Matroids, Eulerian Graphs and Topological Analogues of the Tutte Polynomial

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Declaration

These doctoral studies were conducted under the supervision of Dr Iain Moffat.

The work presented in this thesis is the result of original research I conducted, whilst enrolled in the School of Mathematics and Information Security as a candidate for the degree of Doctor of Philosophy. This work has not been submitted for any other degree or award in any other university or educational establishment.

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Abstract

In this thesis we look at some important properties of graphs embedded into surfaces. The thesis can be divided into three distinct parts.

The first part is a brief introduction to topological graph theory. We finish this section by introducing *coloured ribbon graphs*, $\mathcal{G} = (G_R, \mathcal{P}, \mathcal{Q})$, which are ribbon graphs with partitions on both their vertices and boundary components.

In part two we focus on the relationship between the medial graph and duality. We consider the classical result which relates geometric duality and the medial graph;

$$G_m = H_m \iff G \in \{H, H^*\}. \tag{1}$$

We then look at twisted duality, and include a brief summary of the results by J. A. Ellis-Monaghan and I. Moffat which extend the above result to

$$G_m \cong H_m \iff G \in Orb(H).$$

Where $\cong$ means that two embedded graphs have isomorphic underlying graphs and $Orb(H)$ is the set of all twisted duals of H.

We then consider Hypermaps and introduce the dual, medial and Tait graphs of a hypermap and show that the classic relationship from equation 1 still holds for our new definitions. We define partial duality and partial Petrials for hypermaps. We then combine these operations to define twisted duality for hypermaps, before showing that:

$$G_m \cong H_m \iff G \in Orb(H).$$

Where G_m and H_m in this case are medial graphs of hypermaps and therefore do not have to be 4-regular.

In the final part we provide a brief an introduction to matroids, delta matroids and matroid perspectives before defining delta-matroid perspectives, which are triples of the form (M, D, N) where D is a delta-matroid, M and N are matroids and $(M, D_{\max})$ and $(D_{\min}, N)$ are matroid perspectives. We describe the delta-matroid perspective of a coloured ribbon graph $\mathbf{P}(\mathcal{G}) =$

$(B(\mathcal{G}^*), D(G_R), C(\mathcal{G}))$ and show that

$$\mathbf{P}(\mathcal{G})^* = \mathbf{P}(\mathcal{G}^*),$$

$$\mathbf{P}(\mathcal{G})/e = \mathbf{P}(\mathcal{G}/e)$$

and

$$\mathbf{P}(\mathcal{G})\backslash e = \mathbf{P}(\mathcal{G}\backslash e)$$

We give a brief summary of the Tutte polynomial and describe some of the major existing attempts to extend the polynomial to embedded graphs.

We define the Krushkal Polynomial for Delta-Matroid Perspectives

$$K_{(M,D,M')}(x,y,a,b) := \sum_{A\subseteq E} x^{r'(E)-r'(A)} y^{|A|-r(A)} a^{\rho(A)-r'(A)} b^{r(A)-\rho(A)}$$

and the the Bollobás-Riordan polynomial for DM perspectives

$$R_{(D,M)}(x,y,z) := \sum_{A\subseteq E} x^{r(E)-r(A)} y^{|A|-\rho(A)} z^{\rho(A)-r(A)}.$$

We show that we can recover the Krushkal and Bollobás-Riordan polynomials from these new polynomials:

$$K_{(D_{\max},D,D_{\min})}(x,y,a,b) = b^{\gamma(G)/2} K_G(x,y,a,b^{-1})$$

and

$$R_{(D,D_{\min})}(x,y,z) = R_G(x+1, y, (y^{-1}z)^{\frac{1}{2}}).$$

We then give a full deletion-contraction relationship for each polynomial:

$$\begin{array}{c} K_{(M,D,M')} \\ (x,y,a,b) \end{array} = \begin{cases} K_{(M\backslash e,D\backslash e,M'\backslash e)} + K_{(M/e,D/e,M'/e)} \\ \qquad \text{if } e \text{ is not a loop or a coloop in } M', \\ K_{(M\backslash e,D\backslash e,M'\backslash e)} + yK_{(M/e,D/e,M'/e)} \\ \qquad \text{if } e \text{ is a loop in } M, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)} + aK_{(M/e,D/e,M'/e)} \\ \qquad \text{if } e \text{ is a loop in } M' \text{ and } e \text{ is not a ribbon loop in } D, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)} + (ab)^{\frac{1}{2}}K_{(M/e,D/e,M'/e)} \\ \qquad \text{if } e \text{ is not a loop in } M \text{ and } e \text{ is a non-orientable ribbon loop in } D, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)} + bK_{(M/e,D/e,M'/e)} \\ \qquad \text{if } e \text{ is not a loop in } M \text{ and } e \text{ is an orientable ribbon loop in } D, \\ xK_{(M\backslash e,D\backslash e,M'\backslash e)} + K_{(M/e,D/e,M'/e)} \\ \qquad \text{if } e \text{ is a coloop in } M'. \end{cases}$$

and

$$R_{(D,M)}(x,y,z) = \begin{cases} R_{(D\backslash e, M\backslash e)} + R_{(D/e,M/e)} & \text{if } e \text{ is not a loop or a coloop in } M, \\ R_{(D\backslash e, M\backslash e)} + yR_{(D/e,M/e)} & \text{if } e \text{ is a orientable ribbon loop in } D, \\ R_{(D\backslash e, M\backslash e)} + zR_{(D/e,M/e)} & \text{if } e \text{ is a loop in } M \text{ and } e \text{ is not a ribbon loop in } D, \\ R_{(D\backslash e, M\backslash e)} + (yz)^{\frac{1}{2}} R_{(D/e,M/e)} & \text{if } e \text{ is a non-orientable ribbon loop in } D, \\ xR_{(D\backslash e, M\backslash e)} + R_{(D/e,M/e)} & \text{if } e \text{ is a coloop in } M, \end{cases} \quad (2)$$

as well as a duality and convolution formulae for both new polynomials.

Finally we introduce a Hopf algebra framework and the canonical Tutte polynomial and use this to provide the deletion-contraction, duality and convolution formulas.

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Part I

Introduction

Chapter 1

Introduction

Contents

In this thesis we primarily focus on graphs embedded in surfaces. We start by defining what a graph is, what a surface is and what it means to embed a graph in a surface. We then provide a brief catalogue of various descriptions of cellularly embedded graphs and reviewing the classical constructions of the Petrie dual, $G^{\times}$, geometric dual, G^{*}, and the minors $G \backslash e$ and G/e, of an embedded graph G. The thesis is designed to be self contained but may on occasion assume some basic knowledge of graph theory. In such cases [5] and [8] may prove useful.

1.1 Abstract Graphs

Definition 1.1.1. A *graph* G consists of a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, such that each edge $e \in E(G)$ has an endpoint set associated with it, containing either one or two elements of the vertex set $V(G)$.

If e has only one endpoint we call it a *loop*. We say two vertices x, y are *adjacent* if there exist an edge e in $E(G)$ with endpoints x and y and that x and y are *incident* with the edge e. We say a graph is *even* if every vertex is incident to an even number of edges and a graph is *x-regular* if every vertex is incident to precisely x edges. Two edges are *adjacent* if they have precisely one common endpoint. We call a graph *abstract* if it is not embedded in any surface.

We let $\mathbf{G}$ denote the set of all abstract graphs and $\mathbf{G}_n$ denote the set of all abstract graphs with n vertices. The graph with no edges and n vertices is known as the *null graph* E_n.

We write $v(G)$ for the number of vertices in a graph, G, and $e(G)$ for the number of edges. Where G is clear from context we will often just write v and e.

Definition 1.1.2. A graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$ and each element of E' has the same endpoint set as in E. If $V' = V$ then G' is said to be a *spanning* subgraph of G.

We often want to construct new graphs from existing ones by removing edges the simplest way of doing this is to *delete* edges.

Definition 1.1.3. Let $G = (V, E)$ be a graph and let $A \subseteq E$ then the graph obtained by *deleting* A is defined as $G \backslash A = (V, E \backslash A)$.

The other edge operation we will use throughout this thesis is to *contract* edges.

Definition 1.1.4. Let $G = (V, E)$ be a graph and let $e \in E$ with endpoints x

and y. Then the graph G/e obtained by *contracting* the edge e is formed by identifying the vertices x and y and removing the edge e.

A *walk* on a graph is an alternating sequence of vertices and edges beginning and ending with a vertex, and in which each edge is incident with the vertex immediately preceding it and the vertex immediately following it.

An *(open) path* is a walk where all edges and vertices are distinct. We say that if a path starts at a vertex x and finishes at a vertex y then it is a path from x to y.

A *cycle* is a walk which begins and ends at the same vertex but where all edges and all other vertices are distinct.

A graph is *connected* if for every pair $\{x, y\}$ of distinct vertices there is a path from x to y. Observe that every graph can be partitioned into its *maximal connected subgraphs*, which we call the *components* of the graph. We write $k(G)$ to represent the number of components in a graph G and if $A \subseteq E(G)$ we write $k(A)$ to represent the number of components of the *spanning* subgraph of G with edge set A. An edge is called a *bridge* if removing it will increase the number of components of the graph.

If a graph contains no cycles we call it a *forest* and a *tree* is a connected forest.

We can now define the rank and nullity of a graph.

Definition 1.1.5. Let $G = (V, E)$ be a graph. Then the *rank* of G is defined as $r(G) = v(G) - k(G)$ and the *nullity* of G is defined as $n(G) = e(G) - r(G)$. If $A \subseteq E(G)$ then $r(A) = e(G) - k(A)$ and $n(A) = |A| - r(A)$.

Definition 1.1.6. We say two graphs G and H are *isomorphic*, written $G \cong H$, if there exist bijections

$$\theta : V(G) \to V(H)$$

$$\phi : E(G) \to E(H)$$

such that an edge $e \in E(G)$ has endpoints $v, w \in V(G)$ if and only if the edge

$\phi(e) \in E(H)$ has endpoints $\theta(v), \theta(w) \in V(H)$.

In general we do not distinguish between isomorphic graphs.

1.2 Surfaces

We now briefly review a few basic definitions from surface topology, we refer the reader to [31] and [44] for more detail on the subject.

Definition 1.2.1. A *(topological) surface* is a Hausdorff space in which every point has an open neighbourhood homeomorphic to either the plane $\mathbb{R}^2$ or the upper half-plane $\mathbb{R}^2_+ = \{(x, y) \mid y \geq 0; x, y \in \mathbb{R}\}$, and for which any two distinct points possess disjoint neighbourhoods. A surface is *compact* if it is a compact topological space.

Figure 1.1 shows some common examples of surfaces.

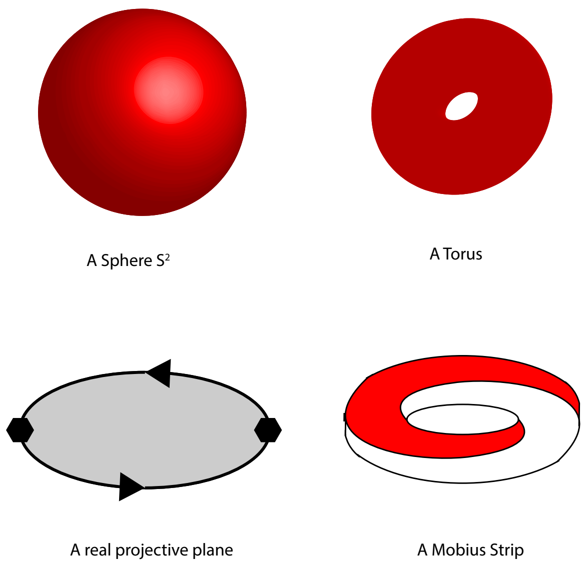


Figure 1.1: Some examples of surfaces

If every point in a surface Σ has an open neighbourhood that is homeomorphic to the plane then we say that Σ is *closed*, for example spaces such as the sphere, the torus and the Klein bottle are all closed. If however, Σ contains points whose neighbourhoods are homeomorphic to the upper half plane then we call Σ a *surface with boundary.*

We call the points with an open neighbourhood that are homeomorphic to $\mathbb{R}^2_+$ the *boundry points* of the surface and the union of these boundary points of Σ is a collection of closed curves. These closed curves are the *boundary components* of Σ.

We can give every point in a surface a *local orientation* by assigning a preferred direction of rotation. We can then generate paths on the surface between points moving the orientation along them. If the orientations at the end points agree we call the path *orientation preserving* otherwise we call them *orientation reversing.* A surface is said to be *orientable* if every closed path is orientation preserving and *non-orientable* if there exists an orientation reversing closed path in the surface.

The connected sum, $\Sigma \# \Sigma'$, of two surfaces Σ and Σ' is obtained by deleting the interior of a disc in each surface and identifying the two boundaries that were created. We can now use the sphere, torus and real projective plane to classify all surfaces.

Theorem 1.2.1. *Let Σ be a connected closed compact surface. Then*

1. *If Σ is orientable, then it is homeomorphic to either a sphere or a connected sum of tori.*
2. *If Σ non-orientable, then it is homeomorphic to a connected sum of real projective planes.*

We can use the above theorem to define the genus, which is an invariant of a closed surface. The genus, $g(\Sigma)$, of a closed surface Σ is defined by

$$g(\Sigma) = \begin{cases} 0 & \text{if } \Sigma \text{ is homeomorphic to the sphere;} \\ n & \text{if } \Sigma \text{ is homeomorphic to the connected sum of } n \text{ tori;} \\ n & \text{if } \Sigma \text{ is homeomorphic to the connected sum of } n \text{ real} \\ & \text{projective planes.} \end{cases}$$

Alternatively the genus of a surface Σ is the number of "handles" or Möbius bands (if the surface is orientable, non-orientable respectively) that must be added to the sphere to obtain a surface which is homeomorphic to Σ. Note in this context Möbius bands are often refered to as *crosscaps.* We can use genus and orientability to completely classify closed surfaces.

Theorem 1.2.2. *Let Σ_1 and Σ_2 be closed connected compact surfaces. Then Σ_1 and Σ_2 are homeomorphic if and only if they are both orientable or both non-orientable and they have the same genus.*

We now describe the classification of surfaces with boundary. Let Σ be a surface with boundary. Each boundary component of Σ is homeomorphic to a closed curve. This means we can obtain a closed surface Σ' from Σ by identifying each boundary component of Σ with the boundary of a (distinct) disc D^2. We say that the closed surface Σ' is obtained from Σ by *capping off the holes.*

The *genus* of a surface with boundary is defined to be the genus of the closed surface obtained by capping off each of the holes.

Theorem 1.2.3. *Let Σ_1 and Σ_2 be connected compact surfaces with boundary, and Σ_1' and Σ_2' be the closed surfaces obtained by capping off the holes. Then Σ_1 and Σ_2 are homeomorphic if and only if they have the same number of boundary components, and Σ_1' and Σ_2' are homeomorphic.*

We can use the number of boundary components along with genus and orientability to provide a complete classification of closed surfaces in a similar way as Theorem 1.2.3

Theorem 1.2.4. *Two connected compact surfaces with boundary are homeomorphic if and only if they have the same number of boundary components,* *the same genus and are either both orientable or both non-orientable.*

1.3 Cellularly Embedded Graphs and Their Representations

The definition of a graph at the start of this chapter is a purely combinatorical object, however in order to manipulate the graph in the way we want we would like to consider a topological representation. To do this we *embed* the graph in a surface. To embed a graph G in a surface Σ we define a mapping i from G to Σ which maps each vertex of G to a distinct point on the surface and maps the edges as continuous paths on the surface between its endpoints in such a way that the edges only meet at the vertices. The graph can then be viewed as a subset of the surface.

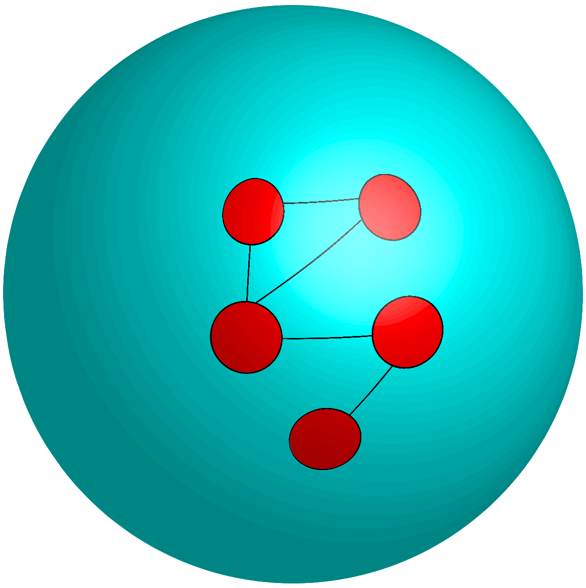


Figure 1.2: A Cellularly Embedded Graph G

Definition 1.3.1. A *cellularly embedded* graph is a graph that has been embedded in a surface so that each connected component of $\Sigma \setminus G$ is homeomorphic to a disc. In this case these components are called the *faces* of G.

Two cellularly embedded graphs $G \subset \Sigma$ and $G' \subset \Sigma'$ are *equivalent*, written $G = G'$ if there is a homeomorphism from Σ to Σ' (which is orientation preserving when Σ is orientable) with the property that when it is restricted to just the subset G of the surface it is an isomorphism of abstract graphs.

There are a number of different ways of representing cellularly embedded graphs and we now define some of the most common.

1.3.1 Band Decomposition

In this thesis we have slightly tweaked the definition of a band decomposition in order to facilitate our discussion of hypergraphs in part II. The band decomposition representation of an embedded graph is a surface divided into three subsets of discs or *bands*, known as the 0-bands, 1-bands and 2-bands and defined as follows.

Definition 1.3.2. A *band decomposition* of a closed surface is a collection of closed discs split into three subsets called the 0-bands, 1-bands and 2-bands such that

1. Different bands only intersect on their boundaries.
2. The union of all the bands is the entire surface.
3. Each subset of discs is pairwise disjoint.

A band decomposition can represent a cellularly embedded graph if certain conditions are met.

Definition 1.3.3. A band decomposition is B of a surface Σ is a *graphical band decomposition* if:

1. The 1-bands are homeomorphic to $I \times I$, where I denotes the unit interval [0,1], where arcs $h(I \times \{j\})$ for $j = 0, 1$ are called the *ends* of the band and the arcs $h(\{j\} \times I)$ for $j = 0, 1$ are called the *sides* of the band.
2. Different bands only intersect on their boundaries.
3. The union of all the bands is the entire surface.

4. The ends of each 1-band are contained in a 0-band.

5. The sides of each 1-band are contained in a 2-band.

Lemma 1.3.1. *Let B be a graphical band decomposition, then B is equivalent to some cellularly embedded graph $G \subset \Sigma$.*

Proof. Let G be a cellularly embedded graph embedded on a surface Σ. We can expand the vertices of G into small discs and the edges (outside of the neighbourhoods of the vertices) into thin bands, which will be the 0-bands and 1-bands respectively. Now since G is a cellularly embedded graph we know that that each connected component of $\Sigma \setminus G$ is homeomorphic to a disc. This means is we remove the edges and vertices of G from Σ we are left with a set of discs, it should be clear therefore that if we remove the 0-bands and 1-bands we will also be left with a set of discs, the 2-bands . These discs correspond to the faces of G hence we have a band decomposition of the surface Σ.

Alternatively if we have a graphical band decomposition of a surface Σ then, by definition, each 1-band has precisely two intersections with 0-bands. So we can obtain a graph by placing a vertex in the centre of each 0-band and drawing edges as follows. For each 1-band draw an edge inside the band from one end to the other then connect the edge to the vertices of the graph which are contained in the 0-bands whose boundaries intersect with the ends of the 1-band. We then remove the bands from the surface and are left with a cellularly embedded graph. □

Two band decompositions are *equivalent* if they represent equivalent cellularly embedded graphs.

1.3.2 Ribbon Graphs

Ribbon graphs are another way of representing cellularly embedded graphs. We will often work in the language of ribbon graphs as a number of the results and proofs are more natural this way.

Definition 1.3.4. A *ribbon graph* $G_R = (V(G_R), E(G_R))$ is a surface with boundary represented as the union of two sets of closed discs, a set $V(G_R)$ of *vertices*, and a set $E(G_R)$ of *edges* such that

1. The vertices and edges intersect in disjoint line segments,
2. Each such line segment lies on the boundary of precisely one vertex and precisely one edge,
3. Every edge contains exactly two such line segments.

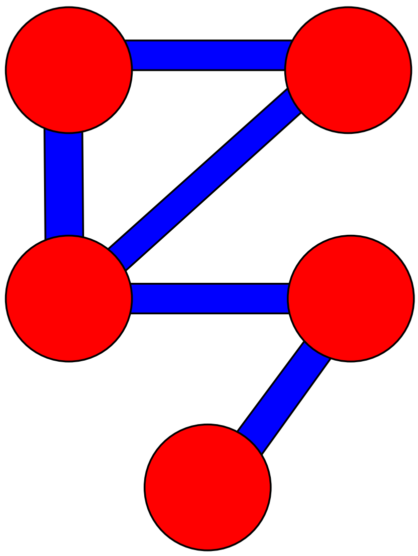


Figure 1.3: The Ribbon Graph G_R of the cellularly embedded graph G shown in Figure 1.2

Every ribbon graph G_R has an associated *underlying abstract graph* $G = (V(G), E(G))$, where $V(G) = V(G_R)$ and $E(G) = E(G_R)$ such that the end-points of $e \in E(G)$ correspond to vertices which intersect with the equivalent edge in G_R.

We can easily move between ribbon graphs and band decomposition and therefore between ribbon graphs and cellularly embedded graphs. If we have a band decomposition, we can obtain a ribbon graph by simply deleting the 2-bands. Then the 0-bands are the vertices and the 1-bands the edges. Conversely if we have a ribbon graph then it is a surface with boundary so by capping of the holes we have a band decomposition.

Two ribbon graphs are *equivalent* if they define equivalent cellularly embedded graphs.

Ribbon graphs have many of the same standard parameters as abstract graphs as well as a few new ones. If G_R is a ribbon graph then $v(G_R) = |V(G_R)|$, $e(G_R) = |E(G_R)|$, $k(G_R)$ is equal to the number of connected components of G_R and $f(G_R)$ is the number of boundary components of the surface defining the ribbon graph. Note that if G_R is connected, then if it is translated to a cellularly embedded graph $f(G_R)$ is the number of faces of the graph.

The rank and nullity of a ribbon graph are calculated in the same way as an abstract graph, that is the rank $r(G_R) = v(G_R) - k(G_R)$ and the nullity $n(G_R) = e(G_R) - r(G_R)$. For example the graph in Figure 1.3 has rank 4 and nullity 1.

If $A \subseteq E(R_G)$ then we define the *spanning ribbon subgraph* of G_R with regards to A as the ribbon graph $(V(G_R), A)$. Throughout this thesis we will write $r(A), k(A), n(A)$ and $f(A)$ to mean the rank, nullity and so forth of the spanning subgraph $(V(G_R), A)$ of G_R. If G_R is not clear from context then we will write r_{G_R} and so on to make it clear.

As is the case in abstract graphs an edge e in a ribbon graph is said to be a *bridge* if removing it increases the number of connected components and a *loop* if it is incident to precisely one vertex. However for ribbon graphs we can further categorise loops. We say that a loop is *non-orientable* if the ribbon subgraph consisting of the loop and its incident edge is homeomorphic to a Möbius band. Otherwise it is orientable.

Two cycles C_1 and C_2 in G_R are said to be *interlaced* if there is a vertex v such that $V(C_1) \cap V(C_2) = \{v\}$ and C_1 and C_2 are met in the cyclic order $C_1C_2C_1C_2$ when travelling round the boundary of the vertex v. A loop is *non-trivial* if it is interlaced with some cycle in G_R. Otherwise the loop is *trivial*.

One advantage of ribbon graphs compared with cellularly embedded graphs, is that removing an edge from a ribbon graph results in another ribbon graph which is not necessarily the case for cellularly embedded graphs. For example if we delete a bridge of a connected ribbon graph the resulting graph is still a ribbon graph it is just no longer connected, however if we deleted the same edge from the cellularly embedded graph, the new graph would no longer be cellularly embedded.

1.3.2.1 Quasi-trees

Quasi-trees are the ribbon graph equivalent of trees and are defined as follows:

Definition 1.3.5. A *quasi-tree* Q is a ribbon graph with exactly the same number of boundary components as connected components. That is every connected component of Q has precisely one boundary component. If G_R is a ribbon graph, a *spanning quasi-tree* Q of G_R is a spanning ribbon subgraph with exactly the same number of boundary components as connected components, it is a ribbon subgraph with $V(G_R) = V(Q)$ and $f(Q) = k(G_R)$.

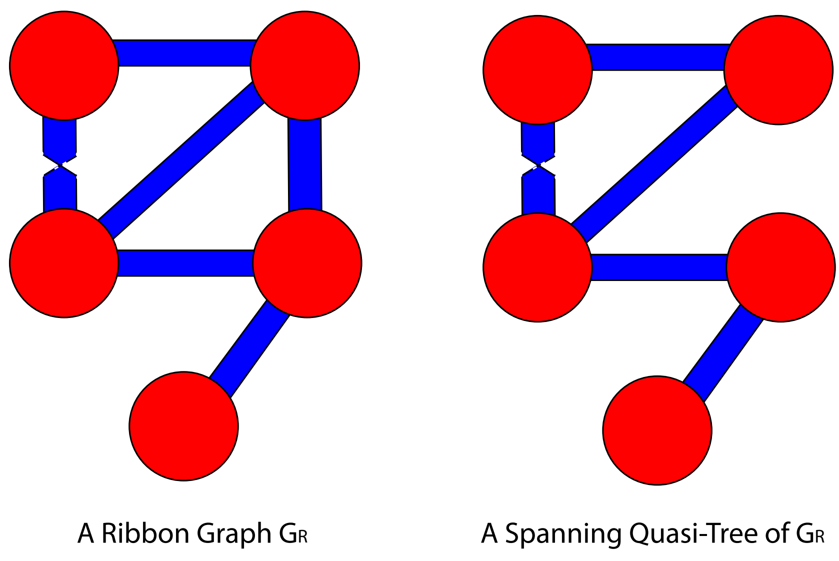


Figure 1.4: A Ribbon Graph G_R and one of its Quasi-Trees

1.3.3 Arrow Presentation

One of the most important representations of embedded graphs for the purposes of this paper is the arrow presentation.

Definition 1.3.6. An *arrow presentation* consists of a set of closed curves with pairs of disjoint labelled arrows on them such that there are exactly two arrows with each label.

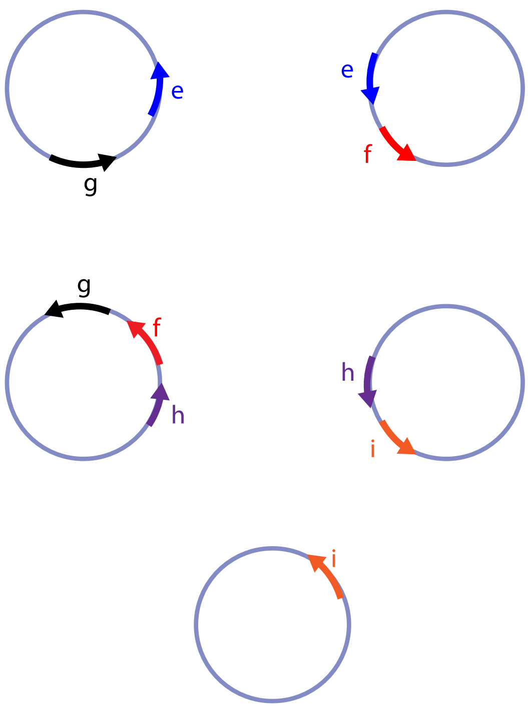


Figure 1.5: The Arrow Presentation of the cellularly embedded graph G shown in Figure 1.2

Arrow presentations are equivalent to ribbon graphs. Let G_R be a ribbon graph. We can obtain an arrow presentation of it as follows; for each edge e of the ribbon graph draw arrows on the intersection of the edges and vertices, such that you can move from the tip of one arrow to the tail of the next by following the boundary of the edge. Label these arrows e. Now delete the edges of the ribbon graph and replace the vertex disc with closed curves which follow the boundary of the discs, keeping the arrows. We now have a set of closed curves with pairs of labelled arrows which is an arrow presentation.

Conversely if we have an arrow presentation we can obtain a ribbon graph.

We start by "filling in" the closed curves to create a disc, these discs are the vertices of the ribbon graph. Then for each pair of labelled arrows e drawing line segments from the head of one arrow to the tail of the other. Then we have a closed curve consisting of the two arrows and two line segments, fill in this closed curve and remove the arrows to create a disc. This disc is the edge e in the ribbon graph. and the closed curves of the arrow presentation become its vertices.

1.3.4 Permutational τ -model

We can also represent an embedded graph G in a purely combinatorial way using the *Permutational τ-model*. This model consists of a set X and three fixed point free involutions τ_0, τ_1 and τ_2 which act on X. When the model is used to represent a graph G we refer to the elements of X as *local flags* of G. A local flag is a triple (v, e, f) consisting of a vertex v, the intersection e of an edge incident to v with a small neighbourhood of v and the intersection f of a face adjacent to v and e with the same neighbourhood of v.

Given any three fixed-point free involution τ_0, τ_1 and τ_2 on a set X we can obtain a graph provided that all the orbits of $\langle \tau_0, \tau_2 \rangle$ contain four elements, this is the case if and only if $\tau_0 \tau_2$ is also an involution. Its vertices correspond to the orbits of $\langle \tau_1, \tau_2 \rangle$, its edges the orbits of $\langle \tau_0, \tau_2 \rangle$ and its faces to the orbits of $\langle \tau_0, \tau_1 \rangle$.

The τ-model is equivalent to an arrow presentation. Given a set X and three three fixed-point free involutions τ_0, τ_1 and τ_2 on X such that $\tau_0 \tau_2$ is also an involution we can obtain an arrow presentation as follows:

1. start by for each orbit of the subgroup $\langle \tau_1, \tau_2 \rangle$ create a closed curve;
2. for each orbit O of $\langle \tau_1, \tau_2 \rangle$ let X_O be the subset of X containing the elements of the orbit O;
3. select an element x of X_O and place it on the closed curve corresponding

to O;

4. now move a short distance anticlockwise (or clockwise if you prefer) along the curve and place the point $y = \tau_2(x)$ on the curve;

5. now move a short distance further around the curve and place the point $z = \tau_1(y)$;

6. repeat this process, alternating between τ_2 and τ_1 until each element of X_O has been placed;

7. repeat this process for each orbit of $\langle \tau_1, \tau_2 \rangle$;

8. we now have a set of closed curves with each point of X place on them;

9. pick a point $a \in X$ and draw an arrow from a to $b = \tau_2(a)$;

10. now let $c = \tau_0(b)$ and draw an arrow from c to $d = \tau_2(c)$, label these two arrows e_1 (note a, b, c, d are the elements of an orbit of $\langle \tau_0, \tau_2 \rangle$);

11. repeat this process until all points are part of an arrow and then remove the points;

12. we now have a set of closed curves with pairs of labelled arrows which is an arrow presentation.

We can also quickly generate the τ-model from an arrow presentation as follows. Given an arrow presentation of a graph we label a point at the head and tail of each arrow then we form τ_0 by pairing the point at the head of one arrow with the tail of the arrow which it was paired with and vice-versa, we form τ_1 by starting at a labelled point and moving around the closed curve away from the arrow until we reach another labelled point and we form τ_2 by simply pairing both ends of the same arrow.

Since the τ-model is equivalent to an arrow presentation it is also equivalent to a band decomposition and a ribbon graph.

