to a previous result of Montgomery and Odlyzko [8], Conrey, Ghosh, and Gonek show

$$\frac{M_2(c_r)}{M_1}=h^{\pm}(c_r)+o(1),$$

where $h(c_r)$ is defined by

$$h^{\pm}(c_r) := c_r \mp \frac{\Re\left(\sum_{kn\le X} \frac{a^{\pm}(n)\overline{a^{\pm}(kn)}g_{c_r}(k)\Lambda(k)}{kn}\right)}{\sum_{n\le X} \frac{|a^{\pm}(n)|^2}{n}} \tag{3}$$

and

$$g_{c_r}(k) = \frac{2\sin\left(\pi c_r \frac{\log k}{\log T}\right)}{\pi \log k}$$

so that $|g_{c_r}(k)| \le 2c_r/\log T$. The function $h^{\pm}(c_r)$ was introduced by Montgomery and Odlyzko to study gaps between consecutive zeros of $\zeta(s)$. In particular, they show that if one is able to find c_r such that $h^+(c_r) < r$ then $\lambda_r > c_r$ and such that if $h^-(c_r) > r$ then $\mu_r < c_r$.

Letting $r = 1$ in (1), it follows from our theorem that $\lambda_1 > 1$ and $\mu_1 < 1$. Quantitative bounds on λ_1 and μ_1 have been obtained using the above approach, with different choices of $a(n)$ leading to improved results. See [2] and subsequently [6] for discussions of these choices. The best current quantitative bounds concerning gaps between consecutive zeros of the Riemann zeta function (under the assumption of the Riemann Hypothesis) are $\lambda_1 > 3.18$, due to Bui and Milinovich [1] , and $\mu_1 < 0.515396$, due to Preobrazhenskii [10]. We note that the method employed in [1], which is based on the work of Hall [7] and different from the method discussed above, is unconditional if one restricts the analysis to critical zeros.

2. Proof of the theorem for fixed $r \ge 1$

For large gaps for any fixed $r \ge 1$, we choose $a^+(n) = d_\ell(n)$, where d_ℓ is multiplicative and defined on prime powers by

$$d_\ell(p^m) = \frac{\Gamma(m+\ell)}{\Gamma(\ell)m!}.$$

Fix $\ell \ge 1$. (In the proof, we will ultimately set ℓ to be an explicit value depending on r.) Similarly, for small gaps for any fixed $r \ge 1$, we choose $a^-(n) = \lambda(n)d_\ell(n)$, where $\lambda(n)$ denotes the Liouville function.

We now prove the result for large gaps for any fixed $r \ge 1$. Take $a^+(n) = d_\ell(n)$ for $\ell \ge 1$ an integer to be determined later. In this case the relevant mean-value to compute is well known:

$$\sum_{n\le x} \frac{d_\ell(n)^2}{n} = C_\ell(\log x)^{\ell^2} + O((\log T)^{\ell^2-1})$$

for fixed $\ell \ge 1$, uniformly for $x \le T$, where C_ℓ is a constant which will not have an effect in our application. It is shown in [3, p.422] that for this choice of $a^+(n)$, the equation $M_2(c_r)/M_1 = h^+(c_r) + o(1)$ reduces to

$$h^+(c_r) = c_r - 2\ell \int_0^1 \frac{\sin(\pi c_r v(1-\delta))}{\pi v}(1-v)^{\ell^2}\, dv + O(1/\log T) \tag{4}$$

where $\delta > 0$ is as in (2) and will be taken to be sufficiently small. To detect large gaps, we must show that $h^+(c_r) < r$ for fixed $r \ge 1$. By the previous discussion, this will imply $\lambda_r > c_r$. For example, using (4) we can compute the following table of values.

r	ℓ	c_r	$h^+(c_r)$
1	2.2	2.337	0.99965
2	2.8	3.708	1.99937
3	3.3	4.994	2.99975
4	3.7	6.235	3.99950
5	4.0	7.448	4.99978

TABLE 1. For fixed r, the table gives values of ℓ, c_r for which $h^+(c_r) < r$, implying $\lambda_r > c_r$.