For example in Figure 1.6

$$\tau_0 = (1,9)(2,10)(3,6)(4,5)(7,11)(8,12)(13,15)(14,16)(17,19)(18,20),$$

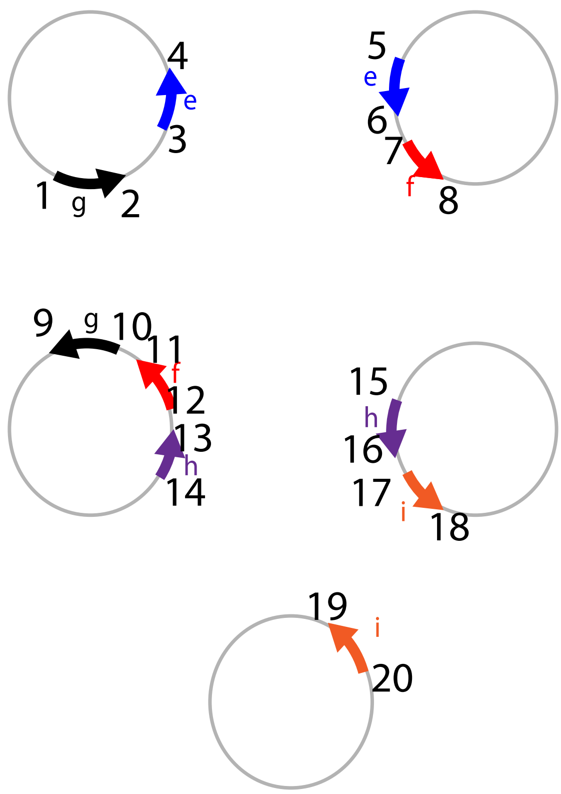


Figure 1.6: The Arrow Presentation of the cellularly embedded graph G shown in Figure 1.2 with the head and tail of each arrow labeled

$$\tau_1 = (1,4)(2,3)(5,8)(6,7)(9,14)(10,11)(12,13)(15,18)(16,17)(19,20)$$

and

$$\tau_2 = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20).$$

Now observe that the orbits of $\langle \tau_0, \tau_1 \rangle$ are $\{1,4,5,8,9,12,13,14,15,16,17,18,19,20\}$ and $\{2,3,6,7,10,11\}$ which form the faces of G, the orbits of $\langle \tau_0, \tau_2 \rangle$ are $\{1,2,9,10\}$, $\{3,4,5,6\}$ $\{7,8,11,12\}$, $\{13,14,15,16\}$ and $\{17,18,19,20\}$ which form the edges of G and the orbits of $\langle \tau_1, \tau_2 \rangle$ are $\{1,2,3,4\}$, $\{5,6,7,8\}$, $\{9,10,11,12,13,14\}$, $\{15,16,17,18\}$ and $\{19,20\}$ which form the vertices of G.

1.4 Orientability and the Genus of Cellularly Embedded Graphs

A cellularly embedded graph $G \subset \Sigma$ is said to be *orientable* if each component of Σ is orientable; otherwise G is said to be *non-orientable.* If G is connected, then the *genus*, $g(G)$, of G is the genus of Σ, otherwise its genus is the sum of the genera of its components. A cellularly embedded graph $G \subset \Sigma$ is a *plane graph* if Σ is the 2-sphere, S^2.

For a ribbon graph G_R we say G_R is *orientable* if it is orientable when viewed as surface with boundary. Similarly the genus, $g(G_R)$, of a ribbon graph G_R is its genus when viewed as a surface with boundary. We define an orientability parameter t for ribbon graphs by setting $t(G_R) = 1$ if G_R is non-orientable, and $t(G_R) = 0$ if G_R is orientable.

The *Euler genus*, $\gamma(G_R)$, of G_R equals the genus of G_R if G_R is non-orientable and is twice the genus otherwise. We say that a ribbon graph G_R is plane if $\gamma(G_R) = 0$.

The *Euler characteristic* χ is a topological invariant, that was originally discovered by Euler as the constant in a formula relating the number of vertices, edges and faces of a polyhedron before being generalised by among others Lhuilier, who showed it for all closed orientable surfaces, [42]. It is defined for a ribbon graph G_R as

$$\chi(G_R) = v(G_R) - e(G_R) + f(G_F).$$

It has since been shown that the Euler genus and the Euler characteristic are related as follows

$$\chi(G_R) = 2k(G_R) - \gamma(G_R).$$

Therefore we have the following formula for γ.

$$\gamma(G_R) = 2k(G_R) + e(G_R) - v(G_R) - f(G_R). \tag{1.1}$$

If $A \subseteq E(R_G)$ then we write $t(A), g(A), \gamma(A)$ and $\chi(A)$ to be the parameters for of the spanning subgraph $(V(G_R), A)$ of G_R.

1.5 Types of Equivalence of Embedded Graphs

In Section 1.3 we described when two cellularly embedded graphs are equivalent. However there are weaker forms of equivalence.

Definition 1.5.1. We say two embedded graphs G and H are *equivalent as abstract graphs* if their underlying abstract graphs are isomorphic and write

$$G \cong H.$$

It should be clear that equivalence of abstract graphs is much more general than equivalence of embedded graphs and that two graphs which are equivalent as abstract graphs can have very different embeddings.

There is also a third form of equivalence which we will use in this thesis. First we need to introduce cyclically ordered graphs.

Definition 1.5.2. A *cyclically ordered graph*, or *cog*, consists of an abstract graph (referred to as the underlying abstract graph) together with a cyclic ordering of the half-edges about each vertex.

We define equivalences for cogs as follows

Definition 1.5.3. We say that two cogs G and H are *equivalent*, if there is an equivalence of the underlying abstract graphs that preserves or reverses the cyclic orders at each vertex.

That is we consider cogs up to *vertex reversals*, which are the reversals of the cyclic order of the edges about some vertices.

Every embedded graph will have an underlying cog defined as follows

Definition 1.5.4. If G is an embedded graph, then its *underlying cog* is the cog that results from the underlying abstract graph of G together with a cyclic order at each vertex induced by a local orientation at that vertex.

Therefore we can define another form of equivalence for embedded graphs.

Definition 1.5.5. We say that two embedded graphs G and H are *equivalent as cyclically ordered graphs*, and write $G \doteq H$, if G and H have equivalent underlying cogs.

Clearly if two embedded graphs are equivalent then they will be equivalent as cogs and if they are equivalent as cogs then they will be equivalent as abstract graphs and so we have a hierarchy of equivalences relating embedded graphs

$$G = H \Longrightarrow G \doteq H \Longrightarrow G \cong H.$$

1.6 Geometric Duality

Geometric duality is a key construction in graph theory and will play a major role in this thesis. Traditionally it is defined as the graph obtained by placing a vertex in each face of G and then embedding an edge between two vertices of G^* if and only if the faces they correspond to are adjacent in G. However since we will be primarily working with ribbon graphs it makes sense to define the dual as follows.

Definition 1.6.1. Topologically a ribbon graph is a surface with boundary. We can cap off the holes using a set of discs, denoted by $V(G_R^*)$, to obtain a surface without boundary. Then *the geometric dual* of G_R is the ribbon graph $G_R^* = (V(G_R^*), E(G_R))$.

Observe that G_R and G_R^* have the same edge set and that each vertex in G_R corresponds to a boundary component in G_R^* and vice versa. An example of a ribbon graph with its dual is shown in Figure 1.7.

Often when dealing with duality it may be beneficial to consider the band decomposition representation since in this case to form its dual we just relabel the 0-bands as 2-bands and the 2-bands as 0-bands. The following proposition follows directly from this definition

Proposition 1.6.1. *Let G be an cellularly embedded graph then*

$$(G^*)^* = G,$$

$$\gamma(G^*) = \gamma(G).$$

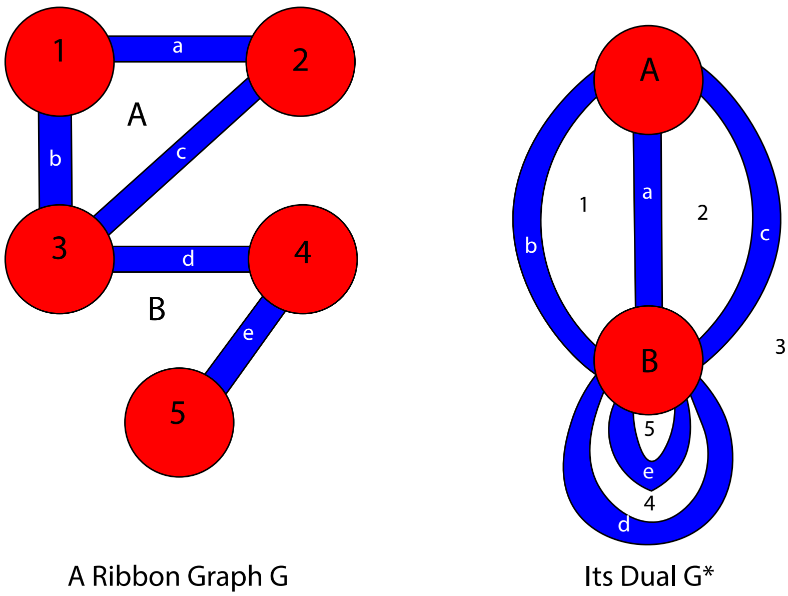


Figure 1.7: A Ribbon Graph and its Dual

1.7 Deletion and Contraction in Ribbon Graphs

Definition 1.7.1. Let G_R be a ribbon graph and let e be an edge of G_R. Then the graph $G_R \backslash e$, obtained by *deleting* e is simply the graph G_R with the edge e removed.

Contracting an edge traditionally means removing the edge and merging the two vertices that formed its endpoints. However with ribbon graphs the process is slightly more complicated.

Definition 1.7.2. Let G_R be a ribbon graph and e be an edge in G_R with endpoints u and v. Now the ribbon graph G_R/e formed by *contracting* e

is obtained as follows: consider the boundary component(s) of $e \cup u \cup v$ as curves on G_R. For each resulting curve, attach a disc to G_R by identifying its boundary component with the curve. These new discs will form vertices in G_R/e. Finally delete e, u and v to obtain G_R/e.

Observe, that if e is not a loop then contracting e works in the traditional manner. However if e is a non-orientable loop then contracting e will result in twisting the adjacent vertex whereas if e is an orientable ribbon loop contracting e results in two new vertices replacing the old one. This is demonstrated in Figure 1.8.

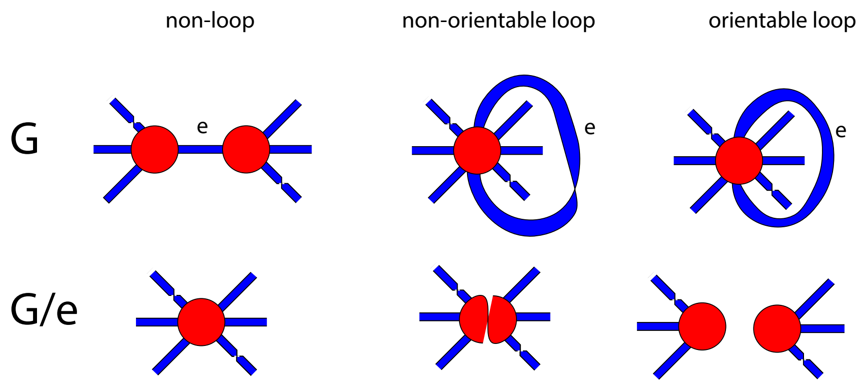


Figure 1.8: Contracting an edge of a Ribbon Graph

Later in this thesis we will discuss the concept of partial duality which provides a way of contracting embedded graphs using the arrow presentation.

Lemma 1.7.1. *Let G be a cellularly embedded graph and let e be an edge of G. Then*

$$(G \backslash e)^* = G^*/e,$$

and

$$(G/e)^* = G^* \backslash e.$$

1.8 Petrials of Embedded Graphs

As well as the geometric dual there is a second operation on embedded graphs that will be important later on, that of forming the Petrie dual. The Petrie dual or Petrial is the graph formed by giving each edge a half twist.

Definition 1.8.1. The *Petrial* of a ribbon graph G, denoted $G^{\times}$, is formed by detaching one end of each edge from its incident vertex disc, giving the edge a half-twist, and reattaching it to the vertex disc.

In the case of the arrow presentation of a graph we can form the Petrial by reversing the direction of one arrow for each label.

1.9 Coloured Ribbon Graphs

Definition 1.9.1. A *vertex partitioned ribbon graph* $(G_R, \mathcal{P})$ consists of a ribbon graph $G_R = (V, E)$ and a partition $\mathcal{P}$ of its vertex set V.

Every vertex partitioned ribbon graph $(G_R, \mathcal{P})$ has three graphs associated with it. The original ribbon graph $G_R = (V(G_R), E(G_R))$, its underlying abstract graph G and the abstract graph $G_{/\mathcal{P}}$, which we will call *the partition graph.*

Definition 1.9.2. Let $(G_R, \mathcal{P})$ be a vertex partitioned ribbon graph then *the partition graph* $G_{/\mathcal{P}}$ is defined as the abstract graph obtained by identifying all the elements of a part of $\mathcal{P}$ as a single vertex.

Unless it is clear from context we will write $r_{\mathcal{P}}, k_{\mathcal{P}}$ and so forth to mean the rank function, number of components etcetera of $G_{/\mathcal{P}}$.

Note that the edge sets of the three graphs are the same as any edge between two vertices in the same part of $\mathcal{P}$ will be a loop in $G_{/\mathcal{P}}$.

We can also define the operations of deletion and contraction for vertex partitioned ribbon graphs as follows.

Definition 1.9.3. Let $(G_R, \mathcal{P})$ be a vertex partitioned ribbon graph and $e \in E(G_R)$, then *deletion* is defined by $(G_R, \mathcal{P})\backslash e := ((G_R)\backslash e, P)$. *Contraction* is defined by $(G_R, \mathcal{P})/e := ((G_R)/e, \mathcal{P}/e)$, where the partition $\mathcal{P}/e$ is induced by

$\mathcal{P}$ as follows. Suppose $e = (u, v)$ and $P_u, P_v \in \mathcal{P}$ are the parts containing u and v respectively (u may equal v and the parts need not be distinct). Then $\mathcal{P}/e$ is obtained from $\mathcal{P}$ by removing parts P_u and P_v, and replacing them with the part $(P_u \cup P_v)\backslash\{u, v\} \cup W$ where W is the set of vertices created by the contraction (so W consists of one or two vertices).

Definition 1.9.4. A *face partitioned ribbon graph* $(G_R, \mathcal{Q})$ consists of a ribbon graph $G_R = (V, E)$ and a partition $\mathcal{Q}$ of its boundary components F.

We then define deletion and contraction in a similar way to how we defined them for a vertex partitioned graph except that we observe that contracting an edge does not change the boundary components of a ribbon graph whereas deleting an edge may increase or decrease the number of boundary components by one.

Definition 1.9.5. Let $(G_R, \mathcal{Q})$ be a face partitioned ribbon graph and $e \in E(G_R)$, then *contraction* is defined by $(G_R, \mathcal{Q})/e := ((G_R/e, \mathcal{Q})$. Deletion is defined by $(G_R, \mathcal{Q})\backslash e := ((G_R)\backslash e, \mathcal{Q}\backslash e)$, where the partition $\mathcal{Q}\backslash e$ is induced by $\mathcal{Q}$ as follows. Suppose u and v are the boundary components of e and $Q_u, Q_v \in \mathcal{Q}$ are the parts containing u and v respectively (u may equal v and the parts need not be distinct). Then $\mathcal{Q}\backslash e$ is obtained from $\mathcal{Q}$ by removing parts Q_u and Q_v, and replacing them with the part $(Q_u \cup Q_v)\backslash\{u, v\} \cup W$ where W is the set of boundary components created by the deletion (so W consists of one or two boundary components).

Definition 1.9.6. A *coloured ribbon graph* $\mathcal{G} = (G_R, \mathcal{P}, \mathcal{Q})$ consists of a ribbon graph $G_R = (V, E)$ a partition $\mathcal{P}$ of its vertex set V and a partition $\mathcal{Q}$ on its boundary components F.

We define deletion and contraction for a coloured ribbon graph as follows.

Definition 1.9.7. Let $\mathcal{G} = (G_R, \mathcal{P}, \mathcal{Q})$ and e be an edge of G_R then we define *deletion* for coloured ribbon graphs as

$$\mathcal{G}\backslash e = (G_R\backslash e, \mathcal{P}, \mathcal{Q}\backslash e)$$

and *contraction* for coloured ribbon graphs as

$$\mathcal{G}/e = (G_R/e, \mathcal{P}/e, \mathcal{Q})$$

Recall that when we take the dual of a ribbon graph G_R each vertex in G_R corresponds to a boundary component in G_R^* and vice versa. This induces a simple way to define the dual of a coloured ribbon graph.

Definition 1.9.8. Given a coloured ribbon graph, $\mathcal{G} = (G_R, \mathcal{P}, \mathcal{Q})$ we define its *dual* $\mathcal{G}^*$ as follows;

$$\mathcal{G}^* = (G_R^*, \mathcal{P}^*, \mathcal{Q}^*)$$

where G_R^* is the dual of the ribbon graph G_R, $\mathcal{P}^*$ is a partition of the vertices of G_R^* such that two vertices are in the same part of $\mathcal{P}^*$ if and only if they correspond to boundary components of G_R which are in the same part of $\mathcal{Q}$ and $\mathcal{Q}^*$ is a partition of the boundary components of G_R^* such that two boundary components are in the same part of $\mathcal{Q}^*$ if and only if they correspond to vertices of G which are in the same part of $\mathcal{P}$.

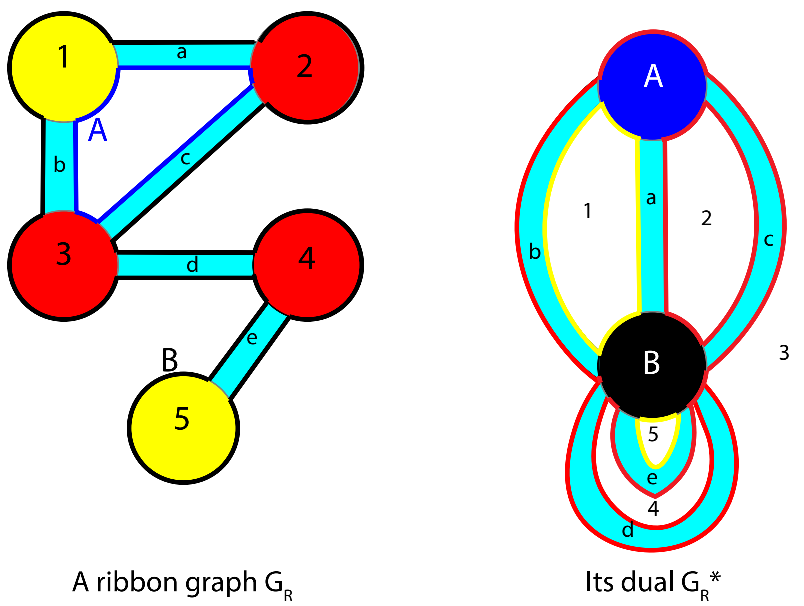


Figure 1.9: A coloured Ribbon Graph with its Dual

If we view G_R as a band decomposition then to get the dual of $(G_R, \mathcal{P}, \mathcal{Q})$ we simple relabel the 0-bands as 2-bands, the 2-bands as 0-bands, the partition P as Q^* and the partition Q as P^*.

For example Figure 1.9 show a coloured ribbon graph $(G_R, \mathcal{P}, \mathcal{Q})$ where $\mathcal{P} = \{\{1,5\},\{2,3,4\}\}$ and $\mathcal{Q} = \{\{A\},\{B\}\}$ and its dual $(G_R^*, \mathcal{P}^*, \mathcal{Q}^*)$ where $\mathcal{P}^* = \{\{A\},\{B\}\}$ and $\mathcal{Q}^* = \{\{1,5\},\{2,3,4\}\}$.

Theorem 1.9.1. *Let $\mathcal{G} = (G_R, \mathcal{P}, \mathcal{Q})$ be a coloured ribbon graph then*

$$(\mathcal{G}/e)^* = (\mathcal{G}^*)\backslash e$$

and

$$(\mathcal{G}\backslash e)^* = (\mathcal{G}^*)/e$$

Proof. We have

$$(\mathcal{G}/e)^* = (G_R/e, \mathcal{P}/e, \mathcal{Q}/e)^* = ((G_R/e)^*, (\mathcal{P}/e)^*, (\mathcal{Q}/e)^*),$$

and

$$(\mathcal{G}^*)\backslash e = ((G_R)^*, \mathcal{P}^*, \mathcal{Q}^*)\backslash e = (((G_R)^*)\backslash e, (\mathcal{P}^*)\backslash e, (\mathcal{Q}^*)\backslash e).$$

Now we know since G_R is a ribbon graph that $(G_R/e)^* = ((G_R)^*)\backslash e$. We also know that $\mathcal{Q}/e = \mathcal{Q}$ hence $(\mathcal{Q}/e)^* = \mathcal{Q}^*$. Now if we consider G_R as a band decomposition $\mathcal{Q}^*$ is a partition on the 2-bands of G_R^* hence since G_R^* is just G_R with the 0-bands relabelled as 2-bands and vice versa then $\mathcal{Q}^*$ is also a partition on the vertices of G_R and in fact by definition is equivalent to $\mathcal{P}$ hence

$$(\mathcal{Q}^*)\backslash e = (\mathcal{P})\backslash e = \mathcal{P} = \mathcal{Q}^*$$

hence

$$(\mathcal{Q}/e)^* = (\mathcal{Q}^*)\backslash e.$$

Similarly, $\mathcal{P}^*$ is equal to$\mathcal{Q}$ and as the action of contraction in a partition of vertices is the same as the action of deletion in faces we have $(\mathcal{P}/e)^* = (\mathcal{P}^*)\backslash e$ and therefore we have

$$(\mathcal{G}/e)^* = (\mathcal{G}^*)\backslash e.$$

The second statement follows from the first since

$$(\mathcal{G}\backslash e)^* = (\mathcal{G}^{**}\backslash e)^* = ((\mathcal{G}^*/e)^*)^* = (\mathcal{G}^*)/e.$$

□

1.10 Graphs in Pseudo-Surfaces

Definition 1.10.1. A *pinch point* is a type of singular point on an algebraic surface, that arises by forming the topological quotient space Σ/P. Where Σ is a closed surface and P is a set of closed curves in Σ. A *pseudo-surface* is a surface with pinch points.

Definition 1.10.2. A *graph in a pseudo-surface*, $G \subseteq \Sigma$, consists of a graph $G = (V, E)$ and an embedding of G on a pseudo-surface Σ such that the edges only intersect at their ends and such that any pinch points are vertices of the graph.

Given a vertex partitioned graph in a surface we can form a graph in a pseudo-surface as follows. If we identify each vertex in a given part to a single point we will form a pinch point in the surface and these pinch points form the vertices of a new graph which is now embedded in a pseudo-surface.

We can also form a vertex partitioned graph from a graph in a pseudo-surface. To do for each pinch point we delete the vertex and its surrounding neighbourhood. This will create a number of boundaries so we then cap these boundaries off and place a new vertex in each cap connecting any edges which were adjacent to the boundary to the new vertex. This gives us a graph embedded in a surface. We then attach a vertex partition by placing vertices that were created from a particular pinch point into the same part.

Part II

Hypermaps, Medial Graphs and Duality

Chapter 2

Medial Graphs and Twisted Duality

Contents

In order to motivate the next section of this thesis we will first introduce the concept of the medial graph, discuss some classical results, including the relationship between the medial graph and geometric dual of a plane graph. We then briefly recap some of the latest developments on generalised dualities for graphs on surfaces, focusing on twisted duality and how this is used to extend the classical relations between a plane graph, its plane dual and its medial graph, to graphs embedded in an arbitrary surface.

2.1 The Medial Graph

We start by defining the medial graph.

Definition 2.1.1. Given a cellularly embedded graph G we construct its *medial graph* G_m by for each edge e of G placing a vertex of degree 4 on e and then drawing the edges of the medial graph by following the face boundaries of G.

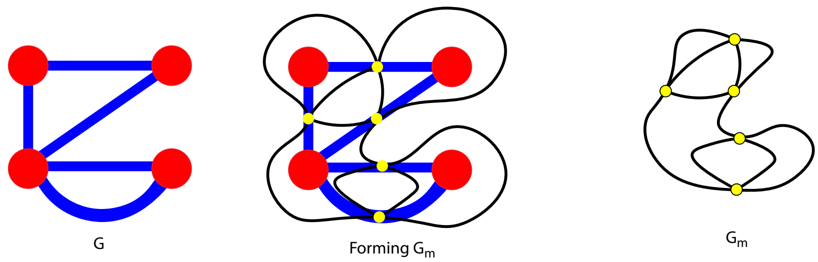


Figure 2.1: Forming the medial graph of a cellularly embedded graph in S^2

Although it is possible to construct the medial graph of a band decomposition or arrow presentation directly it is often simpler to translate the graph into a cellularly embedded graph and then translate back.

Given an embedded 4-regular graph F we may want to know if it is the medial graph of some embedded graph G and if so what are all the embedded graphs G with embedded medial graphs equal to F. To answer this question we need a few more definitions.

Definition 2.1.2. A *checkerboard colouring* of a cellularly embedded graph is an assignment of the colour black or white to each face such that adjacent faces receive different colours. We say that a graph with a checkerboard colouring is *checkerboard coloured.*

This leads to the question of which embedded graphs can be coloured in such a way. Fortunately this is easily answered for plane graphs.

Theorem 2.1.1. *Let F be a plane graph. Then F is checkerboard colourable if and only if F is even.*

Definition 2.1.3. Let F be a checkerboard coloured 4-regular cellularly embedded graph then

1. The *blackface graph* F_{bl} of F is the embedded graph constructed by placing a vertex in each black face then embedding an edge between two of these vertices whenever the corresponding regions meet at a vertex of F.

2. The *whiteface graph* F_{wh} of F is constructed analogously by placing vertices in the white faces.

Every 4-regular graph has two possible colourings and hence two possible blackface graphs and two possible white face graphs. Traditionally we would call a graph canonically checkerboard coloured by considering graphs on the plane and insisting that the unbounded region be white. However since in this thesis we are working in closed surfaces there is no unbounded region and therefore we cannot distinguish the whiteface and blackface graphs of an uncoloured graph and hence we refer to them as a pair as the *Tait graphs*.

Definition 2.1.4. Let F be a checkerboard colourable embedded graph. Then the two *Tait graphs* are the embedded graphs obtained by checkerboard colouring F and forming the whiteface and blackface graphs.

Note when forming the Tait graphs of a disconnected graph, consider each component separately. We can now introduce the two well known results that inspired part of this thesis. We include the proofs as we use similar methods in Chapter 3.

Theorem 2.1.2. *Let F be a 4-regular embedded graph. Then F is the medial graph of some embedded graph G if and only if F is checkerboard colourable. If F is checkerboard colourable, then F is the embedded medial graph of precisely F_{bl} and F_{wh}.*

Proof. First let $F = G_m \subset \Sigma$. In the construction of G_m from F each vertex of G_m lies on a unique edge of F and the edges of G_m follow the face boundaries of G. We can obtain a checkerboard colouring of G_m by colouring each face of G_m that contains a vertex of G black and by colouring each face of G_m that does not contains a vertex of G white or vice versa. These are all of the checkerboard colourings of G_m since an embedded graph can have at most two checkerboard colourings. Conversely, suppose that F is checkerboard colourable. Give F a checkerboard colouring. Then we can form F_{bl} and F_{wh}. By comparing the constructions of medial graphs and Tait graphs, it is easily seen that $(F_{bl})_m = F$ and$(F_{wh})_m = F$, as required.

If the faces of G_m that contain a vertex of G are coloured black, it is readily seen that $G = (G_m)_{bl} = F_{bl}$; and if the faces of G_m that do not contain a vertex of F are coloured black, it is readily seen that $G = (G_m)_{wh} = F_{wh}$. □

Theorem 2.1.3. *Let H be an embedded graph and F be a 4-regular checkerboard coloured embedded graph. Then*

1. $\{F_{bl}, F_{wh}\} = \{H \mid H_m = F\}$, *or equivalently,* $H_m = F \iff H = F_{bl}$ *or* $H = F_{wh}$*;*
2. $(F_{bl})_m = (F_{wh})_m = F$*;*
3. $\{(H_m)_{bl}, (H_m)_{wh}\} = \{H, H^*\}$*;*
4. $F_{bl} = (F_{wh})^*$*;*
5. $\{H, H^*\} = \{G \mid G_m = H_m\}$, *or equivalently,* $G_m = H_m \iff G \in \{H, H^*\}$.

Proof. The first item follows directly from Theorem 2.1.2 and the second item follows from the proof of Theorem 2.1.2.

For the third item if we chose the colouring where the faces of H_m that contain a vertex of H are coloured black then we know that $(H_m)_{bl} = H$ and the white faces of H_m correspond to the faces of H, hence $(H_m)_{wh}$ has vertices which correspond to the faces of H which are the vertices of H^* and the edges of $(H_m)_{wh}$ correspond to the vertices of H_m which correspond to the edges of H and therefore to the edges of H^*. Hence $(H_m)_{wh} = H^*$.

For the fourth item, suppose that $F_{bl} = H$. By Item 2, we have$(F_{bl})_m = H_m = F$. Thus $F_{bl} = (H_m)_{bl}$ and$F_{wh} = (H_m)_{wh}$. By Item 3 one of these two graphs must be H and the other must be H^*, and the result follows upon observing that $(H^*)^* = H$.

Finally, for the fifth item, Item 1 gives that $G_m = H_m$ if and only if $G = (H_m)_{bl}$ or $G = (H_m)_{wh}$. By Item 3 this happens if and only if $G = H$ or $G = H^*$. □

2.2 Vertex States and Graph States

Another construction that will be important later in the paper is that of vertex states.

Definition 2.2.1. Let v be a vertex in a 4-regular graph then we can form a *vertex state* of v by partitioning into pairs the half edges incident to v, then replacing each pair of edges with a single edge which bypasses v and then deleting v.

For example if the 4 edges incident to v are $(u, v), (x, v)$ and a loop (v, v) (which would therefore be incident with v twice) and we partition them so that (u, v) is paired with (x, v) and (v, v) is paired with itself, then the edges (u, v) and (x, v) are replaced with a single edge (u, x) and the loop (v, v) becomes a closed curve. In a 4-regular graph there are 3 possible pairings which are shown in Fig.2.2

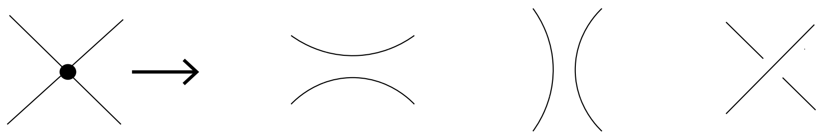


Figure 2.2: The three vertex states of a vertex v of an abstract graph

For an abstract graph there is no way to distinguish between the different vertex states, however if G is a checkerboard coloured 4-regular graph, then we can use the colouring to differentiate the different vertex states, as in Fig. 2.3. We call the three vertex states a white smoothing, a black smoothing, and a crossing as defined in the figure.

Definition 2.2.2. A graph state s of any 4-regular graph F is a choice of vertex state at each of its vertices.

Observe that since forming the vertex state of a loop produces a closed curve

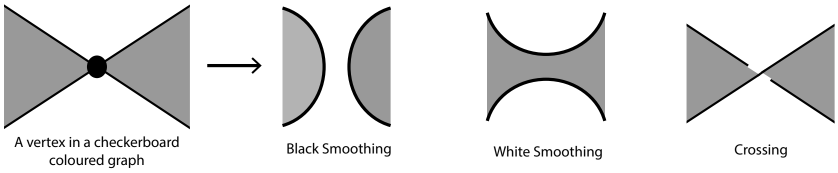


Figure 2.3: The three vertex states of a vertex of a checkerboard coloured embedded graph

a graph state consists of a set of disjoint closed curves (as eventually every edge pairing must be a loop). Therefore if we added a pair of arrows along the new edges for each pair of half edges we end up with a set of closed curves with pairs of marking arrows on their boundaries, which is in fact an arrow presentation. This leads to a further definition.

Definition 2.2.3. Let F be a 4-regular graph and v be a vertex of F then an *arrow marked vertex state* is a vertex state where each pairing is connected with an arrow labeled v.

Observe that for each choice of vertex there are two possible choices for the arrows. They can either agree or disagree with a local orientation of a vertex. Therefore we have six possible arrow marked vertex states for a given vertex, as shown in Figure 2.4. If the arrows agree we call the arrow marked vertex state *consistent* and if they disagree we call the state *inconsistent*.

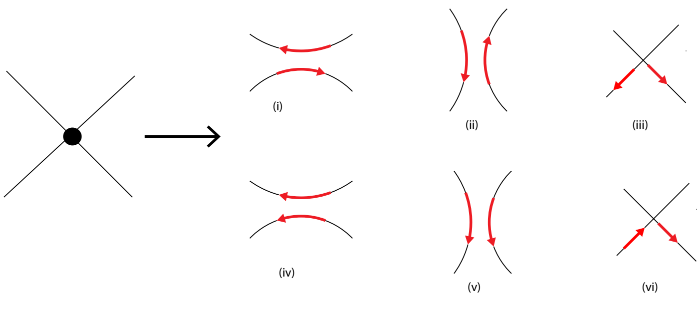
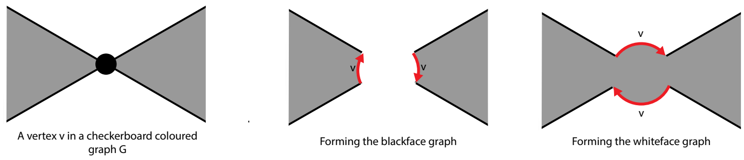


Figure 2.4: The six arrow marked vertex states for a 4 valent vertex

In fact it can be shown that if we choose the consistent black smoothing for each vertex we obtain the arrow presentation of the blackface graph and similarly if we chose the consistent white smoothing we obtain the whiteface graph.

Figure 2.5: Forming the arrow presentation of Tait graphs



Recall that the blackface graph is formed by placing a vertex in each black face and adding an edge whenever these faces meet at a vertex. Now note that if F is a 4-regular graph each vertex will be adjacent to precisely two black faces and hence will account for exactly one edge in the blackface graph of F. As you can see in Figure 2.5 choosing the black smoothing at each edge means that we surround each black face of F with a closed curve and hence the black faces become the closed curves in the arrow presentation and as each vertex of F is replaced with a pair of labelled arrows and since each arrow is on the boundary of one of the black faces they represent edges which connect adjacent black faces. Hence we have an arrow presentation of the blackface graph of F. Similarly if we chose the white smoothing at each vertex we would get the whiteface graph.

2.3 Partial Duality

We now introduce one of the key concepts that inspired our work namely the notion of *partial duality*. First introduced by Chmutov in [17] the partial dual is the result of applying a local operation δ to a subset of the edges of a graph G such that if δ was applied to all the edges of G then we would obtain its geometric dual G^*.

The simplest way to define the partial dual is by using the arrow presentation representation of an embedded graph.

Definition 2.3.1. [17] Let G be an arrow presentation and $A \subseteq E(G)$. Then the *partial dual*, $G^{\delta(A)}$, of G with respect to A is the arrow presentation obtained as follows. For each $e \in A$, let e_1 and e_2 be the two arrows which represent the edge e in the arrow presentation of G. Draw a line segment with an arrow on it directed from the head of e_1 to the tail of e_2 and and a second arrow directed from the head of e_2 to the tail of e_1. Label these new arrows as e. Finally delete e_1 and e_2. The set of all partial duals of an embedded graph G is denoted $Orb_\delta(G)$.

There is a detailed explanation in [25] showing how applying this action to each edge results in the geometric dual. The main idea is that if you take the band decomposition of an embedded graph G and recall that the dual of this is obtained by simply relabeling the 0-bands as 2-bands and vice versa, then if we impose the arrow presentation of G on to this drawing and compare it to another drawing with the arrow presentation of G^* imposed on it we can see that when taking a dual the effect on each edge is to shift the arrows from the boundary between the 0-bands and 1-bands to the boundary between the 2-bands and 1-bands.

We can form the partial dual of any representation of an embedded graph by translating it into an arrow presentation and then performing the operation described above and translating it back to its original form.

There are many other ways of considering partial duality, all of which have their own advantages, but since they are not useful for our purposes we will not describe them here.

There are a number of basic properties of partial duality which follow directly from its definition, as in the following proposition.

Proposition 2.3.1. *[17]. Let G be a ribbon graph and $A, B \subseteq E(G)$.Then the following properties hold:*

1. $G^{\delta(\emptyset)} = G$.
2. $G^{\delta(E(G))} = G^*$, *where G^* is the geometric dual of G.*
3. $G^{\delta(\{e,f\})} = \left(G^{\delta(\{e\})}\right)^{\delta(\{f\})} = \left(G^{\delta(\{f\})}\right)^{\delta(\{e\})}$, *that is partial duals can be formed one edge at a time.*
4. $\left(G^{\delta(A)}\right)^{\delta(B)} = G^{\delta(A \triangle B)}$, *where $A \triangle B := (A \cup B) \setminus (A \cap B)$ is the symmetric difference of A and B.*

5. *G is orientable if and only if* $G^{\delta(A)}$ *is orientable.*

6. *Partial duality acts disjointly on components, i.e. if* P *and* Q*are disjoint subgraphs of* G *then,* $(P \cup Q)^{\delta(A)} = (P^{\delta(A \cap E(P))}) \cup (Q^{\delta(A \cap E(Q))})$.

7. *There is a natural 1-1 correspondence between the edges of* G *and the edges of* $G^{\delta(A)}$.

2.4 Partial Petrial

Recall from Chapter 1.8 that the Petrial of a graph G is obtained by adding a "half-twist" to each edge. It therefore follows that we can introduce a partial Petrial by only twisting some of the edges.

Definition 2.4.1. [24] Let G be the arrow presentation of an embedded graph and let A be a subset of the pairs of arrows of D. Then the *partial Petrial*, $G^{\tau(A)}$, of G with respect to A is the arrow presentation obtained from G by reversing the direction of exactly one of the arrows for each pair in A. The set of all parial Petrials of an embedded graph G is denoted $Orb_{\tau}(G)$.

We can form the partial Petrial of any representation of an embedded graph by translating it into an arrow presentation and then performing the operation described above and translating it back to its original form.

The partial Petrial also satisfies a number of basic properties similar to that of the partial dual, which all follow directly from the definition of τ.

Proposition 2.4.1. *[24]. Let* G *be a ribbon graph and* $A, B \subseteq E(G)$*.Then the following properties hold:*

1. $G^{\tau(\emptyset)} = G$.

2. $G^{\tau(E(G))} = G^{\times}$, *where* $G^{\times}$ *is the Petrial of* G.

3. $G^{\tau(\{e,f\})} = \left(G^{\tau(\{e\})}\right)^{\tau(\{f\})} = \left(G^{\tau(\{f\})}\right)^{\tau(\{e\})}$, *that is partial Petrials can be formed one edge at a time.*

4. $\left(G^{\tau(A)}\right)^{\tau(B)} = G^{\tau(A\triangle B)}$, *where* $A \triangle B := (A \cup B)\backslash(A \cap B)$ *is the symmetric difference of* A *and* B.

5. *Partial Petriality acts disjointly on components,i.e.if* P *and* Q*are disjoint subgraphs of* G *then,* $(P \cup Q)^{\tau(A)} = (P^{\tau(A\cap E(P))}) \cup (Q^{\tau(A\cap E(Q))})$.

6. *There is a natural 1-1 correspondence between the edges of* G *and the edges of* $G^{\tau(A)}$.

2.5 Twisted Duality

Now that we have defined both partial duaility and partial Petriality we can show how the combination of the two gives rise to a generalisation of duality known as *Twisted Duality*, which was first introduced in [24] and then refined in [25].

Definition 2.5.1. [24] Let G and H be embedded graphs. Then we say that G and H are *twisted duals* if and only if we can move from one to the other by applying any combination of δ and τ. The set of all twisted duals of an embedded graph G is denoted $Orb(G)$.

Fig.2.6 shows an example of three graphs that are twisted duals and demonstrates that twisted duals of the same graph can have very different topological and graphical properties.

2.6 Twisted Duality and Equivalence of Embedded Graph

In Section 2.1 we showed the relationship between geometric duality, medial graphs and Tait graphs. So clearly the next logical step is to consider what happens when we extend geometric duality to twisted duality. In this section we examine this question and cite some results which provide a series of relationships between partial duality and embedded graph equivalence.

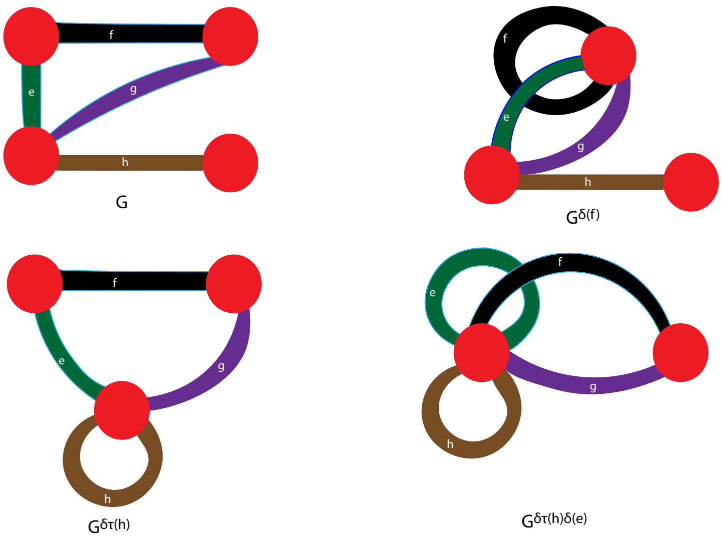


Figure 2.6: Examples of twisted duals

2.7 Cycle Family Graphs

Our first step is to provide an extension of the Tait graphs. To do this, recall from Section 2.2 that the arrow presentations of the Tait graphs of a 4-regular graph F can be obtained by taking the graph state derived from choosing the consistent black (or white) smoothing at each vertex. That is we restrict the options for each vertex state to one of two possibilities so therefore the natural thing to do is to remove these restrictions.

Recall that to form an arrow marked vertex state, of a vertex of degree four, we replace the vertex with two arrows which connect the half edges that were incident to it. Figure 2.4 shows all the possible arrow marked vertex states. The states labelled (i) and (ii) are *consistent smoothings*, while those labelled (iv) and (v) are *inconsistent smoothings*. Similarly, the state labelled (iii) is a *consistent crossing* and that labelled (vi) is an *inconsistent crossing*.

Definition 2.7.1. [24] Let F be a 4-regular cellularly embedded graph. A *cycle family graph* of F is an embedded graph obtained as an arrow presentation given by replacing each vertex with one of the six arrow marked vertex states shown in Figure 2.4. We let $\mathscr{C}(F)$ denote the set of cycle family graphs of F.

Definition 2.7.2. [24] Let F be a 4-regular cellularly embedded graph. A *duality state* of F is a state $\vec{s}$ given by replacing each vertex with one of the two consistent smoothings labelled (i) and (ii) in Figure 2.4. A *smoothing graph* of F is an embedded graph obtained as the arrow presentation resulting from a duality state. We let $\mathscr{C}_{(\delta)}(F)$ denote the set of smoothing graphs of F.

2.8 Twisted Duality and Cycle Family Graphs

We now have the equipment to show the analogues to Theorem 2.1.3 which relates Tait graphs and medial graphs. In this section we will state a number of results which were proved by Ellis-Monaghan and Moffatt in [25] and which form the basis of the work in the following chapter of this thesis. The idea is to use twisted duality and the cycle family graphs to provide the conditions for equivalence of abstract graphs.

Item 1 of Theorem 2.1.3 tell us that if F is a 4-regular embedded graph, then

$$\{F_{bl}, F_{wh}\} = \{G \mid G_m = F\}.$$

That is that a 4-regular embedded graph F is equivalent as an embedded graph to the medial graph G_m of an embedded graph G if and only if G is isomorphic to either F_{bl} or F_{wh}. Now if we consider cycle family graphs instead of Tait graphs we can say that a 4-regular graph F is equivalent as an abstract graph to the medial graph G_m of an embedded graph G if and only if G is a cycle family graph of some for some embedding $\tilde{F}$ of F.

Theorem 2.8.1. *[24] Let F be a 4-regular abstract graph and let $\tilde{F}$ be any embedding of F. Then*

$$\mathscr{C}(\tilde{F}) = \{G \mid G_m \cong F\}$$

i.e.

$$G_m \cong F \iff G \in \mathscr{C}(\tilde{F}).$$

Item 2 of Theorem 2.1.3 states that the medial graph of a Tait graph is the original graph. So the corresponding result would be that the medial graph of a cycle family graph is equivalent as an abstract graph to the original graph.

This follows as an immediate corollary of Theorem 2.8.1.

Corollary 2.8.1. *[24] Let F be a 4-regular embedded graph and $\vec{s}$ be an arrow marked graph state of F. Then $(F_{\vec{s}})_m$ and F are equivalent as abstract graphs, i.e. $(F_{\vec{s}})_m \cong F$.*

We can now give an analogue to Item 3 of Theorem 2.1.3 which states that

$$\{(G_m)_{bl}, (G_m)_{wh}\} = \{G, G^*\}.$$

This theorem gives a corresponding result for twisted duality showing that the set of twisted duals of a graph G is exactly the set of the cycle family graphs of its medial graph.

Theorem 2.8.2. *[24] Let G be an embedded graph. Then the cycle family graphs of its medial graph G_m are exactly its twisted duals, i.e.,*

$$\mathscr{C}(G_m) = Orb(G).$$

Item 4 of Theorem 2.1.3 states that the blackface graph is the dual of the whiteface graph. Therefore the analogous result would be that two cycle family graphs of the same embedded graph are twisted duals and by combining the results of Theorems 2.8.1 and 2.8.2 we have:

Corollary 2.8.2. *[24] If F is a 4-regular embedded graph and $F_{\vec{s}}$ and $F_{\vec{s}'}$ are two cycle family graphs of any embedding of F, then $F_{\vec{s}}$ and $F_{\vec{s}'}$ are twisted duals.*

Finally Item 5 states that two 4-regular embedded graphs G_m and H_m are equivalent if they are the medial graphs of two embedded graphs G and H such that $G = H$ or H^*. If we relaxed the equivalence to only be equivalent as abstract graphs we can reform the statement as follows: two 4-regular embedded graphs G_m and H_m are equivalent as abstract graphs if and only if they are the medial graphs of two embedded graphs G and H such that G and H are twisted duals. That is:

Theorem 2.8.3. *[24] Let G be an embedded graph. Then*

$$Orb(G) = \{H \mid H_m \cong G_m\}$$

i.e.,

$$H_m \cong G_m \iff H \in Orb(G).$$

2.9 Partial Duality and Smoothing Graphs

In Section 2.8. we showed how twisted duality corresponds to equivalence as abstract graphs by using cycle family graphs. In this section by restricting our choice of cycle family graphs to smoothing graphs we will show how partial duality corresponds to equivalence as cyclically ordered graphs, which we defined in Section 1.5. We follow the exact same process as in Section 2.8. We begin with a result stating two 4-regular F and G_m embedded graphs are equivalent as cogs if and only if G is a smoothing graph of some embedding of F.

Theorem 2.9.1. *[24] Let F be a 4-regular abstract graph and let $\tilde{F}$ be any embedding of F. Then*

$$\mathscr{C}_{(\delta)}(\tilde{F}) = \{G \mid G_m \doteq F\}$$

i.e.

$$G_m \doteq F \iff G \in \mathscr{C}_{(\delta)}(\tilde{F})$$

for some embedding $\tilde{F}$ of F.

It then follows that

Corollary 2.9.1. *[24] Let F be a 4-regular embedded graph and $\vec{s}$ be a duality state of F. Then $(F_{\vec{s}})_m$ and F are equivalent as cogs, i.e. $(F_{\vec{s}})_m \doteq F$.*

We then have

Theorem 2.9.2. *[24] Let G be an embedded graph. Then the smoothing graphs* *of its medial graph G_m are exactly its partial duals, i.e.,*

$$\mathscr{C}_{(\delta)}(G_m) = Orb_{(\delta)}(G).$$

and it immediately follows that

Corollary 2.9.2. *[24] If F is a 4-regular embedded graph and $F_{\vec{s}}$ and $F_{\vec{s}'}$ are two smoothing graphs of any embedding of F , then $F_{\vec{s}}$ and $F_{\vec{s}'}$ are partial duals, that is,*

$$F_{\vec{s}} = (F_{\vec{s}})^{\delta(A)}$$

for some $A \subseteq E(G)$.

Finally we have

Theorem 2.9.3. *[24] Let G be an embedded graph. Then*

$$Orb_{(\delta)}(G) = \{H \mid H_m \doteq G_m\}$$

i.e.,

$$H_m \doteq G_m \Longleftrightarrow H \in Orb_{(\delta)}(G).$$

Therefore we now have a full characterisation of the orbits of the ribbon group action (twisted duality) and two of its key subgroups (partial duality and geometric duality). This poses a number of questions not least about how to characterise the many other orbits of the ribbon group action, and some preliminary work has been done on this in [24]. However in this thesis we will look at a different question, how can we expand the scope of these equivalences beyond 4-regular graphs.

Chapter 3

Hypergraphs

Contents

In Chapter 2 we showed the relationship between equivalence and duality and as part of that process we showed that for two 4-regular graphs to be equivalent as abstract graphs they must be medial graphs of twisted duals i.e.

$$H_m \cong G_m \Longleftrightarrow H \in Orb(G).$$

So the question arises how can we tell if two even graphs (which are not necessarily 4-regular) are isomorphic. The solution is to use hypergraphs. Hypergraphs are a much less studied area than graphs and especially embedded hypergraphs. However with some slight tweaking of the definitions we are able to provide analogous definitions and theorems to those that have come before.

3.1 Hypergraphs

Definition 3.1.1. A *hypergraph* H consists of a set $V(H)$ of vertices and a set $E(H)$ of hyperedges. Each hyperedge e has an endpoint multiset containing at least two elements of the vertex set V. These elements are said to be *incident* with the hyperedge e. We say two vertices x, y are *adjacent* if there exist a hyperedge e in $E(H)$ with endpoints x and y. Two hyperedges are *adjacent* if they have at least one common endpoint. We say an hyperedge has *size* k if its endpoint multiset contains k (not necessarily distinct) elements. If a hyperedge's endpoint multiset just contains m copies of the same element we call that hyperedge an *m-loop*.

A hypergraph $H' = (V', E')$ is a *subhypergraph* of $H = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$ and each element of E' has the same endpoint multiset as in E. If $V' = V$ then H' is said to be a *spanning* subhypergraph of H.

Definition 3.1.2. We call a hypergraph *abstract* if it is not embedded in any surface.

Definition 3.1.3. We say two hypergraphs H and G are *isomorphic*, written $H \cong G$, if there exist bijections

$$\theta : V(H) \to V(G)$$

$$\phi : E(H) \to E(G)$$

such that the hyperedge $e \in E(H)$ has endpoint multiset $\{v, \ldots, w\}$ if and only if the hyperedge $\phi(e) \in E(G)$ has endpoint multiset $\{\theta(v), \ldots, \theta(w)\}$.

In general, as in the case with graphs, we do not distinguish between isomorphic hypergraphs.

3.2 Hypermaps

In Chapter 1 we described how to embed a graph into a surface and we can use a similar process for hypergraphs. The only difference is that since each hyperedge may be incident with more than two vertices the line segment may split into separate "forks". Therefore to *embed a hypergraph* H in a surface Σ we define a mapping i from H to Σ which maps each vertex of H to a distinct point on the surface and maps each hyperedge as paths on the surface between its endpoints and a central point, in such a way that each hyperedge is represented by a collection of lines that connect the vertices incident to the hyperedge with a central point and the hyperedges only meet at the vertices.

Definition 3.2.1. A *cellularly embedded hypergraph* or *hypermap* $H \subseteq \Sigma$ is a hypergraph H that has been embedded in a surface Σ, so that the components of $\Sigma \setminus H$ are homeomorphic to an open disc.

Figure 3.1 shows an example of a hypermap. Two hypermaps $H \subset \Sigma$ and $H' \subset \Sigma'$ are *equivalent*, written $H = H'$ if there is a homeomorphism from Σ to Σ' (which is orientation preserving when Σ is orientable) with the property that when it is restricted to just the subset H of the surface it is an isomorphism of abstract hypergraphs.

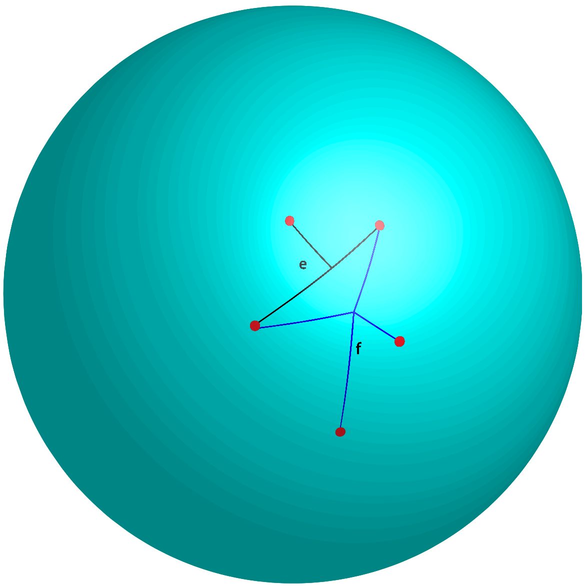


Figure 3.1: A hypermap H

As in the graph case, we can represent Hypermaps in a number of ways, the representations are very similar to those described in the Chapter 1 often with just a few restrictions relaxed.

3.2.1 Band Decomposition Representation of Hypermap

Recall that the band decomposition of a surface is essentially the division of a surface into three subsets known as bands and that it can represent a cellularly embedded graph if specific conditions are met. Similarly a band decomposition can also represent a hypermap.

Proposition 3.2.1. *A band decomposition represents a hypermap if it satisfies the following conditions:*

1. *No two bands of the same type intersect at any point*
2. *Each 1-band intersects at least twice with the 0-bands.*
3. *Each 1-band intersects at least twice with the 2-bands.*

If these conditions are met then we can take the 0-bands as the vertices of a hypermap, the 1-bands as its hyperedges and the 2-bands as its faces.

Proof. Let G be a hypermap embedded on a surface Σ. We can expand the vertices of G into small discs and the edges (outside of the neighbourhoods of the vertices) by thin bands, which will be the 0-bands and 1-bands respectively. The union of these bands will give a neighbourhood of G in Σ. If we take the complement of this neighbourhood we get a family of discs which we call 2-bands. These discs correspond to the faces of G hence we have a band decomposition of the surface Σ.

Alternatively if we have a band decomposition of a surface Σ then each 1-band has at least two intersections with 0-bands so we can obtain a hypermap by placing a vertex in the centre of each 0-band and drawing edges as follows. For each 1-band place point inside it: this will be the central point of the edge.

Then draw paths inside the 1-band from the boundary of each intersection with a 0-band to the central point. Finally, connect the edge to the vertices of the graph. We then remove the bands from the surface and are left with a cellularly embedded graph. □

Figure 3.2 shows the hypermap from Figure 3.1 represented as a band decomposition.

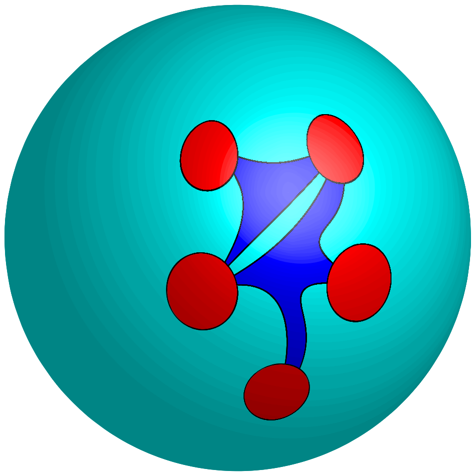


Figure 3.2: A Band Decomposition Representation of the Hypermap H from Figure 3.1

3.2.2 Ribbon Hypermaps

We can also represent hypermaps as ribbon hypermaps.

Definition 3.2.2. A *ribbon hypermap* $H = (V(H), F(H))$ is a (possibly non-orientable) surface with boundary represented as the union of two sets of discs, a set $V(H)$ of *vertices* and a set $E(H)$ of *hyperedges* such that

1. The vertices and hyperedges intersect in disjoint line segments which we will call *joints*
2. Each joint lies on the boundary of precisely one vertex and precisely one hyperedge

3. Every hyperedge contains at least two joints

Figure 3.3 shows the hypermap from Figure 3.1 represented as a ribbon hypermap.

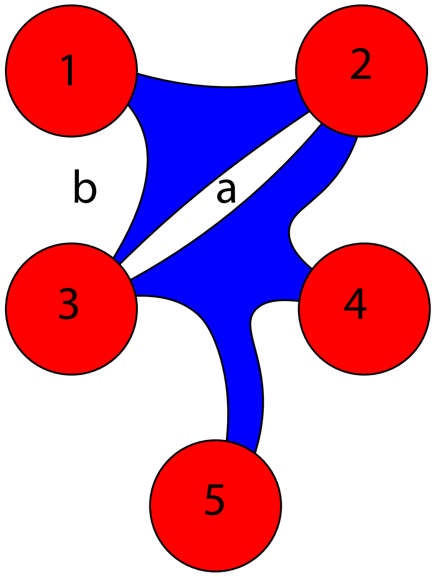


Figure 3.3: The Ribbon Hypermap of a Hypermap H from Figure 3.1

If we have a band decomposition of a hypermap we can obtain a ribbon hypermap by simply deleting the 2-bands, then the 0-bands are the vertices and the 1-bands the hyperedges. Conversely if we have a ribbon hypergraph then it is a surface with boundary so by sewing discs into each boundary component we have a band decomposition. Therefore since a hypermap can be represented by a band decomposition it can also be represented by a ribbon hypermap.

3.2.3 Arrow Presentation of a Hypermap

Definition 3.2.3. An *arrow presentation of a hypermap* consists of a set of closed curves, which represent the vertices and sets of labelled arrows which represent the hyperedges. The set of arrows are partitioned into cyclically ordered subsets such that each subset represents a single hyperedge. We label the arrows that represent a hyperedge e by $e_1, \ldots e_n$, where the index gives its cyclic order.

Figure 3.4 shows the hypermap from Figure 3.1 represented as an arrow presentation.

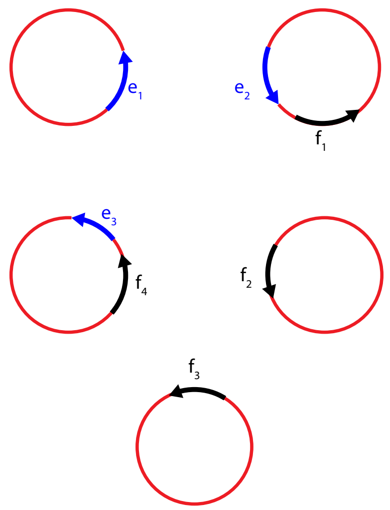


Figure 3.4: Arrow Presentation of the Hypermap H from Figure 3.1

A ribbon hypermap can be obtained from an arrow presentation as follows. Fill in the closed curves to give the vertex discs. Then for each of the cyclically ordered subsets comprising the partition of the arrows draw a line segment from the head of one arrow to the tail of the next arrow in the cyclic order. Now the union of the arrows and line segments give a closed curve. Attach a disc along this closed curve to obtain a hyperedge. Conversely we can obtain an arrow presentation from a ribbon hypermap as follows: for each hyperedge e of the ribbon hypermap draw an arrow along one of its joints and label this arrow e_1 (note the direction of the first arrow does not matter), then follow the boundary of the hyperedge from the head of the arrow to the next vertex (this could be the same vertex if a hyperedge intersects with one vertex more than once) and then draw an arrow along the next joint, starting at the point where the boundary meets the vertex, and label this arrow e_2 then repeat this process until you return to the joint where you started, do the same thing for all the other hyperedges and then delete all the hyperedges and replace the vertex discs with closed curves to obtain an arrow presentation.

Observe that the direction and cyclic order of the arrows describes how the hyperedge is attached to its incident vertices.

Two arrow presentations are said to be *equivalent* if one can be transformed to the other by using any combination of the following moves.

1. Reversing the direction of all the arrows in one set of arrows and reversing the cyclic order.
2. Relabelling a set of arrows. That is we could relabel the set $\{e_1, \ldots, e_n\}$ as $\{f_1, \ldots, f_n\}$. Note that this would not change the cyclic order of the set.

Observe that a hyperedge of size n can be represented by a set of n arrows connecting $2n$ different points on the closed curves. Figure 3.5 shows one example of this. Observe that there will be $\frac{(2n)!}{n!}$ different ways to draw the arrows connecting the points. We will call the way that the arrows are connected the *configuration* of the hyperedge. Note that changing the configuration of a hyperedge will produce a different hypermap.

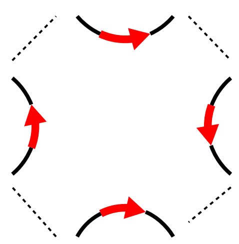


Figure 3.5: Arrow Presentation of a Hyperedge

3.2.4 Permutational τ-model representation of hypermaps

Given any three fixed-point free involutions τ_0, τ_1 and τ_2 on a set X we can obtain a hypermap. Unlike in the graph case we do not need a restriction on $\tau_0\tau_2$. The $\tau - model$ representation of hypermap is equivalent to a band decomposition and the equivalence follows the exact same method as detailed in Section 1.3.4

We can also quickly recover the $\tau - model$ from an arrow presentation of a hypermap as follows. Given an arrow presentation of a hypermap we label the head and tail of each arrow. Then we form τ_0 by pairing the point at the head of one arrow with the tail of the next arrow in its cyclic order, we form τ_1 by starting at a point and moving around the closed curve away from the arrow until we reach another point and we form τ_2 by simply pairing both ends of the same arrow.

3.3 Deletion and Contraction of Hypermaps

In Chapter 1 we defined deleting and contracting edges for ribbon graphs. For hypermaps we will define deleting and contracting hyperedges in terms of the arrow presentation.

Definition 3.3.1. Let H be an arrow presentation of a hypermap and let e be an hyperedge of H represented by a set of n labelled arrows $\{e_1, e_2, \ldots, e_n\}$. Then the arrow presentation $H\backslash e$, obtained by *deleting* e is simply the arrow presentation H with the set of arrows $\{e_1, e_2, \ldots, e_n\}$ removed. The arrow presentation H/e is obtained by *contracting* the hyperedge e. To contract the hyperedge e for $1 \geq i \geq n-1$ simply draw a line segment from the head of arrow e_i to the tail of arrow e_{i+1} and for e_n draw a line segment from its head to the tail of e_1. Then delete the arrows $\{e_1, e_2, \ldots, e_n\}$ to form the new arrow presentation.

To delete a hyperedge of a ribbon hypermap simply remove the edge. To contract a hyperedge of a ribbon hypermap translate the ribbon hypermap into an arrow presentation then perform the contraction and translate back. An example of this is shown in Figure. 3.6.

3.4 Geometric Duality in Hypermaps

In order to investigate twisted duality in hypermaps we first need to define the hypermap equivalents of the major operations which we used when dis-

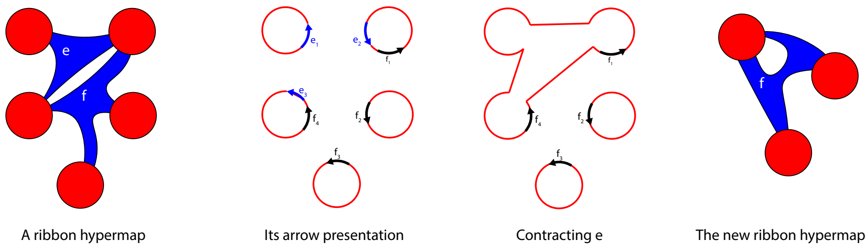


Figure 3.6: Contracting a hyperedge of a ribbon hypermap

cussing duality in graphs. We start by defining geometric duality for ribbon hypermaps.

Definition 3.4.1. Let $H = (V(H), E(H)$ be a ribbon hypermap. Then topologically H is a surface with boundary. We can cap off the holes using a set of discs, denoted by $V(H^*)$, to obtain a surface without boundary. Then the geometric dual of H is the ribbon graph $H^* = (V(H^*), E(H))$.

The simplest way of forming the dual of a hypermap is to form its band decomposition representation, then we can obtain the dual by simply relabelling the $0-$bands as $2-$bands and vice versa and then translate back to a hypermap.