In general, to prove large gaps of the desired shape, we show that $h^+(c_r) < r$ for fixed $r \geq 1$ and $c_r = r + \Theta\sqrt{r}$ with $\Theta > 0$. We estimate the integral appearing in (4) as follows. Let

$$\int_0^1 \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\,dv = I_1 + I_2,$$

where

$$I_1 := \int_0^{1/c_r} \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\,dv \qquad \text{and} \qquad I_2 := \int_{1/c_r}^1 \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\,dv.$$

For I_1, we first observe that the integrand is positive in the range of integration and write

$$I_1 \geq I_{1,a} + I_{1,b}, \tag{5}$$

say, where

$$I_{1,a} := \int_0^{1/4c_r} \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\,dv,$$
$$I_{1,b} := \int_{1/4c_r}^{1/2c_r} \frac{\sin(\pi c_r(1-\delta)v)}{\pi v}(1-v)^{\ell^2}\,dv,$$

and we have discarded the portion of the integral from $1/(2c_r)$ to $1/c_r$. Now we estimate $I_{1,a}$ and $I_{1,b}$. For $I_{1,a}$, we compare $\sin(\pi c_r(1-\delta)v)$ to $2\sqrt{2}c_r(1-\delta)v$ and find

(6)
$$I_{1,a} \geq \int_0^{1/4c_r} \frac{2\sqrt{2}c_r(1-\delta)v}{\pi v}(1-v)^{\ell^2}\,dv = \frac{2\sqrt{2}c_r(1-\delta)}{\pi(\ell^2+1)}\left(1-\left(1-\frac{1}{4c_r}\right)^{\ell^2+1}\right).$$

Similarly for $I_{1,b}$, we compare $\sin(\pi c_r(1-\delta)v)$ to $(4-2\sqrt{2})c_r(1-\delta)v$ and find

$$I_{1,b} \geq \frac{(4-2\sqrt{2})c_r(1-\delta)}{\pi(\ell^2+1)}\left(\left(1-\frac{1}{2c_r}\right)^{\ell^2+1} - \left(1-\frac{1}{4c_r}\right)^{\ell^2+1}\right). \tag{7}$$

Thus by (5), (6), and (7), we have

$$I_1 \geq \frac{2c_r(1-\delta)}{\pi(\ell^2+1)}\left(\sqrt{2} - (2\sqrt{2}-2)\left(1-\frac{1}{4c_r}\right)^{\ell^2+1} - (2-\sqrt{2})\left(1-\frac{1}{2c_r}\right)^{\ell^2+1}\right).$$

Furthermore, since $\exp(-x) \geq 1 - x$ for $x \geq 0$, it follows that
(8)

$$I_1 \geq \frac{2c_r(1-\delta)}{\pi(\ell^2+1)}\left\{\sqrt{2} - (2\sqrt{2}-2)\exp\left(\frac{-(\ell^2+1)}{4c_r}\right) - (2-\sqrt{2})\exp\left(\frac{-(\ell^2+1)}{2c_r}\right)\right\}.$$

We now estimate the second integral I_2. Since $v \geq 0$, we have

$$|I_2| \leq \frac{1}{\pi}\int_{1/c_r}^{1} \frac{(1-v)^{\ell^2}}{v}\,dv \leq \frac{1}{\pi}\int_{1/c_r}^{1} \frac{\exp(-\ell^2 v)}{v}\,dv.$$

Thus, by the change of variable $u = \ell^2 v$, we find

$$I_2 \geq \frac{-1}{\pi}\int_{\ell^2/c_r}^{\infty} \frac{\exp(-u)}{u}\,du. \tag{9}$$