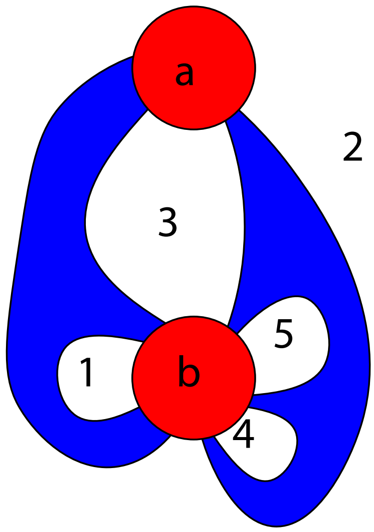


Figure 3.7: The Dual H^* of The Ribbon Hypermap H from Figure 3.3

Figure. 3.7 shows the dual ribbon hypermap of the ribbon hypermap shown earlier in Figue 3.3. Observe that the boundary components of H correspond to the vertices of H^* and vice versa.

3.5 The Medial Graph of a Hypermap

In Chapter 2 we defined the medial graph and gave a brief overview of some of its applications. We can now introduce an extension of this construction to hypermaps.

Definition 3.5.1. Given a hypermap $H \subset \Sigma$ we construct its *medial graph* H_m by, for each hyperedge e of H with size k placing a vertex of degree $2k$ on the hyperedge and then drawing the edges of the medial graph by following the face boundaries of H.

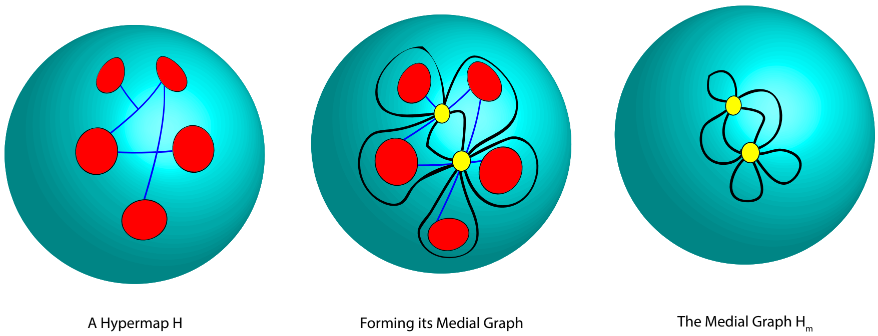


Figure 3.8: Forming the Medial Graph of a Hypermap

Observe that H_m is an embedded graph but unlike the medial graph of a graph does not have to be 4-regular.

3.6 The Tait hypermaps

Let F be a checkerboard coloured cellularly embedded graph then

1. The *Blackface hypermap* F_{hb} of F is the embedded hypermap constructed by placing a vertex in each black face of F then for each vertex of F of degree $2k$ a hyperedge of size k is added to F_{hb} connecting the vertices corresponding to the black faces which meet at this vertex. The edge is embedded by drawing a line segment from the centre of the vertex of F to each of the vertices of F_{hb} that it is incident with.

2. The *Whiteface hypermap* F_{hw} of F is constructed analogously by placing vertices in the white faces.

Note, as when forming the Tait graphs of a disconnected graph, when forming the Tait hypermaps of a disconnected hypermap consider each component separately.

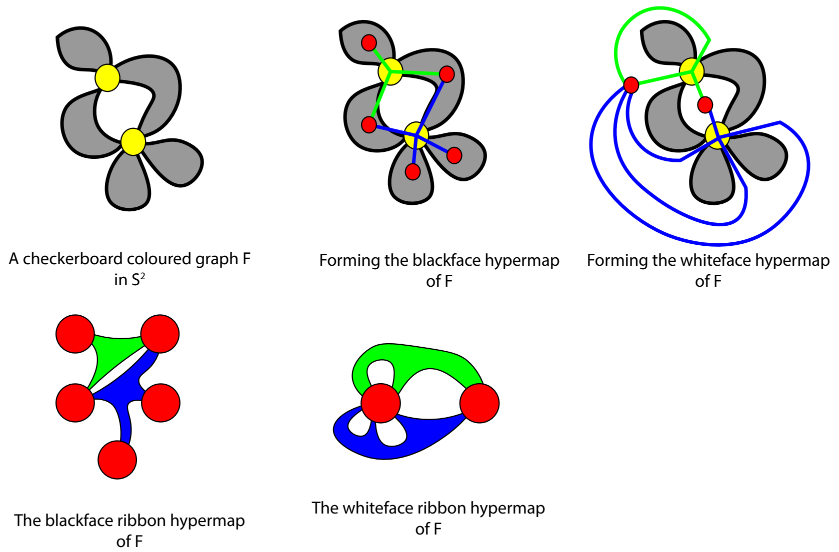


Figure 3.9: Forming the Tait hypermap of a Graph F

Let F be a checkerboard colourable embedded graph. Then the two *Tait hypermaps* are the embedded graphs obtained by checkerboard colouring F and forming the whiteface and blackface hypermaps.

Figure 3.9 shows a checkerboard coloured graph F embedded in S_2. We form the blackface hypermap F_{hb} and whiteface hypermap F_{hw}. It also shows the ribbon hypermaps which correspond to F_{hb} and F_{hw}. Now observe that F is the medial graph of the hypermap H from Figure 3.8 and that F_{hb} is the hypmermap H while F_{hw} is the hypmermap H^*. This suggests that the Theorems 2.1.2 and 2.1.3 have an analogous form for hypermaps. In fact we will now prove that this is the case.

Theorem 3.6.1. *Let F be an even cellularly embedded graph. Then F is the medial graph of some hypermap H if and only if F is checkerboard colourable. If F is checkerboard colourable, then F is the embedded medial graph of precisely F_{hb} and F_{hw}.*

Proof. First let $F = H_m \subset \Sigma$. In the construction of H_m from H, each vertex of H_m lies on a unique hyperedge of H and the hyperedges of H_m follow the face boundaries of H. We can obtain a checkerboard colouring of H_m either by colouring each face of H_m black if and only if it contains a vertex of H or by colouring each face of H_m black if and only if it does not contain a vertex of H. These are all of the checkerboard colourings of H_m since an embedded graph can have at most two checkerboard colourings and recall H_m is just an embedded graph. Conversely, suppose that F is checkerboard colourable. Give F a checkerboard colouring. Then we can form F_{hb} and F_{hw}. By comparing the constructions of the medial graphs and Tait hypermaps, it is easily seen that $(F_{hb})_m = F$ and $(F_{hw})_m = F$, as required.

If the faces of H_m that contain a vertex of H are coloured black, it is readily seen that H $= (H_m)_{hb} = F_{hb}$; and if the faces of H_m that do not contain a vertex of H are coloured black, it is readily seen that H $= (H_m)_{hw} = F_{hw}$.

□

Theorem 3.6.2. *Let H be an embedded hypermap and F be a checkerboard coloured embedded graph. Then*

1. $\{F_{hb}, F_{hw}\} = \{H \mid H_m = F\}$*, or, equivalently,* $H_m = F \iff H = F_{hb}$ *or* $H = F_{hw}$*;*
2. $(F_{hb})_m = (F_{hw})_m = F$*;*
3. $\{(H_m)_{hb}, (H_m)_{hw}\} = \{H, H^*\}$*;*
4. $F_{hb} = (F_{hw})^*$*;*
5. $\{H, H^*\} = \{G \mid G_m = H_m\}$*, or, equivalently,* $G_m = H_m \iff G \in \{H, H^*\}$.

Proof. The first item follows directly from Theorem. 3.6.1 and the second item follows from the proof of Theorem. 3.6.1

For the third item if we chose the canonical checkerboard colouring of H_m then we know that $(H_m)_{hb} = H$ and the white faces of H_m correspond to the faces of H, hence $(H_m)_{hw}$ has vertices which correspond to the faces of H which are the vertices of H^* and the edges of $(H_m)_{hw}$ correspond to the vertices of H_m which correspond to the edges of H and therefore to the edges of H^*. Hence $(H_m)_{hw} = H^*$

For the fourth item, suppose that $F_{hb} = H$. By Item 2, we have $(F_{hb})_m = H_m = F$. Thus $F_{hb} = (H_m)_{hb}$ and$F_{hw} = (H_m)_{hw}$. By Item 3 one of these two graphs must be H and the other must be H^*, and the result follows upon observing that $(H^*)^* = H$.

Finally, for the fifth item, Item 1 gives that $G_m = H_m$ if and only if $G = (H_m)_{hb}$ or $G = (H_m)_{hw}$. By Item 3 this happens if and only if $G = H$ or $G = H^*$.

□

3.7 Edge Operations on Hypermaps

We now define three operations which act independently on the edges of a hypermap, as in the graph case we define these operations for the arrow presentation representation of a hypermap.

3.7.1 Partial Duality in Hypermaps

As in the graph case, we can form the dual of a hypermap with respect to individual edges.

Definition 3.7.1. *The dual with respect to an edge* e of a hypermap H is $H^{\delta(e)} = H'$, where H' is formed from H as follows. Let e be an edge of size k. In the arrow presentation of H there will be k labelled arrows ordered cyclically as $e_1, \cdots, e_k$. For each i with $1 \leq i \leq k-1$, draw a line segment with an arrow from the head e_i to the tail of e_{i+1} and label this new arrow e_i. For $i = k$ draw an arrow from the head of e_k to the tail of e_1 and label this e_k. Then delete the original arrows. The new arrows become arcs of new closed curves in the arrow presentation of H'. We call H' a *partial dual* of H.

Figure 3.10 show the partial dual of a general hyperedge and Figure 3.11 shows how we can form a partial dual of a ribbon hypermap. We let $Orb_{(\delta)}(H)$ be the set of partial duals of a hypermap H.

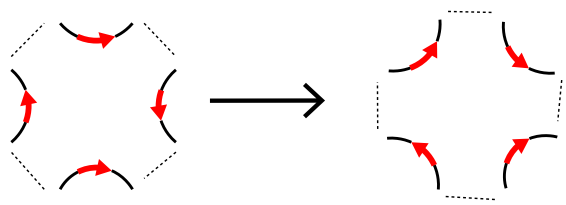


Figure 3.10: Forming a partial dual of a Hyperedge e

The partial dual of a hypermap has many of the same properties of the partial dual of an embedded graph as we will show below. In particular if we apply δ to each edge of a hypermap H we will obtain H^*. We can see this by looking at Figure. 3.12. Diagram b) shows the band decomposition representation of a hypermap H, diagram c) shows the same band decomposition with the arrow presentation of H superimposed on top of it while in d) we have superimposed the arrow presentation of H^*. It should be clear that to obtain d) from c) we simply move the arrows clockwise from the vertex-edge boundaries to the face-edge boundaries and this is what the partial dual operation does. This is because to obtain H^* from H we simply relabel the 0-bands as the 2-bands and vice versa, therefore the arrows of H^* lie on the boundary between the 1-bands as the 2-bands of H.

Theorem 3.7.1. *Let H be a ribbon hypermap and $A, B \subseteq E(H)$.Then the following properties hold:*

1. $H^{\delta(\emptyset)} = H$.
2. $H^{\delta(E(H))} = H^*$, *where H^* is the geometric dual of H.*
3. $H^{\delta(\{e,f\})} = \left(H^{\delta(\{e\})}\right)^{\delta(\{f\})} = \left(H^{\delta(\{f\})}\right)^{\delta(\{e\})}$, *that is partial duals can be formed one edge at a time.*

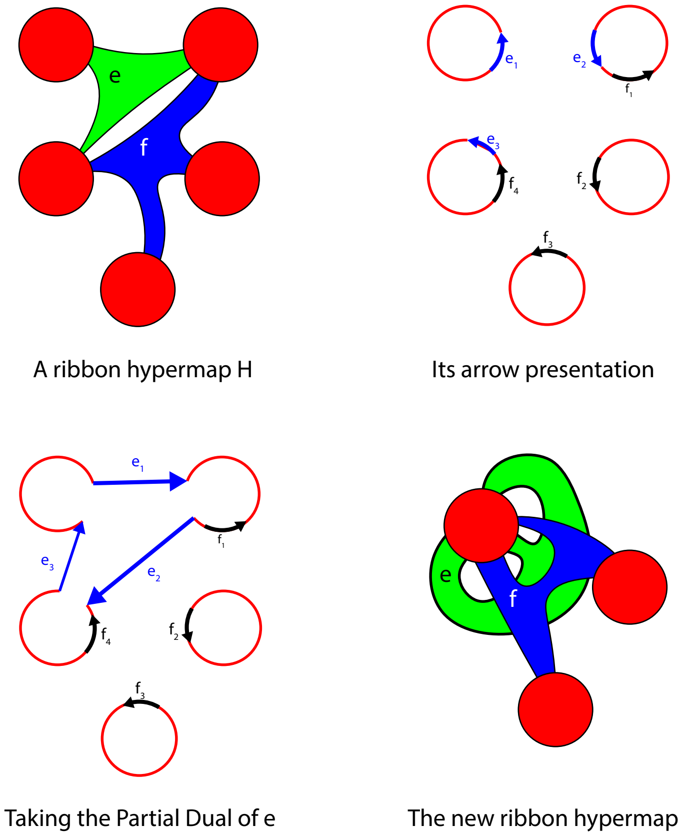


Figure 3.11: Forming a partial dual of a Hypermap H

4. $\left(H^{\delta(A)}\right)^{\delta(B)} = H^{\delta(A \triangle B)}$, *where* $A \triangle B := (A \cup B) \backslash (A \cap B)$ *is the symmetric difference of* A *and* B.

5. *Partial duality acts disjointly on components,i.e.,* $(P \cup Q)^{\delta(A)} = (P^{\delta(A \cap E(P))}) \cup (Q^{\delta(A \cap E(Q))})$.

6. *There is a natural* $1-1$ *correspondence between the edges of* H *and the edges of* $H^{\delta(A)}$.

Proof. Item 1 follows directly from the definition.

For Item 2 as discussed above consider Figure 3.12. Figure c shows the band decomposition of a hypermap with its arrow presentation superimposed on top of it. Observe that the arrows are on the boundary between the 0-bands and 1-bands, and this will always be the case. Now to get the dual of a band decomposition we simply relabel the 0-bands as 2-bands and vice versa. Therefore if we superimposed the arrow presentation of the dual onto the original band decomposition the arrows

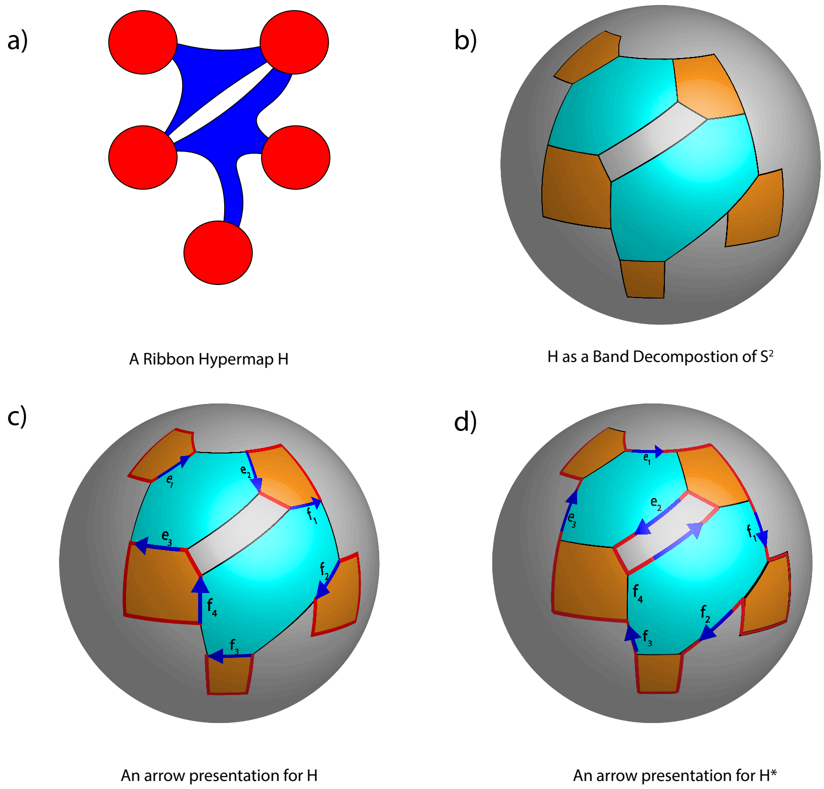


Figure 3.12: The partial dual operation on a band decomposition.

would be on the boundary of the 2-bands and 1-bands which is what would happen if we performed the partial dual operation on every edge.

Item 3 also follows directly from the definition.

For 4 observe that $\left(H^{\delta(\{e\})}\right)^{\delta(\{e\})} = H$ and since we know from 3 that we can form the partial dual one edge at a time we can see that if we apply $\delta(B)$ to $H^{\delta(A)}$ then every edge that is contained in both A and B will be returned to its original form and hence we obtain the partial dual formed by acting on the edges that are in either A or B but not both i.e. $A \triangle B$.

Item 5 follows directly from the definition.

For 6 observe that applying δ to an edge e in H creates a corresponding edge e in $H^{\delta(A)}$ and hence a natural $1-1$ correspondence.

□

Partial duality of hypermaps was also independently found by S. Chmutov and F. Vignes-Tourneret in their paper [19]. In this paper they define partial duality for all three types of cells but for our purposes we only need partial duality relative to the hyperedges.

3.7.2 Partial Petrial of a Hypermap

Recall from Chapter 2 that a partial Petrial of a ribbon graph is the ribbon graph obtained by "twisting" a subset of its edges. We formed the partial Petrial with respect to one edge by reversing the direction of one of the arrows in the arrow presentation. We can do a similar operation for hypermaps. However since each hyperedge can have multiple arrows we need to specify which arrow we are reversing and hence we can produce multiple partial Petrials from acting on one single edge.

Definition 3.7.2. Let H be a ribbon hypermap and e and edge of H the *Partial Petrial* $H^{\tau(e_i)}$ is the hypermap obtained by reversing the direction of the arrow e_i in the arrow presentation of H.

We let $Orb_{(\tau)}(H)$ be the set of partial petrials of a hypermap H.

To make things simpler we introduce some new notation here, let H be an arrow presentation of a hypermap the we say that $Ar(H)$ is the set of all arrows in H. Then if $B \subseteq Ar(H)$ we can write $H^{\tau(B)}$ to represent the arrow presentation obtained by reversing the direction of all the arrows in B. This is well defined because changing the direction of one arrow has no effect on the other arrows, so the order in which τ is applied is irrelevant. Many of the properties of partial Petrials of graphs also hold for hypermaps. These all follow directly from the definition of τ.

Lemma 3.7.1. *Let H be an arrow presentation of a hypermap and $A, B \subseteq Ar(H)$. Then the following properties hold:*

1. $H^{\tau(\emptyset)} = H$.

2. $H^{\tau(\{e_i,f_j\})} = \left(H^{\tau(\{e_i\})}\right)^{\tau(\{f_j\})} = \left(H^{\tau(\{f_j\})}\right)^{\tau(\{e_i\})}$, *that is partial Petrials can be formed one edge at a time.*

3. $\left(H^{\tau(A)}\right)^{\tau(B)} = H^{\tau(A \triangle B)}$, *where* $A \triangle B := (A \cup B) \backslash (A \cap B)$ *is the symmetric difference of* A *and* B.

4. *Partial Petriality acts disjointly on components,i.e.,* $(P \cup Q)^{\tau(A)} = (P^{\tau(A \cap E(P))}) \cup (Q^{\tau(A \cap E(Q))})$.

5. *There is a natural 1-1 correspondence between the edges of* H *and the edges of* $H^{\tau(A)}$.

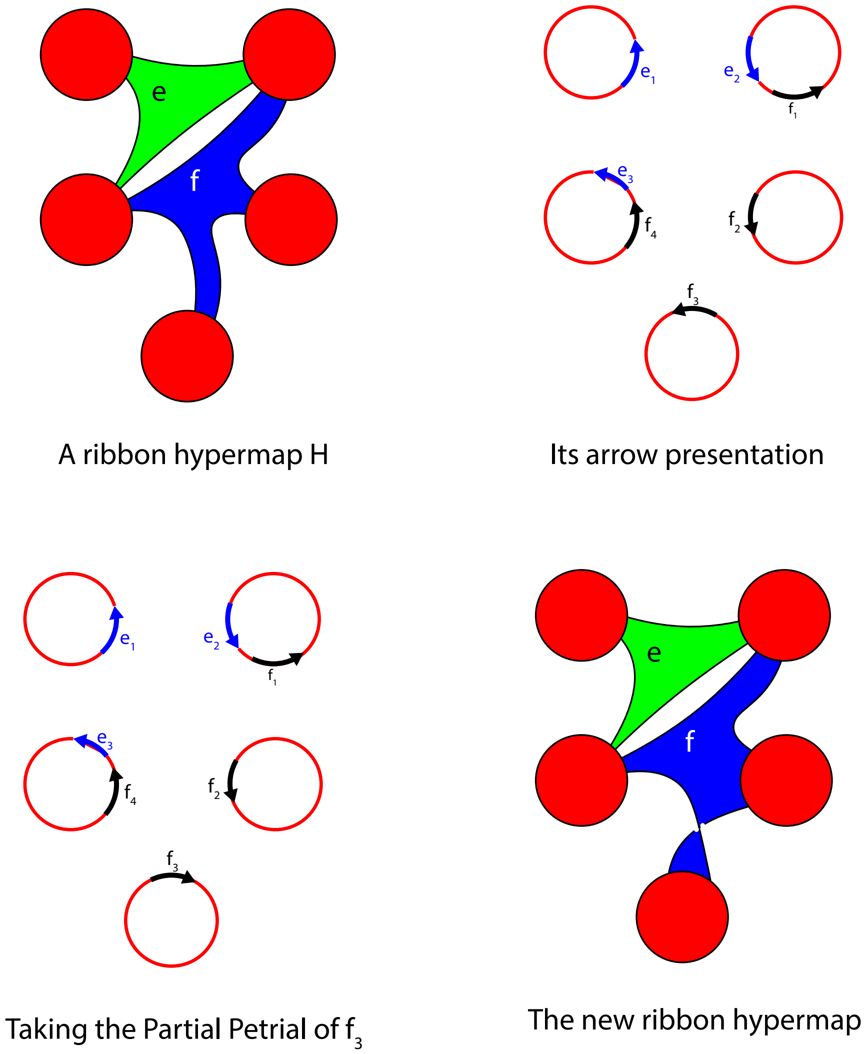


Figure 3.13: Forming a partial petrial of a Hypermap H

3.8 Fixed Point Presentations of Hypermaps

One of the complications with working with hypermaps compared to graphs is that a hyperedge does not have a fixed size and therefore representing it graphically can become tricky. Therefore we now introduce a new representation of hypermaps that will allow us to perform the operations on the hyperedge without drawing a representation of the hypermap at each stage.

We start with an arrow presentation representation of a hypermap. We then label the points at either end of the arrows and represent each arrow as an ordered pair of these points. For example $[i, j]$ would represent an arrow from i to j. Then each edge in the arrow presentation could be written as a word with the pairs of points acting as the alphabet.

This allows us to see the effect of the various operations in a much more compact way and define another construct which can be used to represent hypermaps which we call the *fixed point presentation*.

Definition 3.8.1. A *fixed point presentation* consists of a set of closed curves, a set of labelled points on these closed curves, a set of ordered pairs of these fixed points and a set of words formed from the set of pairs such that:

1. Each label appears in exactly one pair.
2. Each pair of labels contains points on the same closed curve.
3. Each pair appears in exactly one word.

We call each word in a fixed point presentation the *configuration* of a hyperedge.

It is easy to move between a fixed point presentation and an arrow presentation. Figure. 3.14 show an example of this.

Theorem 3.8.1. *Fixed point presentations are equivalent to hypermaps.*

Proof. We have shown already that arrow presentations are equivalent to hy-

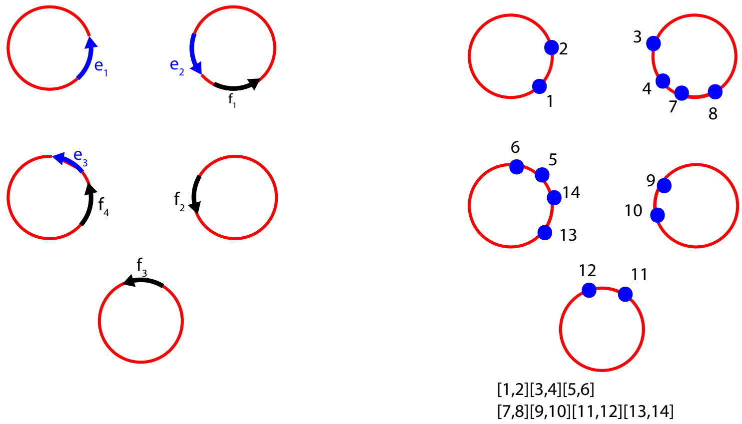


Figure 3.14: The Fixed Point Presentation of an Arrow Presentation

permaps so therefore it is sufficient to show that fixed point presentations are equivalent to arrow presentations.

Given an arrow presentation we can obtain a fixed point presentation as follows. Let G be an arrow presentation and let $e = \{e_1, \ldots e_n\}$ be a subset of arrows which represent an edge in the arrow presentation, place a labelled point at the head and tail of each arrow. Now pair the points such that each point is paired with the other point on its arrow and such that the points at the tails of the arrows are written first. We now have a set of ordered pairs with each pair corresponding to an arrow in G. Form a word corresponding to e by simply writing the pairs down from write to left starting with the pair that corresponds to the arrow e_1 then e_2 and so forth until each pair is recorded. Then remove the arrows(but don't delete the underlying arc). Repeat this process for each other subset of arrows representing an edge. Hence we have a set of labelled points, a set of ordered pairs of these points and a set of words formed from the set of pairs, and furthermore each point appears in exactly one pair, each pair appears in exactly one word and each pair of labels contains points on the same closed curve.

Given a fixed point presentation we can form an arrow presentation by for each pair drawing an arrow from the first point to the second. We then label the arrows as follows. For the first word we label the arrow which corresponds to the first pair when reading from left to right e_1 then label the arrow which corresponds to the next pair e_2 and so forth until we run out of pairs in that word. We then repeat the process for the next word using a different letter. Finally we delete the fixed points. Hence we have a set of closed curves and sets of labelled arrows which are partitioned into cyclically ordered subsets. □

The operations we defined in the previous section can easily be replicated in this new model if we superimpose the fixed point model on top of the arrow presentation and perform the different operations as in Figure. 3.15. It should be clear that applying τ simply changes the order within a pair, whereas δ changes the pairs so that the right point of the first pair in a word is now paired with the left point of the following pair and so forth until the right point of the final pair is then paired with the left point of the first pair. Also note that τ does not change the closed curves but for δ we need to delete the line segments between the original pairs and replace them with line segments between the new pairs.

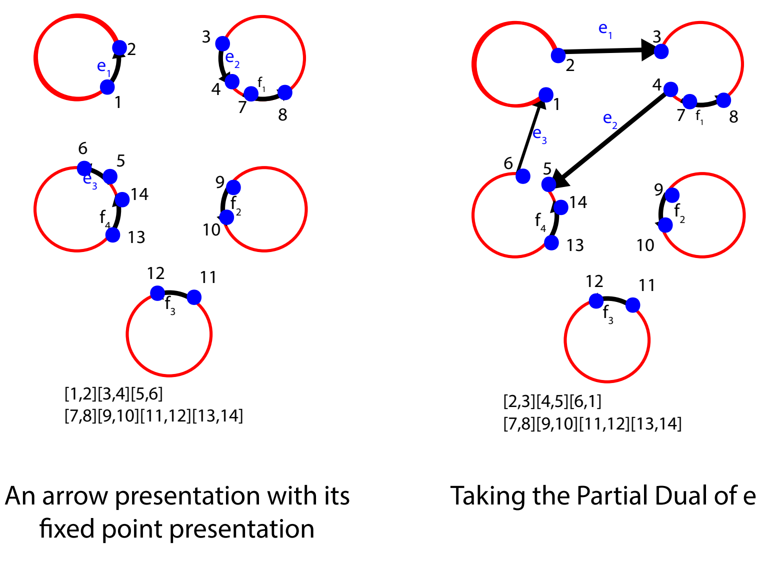


Figure 3.15: The Partial Dual of a Fixed Point Presentation

The main advantage of this construct is that we don't need to draw the diagrams after applying each operation, as we can work on each individual word independently and then form the full picture at the end. This makes it much simpler to prove things about the different operations, especially if it involves large numbers of edge operations.

3.9 Twisted Duality

We have now defined two operations which act in different ways on the edges of a hypermap, and allow us to move between different configurations of hyperedges. In fact they allow us to move between all possible configurations of a hyperedge.

Theorem 3.9.1. *Given an arrow presentation of a hyperedge e it is possible to achieve every configuration of the hyperedge by applying different combinations of δ and τ to e.*

Proof. To prove this we will work with the fixed point presentation. In a fixed point presentation of a hypermap, a configuration of an edge of size n is an assignment of $1, \ldots, 2n$ to a word consisting of n pairs. We want to show that every configuration can be obtained from any given configuration through applications of δ and τ.

To do this it is sufficient to show any element can be switched with any other element whilst leaving every other element unchanged, using only the operations δ and τ.

I.e. given the configuration $[1,2] \ldots [i, i+1] \ldots [i+2j, i+2j+1] \ldots [2n-1, 2n]$, we can obtain the configuration $[1,2] \ldots [i+2j, i+1] \ldots [i, i+2j+1] \ldots [2n-1, 2n]$.

First we show that it is possible to go from $[1,2] \ldots [i, i+1][i+2, j+3] \ldots [2n-1, 2n]$ to $[1,2] \ldots [i+2, i+1][i, j+3] \ldots [2n-1, 2n]$:

$$
\begin{aligned}
X =&[1,2]\dots[i,i+1][i+2,j+3]\dots[2n-1,2n]\\
X_1 =&\tau_{[i,i+1]}(X)\\
=&[1,2]\dots[i+1,i][i+2,j+3]\dots[2n-1,2n]\\
X_2 =&\delta(X_1)\\
=&[2,3]\dots[i-1,i+1][i,i+2][i+3,i+4]\dots[2n,1]\\
X_4 =&\tau_{[i,i+2]}(X_2)\\
=&[2,3]\dots[i-1,i+1][i+2,i][i+3,i+4]\dots[2n,1]\\
X_5 =&\delta(X_4)\\
=&[1,2]\dots[i+1,i+2][i,i+3]\dots[2n-1,2n]\\
X_6 =&\tau_{[i+1,i+2]}(X_5)\\
=&[1,2]\dots[i+2,i+1][i,j+3]\dots[2n-1,2n]
\end{aligned}
$$

This shows that we can swap the positions of any two points in adjacent pairs (as if the point we wish to switch is on the right of the pair we simply apply τ at the start) while keeping the remaining points in the same position. We now show that by repeated applications of this process we can achieve the desire result.

If we denote the process shown above as $\sigma_{(i)}$, i.e. $\sigma_{(i)}(X)$ swaps the position of the element i with the left element of the next pair. Then:

$$
\begin{aligned}
Y =&[1,2]\dots[i,i+1]\dots[i+2j,i+2j+1]\dots[2n-1,2n]\\
Y_1 =&\sigma_{(i)}^{j}(Y)\\
=&[1,2]\dots[i+2,i+1][i+4,i+3]\dots[i+2j,i+2j-1][i,i+2j+1]\\
&\dots[2n-1,2n]\\
Y_2 =&\sigma(i+2j-2)(Y_1)\\
=&[1,2]\dots[i+2,i+1]\dots[i+2j,i+2j-3][i+2j-2,i+2j-1][i,i+2j+1]\\
&\dots[2n-1,2n]\\
&\vdots\\
Y_{j-1} =&\sigma(i+4)(Y_{j-2})\\
=&[1,2]\dots[i+2,i+1][i+2j,i+3][i+4,i+5]\dots[i,i+2j+1]\\
&\dots[2n-1,2n]\\
Y_j =&\sigma(i+2)(Y_{j-1})\\
=&[1,2]\dots[i+2j,i+1][i+2,i+3]\dots[i,i+2j+1]\dots[2n-1,2n].
\end{aligned}
$$

Therefore we can swap the position of any two elements while leaving the remainder unchanged. Hence we can achieve all possible configurations of the hyperedge. □

Now that we have shown that we can move freely between all configurations of a hyperedge we can define the twisted dual of a hypermap.

Definition 3.9.1. Let G and H be hypermaps then we say that G and H are *twisted duals* if and only if we can move from one to the other by applying any combination of δ and τ. The set of all twisted duals of a hypermap G is denoted $Orb(G)$.