Combining the estimates in (8) and (9), we have

$$\begin{aligned} h^+(c_r) \leq c_r - 2\ell\Bigg\{&\frac{2c_r(1-\delta)}{\pi(\ell^2+1)}\left(\sqrt{2} - (2\sqrt{2}-2)\exp\left(\frac{-(\ell^2+1)}{4c_r}\right)\right. \\ &\left. - (2-\sqrt{2})\exp\left(\frac{-(\ell^2+1)}{2c_r}\right)\right) - \frac{1}{\pi}\int_{\ell^2/c_r}^{\infty}\frac{\exp(-u)}{u}\,du\Bigg\} + O\left(1/\log T\right). \end{aligned}$$

In this case, $c_r = r + \Theta\sqrt{r}$ where $\Theta > 0$, and thus $c_r > 1$ for any $r \geq 1$. Thus, letting $\ell = \sqrt{bc_r - 1}$, where $b > 1$ is a real number that will be chosen later, we have

$$\ell = \sqrt{bc_r - 1} \geq \sqrt{br}\sqrt{1 - \frac{1}{b}}$$

for any $r \geq 1$. Furthermore, since we always have $c_r > 1$, for any $r \geq 1$ it follows that

$$\frac{\ell^2}{c_r} > b - 1,$$

and thus we may again increase the length of integration in I_2 to write

$$\int_{\ell^2/c_r}^{\infty}\frac{\exp(-u)}{u}\,du < \int_{b-1}^{\infty}\frac{\exp(-u)}{u}\,du.$$

Combining these estimates, we find

$$\begin{aligned} h^+(c_r) < r + \Theta\sqrt{r} - 2\sqrt{br}\sqrt{1-\frac{1}{b}}\Bigg\{&\frac{2(1-\delta)}{\pi b}\left(\sqrt{2} - (2\sqrt{2}-2)\exp\left(\frac{-b}{4}\right)\right. \\ &\left. - (2-\sqrt{2})\exp\left(\frac{-b}{2}\right)\right) - \frac{1}{\pi}\int_{b-1}^{\infty}\frac{\exp(-u)}{u}\,du\Bigg\} + O\left(\frac{1}{\log T}\right). \end{aligned}$$

To show $h^+(c_r) < r$ and prove the theorem, we set

$$\begin{aligned} \Theta = \max_b\Bigg\{2\sqrt{b}\sqrt{1-\frac{1}{b}}\Bigg(&\frac{2}{\pi b}\left(\sqrt{2} - (2\sqrt{2}-2)\exp\left(\frac{-b}{4}\right)\right. \\ &\left. - (2-\sqrt{2})\exp\left(\frac{-b}{2}\right)\right) - \frac{1}{\pi}\int_{b-1}^{\infty}\frac{\exp(-u)}{u}\,du\Bigg)\Bigg\}. \end{aligned}$$

The choice $b = 5.0107$ yields $\Theta = 0.574271$. With δ sufficiently small and T sufficiently large, these choices guarantee that $h^+(c_r) < r$, as desired.

We now prove the result for small gaps for any fixed $r \geq 1$. The proof for small gaps is similar to the proof for large gaps, so we indicate the necessary changes.

Take $a^-(n) = \lambda(n) d_\ell(n)$ for $\ell \geq 1$ fixed. It is given in [3, p.422] that this choice of $a^-(n)$, yields

$$h^-(c_r) = c_r + 2\ell \int_0^1 \frac{\sin(\pi c_r v(1-\delta))}{\pi v} (1-v)^{\ell^2}\, dv + O\left(1/\log T\right). \tag{10}$$

To detect small gaps, we must show that $h^-(c_r) > r$ for fixed $r \geq 1$. By the previous discussion, this will imply $\mu_r < c_r$. For example, using (10) we can compute the following table of values.

r	ℓ	c_r	$h^-(c_r)$
1	1.1	0.5172	1.00012
2	1.4	1.126	2.00118
3	1.9	1.831	3.00072
4	2.3	2.588	4.00099
5	2.7	3.375	5.00116

TABLE 2. For fixed r, the table gives values of ℓ, c_r for which $h^-(c_r) > r$, implying $\mu_r < c_r$.