3.10 Cycle Family Hypermaps

Now that we have defined the operations which act on the edges of hypermaps we need to define the cycle family hypermaps. Recall that for a vertex of degree 4 its *arrow marked vertex state* is formed by replacing the vertex with a pair of arrows connecting the half edges of the vertex. It is relatively simple to extend this definition to vertices of higher degrees.

Definition 3.10.1. Let v be a vertex in an even graph. Then we can form the *vertex state* of v by partitioning the half edges incident to v into pairs. We then replace each pair of edges with a single edge which bypasses v and then delete v.

Note that while there are only 3 possible states for a vertex of degree 4 this number increases dramatically as the degree of the vertex increases. For example there are 15 possible vertex states for a vertex of degree 6 and 105 for a vertex of degree 8.

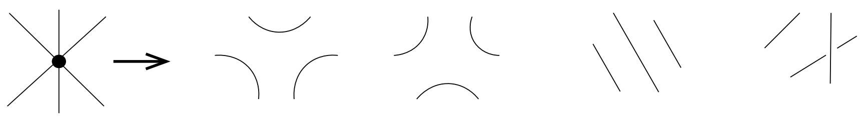


Figure 3.16: Four of the possible vertex states of a vertex v

Definition 3.10.2. A graph state s of any even graph F is a choice of vertex state at each of its vertices.

Definition 3.10.3. Let F be a even graph and v be a vertex of F then an *arrow marked vertex state* is a vertex state together with a cyclic ordering of its pairs. Where each pairing is connected with an arrow and each arrow is given a label v_i where the i denotes its position in the cyclic order. An *arrow marked graph state* $\vec{s}$ of any even graph F is a choice of an arrow marked vertex state at each of its vertices.

Observe that for each choice of vertex state there is a number of choices for the direction of the arrows. There may or may not be a local orientation with which the arrows all agree. There are also a number of ways in which we can order the arrows. Hence for each vertex state there are a number of different possible arrow marked vertex states. If the arrows agree with a local orientation we call the arrow marked vertex state *consistent* and if they disagree we call the state *inconsistent*.

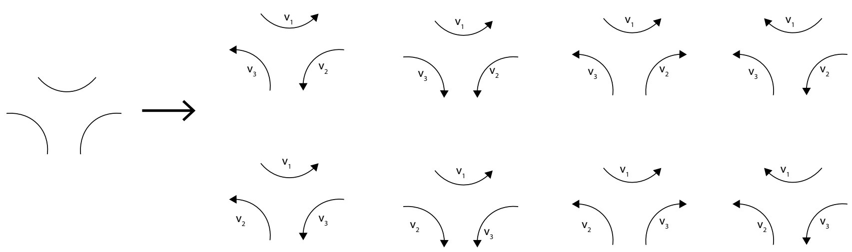


Figure 3.17: Eight of the possible arrow marked vertex states of a vertex state

Now observe if we form an arrow marked graph state of any even graph we have a set of closed curves and a set of labelled arrows, that is we have an arrow presentation of a hypermap. Hence we can now define the cycle family hypermaps.

Definition 3.10.4. Let F be an even cellularly embedded graph. A *cycle family hypermap* of F is an embedded hypermap obtained as the arrow presentation given by replacing each vertex with one of the possible arrow marked vertex states. We let $\mathfrak{C}(F)$ denote the set of Cycle Family hypermaps of F.

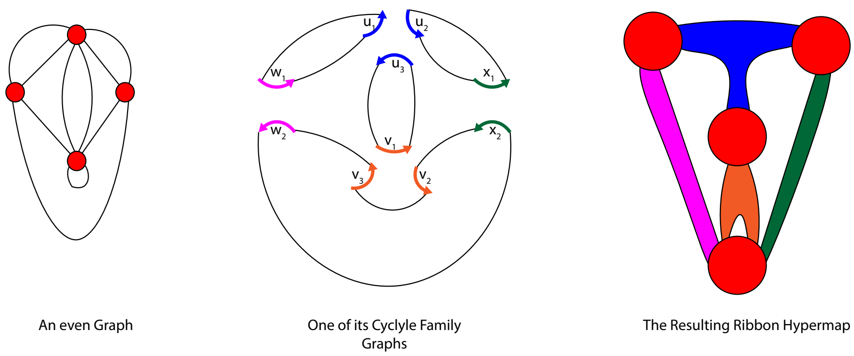


Figure 3.18: Forming a Cycle Family hypermap

3.11 Twisted Duality and Hypermap Equivalence

We now have all the tools we need to prove the analogues of the theorems stated in the previous chapters. We start by showing that two even graphs are equivalent as abstract graphs if and only if one is the medial graph of a cycle family hypermap of the other.

Theorem 3.11.1. *Let F be an even abstract graph and let $\tilde{F}$ be any embedding of F, so $\mathfrak{C}(\tilde{F})$ is the set of cycle family hypermaps of $\tilde{F}$. Then*

$$\mathfrak{C}(\tilde{F}) = \{G \mid G_m \cong F\}$$

i.e.

$$G_m \cong F \iff G \in \mathfrak{C}(\tilde{F})$$

for some embedding $\tilde{F}$ of F.

Proof. We first show that if $G \in \mathfrak{C}(\tilde{F})$, then $G_m \cong F$. The underlying abstract graph of G_m, where G is a cycle family hypermap of $\tilde{F}$, can be constructed as follows.

- G is obtained by, for each vertex v of $\tilde{F}$, choosing an arrow marked state and then forming the cycle family hypermap.
- We can form the medial graph of a hypermap directly from its arrow presentation as follows. At each set of n v-labelled arrows in the arrow presentation, replace the arrows with an n-valent vertex as follows: add a vertex; connect this vertex to the arrow presentation by adding an arc between the vertex and the tip and tail of each v-labelled arrow. Since we are only concerned with abstract graphs the order in which they are attached does not mater.
- Delete all of the arrows and the arcs on which they lie from the resulting diagram.

An example of this construction, for when the vertex is of degree six is shown in Figure 3.19.

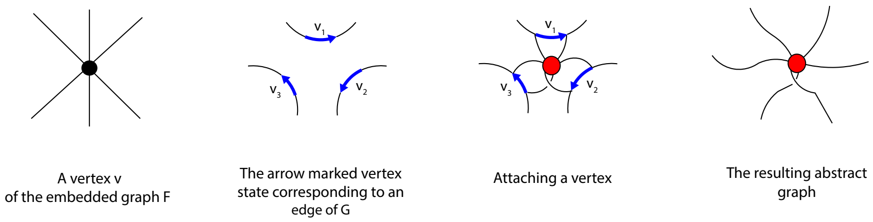


Figure 3.19: Forming the underlying abstract graph of G_m

This results in an even abstract graph that is clearly the underlying abstract graph of the medial graph G_m. For the arrow marked state shown in Figure 3.19, we can clearly see that the vertex v in $\tilde{F}$ and the corresponding vertex in G_m have the same adjacency information which is all we are interested in since we wish to show equivalence as abstract graphs. Moreover, doing the same construction with any of the other possible arrow marked vertex states also results in the vertex v in $\tilde{F}$ and the corresponding vertex in G_m having the same adjacency information, since the vertex will always connect to the same half edges regardless of which arrow marked vertex state is chosen. The choice will only effect the cyclic order. Thus G_m and $\tilde{F}$ have the same underlying abstract graphs, giving that $G_m \cong F$ as required. Note that although Figure 3.19 shows a vertex of degree six we could use a vertex of any even degree and achieve the same result.

Conversely, suppose that G is a hypermap such that $G_m \cong F$. Then $G_m \cong \tilde{F}$. Consider an edge e and think of G as both a ribbon hypermap and an arrow presentation. Figure. 3.20 gives an example of this. Let v_e be the vertex of G_m which corresponds to e and suppose the edges incident to v_e are in the cyclic order $(a\ b \dots n)$.

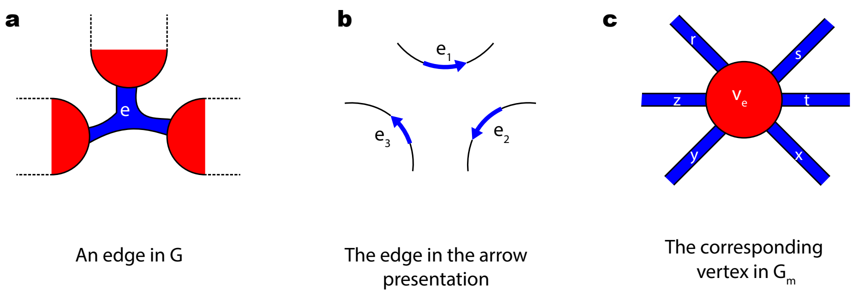


Figure 3.20: An edge of a ribbon hypermap and the corresponding vertex of its medial graph

We have that $G_m \cong \tilde{F}$, so let $\tilde{v}$ be the vertex of $\tilde{F}$ that identified with v_e under the equivalence, and $\tilde{a}\tilde{b}\ldots\tilde{n}$ be the edges corresponding to $(ab\ldots n)$. Then $\tilde{F}$ and G_m can only differ at $\tilde{v}$ and v_e in the number of half twists in the edges or in the cyclic ordering of their incident edges.

We need to show that G arises as a cycle family hypermap of $\tilde{F}$. That is, we need to show that regardless of the number of half-twists and the cyclic order at $\tilde{v}$, the state which gives the arrow presentation of the edge e can always be formed as an arrow marked state of $\tilde{v}$. First observe that adding half twists to the edges $\tilde{a}\tilde{b}\ldots\tilde{n}$ does not affect the equivalence class of the cycle family hypermap given by a particular set of arrow marked vertex states. Therefore for any cyclic ordering of the edges $\tilde{a}\tilde{b}\ldots\tilde{n}$ we need to show that there is an arrow marked vertex state that will connect the half edges in the same way as in the arrow presentation of the edge e. However we defined the arrow marked vertex state of a vertex to cover all possible permutations of connecting half edges so the appropriate choice of arrow marked vertex state must exist. If we do this at each vertex of F then we will recover G.

□

The next step is to show that cycle family hypermaps can be used to give an alternative construction of twisted duals and that $Orb(G)$ is equal to $\mathfrak{C}(G_m)$.

Theorem 3.11.2. *Let G be an embedded hypermap. Then the cycle family hypermaps of its medial hypergraph (G_m) are exactly its twisted duals, i.e.*

$$\mathfrak{C}(G_m) = Orb(G).$$

Proof. Let $H \in Orb(G)$. Then the arrow representation of each hyperedge of H will be a set of arrows connecting the half edges of H, that is some variation of the picture in Figure 3.5. But these are exactly the arrow presentations of the cycle family hypermaps of G_m that arise by replacing each vertex of G_m with one of the arrow marked states defined at the beginning of Section 3.10.

Hence $H \in \mathfrak{C}(G_m)$.

Now let $H \in \mathfrak{C}(G_m)$. Then it is obtained by replacing each vertex with one of the arrow marked states and then viewing these states as edges in an arrow presentation. Therefore to show that $H \in Orb(G)$ we need to show that for any arrow marked vertex state we can get to any other state by using the operations τ and δ. However we showed this in Theorem 3.9.1. Hence the arrow marked states of a cycle family hypermap are the arrow presentations of the twisted duals of G, so $H \in Orb(G)$. □

Therefore putting these two theorems together we have

Theorem 3.11.3. *Let G be an embedded hypermap. Then*

$$Orb(G) = \{H \mid H_m \cong G_m\}$$

i.e.

$$H_m \cong G_m \iff H \in Orb(G)$$

Proof. $Orb(G) = \mathfrak{C}(G_m) = \{H \mid H_m \cong G_m\}$ where the first equality follows from Theorem 3.11.2 and the second from Theorem 3.11.1 □

3.12 Partial Duality and Equivalence of Cyclically Ordered Graphs

In this section we show that as in the case of regular graphs two hypermaps are partial duals if and only if their medial graphs are equivalent as cyclically ordered graphs. Recall that a *Cyclically Ordered Graph*, or *cog*, is an abstract graph together with a cyclic ordering of the half-edges about each vertex and that two cogs G and H are *equivalent*, if there is an equivalence of the underlying abstract graphs that preserves or reverses the cyclic orders at the vertices.

We also need to define a specialisation of the cycle family hypermaps. For this we again take our cue from the graph case.

Definition 3.12.1. *The smoothing* of a vertex is formed by replacing the vertex with a set of arrows connecting adjacent half edges. A *consistent smoothing* is a smoothing in which the arrows are all pointing clockwise or all pointing anti-clockwise. Each vertex will have exactly two possible consistent smoothings.

If a graph is a checkerboard coloured, then we can use the colouring to differentiate the different smoothings of a vertex. We call a smoothing a white smoothing if the arrows are on the boundary of the white faces and a black smoothing if the arrows are on the boundary of the black faces.

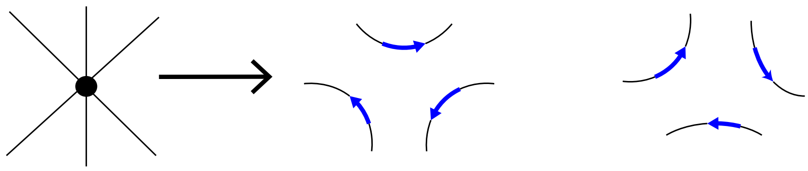


Figure 3.21: The Smoothings of a 6 Valent Vertex

Definition 3.12.2. Let F be an even cellularly embedded graph. *A duality state* of F is a state $\overrightarrow{S}$ formed by replacing each vertex of F with one of its consistent smoothings.

Definition 3.12.3. A *smoothing hypermap* of F is a hypermap obtained as the arrow presentation given by replacing each vertex by one of the two consistent smoothings. We let $\mathfrak{C}_{(\delta)}(F)$ denote the set of smoothing hypermaps of F.

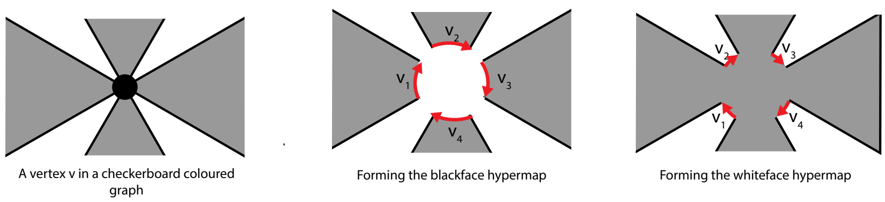


Figure 3.22: Forming the arrow presentation of Tait hypermaps

Recall that given a checkerboard coloured graph, the blackface hypermap is formed by placing a vertex in each black face and adding an edge whenever these faces meet at a vertex. Now note that if F is an even graph each vertex of degree $2n$ will be adjacent to precisely n black faces and hence will account for a hyperedge of size n in the blackface hypermap of F. As you can see in Fig.3.22 choosing the black smoothing at each edge means that we surround each black face of F with a closed curve and hence the black faces become the closed curves in the arrow presentation and as each vertex of F is replaced with a set of labelled arrows and since each arrow is on the boundary of one of the black faces they represent the hyperedges which connect adjacent black faces. Hence we have an arrow presentation of the blackface graph of F. Similarly if we chose the white smoothing at each vertex we would get the whiteface graph.

We can now show that $\{G \mid G_m \doteq H_m\} = Orb_{(\delta)}(H)$ using a similar process to how we showed $\{G \cong G_m \doteq H_m\} = Orb(H)$.

First observe that when considering equivalence of graphs as cogs we can ignore twists in the edges.

Lemma 3.12.1. *[25]*

1. *Let G be an embedded graph and $A \subseteq E(G)$. Then $G \doteq G^{\tau(A)}$*
2. *Let G and H be embedded graphs such that $G \doteq H$. Then $G = H^{\tau(A)}$ for some $A \subseteq E(G)$.*

This lemma is proved in [25]. However the main idea is that τ does not change the cyclic order of the edges and that if two graphs are equivalent as cogs then corresponding edges can only differ by the number of half twists.

We can now begin the process of showing that $\{G \mid G_m \doteq H_m\} = Orb_{(\delta)}(H)$, as before we start with cycle family hypermaps but in this case restrict ourselves to the set of smoothing hypermaps $\mathfrak{C}_{(\delta)}(F)$.

Theorem 3.12.1. *Let F be an even regular abstract graph and let $\tilde{F}$ be any embedding of F then*

$$\mathfrak{C}_{(\delta)}(\tilde{F}) = \{G \mid G_m \doteq F\}.$$

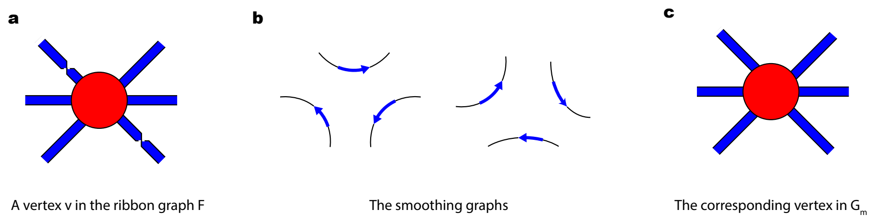


Figure 3.23: Forming the medial graph of a smoothing hypermap

Proof. Let G be a smoothing hypermap of $\tilde{F}$, that is $G \in \mathfrak{C}_{(\delta)}(\tilde{F})$ We need to show that $G_m \doteq \tilde{F}$. We start by viewing $\tilde{F}$ as a ribbon graph and considering a vertex v. Figure 3.23a. shows an example of this for a 6 valent vertex, however the same principle applies for any vertex with an even number of incident edges. Then forming the smoothing hypermap of $\tilde{F}$ results in v becoming one of the two arrow presentations shown in Figure. 3.23b. Finally forming the medial graph of of either of these arrow presentations will produce the ribbon graph shown in Figure. 3.23c.

We can see that the adjacency information of G_m and $\tilde{F}$, and the cyclic orders at vertices, is preserved. All that has possibly changed is the number of half twists in the edges. Hence $G_m = \tilde{F}^{\tau_A}$, where A is a subset of the arrows in the arrow presentation of $\tilde{F}$. But we know from Lemma. 3.12.1 that $\tilde{F} \doteq \tilde{F}^{\tau(A)}$ and hence we have $G_m \doteq \tilde{F}$ as required.

Conversly let $G_m \doteq \tilde{F}$. Then G_m and $\tilde{F}$ are partial petrials. As G_m is a medial graph G can be recovered from it as a Tait hypermap by choosing one of the duality states. Choosing the same duality state for $\tilde{F}$ then also results in G (as G_m and $\tilde{F}$ differ only in the number of half-twists on their edges). Thus G is a smoothing hypermap of $\tilde{F}$ and we are done. □

Theorem 3.12.2. *Let G be an embedded hypermap Then*

$$\mathfrak{C}_{(\delta)}(G_m) = Orb_{(\delta)}(G)$$

Proof. Let $H = G^{\delta(A)}$. Then the arrow presentation for each edge of H is one of those shown in Figure. 3.10 but these are exactly the arrow presentations of the smoothing hypermaps of G_m that arise by replacing each vertex with one of the duality states. Thus $H \in \mathfrak{C}_{(\delta)}(G_m)$.

Conversly if $H \in \mathfrak{C}_{(\delta)}(\tilde{G_m})$ then it is obtained by replacing each vertex of G_m with one of the duality states but these are exactly the arrow presentations of the twisted duals of G. Thus $H = G^{\delta(A)}$. □

We have shown that $G_m = H_m$ if and only if $H \in \{G, G^*\}$ and that $H_m \cong G_m$ if and only if $H \in Orb(G)$. We can now give the corresponding result for partial duality.

Theorem 3.12.3. *Let G be an embedded hypermap. Then*

$$Orb_{(\delta)}(G) = \{H \mid H_m \doteq G_m\}$$

i.e.

$$H_m \doteq G_m \iff H \in Orb_{(\delta)}(G)$$

Proof. This follows directly from combining the previous two theorems. □

We have now proved a hierarchy of equivalences for hypermaps and shown that all the results that apply for graphs have an equivalent result for hypermaps.

There are however a number of unanswered questions such as what happens if we restrict to partial petrials, does this give us another form of equivalence? Also if we allow edges to have unnatatched ends then this induces other forms of duality where we can interchange edges and vertices and edges and faces, as in this case there are no special conditions for the edges.

Part III

Matroids and Polynomials

Chapter 4

Matroids

Contents

4.1 Matroids

Matroids were first introduced in 1935 by Hassler Whitney [58] to provide a unifying abstract treatment of dependence in linear algebra. The subject was then relatively dormant until 1958 when Tutte published a series of papers [52], [53] on matroids and graphs, since then the subject has grown and matroids can be used to solve a wide variety of combinatorial problems.

We are interested in matroids as they can be used to prove a number of results about graphs and many well known graph polynomial are in fact special cases of matroid polynomials. Before we define what a matroid is we will need to introduce some basic definitions from set theory. Our terminology follows [9] and [47] except where specifically stated.

Definition 4.1.1. A *set system* is a pair $D = (E, \mathcal{F})$ where E is a set, which we call the ground set, and $\mathcal{F}$ is a collection of subsets of E, called *feasible sets*. A set system is *proper* if $\mathcal{F}$ is non empty.

Definition 4.1.2. The *symmetric difference* of two sets X and Y, denoted $X \triangle Y$, is defined as

$$X \triangle Y = (X \cup Y) - (X \cap Y)$$

Definition 4.1.3. A set system $D = (E, \mathcal{F})$ satisfies the *symmetric exchange axiom* if for all X and Y in $\mathcal{F}$, and for all $u \in X \triangle Y$ then there exists $v \in X \triangle Y$ such that $X \triangle \{u, v\} \in \mathcal{F}$.

We can now give the definition of a matroid.

Definition 4.1.4. A *matroid* is a proper set system $M = (E, \mathcal{B})$ that satisfies the symmetric exchange axiom and all the feasible sets or *bases* are equicardinal.

A subset I of E is an *independent set* of M if and only if it is a subset of a basis of M otherwise it is a *dependent set*.

An element $x \in E$ which is not contained in any bases is called a *loop* and an element $y \in E$ that is in every basis is called a *coloop*.

Definition 4.1.5. The *rank function* of a matroid $M = (E, \mathcal{B})$ is a function $r_M : 2^E \to \mathbb{Z}$ such that for all $A \subseteq E$

$$r_M(A) = \max\{|X \cap A| \mid X \in \mathcal{B}\}.$$

Note we will often just write r for r_M when the context is clear.

That is the rank of a subset A is is the cardinality of the largest independent set contained in A.

4.1.1 Duality, Deletion and Contraction

Definition 4.1.6. Given a matroid $M = (E, \mathcal{B})$ then the *dual*, M^*, of M, is defined as $M^* = (E, \mathcal{B}^*)$, where $\mathcal{B}^* = \{E \backslash A \mid A \in \mathcal{B}\}$.

Lemma 4.1.1. *Let* $M = (E, \mathcal{B})$ *be a matroid then its dual* M^* *is also a matroid and its rank function is given by*

$$r_{M^*}(A) = |A| + r_M(E \backslash A) - r_M(E).$$

Definition 4.1.7. Let $M = (E, \mathcal{B})$ be a matroid and let $e \in E$. Then if e is not a coloop of M we define M *delete* e, denoted $M \backslash e$, as

$$M \backslash e = (E \backslash e, \{X \mid X \in \mathcal{B} \text{ and } X \subseteq E \backslash e\}).$$

If e is not a loop of M we define M *contract* e, denoted M/e, as

$$M/e = (E \backslash e, \{X \backslash e \mid X \in \mathcal{B} \text{ and } e \in X\}).$$

If e is a loop or coloop, then $M/e = M \backslash e$.

Definition 4.1.8. Let $M = (E, \mathcal{B})$ be a matroid and let $A \subseteq E$ then $A^c = \{x \mid x \in E \text{ and } x \notin A\}$.

Lemma 4.1.2. *Let* $M = (E, \mathcal{B})$ *be a matroid with rank function* r *and let* $e \in E$*. Then the rank function of the contraction is*

$$r_{M/e}(A) = r(A \cup e) - r(e).$$

Both $M \backslash e$ and M/e are matroids.

The following Lemma follows directly from the definitions.

Lemma 4.1.3.

$$r(E) = \begin{cases} r(M/e) & \textit{if } e \textit{ is a loop,} \\ r(M/e) + 1 & \textit{otherwise} \end{cases}$$

4.1.2 Examples

We will now give a couple of examples of common matroids.

Example 1. Let E be a set of cardinality n and let $\mathcal{B}$ be all subsets of E of cardinality equal to k. Then $(E, \mathcal{B})$ is a matroid called the *uniform matroid* of rank k and denoted $U_{k,n}$.

We can see that there are two matroids on the set of one element $U_{0,1}$ and $U_{1,1}$ and in fact we can tell if an edge is a loop or coloop by reducing a matroid down to one of these by either deletion or contraction.

Lemma 4.1.4. *Let $M = (E, \mathcal{B})$ be a matroid and $e \in E$ then*

1. *e is a loop if and only if $M \backslash e^c = U_{0,1}$*

2. *e is a coloop if and only if $M/e^c = U_{1,1}$.*

Example 2. Let $G = (V(G), E(G))$ be a graph. Let $\mathcal{B}$ be the edge set of maximal spanning forests of G, then $C(G) = (E(G), \mathcal{B})$ is a matroid on $E(G)$ known as the *cycle matroid* of the graph G.

The independent sets of $C(G)$ are the the edge sets of the forests of G. Hence the dependent sets are the edge sets of the spanning subgraphs of G which contain cycles.

It is a well known fact that if G is connected any maximal spanning forest of G must have $|V(G)| - 1$ edges so if G has k connected components each maximal spanning subgraph must have $|V(G)| - k$ edges, hence the rank of $C(G)$ is $|V(G)| - k = r(G)$.

If an edge e is a loop in $C(G)$ then it is an edge which is not in any independent set and therefore must be a dependent set and hence must be a cycle so therefore e is a loop in the graph G.

If e is a coloop in $C(G)$ then e must be in every basis and hence e must be in every maximal spanning forest and therefore e is a bridge in G.

The dual of the cycle matroid is called the *bond matroid* $B(G) = (C(G))^*$.

Note that if G_R is a ribbon graph we write $C(G_R)$ to mean the cycle matroid of the underlying abstract graph of G_R, and $B(G_R) = (C(G_R))^*$ is the dual of the cycle matroid of the underlying abstract graph of G_R. Note that $B(G_R)$ does not necessarily equal $C(G_R^*)$.

4.2 Delta-Matroids

Delta-matroids were first introduced by Bouchet in [9]. They can be seen as a generalisation of matroids and can be used to represent embedded graphs in a similar way as matroids can be used to represent abstract graphs.

4.2.1 Introduction

Definition 4.2.1. A *delta-matroid* is a proper set system $D = (E, \mathcal{B})$ that satisfies the symmetric exchange axiom.

Seeing as matroids are specialisations of delta-matroids, many of the definitions and operations defined for matroids can also be applied to delta-matroids.

Definition 4.2.2. Let $D = (E, \mathcal{F})$ be a delta-matroid. An element of E which is not contained in any feasible sets is called a *loop* and an element of E that is in every feasible set is called a *coloop*.

Definition 4.2.3. If $D = (E, \mathcal{F})$ is a delta-matroid and $e \in E$, then if e is not a coloop of D we define D *delete* e, denoted $D\backslash e$, as

$$D\backslash e = (E\backslash e, \{X \mid X \in \mathcal{F} \text{ and } X \subseteq E\backslash e\}).$$

If e is not a loop of D we define D *contract* e, denoted D/e, as

$$D/e = (E\backslash e, \{X\backslash e \mid X \in \mathcal{F} \text{ and } e \in X\}).$$

If e is a loop or coloop, then $D/e = D\backslash e$.

Lemma 4.2.1. *[12] Let D be a delta-matroid. If D' is a delta-matroid obtained from D by a sequence of edge deletions and edge contractions, then*

D' is independent of the order of the deletions and contractions used in its construction.

We write $D\backslash A$ (respectively D/A) to mean the delta-matroid obtained by deleting (respectively contracting) all elements of A.

In [10] Bouchet defined two matroids which can be derived from a delta-matroid. Given a delta-matroid $D = (E, \mathcal{F})$ let $\mathcal{F}_{\min}$ and $\mathcal{F}_{\max}$ be the sets of feasible sets with minimum and maximum cardinality, respectively. We can define two matroids $D_{\min} = (E, \mathcal{F}_{\min})$ and $D_{\max} = (E, \mathcal{F}_{\max})$ which we call the *upper and lower matroids* of the delta-matroid D, and let $r_{\max}$ and $r_{\min}$ be the rank functions of the upper and lower matroids of D. We can now define two functions for delta-matroids.

Definition 4.2.4. Let $D = (E, \mathcal{F})$ be a delta-matroid and $A \subseteq E$. Then the *width* $w(D)$ of D is defined as

$$w(D) = r_{\max}(E) - r_{\min}(E)$$

and the width of the subset A is defined as

$$w(A) = w(D\backslash A^c).$$

We can also define a delta-matroid "rank" function as follows

Definition 4.2.5. [34] Let $D = (E, \mathcal{F})$ be a delta-matroid and $A \subseteq E$ then

$$\rho(D) = \frac{1}{2}(r_{\max}(E) + r_{\min}(E))$$

$$\rho_D(A) = \rho(D\backslash A^c).$$

We will often write ρ_D as ρ if the context is clear. Note that if D is a matroid then $D_{\max} = D_{\min}$ and so ρ is precisely the the rank function of D, however in general $\rho_D(A) \neq \frac{1}{2}(r_{D_{\max}}(A) + r_{D_{\min}}(A))$.

Definition 4.2.6. Given a delta-matroid $D = (E, \mathcal{F})$ and $T \subseteq E$, the *twist* of D with respect to T, denoted $D * T$ is defined as

$$D * T = (E, \{T \triangle X \mid X \in \mathcal{F}\}).$$

The *dual* of D, denoted D^* is defined to be $D * E$.

The twist of a delta-matroid is always a delta-matroid.

Lemma 4.2.2. *[9] Let D be a delta-matroid and let A be a subset of $E(D)$. Then $D * A$ is a delta-matroid.*

Also observe that if e is a loop of D then e is a coloop of D^* and vice versa.

Lemma 4.2.3. *Let $D = (E, \mathcal{F})$ be a delta-matroid and $e \in E$. Then:*

1. *if e is a loop in D then e is a coloop in D^*,*
2. *if e is a coloop in D then e is a loop in D^*.*

And note that the upper matroid of D^* is equal to the dual of the lower matroid and vice versa i.e.

Lemma 4.2.4. *Let $D = (E, \mathcal{F})$ be a delta-matroid then*

1. $(D^*)_{\max} = (D_{\min})^*$,
2. $(D^*)_{\min} = (D_{\max})^*$.

Just as edges in ribbon graphs could be further categorised so can elements in the ground set of a delta-matroid.

Definition 4.2.7. Let $D = (E, \mathcal{F})$ be a delta-matroid and $e \in E$ then we call e a *ribbon loop* if e is a loop in $D_{\min}$. A ribbon loop is *orientable* if e is not a loop in $(D * e)_{\min}$ otherwise it is *non-orientable*. A ribbon loop e is trivial if either e is a loop in D or $D = D * e$, and is non-trivial otherwise.

Lemma 4.2.5. *[34] Let $D = (E, \mathcal{F})$ be a delta-matroid and $e \in E$ then*

$$\rho(D) = \begin{cases} \rho(D/e) + 1 & \text{if } e \text{ is not a ribbon loop,} \\ \rho(D/e) & \text{if } e \text{ is an orientable ribbon loop,} \\ \rho(D/e) + \frac{1}{2} & \text{if } e \text{ is a non orientable ribbon loop.} \end{cases}$$

It has also been shown that there are relationships between duality and deletion/contraction.

Lemma 4.2.6. *Let $D = (E, \mathcal{F})$ be a delta-matroid and let $e \in E$ then*

$$D^*/e = (D\backslash e)^*$$

$$D^*\backslash e = (D/e)^*$$

Example 3. [34] There exist, up to isomorphism, exactly three delta-matroids over a single element. They are:

$$D_0 := (\{e\}, \{\emptyset\})$$
$$D_c := (\{e\}, \{\{e\}\})$$
$$D_n := (\{e\}, \{\emptyset, \{e\}\})$$

We can tell what type of edge and element e is by seeing which one of the single element delta-matroids the delta-matroid $D\backslash e^c$ is isomorphic to.

Lemma 4.2.7. *[34] Let $D = (E, \mathcal{F})$ be a delta-matroid and $e \in E$ then*

1. *e is not a ribbon loop if and only if $D\backslash e^c$ is isomorphic to D_c,*
2. *e is an orientable ribbon loop if and only if $D\backslash e^c$ is isomorphic to D_0,*
3. *e is a non-orientable ribbon loop if and only if $D\backslash e^c$ is isomorphic to D_n.*

The greater generality of delta-matroids compared to matroids allows us to capture extra information. In particular we can capture information on how a graph is embedded in a surface. That is we can use delta-matroids to represent embedded or ribbon graphs.

Theorem 4.2.1. *[20] Let $G_R = (V, E)$ be a ribbon graph and let $\mathcal{F}$ be the collection of the edge-sets of the spanning quasi-trees of G_R. Then $D(G_R) := (E, \mathcal{F})$ is a delta-matroid.*

Definition 4.2.8. Let D be a delta-matroid. If there exists a ribbon graph G_R such that $D = D(G_R)$ then we call D a *graphic delta-matroid.*

Lemma 4.2.8. *[20] Let G_R be a ribbon graph. Then*

1. $D(G_R)_{\min} = C(G_R)$,
2. $D(G_R)_{\max} = C(G_R^*)^* = B(G_R^*)$,
3. $D(G_R) = C(G_R)$ *if and only if* G_R *is a plane ribbon graph.*

We can also calculate ρ for a graphic delta-matroid directly from the ribbon graph.

Lemma 4.2.9. *[34] Let G_R be a ribbon graph and let $D(G_R) = (E, \mathcal{F})$ be its graphic delta-matroid. Then*

$$\rho(A) = \frac{1}{2}(|A| + v(A) - f(A)).$$

Just as bridges (loops) in a graph G correspond to coloops (loops) in $C(G)$ there is a similar correspondence between ribbon graphs and graphic delta-matroids. Recall that a loop in a ribbon graph is said to be non-orientable if it is homeomorphic to a Möbius band. Otherwise it is orientable. A loop is non-trivial if it is interlaced with some cycle in G. Otherwise the loop is trivial.

Lemma 4.2.10. *[20] Let G be a ribbon graph, $D(G) = (E, F)$, and $e \in E(G)$. Then*

- *e is a coloop in $D(G)$ if and only if e is a bridge in G_R;*

- *e is a loop in $D(G)$ if and only if e is a trivial orientable loop in G_R.*

Theorem 4.2.2. *[20] Let G_R be a ribbon graph. Then $D(G_R^*) = D(G_R)^*$.*

Note however that in general $C(G^*) \neq C(G)^*$. Similarly minors of a ribbon graph correspond directly with the minors of its graphic delta matroid.

Lemma 4.2.11. *[20] Let G_R be a ribbon graph with $e \in E(G_R)$. Then*

1. $D(G_R \backslash e) = D(G_R) \backslash e$
2. $D(G_R/e) = D(G_R)/e$

4.3 Matroid Perspectives

Definition 4.3.1. [4], [39] Let M and M' be two matroids on a set E we say that (M, M') is a *matroid perspective* if for any A, B such that $B \subseteq A \subseteq E$

$$r_{M'}(A) - r_{M'}(B) \leq r_M(A) - r_M(B).$$

If two matroids form a matroid perspective then their minors will also be matroid perspectives i.e.

Lemma 4.3.1. *[39] If (M, M') is a matroid perspective then for any $A \subseteq E$ we also have $(M \backslash A, M' \backslash A)$ and $(M/A, M'/A)$ are matroid perspectives.*

Lemma 4.3.2. *[39] If (M, M') is a matroid perspective then $(M, M')^* = (M'^*, M^*)$ is a matroid perspective.*

Lemma 4.3.3. *Let (M, N) and (N, P) be matroid perspectives then (M, P) is a matroid perspective.*

Theorem 4.3.1. *[47] Let $M = (E, \mathcal{B})$ be a matroid and let $e \in E$. Then $(M/e, M \backslash e)$ is a matroid perspective.*

Theorem 4.3.2. *Let (M, M') be a matroid perspective on a set E and let $e \in E$ then*

- *If e is a loop in M then e is also a loop in M'.*

- *If e is a coloop in M' then e is also a coloop in M.*

Proof. We know from Lemma 4.3.1 that we can take minors of matroid perspectives and in particular we know that $(M\backslash e^c, M'\backslash e^c)$ is a matroid perspective. Now if e is a loop in M then $M\backslash e^c = U_{0,1}$ and if e is not a loop in M' then $M'\backslash e^c = U_{1,1}$. Now if we let $A = \{e\}$ and $B = \emptyset$ then we have $r_{M\backslash e^c}(A) - r_{M\backslash e^c}(B) = 0 - 0 < 1 - 0 = r_{M'\backslash e^c}(A) - r_{M'\backslash e^c}(B)$ which is a contradiction to $(M\backslash e^c, M'\backslash e^c)$ being a matroid perspective. Hence e must be a loop in M'. Similarly if e is a coloop in M' then $M'/e^c = U_{1,1}$ and if e is not a coloop in M then $M'/e^c = U_{0,1}$. Now if we let $A = \{e\}$ and $B = \emptyset$ then we have $r_{M/e^c}(A) - r_{M/e^c}(B) = 0 - 0 < 1 - 0 = r_{M'/e^c}(A) - r_{M'/e^c}(B)$ which is a contradiction to $(M/e^c, M'/e^c)$ being a matroid perspective hence e must be a coloop in M'. □

Example 4. Let G_R be a ribbon graph then $B(G_R^*) \to C(G_R)$ is a matroid perspective.

Example 5. [11] Let D be a delta-matroid then $(D_{\max}, D_{\min})$ is a matroid perspective.

Chapter 5

Delta-matroid perspectives

Contents

5.1 Delta-matroid perspectives

We extend the notion of a matroid perspective to delta-matroids.

Definition 5.1.1. A *delta-matroid perspective* is a triple (M, D, M') where M and M' are matroids, D is a delta-matroid, and M, M' and D are over the same ground set, such that $(M, D_{\max})$ and $(D_{\min}, M')$ are matroid perspectives. We will often use $\mathbf{P}$ to denote a delta-matroid perspective.

Example 6. Recall that $U_{k,n}$ denotes a *uniform matroid* of rank k and the delta-matroids with one element are $D_c := (\{e\}, \{\{e\}\})$, $D_o := (\{e\}, \{\emptyset\})$, and $D_n := (\{e\}, \{\emptyset, \{e\}\})$. The *trivial* delta-matroid perspective $(U_{0,0}, U_{0,0}, U_{0,0})$ is the unique delta-matroid perspective over the empty set. Up to isomorphism, there are exactly five delta-matroid perspective over a one element set:

$$(U_{1,1}, D_c, U_{0,1}), (U_{1,1}, D_c, U_{1,1}), (U_{1,1}, D_o, U_{0,1}), (U_{0,1}, D_o, U_{0,1}), (U_{1,1}, D_n, U_{0,1}).$$

Example 7.

1. M is a matroid if and only if (M, M, M) is a delta-matroid perspective.

2. D is a delta-matroid if and only if $(D_{\max}, D, D_{\min})$ is a delta-matroid perspective.

3. (M, M') is a matroid perspective if and only if (M, M, M') is a delta-matroid perspective if and only if (M, M', M') is a delta-matroid perspective.

4. If (M, D, M') is a delta-matroid perspective then (M, M') is a matroid perspective.

Proof. Items one to three are trivial. For item four recall that $(D_{\max}, D_{\min})$ is a matroid perspective. As $(M, D_{\max})$ and $(D_{\max}, D_{\min})$ are matroid perspectives by Lemma 4.3.3 $(M, D_{\min})$ is a matroid perspective. But we also know that $(D_{\min}, M')$ is a matroid perspective by applying Lemma 4.3.3 again so (M, M') is a matroid perspective. □

Lemma 5.1.1. *Let (M, D, M') be a delta-matroid perspective. Then $(M, D, M')^* = (M'^*, D^*, M^*)$ is also a delta-matroid perspective.*

Proof. Since (M, D, M') is a delta-matroid perspective we know that $(M, D_{\max})$ and $(D_{\min}, M')$ are matroid perspectives. Lemma 4.3.2 tells us that $(D^*_{\max}, M^*)$ and $(M'^*, D^*_{\min})$ are matroids perspectives and then applying Lemma 4.2.4 tells us $((D^*)_{\min}, M^*)$ and $(M'^*, (D^*)_{\max})$ are matroid perspectives. Hence, (M'^*, D^*, M^*) is a delta-matroid perspective. □

Lemma 5.1.2. *Let $M = (E, \mathcal{B})$ and $M' = (E, \mathcal{B}')$ be matroids and let $D = (E, \mathcal{F})$ be a delta-matroid. Let (M, D, M') be a delta-matroid perspective and let $e \in E$ then*

1. *if e is a loop in M then e is a loop in D and M'*

2. *if e is a coloop in M' then e is a coloop in D and M*

3. if e is a ribbon loop in D then e is a loop in M'.

Proof. For item 1 we know that $(M, D_{\max}), (D_{\max}, D_{\min}), (D_{\min}, M')$ and (M, M') are matroid perspectives and therefore by Theorem 4.3.2 we know that if e is a loop in M then it must also be a loop in $D_{\max}, D_{\min}$ and M', therefore it just remains to show that e is a loop in D.

Assume e is not a loop. Then there exists a set $X \in \mathcal{F}$ such that $e \in X$. Let $Y \in \mathcal{F}_{\max}$. Now e is a loop in $D_{\max}$ so $e \notin Y$ hence $e \in Y \triangle X$, therefore by the symmetric exchange axiom there exists $u \in Y \triangle X$ such that $Y \triangle \{u, e\} \in \mathcal{F}$ but since $e \in Y \triangle \{u, e\}$ as $e \notin Y$, we have $Y \triangle \{u, e\} \in \mathcal{F}_{\max}$, so we have a contradiction so e must be a loop in D.

For item 2 if e is a coloop in M' then from Lemma 4.2.3 we know that e is a loop in M'^*. Now we know from Lemma 5.1.1 that as (M, D, M') is a delta-matroid perspective, then so is (M'^*, D^*, M^*) and since e is a loop in M'^* we know from item 1 that it is also a loop in D^* and M^*. Therefore, by applying Lemma 4.2.3 again we have that e is a coloop in D and M as required.

Item 3 follows directly from Theroem 4.3.2 since if e is a ribbon loop in D then e is a loop in $D_{\min}$ so since $(D_{\min}, M')$ is a matroid perspective, e is a loop in M'. □

We now want to show that delta-matroid perspectives are closed under deletion and contraction. However this does not follow directly from the fact that matroid perspectives are closed under deletion and contraction since in general $D_{\max} \backslash e \neq (D \backslash e)_{\max}$ and $D_{\min}/e \neq (D/e)_{\min}$. Therefore we need a new definition and the following lemmas to complete the proof.

Definition 5.1.2. Let $D = (E, \mathcal{F})$ be a delta-matroid. Then we define $\mathcal{F}_{\min + i} = \{X \mid X \in \mathcal{F} \text{ and } |X| = r(D_{\min}) + i\}$.

Lemma 5.1.3. *[20] Let $D = (E, \mathcal{F})$ be a delta-matroid and $e \in E$ be a non-orientable ribbon loop. Let $F \subseteq E \backslash e$. Then $F \in \mathcal{F}_{\min}$ if and only if $F \cup e \in \mathcal{F}_{\min + 1}$.*

Lemma 5.1.4. *Let $D = (E, \mathcal{F})$ be a delta-matroid, $M = (E, \mathcal{B})$ be a matroid and $e \in E$. Then if $(D_{\min}, M)$ is a matroid perspective then so are $((D\backslash e)_{\min}, M\backslash e)$ and $((D/e)_{\min}, M/e)$.*

Proof. We know from Lemma 4.3.1 that if $(D_{\min}, M)$ is a matroid perspective then so are $((D_{\min})\backslash e, M\backslash e)$ and $((D_{\min})/e, M/e)$. So we now need to show that if $((D_{\min})\backslash e, M\backslash e)$ and $((D_{\min})/e, M/e)$ are matroid perspectives then so are $((D\backslash e)_{\min}, M\backslash e)$ and $((D/e)_{\min}, M/e)$.

We start with the deletion case. If e is a coloop of D then by definition e is in every set in $\mathcal{F}(D)$ and $\mathcal{F}(D\backslash e) = \mathcal{F}(D/e) = \{X\backslash e \mid X \in \mathcal{F}(D) \text{ and } e \in X\}$. Therefore $(\mathcal{F}(D\backslash e))_{\min}$ is obtained by taking the minimal sets of $\{X\backslash e \mid X \in \mathcal{F}(D) \text{ and } e \in X\}$. Since e is in every set of $\mathcal{F}(D)$ this is just the minimal sets of $\mathcal{F}(D)$ with the element e removed from each of them. However this is also how we would obtain $\mathcal{F}((D_{\min})\backslash e)$. Hence $((D_{\min})\backslash e = (D\backslash e)_{\min}$ and therefore

$$((D\backslash e)_{\min}, M\backslash e) = ((D_{\min})\backslash e, M\backslash e)$$

and so if $(D_{\min}, M)$ is a matroid perspective then so is $((D\backslash e)_{\min}, M\backslash e)$ when e is a coloop.

Now suppose e is not a coloop. Then we need to show there is a set in $\mathcal{F}(D)_{\min}$ that does not contain e. Let $X \in \mathcal{F}(D)_{\min}$. Then either $e \notin X$ and we are done or $e \in X$. If $e \in X$ then since e is not a coloop there exists a set $Y \in \mathcal{F}(D)$ such that $e \notin Y$. Therefore $e \in X \triangle Y$ and the symmetric exchange axiom gives that there exists $v \in X \triangle Y$ such that $X \triangle \{e, v\} \in \mathcal{F}(D)$. Therefore since $e \in X$ and $X \in \mathcal{F}(D)_{\min}$ then $e \notin X \triangle \{e, v\}$ and $X \triangle \{e, v\} \in \mathcal{F}(D)_{\min}$.

Now since e is not a coloop $\mathcal{F}(D\backslash e) = \{X \mid X \in \mathcal{F}(D) \text{ and } X \subseteq E\backslash e\}$. Now since e is not in some element of $\mathcal{F}(D)_{\min}$ we can see that $\mathcal{F}(D\backslash e)_{\min} = \{X \mid X \in \mathcal{F}(D)_{\min} \text{ and } e \notin X\}$. However this is exactly $\mathcal{F}((D_{\min})\backslash e)$. Hence $((D_{\min})\backslash e = (D\backslash e)_{\min}$ and therefore

$$((D\backslash e)_{\min}, M\backslash e) = ((D_{\min})\backslash e, M\backslash e)$$

and so if if $(D_{\min}, M)$ is a matroid perspective then so is $((D\backslash e)_{\min}, M\backslash e)$

when e is not a coloop.

We now prove the contraction case. We will consider three cases: when e is not a ribbon loop, when e is a non-orientable ribbon loop and when e is an orientable ribbon loop. The first two cases are reasonably straightforward but the final case is more complicated.

First suppose e is not a ribbon loop in D. Then e is not a loop in $D_{\min}$ and so is not a loop in D. Therefore $\mathcal{F}(D/e) = \{X\backslash e \mid X \in \mathcal{F}(D) \text{ and } e \in X\}$. Now it is clear that $\mathcal{F}(D/e)_{\min} = \{X\backslash e \mid X \in \mathcal{F}(D)_{\min} \text{ and } e \in X\}$, but these are also the feasible sets of $(D_{\min})/e$. Therefore $(D_{\min})/e = (D/e)_{\min}$ and hence

$$((D/e)_{\min}, M/e) = ((D_{\min})/e, M/e).$$

Therefore if $(D_{\min}, M)$ is a matroid perspective then so is $((D/e)_{\min}, M/e)$ when e is a not a ribbon loop.

Next suppose that e is a non-orientable ribbon loop. By definition e is not in any set of $\mathcal{F}(D)_{\min}$ but from Lemma 5.1.3 we know that e is in some set of $\mathcal{F}(D)_{\min+1}$ and hence e is not a loop in D. Therefore $\mathcal{F}(D/e) = \{X\backslash e \mid X \in \mathcal{F}(D) \text{ and } e \in X\}$. Since e is not in any sets of $\mathcal{F}(D)_{\min}$ but e is in some sets of $\mathcal{F}(D)_{\min+1}$ then it is clear that $\mathcal{F}(D/e)_{\min} = \{X\backslash e \mid X \in \mathcal{F}(D)_{\min+1} \text{ and } e \in X\}$.

Now Lemma 5.1.3 tells us that a set $X \in \mathcal{F}(D)_{\min}$ if and only if $X \cup e \in \mathcal{F}(D)_{\min+1}$, therefore we can obtain the sets of $\mathcal{F}(D)_{\min}$ by removing e from the sets of $\mathcal{F}(D)_{\min+1}$ that contain e. Hence $\mathcal{F}(D/e)_{\min} = \mathcal{F}(D)_{\min}$. Since e is a loop in $D_{\min}$ then it follows that $\mathcal{F}(D)_{\min} = \mathcal{F}(D_{\min}/e)$ and so $\mathcal{F}(D/e)_{\min} = \mathcal{F}(D_{\min}/e)$. Hence

$$((D/e)_{\min}, M/e) = ((D_{\min})/e, M/e).$$

Therefore if $(D_{\min}, M)$ is a matroid perspective then so is $((D/e)_{\min}, M/e)$ when e is a a non-orientable ribbon loop.

Finally suppose that e is an orientable ribbon loop. There are two subcases, either e is a loop in D or it isn't. If e is a loop in D then $\mathcal{F}(D/e) = \mathcal{F}(D)$, and so $(D/e)_{\min} = (D_{\min}/e)$ and therefore if $(D_{\min}, M)$ is a matroid perspective then so is $((D/e)_{\min}, M/e)$.

Now suppose that e is not a loop in D. We know e is not in any set of $\mathcal{F}(D)_{\min}$ or $\mathcal{F}(D)_{\min+1}$ but e must be in some set of $\mathcal{F}(D)$. In fact we can show that there exists a set in $\mathcal{F}(D)_{\min+2}$ which contains e. Let $X \in \mathcal{F}(D)_{\min}$ and $Y \in \mathcal{F}$ such that $e \in Y$. Then $e \in X \triangle Y$, and so by the symmetric exchange axiom there exists $v \in X \triangle Y$ such that $X \triangle \{e, v\} \in \mathcal{F}$. Now since e is not in any set of $\mathcal{F}(D)_{\min}$ or $\mathcal{F}(D)_{\min+1}$ then $X \triangle \{e, v\} \in \mathcal{F}(D)_{\min+2}$ as required. In fact we have

$$X \in \mathcal{F}(D)_{\min} \implies X \triangle \{e, v\} \in \mathcal{F}(D)_{\min+2} \quad \text{for some } v \in E. \tag{5.1}$$

As e is not a loop, $\mathcal{F}(D/e) = \{X \backslash e \mid X \in \mathcal{F}(D) \text{ and } e \in X\}$. Since e is not in any sets of $\mathcal{F}(D)_{\min}$ or $\mathcal{F}(D)_{\min+1}$ but is in some sets of $\mathcal{F}(D)_{\min+2}$ it follows that

$$\mathcal{F}(D/e)_{\min} = \{X \backslash e \mid X \in \mathcal{F}(D)_{\min+2} \text{ and } e \in X\}. \tag{5.2}$$

Now consider the matroid $N = (D * e)_{\min}$. Recall that $D * e = (E, \{X \triangle e \mid X \in \mathcal{F}\})$ and note that if the size of a set in $\mathcal{F}_{\min}$ is y then

$$X \in \mathcal{F}_{\min} \implies X \triangle e = X \cup e \implies |X \triangle e| = y + 1$$

,

$$X \in \mathcal{F}_{\min+1} \implies X \triangle e = X \cup e \implies |X \triangle e| = y + 2$$

,

$$X \in \mathcal{F}_{\min+2} \text{ and } e \in X \implies X \triangle e = X \backslash e \implies |X \triangle e| = y + 1$$

,

$$X \in \mathcal{F}_{\min+2} \text{ and } e \notin X \implies X \triangle e = X \cup e \implies |X \triangle e| = y + 3$$

.

Therefore, it should be clear that the minimum sets of $D * e$ are the sets of

$\mathcal{F}_{\min}$ with an e added and the sets of $\mathcal{F}_{\min+2}$ which contain e with that e removed. That is

$$N = (D * e)_{\min} = (E, \{X \cup e \mid X \in \mathcal{F}_{\min}\} \cup \{Y \backslash e \mid Y \in \mathcal{F}_{\min+2} \text{ and } e \in Y\}) \tag{5.3}$$

Therefore

$$N \backslash e = (E \backslash e, \{Y \backslash e \mid Y \in \mathcal{F}_{\min+2} \text{ and } e \in Y\}) = \mathcal{F}(D/e)_{\min}$$

and

$$N/e = (E \backslash e, \{X \mid \in \mathcal{F}_{\min}\}) = D_{\min}$$

Now, from Theorem 4.3.1 we know that $(N\backslash e, N/e)$ is a matroid perspective i.e.

$$(\mathcal{F}(D/e)_{\min}, D_{\min})$$

is a matroid perspective and hence for $A \subseteq B \subseteq E$

$$r_{(D/e)_{\min}}(B) - r_{(D/e)_{\min}}(A) \geq r_{D_{\min}}(B) - r_{D_{\min}}(A) \tag{5.4}$$

Now we know from Lemma 4.1.2 that $r_{M/e}(A) = r(A \cup e) - r(e)$ so $r_{M/e}(B) - r_{M/e}(A) = r(B \cup e) - r(A \cup e)$. Therefore since $(D_{\min}, M)$ is a matroid perspective and if $A \subseteq B \subseteq E\backslash e$ we have

$$r_{D_{\min}}(B \cup e) - r_{D_{\min}}(A \cup e) \geq r_M(B \cup e) - r_M(A \cup e) = r_{M/e}(B) - r_{M/e}(A). \tag{5.5}$$

Now since e is not in any of the sets of $\mathcal{F}(D)_{\min}$, we can see that for all $F \subseteq E$ we have

$$r_{D_{\min}}(F) = \max_{X \in \mathcal{F}(D)_{\min}} \{|F \cap X|\} = \max_{X \in \mathcal{F}(D)_{\min}} \{|(F \cup e) \cap X|\} = r_{D_{\min}}(F \cup e). \tag{5.6}$$

Now combining Equations 5.4, 5.5 and 5.6 gives

$$\begin{aligned} r_{(D/e)_{\min}}(B) - r_{(D/e)_{\min}}(A) &\geq r_{D_{\min}}(B) - r_{D_{\min}}(A) \\ &= r_{D_{\min}}(B \cup e) - r_{D_{\min}}(A \cup e) \\ &\geq r_{M/e}(B) - r_{M/e}(A). \end{aligned}$$

Hence $((D/e)_{\min}, M/e)$ is a matroid perspective. □

Lemma 5.1.5. *Let $D = (E, \mathcal{F})$ be a delta-matroid, $M = (E, \mathcal{B})$ be a matroid and $e \in E$. Then if $(M, D_{\max})$ is a matroid perspective then so are $(M\backslash e, (D\backslash e)_{\max})$ and $(M/e, (D/e)_{\max})$.*

Proof. Recall Lemmas 4.2.4 and 4.2.6 which tell us that $(D^*)_{\max} = (D_{\min})^*$, $(D^*)_{\min} = (D_{\max})^*$, $D^*/e = (D\backslash e)^*$ and $D^*\backslash e = (D/e)^*$. Now if $(M, D_{\max})$ is a matroid perspective then by Lemma 4.3.2 so is $(M, D_{\max})^* = ((D_{\max})^*, M^*)$. As $(D_{\max})^* = (D^*)_{\min}$ we have $((D^*)_{\min}, M^*)$ is a matroid perspective. We can therefore apply Lemma 5.1.4 to get that $((D^*/e)_{\min}, M^*/e)$ is a matroid perspective. Now by applying the above identities we have

$$\begin{aligned}((D^*/e)_{\min}, M^*/e) &= ((D\backslash e)^*_{\min}, (M\backslash e)^*) \\ &= (((D\backslash e)_{\max})^*, (M\backslash e)^*) \\ &= (M\backslash e, (D\backslash e)_{\max})^*.\end{aligned}$$

Therefore $(M\backslash e, (D\backslash e)_{\max})^*$ is a matroid perspective hence $(M\backslash e, (D\backslash e)_{\max})$ is a matroid perspective.

Similarly we have that if $(M, D_{\max})$ is a matroid perspective then $((D^*\backslash e)_{\min}, M^*\backslash e)$ is a matroid perspective. Furthermore

$$\begin{aligned}((D^*\backslash e)_{\min}, M^*\backslash e) &= ((D/e)^*_{\min}, (M/e)^*) \\ &= (((D/e)_{\max})^*, (M/e)^*) \\ &= (M/e, (D/e)_{\max})^*.\end{aligned}$$

Therefore $(M/e, (D/e)_{\max})^*$ is a matroid perspective. Hence $(M/e, (D/e)_{\max})$ is a matroid perspective. □

Theorem 5.1.1. *Let (M, D, M') be a delta-matroid perspective over E, and let $e \in E$. Then $(M, D, M')\backslash e = (M\backslash e, D\backslash e, M'\backslash e)$ and $(M, D, M')/e = (M/e, D/e, M'/e)$ are both delta-matroid perspectives.*

Proof. Since (M, D, M') is a delta-matroid perspective we know that $(M, D_{\max})$ and $(D_{\min}, M')$ are matroid perspectives. Then Lemma 5.1.4 tells us that

$((D\backslash e)_{\min}, M'\backslash e)$ and $((D/e)_{\min}, M'/e)$ are matroid perspectives and Lemma 5.1.5 tells us that $(M\backslash e, (D\backslash e)_{\max})$ and $(M/e, (D/e)_{\max})$ are also matroid perspectives

Therefore we have $(M, D, M')\backslash e$ and $(M, D, M')/e$ are both delta-matroid perspectives. □

Theorem 5.1.2. *Let* (M, D, M') *be a delta-matroid perspective then* $(M, D, M')^* = ((M')^*, D^*, M^*)$ *is a delta-matroid perspective.*

Proof. We have that (M, D, M') is a delta-matroid perspective so we know that $(M, D_{\max})$ and $(D_{\min}, M')$ are matroid perspectives. Now Lemma 4.3.2 tells us that $((D_{\max})^*, M^*)$ and $((M')^*, (D_{\min})^*)$ are matroid perspectives. But we know from Lemma 4.2.4 that $(D^*)_{\max} = (D_{\min})^*$ and $(D^*)_{\min} = (D_{\max})^*$. Hence $((D^*)_{\min}, M^*)$ and $((M')^*, (D^*)_{\max})$ are matroid perspectives and therefore $(M, D, M')^* = ((M')^*, D^*, M^*)$ is a delta-matroid perspective. □

5.2 DM-perspectives and MD-perspectives

Two natural objects arise by considering pairs of elements of a delta-matroid perspective:

Definition 5.2.1. Let $M = (E, r)$ be a matroid, and $D = (E, \mathcal{F})$ be a delta-matroid. Then

1. the pair (D, M) is an *DM-perspective* if $(D_{\min}, M)$ is a matroid perspective, and
2. the pair (M, D) is an *MD-perspective* if $(M, D_{\max})$ is a matroid perspective.

DM-perspectives will prove to be important later when studying the Bollobás-Riordan polynomial. We can define deletion and contraction for DM-perspectives in a similar way to how we defined it for delta-matroid perspectives.

Theorem 5.2.1. *Let (D, M) be a DM-perspective over E, and let $e \in E$. Then $(D, M)\backslash e = (D\backslash e, M\backslash e)$ and $(D, M)/e = (D/e, M/e)$ are both DM-perspectives.*

Proof. Since (D, M) is a DM-perspective we know that $(D_{\min}, M')$ is a matroid perspective. Now we know from Lemmas 5.1.4 and 5.1.5 that if $(D, M)\backslash e$ and $(D, M)/e$ are both delta-matroid perspectives. □

We also provide an analogous result for MD-perspectives.

Theorem 5.2.2. *Let (M, D) be a MD-perspective over E, and let $e \in E$. Then $(M, D)\backslash e = (M\backslash e, D\backslash e)$ and $(M, D)/e = (M/e, D/e)$ are both MD-perspectives.*

Recall that when we take the dual of a delta-matroid perspective we reverse the order of the matroids. Therefore it follow that when we take the dual of a DM-perspective we obtain an MD-perspective.

Definition 5.2.2. Let (D, M) be a DM-perspective. Then $(D, M)^* = (M^*, D^*)$

Theorem 5.2.3. *Let (D, M) be a DM-perspective. Then $(D, M)^*$ is an MD-perspective.*

Proof. We have that (D, M) is a DM-perspective so we know that $(D_{\min}, M)$ is a matroid perspective. Now Lemma 4.3.2 tells us that $(M^*, (D_{\min})^*)$ is a matroid perspective. But we know from Lemma 4.2.4 that $(D^*)_{\max} = (D_{\min})^*$. Hence $(M^*, (D^*)_{\max})$ is a matroid perspective and therefore $(D, M)^* = (M^*, D^*)$ is an MD-perspective. □

Similarly the dual of a MD-perspective is a DM-perspective.

Theorem 5.2.4. *Let (M, D) be a MD-perspective. Then $(M, D)^* = (D^*, M^*)$ and (D^*, M^*) is an DM-perspective.*

5.3 Delta-Matroid Perspectives and Coloured Ribbon Graphs

Recall that a coloured ribbon graph $\mathcal{G} = (G_R, \mathcal{P}, \mathcal{Q})$ consists of a ribbon graph $G_R = (V, E)$ a partition $\mathcal{P}$ of its vertex set V and a partition $\mathcal{Q}$ on its boundary components F. Also recall that $\mathcal{G}^* = (G_R^*, \mathcal{P}^*, \mathcal{Q}^*)$ and that $G_{/\mathcal{P}}$ is defined as the abstract graph obtained by identifying all the elements of a part of $\mathcal{P}$ as a single vertex.

Definition 5.3.1. Given a coloured ribbon graph $\mathcal{G} = (G_R, \mathcal{P}, \mathcal{Q})$ we define the matroid $C(\mathcal{G})$ as the graphic matroid $C(G_{/\mathcal{P}})$ that is the matroid of the abstract graph $G_{/\mathcal{P}}$. Similarly we define the bond matroid $B(\mathcal{G})$ as the dual of $C(\mathcal{G})$ i.e. $B(\mathcal{G}) = C(\mathcal{G})^*$.

Note when we are considering the bond matroid of a coloured ribbon graph we have

$$B(\mathcal{G}^*) = C(\mathcal{G}^*)^* = C(G_R^*, \mathcal{P}^*, \mathcal{Q}^*)^* = C((G_R^*)_{/\mathcal{P}^*})^*$$

Theorem 5.3.1. *Let $\mathcal{G} = (G_R, \mathcal{P}, \mathcal{Q})$ be a coloured ribbon graph. Then $(B(\mathcal{G}^*), D(G_R), C(\mathcal{G}))$ is a delta-matroid perspective.*

Proof. We need to show that $(D_{\min}(G_R), C(\mathcal{G}))$ and $(B(\mathcal{G}^*), D_{\max}(G_R))$ are matroid perspectives. First we will show $(D_{\min}(G_R), C(\mathcal{G}))$ is a matroid perspective.