In general, to prove small gaps of the desired shape, we show that $h^-(c_r) < r$ for fixed $r \geq 1$ and $c_r = r - \Theta\sqrt{r}$ with $\Theta > 0$. We estimate the integral appearing in (10) as before, however for brevity we will perform the calculation without writing I_1 as the sum of two integrals of equal length.[1] We find

$$h^-(c_r) \geq c_r + 2\ell \left\{ \frac{2c_r(1-\delta)}{\pi(\ell^2+1)} \left(1 - \exp\left(\frac{-(\ell^2+1)}{c_r}\right)\right) - \frac{1}{\pi}\int_{\ell^2/c_r}^{\infty} \frac{\exp(-u)}{u}\, du \right\} + O\left(\frac{1}{\log T}\right).$$

Let $\ell = \sqrt{bc_r - 1}$ and $c_r = r - \vartheta\sqrt{r}$, with $\vartheta > 0$. In this case, we do not always have $c_r > 1$. Indeed, since $\vartheta > 0$, if $r = 1$ then $0 < c_r < 1$. However, if we require that $\vartheta \leq 0.5$, the estimate

$$\ell = \sqrt{bc_r - 1} > \sqrt{br}\sqrt{\frac{1}{2} - \frac{1}{b}}$$

holds for any $r \geq 1$. The requirement that $\vartheta \leq 0.5$ also implies

$$\frac{\ell^2}{c_r} \geq b - 2$$

for any $r \geq 1$, and we may increase the length of integration in I_2 to write

$$\int_{\ell^2/c_r}^{\infty} \frac{\exp(-u)}{u}\, du \leq \int_{b-2}^{\infty} \frac{\exp(-u)}{u}\, du.$$

Thus, requiring that $\vartheta \leq 0.5$, we may put these estimates together to write

$$h^-(c_r) > r - \vartheta\sqrt{r} + 2\sqrt{br}\sqrt{\frac{1}{2} - \frac{1}{b}} \left\{ \frac{2(1-\delta)}{\pi b}\Big(1 - \exp\left(-b\right)\Big) - \frac{1}{\pi}\int_{b-2}^{\infty} \frac{\exp(-u)}{u}\, du \right\} + O\left(\frac{1}{\log T}\right).$$

To show $h^-(c_r) > r$ and thus prove the theorem, we set

[1] To see how these choices affect the size of ϑ here and in the large gaps setting, please refer to the remark following the proof.

$$\vartheta = \max_b \left\{ 2\sqrt{b}\sqrt{\frac{1}{2}-\frac{1}{b}}\left(\frac{2}{\pi b}(1-\exp(-b)) - \frac{1}{\pi}\int_{b-2}^{\infty}\frac{\exp -u}{u}\,du\right)\right\}.$$

The choice $b = 5.17305$ yields $\vartheta = 0.299856$. (We note that the condition $\vartheta < 0.5$ is satisfied.) With δ sufficiently small and T sufficiently large, these choices guarantee that $h^+(c_r) < r$, as desired.

Remark 1. *In the argument above for large gaps, if we had not divided the remaining portion of* I_1 *into two smaller integrals and instead compared* $\sin(\pi c_r(1-\delta)v)$ *to* $2c_r(1-\delta)v$ *over the interval* $[0, 1/2c_r]$*, we would have ultimately found that one can take* $\Theta = 0.447$*. Instead, by carrying out the analysis on* $I_1 > I_{1,a} + I_{1,b}$ *(see* (5)*) and estimating* $I_{1,a}$ *and* $I_{1,b}$ *separately, we were able to provide the stronger constant* $\Theta = 0.570717$*. One could thus slightly improve the absolute constant* Θ *by breaking up* I_1 *into smaller pieces over the interval* $[0, 1/2c_r]$*, and estimating each piece accordingly. For example, writing* $I_1 > I_{1,a'} + I_{1,b'} + I_{1,c'} + I_{1,d'}$ *where each integral has equal length of integration over the interval* $[0, 1/2c_r]$*, one can obtain* $\Theta = 0.593234$*, and comparing* I_1 *to the sum of sixteen such smaller integrals* $\Theta = 0.599648$*. Similarly, for small gaps, comparing* I_1 *to the sum of two smaller integrals of equal length over the interval* $[0, 1/2c_r]$ *yields* $\vartheta = 0.359222$*; using sixteen smaller integrals of equal length of integration over the interval* $[0, 1/2c_r]$ *yields* $\vartheta = 0.379674$*.*