We know from Lemma 4.2.8 that $D_{\min}(G_R) = C(G_R)$ and by definition $C(\mathcal{G}) = C(G_{/\mathcal{P}})$ so therefore we need to show that $(C(G_R), C(G_{/\mathcal{P}}))$ is a matroid perspective, that is, for all $A \subseteq B \subseteq E$ we have

$$r(B) - r(A) \geq r_{\mathcal{P}}(B) - r_{\mathcal{P}}(A).$$

But $r(A) = v(A) - k(A)$ and $v(A) = v(B)$ for all A and B so

$$r(B) - r(A) = k(A) - k(B)$$

and similarly

$$r_{\mathcal{P}}(B) - r_{\mathcal{P}}(A) = k_{\mathcal{P}}(A) - k_{\mathcal{P}}(B).$$

Now let $k(B) = x$ and $k_{\mathcal{P}}(B) = y$

If $A = B$ then

$$k(A) - k(B) = 0 = k_{\mathcal{P}}(A) - k_{\mathcal{P}}(B)$$

so

$$r(B) - r(A) \geq r_{\mathcal{P}}(B) - r_{\mathcal{P}}(A).$$

If $A = B - \{e\}$ then

$$\text{either } k(A) = x \text{ or } k(A) = x + 1$$
$$\text{and either } k_{\mathcal{P}}(A) = y \text{ or } k_{\mathcal{P}}(A) = y + 1$$

since removing a single edge from a graph either increases the number of components by 1 (if the edge is a bridge) or leaves the number of components unchanged (if the edge is not a bridge). If $k_{\mathcal{P}}(A) = y$ then $r_{\mathcal{P}}(B) - r_{\mathcal{P}}(A) = 0$ and clearly

$$r(B) - r(A) \geq r_{\mathcal{P}}(B) - r_{\mathcal{P}}(A).$$

If $k_{\mathcal{P}}(A) = y + 1$ then e is a bridge in $G_{/\mathcal{P}}$ but if e is a bridge in $G_{/\mathcal{P}}$ it must also be a bridge in G, as any path in G has a corresponding path in $G_{/\mathcal{P}}$. Suppose e is a bridge in $G_{/\mathcal{P}}$ joining vertices u and v representing sets A_u and A_v of $\mathcal{P}$. Then e corresponds to an edge e' of G joining vertices $u' \in A_u$ and $v' \in A_v$ and every $u'v'$ path in G passes through e'. Hence $k(A) = x + 1$. Therefore

$$k(A) - k(B) = 1 = k_{\mathcal{P}}(A) - k_{\mathcal{P}}(B)$$

hence

$$r(B) - r(A) \geq r_{\mathcal{P}}(B) - r_{\mathcal{P}}(A).$$

Now since we can get any subset A of B by removing one edge at a time and since removing an edge will never result in the number of components of $G_{/\mathcal{P}}$ increasing by more than the number of components of G we have

$$r(B) - r(A) \geq r_{\mathcal{P}}(B) - r_{\mathcal{P}}(A)$$

and hence $(C(G), C(G_{/\mathcal{P}}))$ is a matroid perspective.

For $(B(\mathcal{G}^*), D_{\max}(G))$ we know

$$D_{\max}(G) = B(G_R^*) = C(G_R^*)^*$$

and that

$$B(\mathcal{G}^*) = C((G_R^*)_{/\mathcal{P}^*})^*.$$

We also showed earlier in this proof that

$$\Big(C(G), C(G_{/\mathcal{P}})\Big)$$

is a matoid perspective so

$$\Big(C(G_R^*), C((G_R^*)_{/\mathcal{P}^*})\Big)$$

is also a matroid perspective so

$$\Big(C(G_R^*), C((G_R^*)_{/\mathcal{P}^*})^*\Big) = \Big((C(G_R^*)_{/\mathcal{P}^*})^*, C(G_R^*)^*\Big) = \Big(B(\mathcal{G})^*, D_{\max}(G)\Big)$$

is a matroid perspective. Hence $\Big(B(\mathcal{G}^*), D(G_R), C(\mathcal{G})\Big)$ is a delta-matroid perspective. □

Theorem 5.3.2. *Let $\mathcal{G} = (G_R, \mathcal{P}, \mathcal{Q})$ be a coloured ribbon graph and let*

$$\mathbf{P}(\mathcal{G}) = \Big(B(\mathcal{G}^*), D(G_R), C(\mathcal{G})\Big),$$

then

$$\mathbf{P}(\mathcal{G})^* = \mathbf{P}(\mathcal{G}^*).$$

Proof. Recall that $\mathcal{G}^* = (G_R^*, \mathcal{P}^*, \mathcal{Q}^*)$ so $C(\mathcal{G}^*) = C((G_R)^*_{/\mathcal{P}^*})$. Therefore

$$\begin{aligned}\mathbf{P}(\mathcal{G}^*) &= \Big(B((\mathcal{G}^*)^*), D(G_R^*), C(\mathcal{G}^*)\Big)\\ &= \Big(B(\mathcal{G}), D(G_R^*), C((G_R)^*_{/\mathcal{P}^*})\Big)\\ &= \Big(B(G_{/\mathcal{P}}), D(G_R^*), C((G_R)^*_{/\mathcal{P}^*})\Big).\end{aligned}$$

$$\begin{aligned}\mathbf{P}(\mathcal{G})^* &= \Big(B(\mathcal{G}^*), D(G_R), C(\mathcal{G})\Big)^*\\ &= \Big(C(\mathcal{G})^*, D(G_R)^*, (B(\mathcal{G}^*)^*\Big)\\ &= \Big(C(G_{/\mathcal{P}})^*, D(G_R)^*, B((G_R^*)_{/\mathcal{P}^*})^*\Big)\\ &= \Big(B(G_{/\mathcal{P}}), D(G_R^*), C((G_R)^*_{/\mathcal{P}^*})\Big)\\ &= \mathbf{P}(\mathcal{G}^*)\end{aligned}$$

□

Theorem 5.3.3. *Let $\mathcal{G} = (G_R, \mathcal{P}, \mathcal{Q})$ be a coloured ribbon graph and let*

$$\mathbf{P}(\mathcal{G}) = ((B(\mathcal{G}^*), D(G_R), C(\mathcal{G}))$$

then

$$\mathbf{P}(\mathcal{G})/e = \mathbf{P}(\mathcal{G}/e)$$

and

$$\mathbf{P}(\mathcal{G})\backslash e = \mathbf{P}(\mathcal{G}\backslash e)$$

Proof. We have

$$\mathbf{P}(\mathcal{G})\backslash e = \Big(B(\mathcal{G}^*)\backslash e, D(G_R)\backslash e, C(\mathcal{G})\backslash e\Big)$$

and

$$\mathcal{G}\backslash e = (G_R\backslash e, \mathcal{P}, \mathcal{Q}\backslash e)$$

so

$$\mathbf{P}(\mathcal{G}\backslash e) = \Big(B\big((\mathcal{G}\backslash e)^*\big), D(G_R\backslash e), C(\mathcal{G}\backslash e)\Big).$$

We also have

$$\mathbf{P}(\mathcal{G})/e = \Big(B(\mathcal{G}^*)/e, D(G_R)/e, C(\mathcal{G})/e\Big)$$

and

$$\mathcal{G}/e = (G_R/e, \mathcal{P}/e, \mathcal{Q})$$

so

$$\mathbf{P}(\mathcal{G}/e) = \Big(B\big((\mathcal{G}/e)^*\big), D(G_R/e), C(\mathcal{G}/e)\Big).$$

Therefore we need to show that

$$\begin{aligned} B(\mathcal{G}^*)\backslash e &= B\big((\mathcal{G}\backslash e)^*\big) & B(\mathcal{G}^*)/e &= B\big((\mathcal{G}/e)^*\big) \\ D(G_R)\backslash e &= D(G_R\backslash e) & D(G_R)/e &= D(G_R/e) \\ C(G_{/\mathcal{P}})\backslash e &= C(\mathcal{G}\backslash e) & C(G_{/\mathcal{P}})/e &= C(\mathcal{G}/e). \end{aligned}$$

We know from Lemma 4.2.11 that since $D(G_R)$ is a graphic delta-matroid then $D(G_R)\backslash e = D(G_R\backslash e)$ and $D(G_R)/e = D(G_R/e)$.

Now since $C(G_{/\mathcal{P}})$ is the cycle matroid of $(G_{/\mathcal{P}})$ then $C(G_{/\mathcal{P}})\backslash e = C((G_{/\mathcal{P}})\backslash e)$. Therefore to show that $C(G_{/\mathcal{P}})\backslash e = C(\mathcal{G}\backslash e)$ we need to show that $(G_{/\mathcal{P}})\backslash e = (G_R\backslash e)_{/(\mathcal{P})}$. That is if we form $G_{/\mathcal{P}}$ and then delete the edge e we will obtain the same graph as if we delete the edge e from the ribbon graph G_R and then form the partition graph $(G_R\backslash e)_{/(\mathcal{P}\backslash e)}$. But this should be clear.

Therefore

$$(G_{/\mathcal{P}})\backslash e = (G_R\backslash e)_{/(\mathcal{P}\backslash e)}. \tag{5.7}$$

Hence $C(G_{/\mathcal{P}})\backslash e = C(\mathcal{G}\backslash e)$.

We also have that $C(G_{/\mathcal{P}})/e = C((G_{/\mathcal{P}})/e)$. Therefore to show $C(G_{/\mathcal{P}})/e = C(\mathcal{G}/e)$ we need to show that $(G_{/\mathcal{P}})/e = (G_R/e)_{/(\mathcal{P}/e)}$. That is if we form $G_{/\mathcal{P}}$ and then contract the edge e we will obtain the same graph as if we contract the edge e from the ribbon graph G_R, generate the partition $\mathcal{P}/e$ and then form the partition graph $(G_R/e)_{/(\mathcal{P}/e)}$.

To show this we will consider three cases: when e is not a loop, an orientable loop and a non-orientable loop.

First suppose e is not a loop of G_R. Then e has two endpoints. Let u and v be its endpoints and let U and V be the parts of $\mathcal{P}$ containing u and v respectively (U and V are not necessarily distinct). Now to contract e we delete e and replace the discs u and v with a new disc z such that any other edge that was adjacent to u or v is now adjacent to z (if there is another edge between u and v this become a loop of z). We form the new partition be removing parts U and V and replacing them with a new part $W = (U \cup V \cup \{z\}) \backslash \{u, v\}$. We then form the partition graph $(G_R/e)_{/(\mathcal{P}/e)}$. Now observe that every vertex in parts $U$and V in $\mathcal{G}$ has been identified to a single vertex in $(G_R/e)_{/(\mathcal{P}/e)}$, but this is exactly what happens if we form the partition graph $(G_{/\mathcal{P}})$ and then contract e, since forming the partition graph identifies all the vertices in U into one vertex and all the vertices in V into one vertex and contracting e identifies U and V as one vertex.

Next suppose e is an orientable loop. Then e has one endpoint. Let u be its endpoint and let U be the part of $\mathcal{P}$ containing u. Now when we contract u, since it is an orientable loop we delete e and replace u with two new discs v and z such that any other edge that was adjacent to u is now adjacent to v or z (we are not concerned with the exact details of this as we will soon identify v and z as one vertex again). We form the new partition by removing part U and replacing it with a new part $W = (U \cup \{v, z\}) \backslash \{u\}$. We then form the partition graph $(G_R/e)_{/(\mathcal{P}/e)}$. Now observe that since v and z are in the same part of $\mathcal{P}/e$ they are identified to the same vertex, in fact observe that we would obtain the same abstract graph if we formed the partition graph deleted the edge e, which is exactly the graph $(G_{/\mathcal{P}})/e$ since contracting a loop in an abstract graph is the same as deleting it.

Finally suppose e is a non-orientable loop. Now observe that contracting a non-orientable loop does not change the adjacency information of a ribbon graph. Hence if we contract e and then form the partition graph we will obtain the same abstract graph as if we formed the partition graph then contracted e.

Therefore

$$(G_{/\mathcal{P}})/e = (G_R/e)_{/(\mathcal{P}/e)} \tag{5.8}$$

hence $C(G_{/\mathcal{P}})/e = C(\mathcal{G}/e)$.

For the final parts we want

$$B(\mathcal{G}^*)\backslash e = B\big((\mathcal{G}\backslash e)^*\big)$$

and

$$B((\mathcal{G}/e)^*) = B(\mathcal{G}^*)/e$$

and we know from Theorem 1.9.1 that

$$(\mathcal{G}\backslash e)^* = (\mathcal{G}^*)/e$$

and

$$(\mathcal{G}/e)^* = (\mathcal{G}^*)\backslash e.$$

Hence

$$B((\mathcal{G}\backslash e)^*) = B((\mathcal{G}^*)/e) = C((\mathcal{G}^*)/e)^* = C(G_R^*/e, \mathcal{P}^*/e, \mathcal{Q}^*)^* = C((G_R^*/e)_{/(\mathcal{P}^*/e)})^*.$$

Now from Equation 5.8 we know

$$(G_R^*/e)_{/(\mathcal{P}^*/e)} = ((G_R^*)_{/\mathcal{P}^*})/e$$

so we have

$$B((\mathcal{G}\backslash e)^*) = C((G_R^*/e)_{/(\mathcal{P}^*/e)})^* = C((G_R^*)_{/\mathcal{P}^*}/e)^*$$

and since $C((G_R^*)_{/\mathcal{P}^*})$ is a graphic matroid of an abstract graph then

$$(C((G_R^*)_{/\mathcal{P}^*}/e))^* = (C((G_R^*)_{/\mathcal{P}^*})/e)^* = (C((G_R^*)_{/\mathcal{P}^*})^*)\backslash e = B(\mathcal{G}^*)\backslash e.$$

Therefore

$$B(\mathcal{G}^*)\backslash e = B\big((\mathcal{G}\backslash e)^*\big)$$

Similarly

$$B((\mathcal{G}/e)^*) = B((\mathcal{G}^*)\backslash e) = C((\mathcal{G}^*)\backslash e)^* = C(G_R^*\backslash e, \mathcal{P}^*, \mathcal{Q}^*\backslash e)^* = C((G_R^*\backslash e)_{/\mathcal{P}^*})^*.$$

Now from Equation 5.7 we know

$$(G_R^*\backslash e)_{/\mathcal{P}^*} = (G_R^*)_{/\mathcal{P}^*}\backslash e$$

so we have

$$B((\mathcal{G}/e)^*) = C((G_R^*\backslash e)_{/\mathcal{P}^*})^* = C((G_R^*)_{/\mathcal{P}^*}\backslash e)^*$$

and since $C((G_R^*)_{/\mathcal{P}^*})$ is a graphic matroid of an abstract graph then

$$C((G_R^*)_{/\mathcal{P}^*}\backslash e)^* = (C((G_R^*)_{/\mathcal{P}^*})\backslash e)^* = C((G_R^*)_{/\mathcal{P}^*})^*/e = B(\mathcal{G}^*)/e.$$

Therefore

$$B((\mathcal{G}/e)^*) = B(\mathcal{G}^*)/e.$$

Hence

$$\mathbf{P}(\mathcal{G})\backslash e = \mathbf{P}(\mathcal{G}\backslash e).$$

and

$$\mathbf{P}(\mathcal{G})/e = \mathbf{P}(\mathcal{G}/e).$$

□

Chapter 6

Graph polynomials

Contents

6.1 The Tutte Polynomial

The Tutte Polynomial was original defined by W. T. Tutte, in his paper [51] in 1954 where he referred to it as the dichromate of a graph. Since then it has become one of the most widely studied and most important graph polynomials. This is because many important polynomials from a variety of areas of mathematics, such as the Jones polynomial from knot theory and the Potts model from statistical mechanics can be shown to be specialisations of the Tutte polynomial.

There are numerous ways of defining the Tutte polynomial, including the spanning trees expansion formulation which was the method originally used by Tutte in [51]. However for our purposes two formulations are going to be important.

6.1.1 Rank Nullity Definition of the Tutte Polynomial

Recall that the *rank* $r(E)$ of a graph $G = (V, E)$ is equal to the number $v(E) of vertices of$ G$minus the number$ k(E)$of components and the nullity$ n(E)of G$is equal to the number e$ e(of edges of G minus the rank $r(E)$ of G. Then we can define the Tutte polynomial of G as follows:

Definition 6.1.1. Let $G = (V, E)$ be a graph then

$$T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{n(A)}.$$

6.1.2 Linear Recursion Definition

A linear recursion is a process by which we can reduce an object to a weighted sum of "simpler" objects which can then similarly be reduced to even simpler objects until a terminal form is reached. The Tutte Polynomial can be defined by a linear recursion relation given by deleting and contracting edges. We call such a recursion a *deletion-contraction relation.*

Theorem 6.1.1. *Let $G = (V, E)$ be a graph and let $e \in E$ then*

$$T'(G; x, y) = \begin{cases} xT(G \backslash e; x, y) & \textit{if } e \textit{ is a bridge} \\ yT(G \backslash e; x, y) & \textit{if } e \textit{ is a loop} \\ TG \backslash e; x, y) + T(G/e; x, y) & \textit{if } e \textit{ not a bridge or a loop.} \end{cases}$$

If G is equal to the null graph E_n then

$$T(E_n; x, y) = 1$$

The proof of this is shown in [15] for example.

In other words we can find the Tutte polynomial of a graph by repeatedly deleting and contracting edges until there are no edges left and we just have the null graph. Importantly the polynomial we obtain is independent of the order in which we choose the edges. One way of showing this is to prove that the linear recursion definition of the Tutte polynomial is equal to the rank nullity definition.

6.1.3 The Universal Form of the Tutte Polynomial

While the various evaluations and specialisations are important, the universality of the Tutte polynomial is probably its most powerful aspect. It effectively means that any graph invariant that satisfies a deletion-contraction relation must be an evaluation of the Tutte polynomial.

Theorem 6.1.1. *[5] There is a unique map $U : \mathbf{G} \to \mathbb{Z}[x, y, \alpha, \sigma, \tau]$ such that*

$$U(E_n) = U(E_n; x, y, \alpha, \sigma, \tau) = \alpha^n$$

for every $n \geq 1$, and for every $G \in \mathbf{G}$ with $e \in E(G)$ we have

$$U(G; x, y, \alpha, \sigma, \tau) = \begin{cases} xU(G\backslash e; x, y) & \text{if } e \text{ is a bridge} \\ yU(G\backslash e; x, y) & \text{if } e \text{ is a loop} \\ \sigma U(G\backslash e; x, y) + \tau U(G/e; x, y) & \text{if } e \text{ not a bridge or a loop.} \end{cases}$$

Futheremore,

$$U(G; x, y, \alpha, \sigma, \tau) = \alpha^{k(E)} \sigma^{n(E)} \tau^{r(E)} T(G; \alpha x/\tau, y/\sigma).$$

The proof of this is shown in [5] for example.

Example 8. The *Chromatic Polynomial* $p_G(x)$ is the polynomial which counts the number of proper vertex colourings of a graph. It satisfies a deletion-contraction recurrence

$$p_G(x) = p_{G\backslash e} - p_{G/e}$$

and therefore is a specialisation of the Tutte polynomial. That is

$$p_G(x) = (-1)^{v(E)} x^{n(E)} T_G(1 - x, 0).$$

6.2 Topological Graph Polynomials

The Tutte polynomial is defined for abstract graphs and so does not include any topological information. Therefore it is a logical step to see if it can be extended to create a new polynomial which contains all the information about the graph provided by the Tutte polynomial, satisfies a deletion-contraction relationship and that also contains topological information about how the graph is embedded in a surface. We will first introduce and discuss the existing topological graph polynomials.

6.2.1 Las Vergnas Polynomial

The Las Vergnas polynomial, which was first introduced by Michel Las Vergnas in his paper [37], appears to be the first example of an extension of the Tutte polynomial to embedded graphs.

The polynomial was initially defined as a specialisation of the Tutte polynomial of a matroid perspective, which we will discuss in section 6.3. The formulation we give here was shown by Joanna A. Ellis-Monaghan and Iain Moffatt in [27].

Definition 6.2.1. Let G_R be a ribbon graph then

$$L_{G_R}(x,y,z) = \sum_{A \subseteq E(G_R)} (x-1)^{r_{G_R}(G_R) - r_{G_R}(A)} \cdot (y-1)^{n_{G_R}(A) - (\gamma_{G_R}(G_R) + \gamma_{G_R}(A) - \gamma_{G_R^*}(A^c))/2} \cdot z^{(\gamma_{G_R}(G_R) - \gamma_{G_R}(A) + \gamma_{G_R^*}(A^c))/2}, \quad (6.1)$$

where $A^c := E(G_R) \backslash A$.

The Las Vergnas polynomial for ribbon graphs (or cellularly embedded graphs) does not have a deletion-contraction relation that applies to all types of edges. However it was shown in [27] that if we instead look at graphs embedded in pseudo-surfaces a deletion-contraction relationship can be found.

6.2.2 Bollobás-Riordan Polynomial

Following the introduction of the Las Vergnas polynomial in 1978 there seems to have been very little development in topological graph polynomials until Béla Bollobás and Oliver Riordan published their papers [6], [7] in 2001 and 2002 in which they define the ribbon graph polynomial or Bollobás-Riordan polynomial as it has since come to be called.

The idea was to construct a polynomial similar to the Tutte polynomial, that is a sum over the subgraphs, but which also included information about the topology of the surface naturally associated with each subgraph. Recall that $t(A) = 0$ if $(V(G_R), A)$ is orientable and $t(A) = 1$ otherwise and that $f(A)$ is the number of boundary components of $(V(G_R), A)$.

Definition 6.2.2. Let G_R be a ribbon graph. Then the *Bollobás-Riordan* polynomial is defined as

$$R(G_R; x, y, z, w) = \sum_{A \subseteq E(G_R)} (x-1)^{r(G_R)-r(A)} y^{n(A)} z^{k(A)-f(A)+n(A)} w^{t(A)}.$$

It is clear that if we set z and w to 1 we can recover the Tutte polynomial. Also since

$$k(A) - f(A) + n(A) = \gamma(A)$$

then if G_R is a plane graph $R(G_R; x, y-1, z, w) = T(G_R; x, y)$, as the Euler genus of a plane graph is zero.

One of the most important properties of the Tutte polynomial is that it satisfies a deletion-contraction relationship on all its edges. Unfortunately no such relationship exist for the Bollobás-Riordan polynomial for all possible ribbon graphs. However it has been shown that:

Theorem 6.2.1. *[7] Let G_R be a ribbon graph then*

$$R(G_R;x,y,z,w)=\begin{cases} xR(G_R/e;x,y,z,w) & \text{if } e \text{ is a bridge} \\ R(G_R\backslash e;x,y,z,w)+R(G_R/e;x,y,z,w) & \text{if } e \text{ not a bridge or a loop} \\ (1+y)R(G_R\backslash e;x,y,z,w) & \text{if } e \text{ is a trivial orientable loop} \\ R(G_R\backslash e;x,y,z,w)+yzwR(G_R/e;x,y,z,w) & \text{if } e \text{ is a non-orientable loop.} \end{cases} \tag{6.2}$$

However no deletion-contraction relationship exists for non-trivial orientable loops.

Note that often when we refer to the Bollobás-Riordan polynomial we refer the three variable version where the final term is omitted;

$$R(G_R;x,y,z)=R(G_R;x,y,z,1)=\sum_{A\subseteq E(G_R)}(x-1)^{r(G_R)-r(A)}y^{n(A)}z^{\gamma(A)}$$

6.2.3 Krushkal polynomial

The final polynomial that we will consider is the Krushkal polynomial that was discovered by Vyacheslav Krushkal in [35] whilst he was researching the Potts model.

The Krushkal polynomial is an invariant of an embedded graph G which satisfies a deletion-contraction rule and a duality relation. It can be seen as a generalisation of both the Tutte polynomial and the Bollobás-Riordan polynomial. It was originally defined for graphs in orientable surfaces, and extended by Clark Butler in [16] to graphs in non-orientable surfaces. However we will define it in terms of ribbon graphs.

Definition 6.2.3. Let $G_R=(V,E)$ be a ribbon graph then the *Krushkal* *polynomial* is defined by

$$K(G_R;x,y,a,b) := \sum_{A\subseteq E} x^{r_{G_R}(E)-r(-G_R A)} y^{r_{G_{R^*}}(E)-r_{G_{R^*}}(A^c)} a^{\gamma_{G_R}(A)/2} b^{\gamma_{G_R^*}(A^c)/2}. \tag{6.3}$$

This polynomial also satisfies a deletion-contraction rule, however it also cannot be applied to every possible type of edge.

Lemma 6.2.1. *[35] Let G_R be a ribbon graph and e be an edge of G_R. Then*

$$K(G;x,y,a,b) = \begin{cases} (x+1)K(G/e;x,y,a,b) & \\ \qquad \text{if } e \text{ is a bridge} \\ K(G\backslash e;x,y,a,b) + K(G/e;x,y,a,b) & \\ \qquad \text{if } e \text{ is not a bridge or a loop} \\ (1+y)K(G\backslash e;x,y,a,b) & \\ \qquad \text{if } e \text{ is a trivial loop.} \end{cases} \tag{6.4}$$

Krushkal also showed that the Tutte polynomial is a specialisation of the Krushkal polynomial.

Lemma 6.2.2. *[35] Let G_R be a ribbon graph. Then*

$$T(G_R;x,y) = K(G_R;x-1,y-1,1,1).$$

Since we are working with ribbon graphs we have a duality result for K_{G_R}.

Theorem 6.2.2. *[35] Let G_R be a ribbon graph and let G_R^* be its dual. Then*

$$K(G_R;x,y,a,b) = K(G_R^*;y,x,b,a).$$

The Krushkal polynomial can be specialised to the Bollobás-Riordan polynomial.

Theorem 6.2.3. *[35] Let G_R be a ribbon graph. Then*

$$R(G_R; x, y, z) = y^{\gamma(G_R)/2} K(G_R; x-1, y, yz^2, y^{-1}),$$

6.3 Matroid Polynomials

Many of the graph polynomials that we have discussed above have equivalent polynomials for matroids.

6.3.1 Tutte polynomial of a matroid

Definition 6.3.1. Given a matroid $M = (E, r)$ the Tutte polynomial of M is defined as

$$T(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(M) - r_M(A)} (y-1)^{|A| - r_M(A)}.$$

Trivially if G is a graph and $C = C(G)$ is the cycle matroid of G then

$$T(C; x, y) = T(G; x, y).$$

In [36] Michel Las Vergnas defined the *Tutte polynomial of the matroid perspective.*

Definition 6.3.2. Let (M, M') be a matroid perspective, where $M = (E, r)$ and $M' = (E, r')$. Then

$$T_{(M,M')}(x, y, z) := \sum_{A \subseteq E} (x-1)^{r'(E) - r'(A)} (y-1)^{|A| - r(A)} z^{(r(E) - r(A)) - (r'(E) - r'(A))}.$$

Note that if $M' = M$ then $T_{(M,M')}$ equals T_M.

6.3.2 The Bollobás-Riordan and Krushkal polynomials of a delta-matroid

The Tutte polynomial is a polynomial for abstract graphs. Therefore it follows that there should be an equivalent polynomial for matroids. Similarly the the Bollobás-Riordan and Krushkal polynomials are polynomials which can be defined for ribbon graphs. Therefore it makes sense that there would be equivalent polynomials for delta-matroids, which in fact there are.

Definition 6.3.3. [20] Let $D = (E, \mathcal{F})$ be a delta-matroid. Then the Bollobás-Riordan polynomial of D is

$$R(D; x, y, a, b) := \sum_{A \subseteq E} (x-1)^{r_{D_{\min}}(E) - r_{D_{\min}}(A)} y^{n_{D_{\min}}(A)} a^{w_D(A)} b^{t(A)}. \tag{6.5}$$

Definition 6.3.4. [20] Let $D = (E, \mathcal{F})$ be a delta-matroid. Then the Krushkal polynomial of D is

$$K(D; x, y, a, b) := \sum_{A \subseteq E} x^{r_{D_{\min}}(E) - r_{D_{\min}}(A)} y^{r_{(D^*)_{\min}}(E) - r_{(D^*)_{\min}}(A^c)} a^{w_D(A)} b^{w_{D^*}(A^c)}. \tag{6.6}$$

These polynomials were constructed to coincide with their ribbon graph counterparts.

Theorem 6.3.1. *[20] Let G_R be a ribbon graph and let $D(G_R)$ be its graphic delta-matroid. Then*

$$R(D(G_R); x, y, a, b) = R(G_R; x, y, a, b),$$

and

$$K(D(G_R); x, y, a, b) = K(G_R; x, y, a, b).$$

Chapter 7

Delta-Matroid Perspective Polynomials

Contents

In chapter 5 we introduced delta-matroid perspective and in chapter 6 we showed that a number of graphical polynomials have equivalent polynomials defined for matroids or delta-matroids. In this chapter we introduce new polynomials which are defined over delta-matroid perspectives.

7.1 The Krushkal Polynomial for Delta-Matroid Perspectives

Definition 7.1.1. Let (M, D, M') be a delta-matroid perspective, where $M = (E, r)$, $M' = (E, r')$, and $\rho = \rho_D$. Then we define $K_{(M,D,M')}(x, y, a, b)$ *the Krushkal Polynomial for a delta-matroid perspective* (M, D, M') as:

$$K_{(M,D,M')}(x, y, a, b) := \sum_{A \subseteq E} x^{r'(E)-r'(A)} y^{|A|-r(A)} a^{\rho(A)-r'(A)} b^{r(A)-\rho(A)}.$$

Similarly we can create a polynomial for a DM-perspective.

Definition 7.1.2. Let (D, M) be a DM-perspective, where $M = (E, r)$, and $\rho = \rho_D$. Then we define $R_{(D,M)}$ *the Bollobás-Riordan polynomial for a DM-perspective* (D, M) as:

$$R_{(D,M)}(x, y, z) := \sum_{A \subseteq E} x^{r(E)-r(A)} y^{|A|-\rho(A)} z^{\rho(A)-r(A)}.$$

It is simple to move between $T_{(M,M')}, R_{(D,M)}$ and $K_{(M,D,M')}$.

Theorem 7.1.1. *Let M and M' be matroids and let D be a delta matroid then*

1. $K_{(M,M,M)}(x - 1, y - 1, a, b) = T_M(x, y)$
2. $T_{(M,M')}(x, y, z) = z^{r(E)-r'(E)} K_{(M,M,M')}(x - 1, y - 1, z^{-1}, b)$
3. $T_{(M,M')}(x, y, z) = z^{r(E)-r'(E)} K_{(M,M',M')}(x - 1, y - 1, a, z^{-1})$
4. $T_{(M,M')}(x, y, z) = z^{r(E)-r'(E)} K_{(M,D,M')}(x - 1, y - 1, z^{-1}, z^{-1})$
5. $K_{(M',D,M)}(x, y, z, y) = R_{(D,M)}(x, y, z)$

Proof. For the first item recall that

$$T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{r(E)-r(A)} (y - 1)^{|A|-r(A)}.$$

If M is a matroid then $\rho_M(A) = r(A)$, since $M_{\min} = M_{\max} = M$ so $r_{\max}(E) = r_{\min}(E) = r(E)$ hence $\frac{1}{2}(r_{\max}(E) + r_{\min}(E)) = r(E)$. Therefore,

$$\begin{aligned}
K_{(M,M,M)}(x-1, y-1, a, b) &= \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} a^{r(A)-r(A)} b^{r(A)-r(A)} \\
&= \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)} \\
&= T_M(x, y).
\end{aligned}$$

For the second, third and forth items items we know that $\rho_M(A) = r(A)$ and $\rho_{M'}(A) = r'(A)$ so

$$K_{(M,M,M')}(x-1, y-1, z^{-1}, b) = \sum_{A \subseteq E} (x-1)^{r'(E)-r'(A)}(y-1)^{|A|-r(A)} z^{r'(A)-r(A)}$$

$$K_{(M,M',M')}(x-1, y-1, a, z^{-1}) = \sum_{A \subseteq E} (x-1)^{r'(E)-r'(A)}(y-1)^{|A|-r(A)} z^{r'(A)-r(A)}.$$

$$K_{(M,D,M')}(x-1, y-1, z^{-1}, z^{-1}) = \sum_{A \subseteq E} (x-1)^{r'(E)-r'(A)}(y-1)^{|A|-r(A)} z^{r'(A)-r(A)}.$$

Hence

$$\begin{aligned}
T_{(M,M')}(x, y, z) &= \sum_{A \subseteq E} (x-1)^{r'(E)-r'(A)}(y-1)^{|A|-r(A)} z^{(r(E)-r(A))-(r'(E)-r'(A))} \\
&= z^{r'E)-r'(E)} \sum_{A \subseteq E} (x-1)^{r'(E)-r'(A)}(y-1)^{|A|-r(A)} z^{r'(A)-r(A)} \\
&= z^{r(E)-r('E)} K_{(M,M,M')}(x-1, y-1, z^{-1}, b) \\
&= z^{r(E)-r'(E)} K_{(M,M',M')}(x-1, y-1, a, z^{-1}) \\
&= z^{r(E)-r'(E)} K_{(M,D,M')}(x-1, y-1, z^{-1}, z^{-1}).
\end{aligned}$$

Finally for the Fifth item observe that

$$\begin{aligned}
K_{(M',D,M)}(x, y, z, y) &= \sum_{A \subseteq E} x^{r(E)-r(A)} \cdot y^{|A|-r'(A)} \\
&\qquad \cdot z^{\rho(A)-r(A)} \cdot y^{r'(A)-\rho(A)} \\
&= \sum_{A \subseteq E} x^{r(E)-r(A)} y^{|A|-\rho(A)} z^{\rho(A)-r(A)} \\
&= R_{(D,M)}(x, y, z).
\end{aligned}$$

□

7.2 The relationship between Graph and Matroid Polynomials

In the previous section we introduced two new matroidal polynomials. We can now show that by carefully choosing the matroids we use we can recover some of the topological graph polynomials we discussed earlier.