3. Proof of the theorem for r sufficiently large

We can improve the constants Θ and ϑ appearing in the theorem if we take r to be large. In fact, we will see that in this setting, we may take $\Theta = \vartheta = 0.9065$.

We first consider large gaps for sufficiently large r. Starting with (4), to detect large gaps of the desired size, we must show that $h^+(c_r) < r$ for sufficiently large r and $c_r = r + \Theta\sqrt{r}$ with $\Theta > 0$. Choosing $\ell = B\sqrt{r}$, we have

$$h^+(c_r) < c_r - 2B\sqrt{r}\int_0^1 \frac{\sin(\pi r v(1-\delta))}{\pi v}(1-v)^{B^2 r}\,dv + O(1/\log T)$$

for sufficiently large r. Making the change of variable $rv = w$, the above inequality becomes

(11)

$$\begin{aligned} h^+(c_r) &< c_r - 2B\sqrt{r}\int_0^r \frac{\sin(\pi w(1-\delta))}{\pi w}\left(1-\frac{w}{r}\right)^{B^2 r} dw + O(1/\log T) \\ &< c_r - 2B\sqrt{r}\int_0^r \frac{\sin(\pi w(1-\delta))}{\pi w}\exp\left(-B^2 w\right) dw + O(1/\log T) \\ &= c_r - 2B\sqrt{r}\int_0^\infty \frac{\sin(\pi w(1-\delta))}{\pi w}\exp\left(-B^2 w\right) dw - 2B\sqrt{r}E(r) + O(1/\log T), \end{aligned}$$

where

$$E(r) = \int_r^\infty \frac{\sin(\pi w(1-\delta))}{\pi w}\exp\left(-B^2 w\right) dw.$$

Note that as $r \to \infty$, $\sqrt{r}E(r) \to 0$, so for sufficiently large r this term is negligible. Thus we set

$$\Theta = \max_B \left\{ 2B\int_0^\infty \frac{\sin(\pi w)}{\pi w}\exp\left(-B^2 w\right) dw \right\} = \max_B \left\{ \frac{2B}{\pi}\arctan\left(\frac{\pi}{B^2}\right)\right\}.$$

The choice $B = 1.502243$. yields $\Theta = 0.9065$. With δ sufficiently small, T and r sufficiently large, these choices guarantee that $h^+(c_r) < r$.

We now consider small gaps for r sufficiently large. We begin with (10) and let $\ell = B\sqrt{r-\sqrt{r}}$. If we assume $\vartheta < 1$, then $r - \vartheta\sqrt{r} > r - \sqrt{r}$ for all r, and we have

$$h^-(c_r) > c_r + 2B\sqrt{r-\sqrt{r}} \int_0^1 \frac{\sin(\pi(r-\sqrt{r})v(1-\delta))}{\pi v}(1-v)^{B^2(r-\sqrt{r})}\,dv + O(1/\log T).$$

Using the change of variable $(r-\sqrt{r})v = w$, we follow an analogous argument as in the previous subsection and ultimately set

$$\vartheta = \max_B \left\{ 2B\int_0^\infty \frac{\sin(\pi w)}{\pi w}\exp\left(-B^2 w\right)\,dw \right\} = \max_B \left\{ \frac{2B}{\pi}\arctan\left(\frac{\pi}{B^2}\right) \right\}.$$

As before, the choice $B = 1.502243$ yields $\vartheta = 0.9065$. With δ sufficiently small, T and r sufficiently large, these choices guarantee that $h^-(c_r) > r$.

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