Theorem 7.2.1. *Let $G_R = (V, E)$ be a ribbon graph and let $D = D(G)$ be its delta-matroid. Then*

$$K_{(D_{\max},D,D_{\min})}(x, y, a, b) = b^{\gamma(G)/2} K_G(x, y, a, b^{-1})$$

Proof. We want to show

$$b^{\gamma(G)/2} \sum_{A\subseteq E} x^{r_{G_R}(E)-r_{G_R}(A)} y^{r_{G_R^*}(E)-r_{G_R^*}(A^c)} a^{\gamma_{G_R}(A)/2} b^{-\gamma_{G_R^*}(A^c)/2}$$
$$= \sum_{A\subseteq E} x^{r_{M'}(E)-r_{M'}(A)} y^{|A|-r_M(A)} a^{\rho(A)-r_{M'}(A)} b^{r_M(A)-\rho(A)},$$

where r_M is the rank function for $D_{\max}$ and $r_{M'}$ is the rank function for $D_{\min}$. We will prove this term by term.

We start with the x term. We need to show that $r_{G_R}(E) - r_{G_R}(A) = r_{M'}(E) - r_{M'}(A)$. But $r_{M'}$ is the rank function of $D_{\min}$ and $D_{\min} = C(G_R)$ since $D(G_R)$ is the graphic delta-matroid of R_G. Therefore $r_{G_R}(A) = r_{M'}(A)$ for all A. Hence the x terms agree.

For the y terms we need to show that

$$r_{G_R^*}(E) - r_{G_R^*}(A^c) = |A| - r_M(A)$$

Since $D(G_R)$ is the graphic delta-matroid of R_G we know that $D_{\max} = B(G_R^*) = (C(G_R^*))^*$. Now $C(G_R^*)$ is the cycle matroid of G_R^* so its rank function is equal to r_{G_R*}. Recall that if N is a matroid and N^* is its dual then $r_{N^*}(A) = |A| + r_N(E\backslash A) - r_N(E)$. Therefore since r_M is the rank function for $(C(G_R^*))^*$ we have

$$\begin{aligned}|A| - r_M(A) &= |A| - (r_{C(G_R^*)}(E\backslash A) + |A| - r_{C(G_R^*)}(E)) \\ &= r_{G_R^*}(E) - r_{G_R^*}(A^c).\end{aligned}$$

For the a term we need to show that

$$\frac{1}{2}(\gamma_{G_R}(A)) = \rho(A) - r_{M'}(A).$$

We know that $r_{M'}(A) = v(A) - k(A)$ and from Lemma 4.2.9 that $\rho(A) = \frac{1}{2}(|A| + v(A) - f(A))$. Finally Equation 1.1 tells us that $\gamma(A) = 2k(A) + (|A| - v(A) - f(A))$.

Hence we have

$$\begin{aligned}\rho(A) - r_{M'}(A) &= \frac{1}{2}(|A| + v(A) - f(A)) - v(A) + k(A) \\ &= k(A) + \frac{1}{2}(|A| - v(A) - f(A)) \\ &= \gamma(A)/2\end{aligned}$$

For b we need to we need to show that

$$\frac{1}{2}(\gamma(G) - \gamma_{G_R^*}(A^c)) = r_M(A) - \rho(A).$$

We have

$$\begin{aligned}r_M(A) - \rho(A) &= |A| + r_{G_R^*}(A^c) - r_{G_R^*}(E) - \frac{1}{2}(|A| + v_{G_R}(A) - f_{G_R}(A)) \\ &= \frac{1}{2}(f_{G_R}(A) + |A| - v_{G_R}(A)) + k_{G_R^*}(E) - k_{G_R^*}(A^c).\end{aligned}$$

We also know that $\gamma(A) = 2k(A) + (|A| - v(A) - f(A))$ so $\frac{1}{2}(\gamma_{G_R^*}(A^c)) = k_{G_R^*}(A^c) + \frac{1}{2}(|A^c| - v_{G_R^*}(A^c) - f_{G_R^*}(A^c))$. Therefore

$$\begin{aligned}\frac{1}{2}(\gamma(G) - \gamma_{G_R^*}(A^c)) = k_{G_R}(E) &+ \frac{1}{2}(|E| - v_{G_R}(G_R) - f_{G_R}(G_R)) \\ &- k_{G_R^*}(A^c) - \frac{1}{2}(|A^c| - v_{G_R^*}(A^c) - f_{G_R^*}(A^c)).\end{aligned}$$

But we know that $|A^c| = |E| - |A|$ and that $v_{G_R^*}(A^c) = v(G_R^*) = f(G_R^*)$. Therefore we have

$$\frac{1}{2}(\gamma(G) - \gamma_{G_R^*}(A^c)) = k_{G_R}(E) - k_{G_R^*}(A^c) + \frac{1}{2}(|A| - v_{G_R}(G_R) + f_{G_R^*}(A^c)).$$

We also know that the taking the dual of a graph will not change the number of its components so $k_{G_R}(E) = k_{G_R^*}(E)$.

Now recall that a ribbon graph $R_G = (V(G_R), E(G_R))$ is a surface with boundary. We can cap off the holes using a set of discs, denoted by $V(G_R^*)$, to obtain a surface Σ without boundary. Also recall that the dual of G_R is the ribbon graph $G_R^* = (V(G_R^*), E(G_R))$. Therefore the ribbon subgraph $(V(G_R^*), A^c) = \Sigma \backslash (V(G_R) \cup A)$. Clearly $\Sigma \backslash (V(G_R) \cup A)$ must have the same number of boundary components as $(V(G_R) \cup A)$. Now observe that $(V(G_R) \cup A)$ is the spanning ribbon subgraph $(V(G_R), A)$ so $f_{G_R^*}(A^c) = f_{G_R}(A)$. Therefore we have

$$\begin{aligned}
\frac{1}{2}(\gamma(G) - \gamma_{G_R^*}(A^c)) &= k_{G_R}(G_R) - k_{G_R^*}(A^c) + \frac{1}{2}(|A| - v_{G_R}(G_R) + f_{G_R^*}(A^c)) \\
&= k_{G_R^*}(E) - k_{G_R^*}(A^c) + \frac{1}{2}(|A| + f_{G_R}(A) - v_{G_R}(G_R)) \\
&= r_M(A) - \rho(A).
\end{aligned}$$

Therefore we have

$$K_{(D_{\max}, D, D_{\min})}(x, y, a, b) = b^{\gamma(G)/2} K_G(x, y, a, b^{-1}).$$

□

We can also show that we can recover the Bollobás-Riordan polynomial for a Ribbon Graph from the Bollobás-Riordan polynomial for a DM-perspective.

Theorem 7.2.2. *Let G be a ribbon graph and $D = D(G)$ be its delta-matroid. Then*

$$R_{(D, D_{\min})}(x, y, z) = R_G(x + 1, y, (y^{-1} z)^{\frac{1}{2}})$$

Proof. We can prove this using previous results.

$$\begin{aligned} R_{(D,D_{\min})}(x,y,z) &= K_{(D_{\max},D,D_{\min})}(x,y,z,y) && \text{by Theorem 7.1.1} \\ &= y^{\gamma(G)/2} K_G(x,y,z,y^{-1}) && \text{by Theorem 7.2.1} \\ &= R_G(x+1,y,(y^{-1}z)^{\frac{1}{2}}) && \text{by Theorem 6.2.3.} \end{aligned}$$

□

7.3 Deletion-contraction, duality and convolution formulae for $K_{(M,D,M')}$

We now state some results for $K_{(M,D,M')}$. The proof will follow in Chapter 9. We first state one of the key results of this thesis that $K_{(M,D,M')}$ satisfies a full deletion contraction formula.

Theorem 7.3.1. *Let $D = (M, D, M')$ be a delta-matroid perspective where $D = (E, \mathcal{F})$ is a delta-matroid and $M = (E, r)$ and $M' = (E, r')$ are matroids. Then*

$$K_{(M,D,M')}(x,y,a,b) = \begin{cases} K_{(M\backslash e,D\backslash e,M'\backslash e)} + K_{(M/e,D/e,M'/e)} \\ \qquad \text{if } e \text{ is not a loop or a coloop in } M', \\ K_{(M\backslash e,D\backslash e,M'\backslash e)} + yK_{(M/e,D/e,M'/e)} \\ \qquad \text{if } e \text{ is a loop in } M, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)} + aK_{(M/e,D/e,M'/e)} \\ \qquad \text{if } e \text{ is a loop in } M' \text{ and } e \text{ is not a ribbon loop in } D, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)} + (ab)^{\frac{1}{2}}K_{(M/e,D/e,M'/e)} \\ \qquad \text{if } e \text{ is not a loop in } M \text{ and } e \text{ is a non-orientable ribbon loop in } D, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)} + bK_{(M/e,D/e,M'/e)} \\ \qquad \text{if } e \text{ is not a loop in } M \text{ and } e \text{ is an orientable ribbon loop in } D, \\ xK_{(M\backslash e,D\backslash e,M'\backslash e)} + K_{(M/e,D/e,M'/e)} \\ \qquad \text{if } e \text{ is a coloop in } M'. \end{cases}$$

We also have a duality result for $K_{(M,D,M')}$.

Theorem 7.3.2. *Let $D = (M, D, M')$ be a delta-matroid perspective where*

$D = (E, \mathcal{F})$ is a delta-matroid and $M = (E, r)$ and $M' = (E, r')$ are matroids. Then

$$K_{(M,D,M')}(x, y, a, b) = b^{\rho_{D^*}(E)-r'(E)}a^{r(E)-\rho_{D^*}(E)}K_{((M')^*,D^*,M^*)}(y, x, b^{-1}, a^{-1}),$$

and

$$K_{((M')^*,D^*,M^*)}(x, y, a, b) = b^{\rho_D(E)-r'(E)}a^{r(E)-\rho_D(E)}K_{(M,D,M')}(y, x, b^{-1}, a^{-1}).$$

Finally we have a convolution formula for $K_{(M,D,M')}$.

Theorem 7.3.3. *Let $D = (M, D, M')$ be a delta-matroid perspective where $D = (E, \mathcal{F})$ is a delta-matroid and $M = (E, r)$ and $M' = (E, r')$ are matroids. Then*

$$K_{(M,D,M')}(x, y, a, ab^2) = \sum_{A \subseteq E} K_{(M,D,M')\backslash A^c}(-1, y, a, ab^2) \cdot K_{(M,D,M')/A}(x, -1, -1, -1).$$

7.4 Deletion-contraction, duality and convolution formulae for $R_{(D,M)}$

In Section 7.3 we stated a number of results for $K_{(M,D,M')}$. in this section we state the corresponding results for $R_{(D,M)}$, which again we prove in Chapter 9. We again start with deletion-contraction.

Theorem 7.4.1. *Let (D, M) be a DM-perspective where $D = (E, \rho)$ is a* *delta-matroid and $M = (E, r)$ is a matroid. Then*

$$R_{(D,M)}(x,y,z) = \begin{cases} R_{(D\backslash e, M\backslash e)} + R_{(D/e,M/e)} & \text{if } e \text{ is not a loop or a coloop in } M, \\ R_{(D\backslash e, M\backslash e)} + yR_{(D/e,M/e)} & \text{if } e \text{ is a orientable ribbon loop in } D, \\ R_{(D\backslash e, M\backslash e)} + zR_{(D/e,M/e)} & \text{if } e \text{ is a loop in } M \text{ and } e \text{ is not a ribbon loop in } D, \\ R_{(D\backslash e, M\backslash e)} + (yz)^{\frac{1}{2}}R_{(D/e,M/e)} & \text{if } e \text{ is a non-orientable ribbon loop in } D, \\ xR_{(D\backslash e, M\backslash e)} + R_{(D/e,M/e)} & \text{if } e \text{ is a coloop in } M \end{cases} \tag{7.1}$$

We now can use previous results to define a duality result for $R_{(D,M)}$.

Theorem 7.4.2. *Let (D, M) be a DM-perspective where $D = (E, \mathcal{F})$ is a delta-matroid and $M = (E, r)$ is a matroid. Then*

$$R_{(D^*,M^*)}(x,y,z) = z^{r(E)-\rho(E)} \sum_{A\subseteq E} y^{\rho(E)-\rho(A)} x^{|A|-r(A)} z^{\rho(A)-r(A)}$$

Proof. Given a matroid M' we know from Theorem 7.1.1 and Theorem 5.1.2 that

$$R_{(D^*,M^*)}(x,y,z) = K_{((M')^*,D^*,M^*)}(x,y,z,y) = K_{(M,D,M')^*}(x,y,z,y)$$

and from Theorem 7.3.2 we have that

$$K_{(M,D,M')^*}(x,y,a,b) = b^{\rho(E)-r'(E)} a^{r(E)-\rho(E)} K_{(M,D,M')}(y,x,b^{-1},a^{-1}).$$

hence

$$\begin{aligned} R_{(M,D)}(x,y,z) &= K_{(M,D,M')^*}(x,y,z,y) \\ &= y^{\rho(E)-r'(E)} z^{r(E)-\rho(E)} K_{(M,D,M')}(y,x,y^{-1},z^{-1}) \\ &= y^{\rho(E)} z^{r(E)-\rho(E)} \sum_{A\subseteq E} y^{-\rho(A)} x^{|A|-r(A)} z^{\rho(A)-r(A)} \\ &= z^{r(E)-\rho(E)} \sum_{A\subseteq E} y^{\rho(E)-\rho(A)} x^{|A|-r(A)} z^{\rho(A)-r(A)}. \end{aligned}$$

□

Note since the dual of an DM-perspective is an MD-perspective we can use the the above result to define a new polynomial $R'_{(M,D)}$ for MD-perspectives which we call *the Bollobás-Riordan polynomial for a MD-perspectives.*

Definition 7.4.1. Let (M,D) be a MD-perspective $D=(E,\rho)$ is a delta-matroid and $M=(E,r)$ is a matroid. Then

$$R'_{(M,D)}(x,y,z) = R_{(D^*,M^*)}(x,y,z). \tag{7.2}$$

Note this is well defined since if (M,D) is a MD-perspective then (D^*,M^*) is a DM-perspective.

We also have a convolution theorem for $R_{(D,M)}$ which again we prove in Chapter 9.

Theorem 7.4.3. *Let (D,M)be a DM-perspective where $M=(E,r)$ is a matroid and $D=(E,\mathcal{F})$ a delta-matroid. Then*

$$R_{(D,M)}(x,y,yz^2) = \sum_{A\subseteq E} R_{(D,M)\backslash A^c}(-1,y,yz^2)\cdot R_{(D,M)/A}(x,-1,-1)$$

Chapter 8

Hopf Algebras

Contents

In the paper [34] the authors use a Hopf algebra framework to show a range of results about the Tutte polynomial in a general setting. In this section we give a brief introduction to a Hopf algebra and define its "Tutte polynomial".

8.1 Hopf Algebras

Definition 8.1.1. Let K be a field, and let A be a vector space over K with an additional binary operation $m : A \times A \to A$. Then A is an *algebra* over K if and only if for all $x, y, z \in A$, and $a, b \in K$ we have:

1. $m((x + y), z) = m(x, z) + m(y, z)$,
2. $m(z, (x + y)) = m(z, x) + m(z, y)$,
3. $m(ax, by) = (ab)m(x, y)$.

That is, simply speaking, an algebra is just a vector space with tan additional operation.

We say that an algebra A over a field K is a *unital associative algebra* if it also satisfies

1. $m(m(x,y),z) = m(x,m(y,z))$.
2. There exist a unique element $1_A \in A$ such that $m(1_A,x) = x = m(x,1_A)$.

Definition 8.1.2. We define a function $\eta : K \to A$, where A is a unital associative algebra, such that for $\lambda \in K$ we have

1. $\eta(1_K) = 1_A$.
2. $\eta(\lambda) = \lambda\eta(1_K) = \lambda 1_A$.

Definition 8.1.3. A *coalgebra* (B,Δ,ϵ) over a field K is a vector space B together with K-linear maps $\Delta : B \to B \otimes B$ and $\epsilon : B \to K$ such that

1. $(id_B \otimes \Delta) \circ \Delta = (\Delta \otimes id_B) \circ \Delta$,
2. $(id_B \otimes \epsilon) \circ \Delta = id_B = (\epsilon \otimes id_B) \circ \Delta$,

where $\otimes$ is the tensor product over K and id_B is the identity function.

Figure 8.1 shows this pictorially.

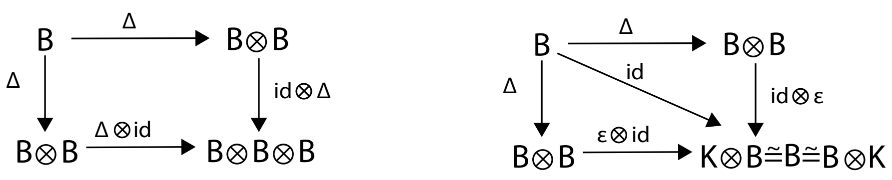


Figure 8.1: The operations of a coalegbra

A coalgebra can be seen as a dual to a unital associative algebra.

We can now define a *bialgebra*.

Definition 8.1.4. Let A be a vector space over a field K. Then A is a *bialgebra* over K if there are K-linear functions $m : A \times A \to A, \Delta : A \to A \otimes A$ and $\epsilon : A \to K$ such that

1. (A, m, η) forms an algebra,
2. (A, Δ, ϵ) forms a coalgebra,
3. $\Delta \circ m = (m \otimes m)(id_B \otimes \tau \otimes id_B)(\Delta \otimes \Delta)$,

   where $\tau : B \otimes B \to B \otimes B$ is the linear map defined by $\tau(x \otimes y) = y \otimes x$ for all $x, y \in B$,
4. $\epsilon \circ m = \epsilon \otimes \epsilon$,
5. $\eta \otimes \eta = \Delta \circ \eta$,
6. $\epsilon \circ \eta = id_K$.

A Hopf algebra is a bialgebra equipped with an antiautomorphism (a function that reverses the order of multiplication).

Definition 8.1.5. Let $(H, m, \eta, \Delta, \epsilon)$ be a bialgebra over a field K then it is a *Hopf algebra* if there exists a K-linear map $S : H \to H$ such that for all $x, y \in H$

$$S(xy) = S(y)S(x)$$

and

$$m \circ (S \otimes id_H) \circ \Delta = m \circ (id_H \otimes S) \circ \Delta.$$

We say that a Hopf algebra is *Combinatorial* if it is graded and the smallest component has 1 element.

We can now show that there exist a bialgebra for graphs.

Definition 8.1.6. Define $\mathbf{G}$ be the vector space of graphs, considered up to 1-point joins, over $\mathbb{R}$. Then for every $G, H \in \mathbf{G}$ define $G \cup H$ to be the graph formed by taking the disjoint union of G and H and identifying any isolated vertices. Define G_0 to be the graph with no edges and one vertex.

Theorem 8.1.1. *Let $G, H \in \mathbf{G}$ and define*

1. $m(G, H) = G \cup H$*,*

2. $\eta(1) = G_0$*,*

3. $\Delta(G) = \sum_{A \subseteq E(G)} G \backslash A^c \otimes G/A$,

4. $\epsilon = \begin{cases} 1 & \text{if } G = G_0 \\ 0 & \text{otherwise} \end{cases}$.

Then $(\mathbf{G}, m, \eta, \Delta, \epsilon)$ *is a bialgebra.*

In fact given a set and operations that satisfies certain properties we can find a bialgebra. This was formalised in [34] as follows:

Definition 8.1.7. A *minor system* consists of the following:

1. A graded set $\mathcal{S} = \bigoplus_{i \geq 0} S_i$ of finite combinatorial objects such that each $S \in \mathcal{S}_i$ has a set $E(S)$ of exactly i sub-objects associated with it and such that there is a unique element $1_{\mathcal{S}} \in \mathcal{S}_0$.

2. Two minor operations, deletion and contraction which we denote by $\backslash\!\backslash$ and $/\!/$, that associate elements $S \backslash\!\backslash e$ and $S /\!/ e$ to each pair $(S \in \mathcal{S}_n, e \in E(S))$ with $E(S \backslash\!\backslash e) = E(S /\!/ e) = E(S) \backslash e$ and if $e \neq f$ then

$$(S \backslash\!\backslash e) \backslash\!\backslash f = (S \backslash\!\backslash f) \backslash\!\backslash e, (S /\!/ e) /\!/ f = (S /\!/ f) /\!/ e, (S \backslash\!\backslash e) /\!/ f = (S /\!/ f) \backslash\!\backslash e$$

For example the set of matroids, with the cardinality of the ground set of the matroids, and with the usual deletion and contraction of matroids forms a minor system, as does the equivalence classes of graphs up to 1-point joins with the cardinality of the edge set of the graphs and with the usual deletion and contraction of graphs. We have already shown that $\mathbf{G}$ forms a bialgebra and it is fairly straightforward to see that the set of matroids forms a bialgebra in a very similar way. In fact given any minor system we can form a bialgebra as follows.

Theorem 8.1.2. *Let* $(\mathcal{S}, \backslash\!\backslash, /\!/)$ *be a minor system and* K *a field. Define* $\mathcal{H}$ *as the module of formal* K*-linear combinations of elements of* $\mathcal{S}$. *Then if*

1. $m : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ *such that* $m(G, H) = G \cup H$ *where* $\cup$ *is the disjoint union,*

2. $\eta : K \to \mathcal{H}$ *such that* $\eta(1_K) = 1_{\mathcal{H}}$ *where* $1_{\mathcal{H}}$ *is the unique element in* H_0,

3. $\Delta : \mathcal{H} \to: \mathcal{H} \otimes \mathcal{H}$ *such that* $\Delta(G) = \sum_{A \subseteq E(G)} G \backslash\backslash A^c \otimes G /\!/ A$,

4. $\epsilon : \mathcal{H} \to K$ *such that* $\epsilon = \begin{cases} 1 & \text{if } G = 1_{\mathcal{H}} \\ 0 & \text{otherwise,} \end{cases}$

then $(\mathcal{H}, m, \eta, \Delta, \epsilon)$ *is a bialgebra.*

In fact it can be shown using the *Milnor-Moore theorem* [45] that a minor system $\mathcal{H}$ that satisfies the above conditions forms a combinatorial Hopf algebra, which we call a *Hopf algebra of the minor system* $\mathcal{H}$. This allows us to form Hopf algebras for a range of combinatorial objects including abstract graphs, ribbon graphs, matroids and delta matroids. It also means that we can form Hopf algebras for delta-matroid perspectives and DM-perspective.

Definition 8.1.8. Define $\mathbf{D}$ be the Hopf algebra of the minor system of delta-matroids perspectives. Then for every $G = (M, D, N), H = (M', D', N') \in \mathbf{D}$ define $G \cup H = (M \cup N, D \cup B, M' \cup N')$ where $\cup$ is the disjoint union of delta-matroids.

Corollary 8.1.1. *Let* $G, H \in \mathbf{D}$ *where* $G = (M, D, M')$ *and* $H = (N, B, N')$. *Then define*

1. $m : \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ *such that* $m(G, H) = G \cup H$

2. $\eta : K \to \mathbf{D}$ *such that* $\eta(1_K) = (U_{0,0}, D_0, U_{0,0})$

3. $\Delta : \mathbf{D} \to: \mathbf{D} \otimes \mathbf{D}$ *such that* $\Delta(G) = \sum_{A \subseteq E(G)} G \backslash A^c \otimes G / A$

4. $\epsilon : \mathbf{D} \to K$ *such that* $\epsilon = \begin{cases} 1 & \text{if } G = (U_{0,0}, D_0, U_{0,0}) \\ 0 & \text{otherwise} \end{cases}$

then $(\mathbf{D}, m, \eta, \Delta, \epsilon)$ *is a Hopf algebra.*

Definition 8.1.9. Define $\mathbf{DM}$ be the Hopf algebra of the minor system of DM perspectives. Then for every $G = (M, D, N), H = (M', D', N') \in \mathbf{DM}$ define $G \cup H = (M \cup N, D \cup B, M' \cup N')$ where $\cup$ is the disjoint union of delta-matroids.

Corollary 8.1.2. *Let $G, H \in \mathbf{DM}$ where $G = (D, M)$ and $H = (B, N)$. Then define*

1. *$m : \mathbf{DM} \times \mathbf{DM} \to \mathbf{DM}$ such that $m(G, H) = G \cup H = (D \cup B, M \cup N)$ where $\cup$ is the disjoint union of delta-matroids.*
2. *$\eta : K \to \mathbf{DM}$ such that $\eta(1_K) = (D_0, U_{0,0})$*
3. *$\Delta : \mathbf{DM} \to: \mathbf{DM} \otimes \mathbf{DM}$ such that $\Delta(G) = \sum_{A \subseteq E(G)} G \backslash A^c \otimes G/A$*
4. *$\epsilon : \mathbf{DM} \to K$ such that $\epsilon = \begin{cases} 1 & \text{if } G = (D_0, U_{0,0}) \\ 0 & \text{otherwise} \end{cases}$*

then $(\mathbf{DM}, m, \eta, \Delta, \epsilon)$ is a Hopf algebra.

The proof of both these corollaries comes from the fact that both **D** and **DM** form minor systems and Theorem 8.1.2.

8.2 The Tutte Polynomial of a Hopf Algebra

We can now introduce the Tutte polynomial of a Hopf algebra. To do this we need to introduce a few more functions.

Definition 8.2.1. Let $\mathcal{H} = \bigoplus_{i \geq 0} \mathcal{H}_i$ be a *combinatorial Hopf algebra* (so, in particular, $\mathcal{H}$ is graded with $|\mathcal{H}_0| = 1$) and each $\mathcal{H}_i$ is a vector space over a field K and let $\{S_i\}_{i \in I}$ be a basis for $\mathcal{H}_1$ then we define the function $\delta_i : \mathcal{H} \to K$ to be

$$\delta_i(S) = \begin{cases} 1 & \text{if } S = S_i \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{a} = \{a_i\}_{i \in I}$ we then define the *selector* $\delta_{\mathbf{a}} : \mathcal{H} \to F$ as

$$\delta_{\mathbf{a}} = \sum_{i \in I} a_i \delta_i$$

We say that a selector $\delta_{\mathbf{a}}$ is *uniform* if, for each $S \in \mathcal{H}$, the evaluations of $\delta^{\otimes m}$ for each summand of $\triangle^{(m-1)}(S)$ are equal.

Definition 8.2.2. Let $(\mathcal{H}, \Delta, \epsilon)$ be a coalgebra and let f and g be functions from $\mathcal{H}$ to an algebra with product m then we define the *convolution product* $f * g$ as

$$f * g = m \circ (f \otimes g) \circ \Delta.$$

and hence the $*$-*exponential* as

$$\exp_*(f) = \sum_{m \geq 0} \frac{f^{*m}}{m!} = \epsilon + f + \frac{1}{2}(f * f) + \dots$$

Lemma 8.2.1. *Let $H \in H_i$ and consider $\exp_*(\delta_{\mathbf{a}})$. Then only the $\delta_{\mathbf{a}}^{*i}$ term is non-zero.*

Proof. We know that $\delta_{\mathbf{a}}(G) = 0$ if $G \neq S_i$ and therefore $\delta_{\mathbf{a}}(G_1) \cdot \delta_{\mathbf{a}}(G_2) \cdot \dots \cdot \delta_{\mathbf{a}}(G_n) = 0$ unless each G_i is an element of H_1. Now observe that when we apply Δ to $\mathcal{H}$ we have a sum of tensor products of the form $G_j \otimes G_k$ where $G_j \in H_j$ and $G_k \in H_k$ such that $j + k = i$. However we can discard all the summands where $k \neq 1$ as they will all be zero. We then apply $\Delta \circ id$ to the remaining summands which gives a sum of tensor products of the form $G_l \otimes G_q \otimes G_k$ $G_l \in H_l$ and $G_q \in H_q$ such that $l + q = i - 1$. We can similarly discard all the summands where $q \neq 1$ and if we repeat this process i times we will eventually have a sum where all the terms are elements of H_1 and hence non-zero after applying δ. □

Example 9. Let $(\mathcal{M}, \cdot, \eta, \Delta, \epsilon)$ be the combinatorial Hopf algebra formed by the vector space of matroids over a field K, then $M_1 = \{U_{0,1}, U_{1,1}\}$, and let $\mathbf{a} = (a, b)$ so

$$\delta_0(M) = \begin{cases} 1 & \text{if } M = U_{0,1} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_1(M) = \begin{cases} 1 & \text{if } M = U_{1,1} \\ 0 & \text{otherwise.} \end{cases}$$

Hence if $M \in \bigoplus_{i \geq 2} M_i$ and $w, x, y, z \in K$ then

$$\delta_{\mathbf{a}}(wU_{0,0} + xU_{0,1} + yU_{1,1} + zM) = ax + by.$$

Now consider $U_{1,2} = (\{1,2\}, (\{1\},\{2\}))$ and note that

$$U_{1,2}\backslash\{1\} = (\{2\}, (\{2\})) = U_{1,1}$$

and

$$U_{1,2}/\{1\} = (\{2\}, (\emptyset)) = U_{0,1}.$$

Similarly $U_{1,2}\backslash\{2\} = U_{1,1}$ and $U_{1,2}/\{1\} = U_{0,1}$. Also note $U_{1,2}/\{1,2\} = U_{1,2}\backslash\{1,2\} = U_{0,0}$. Hence

$$\begin{aligned}
\Delta(U_{1,2}) &= \sum_{A\subseteq\{1,2\}} U_{1,2}\backslash A^c \otimes U_{1,2}/A \\
&= U_{1,2}\backslash\emptyset \otimes U_{1,2}/\{1,2\} + U_{1,2}\backslash\{1\} \otimes U_{1,2}/\{2\} + U_{1,2}\backslash\{2\} \otimes U_{1,2}/\{1\} \\
&+ U_{1,2}\backslash\{1,2\} \otimes U_{1,2}/\emptyset \\
&= U_{1,2} \otimes U_{0,0} + 2U_{1,1} \otimes U_{0,1} + U_{0,0} \otimes U_{1,2}
\end{aligned}$$

and

$$\begin{aligned}
(\delta_{\mathbf{a}} * \delta_{\mathbf{a}})(U_{1,2}) &= m \circ (\delta_{\mathbf{a}} \otimes \delta_{\mathbf{a}}) \circ \Delta(U_{1,2}) \\
&= m \circ (\delta_{\mathbf{a}} \otimes \delta_{\mathbf{a}})(U_{1,2} \otimes U_{0,0} + 2U_{1,1} \otimes U_{0,1} + U_{0,0} \otimes U_{1,2}) \\
&= m(0 \otimes 0 + 2b \otimes a + 0 \otimes 0) \\
&= 2ab.
\end{aligned}$$

Therefore

$$\exp_*(\delta_{\mathbf{a}})(U_{1,2}) = ab.$$

We can now define the Tutte polynomial of a Hopf algebra.

Definition 8.2.3. Let $\mathcal{H} = \bigoplus_{i\geq 0} H_i$ be a combinatorial Hopf algebra and let $\mathbf{a} = \{a_i\}_{i\in I}$ and $\mathbf{b} = \{b_i\}_{i\in I}$ then the *Tutte polynomial* $\alpha(\mathbf{a},\mathbf{b})$ of $\mathcal{H}$ is defined as

$$\alpha(\mathbf{a},\mathbf{b}) = \exp_*(\delta_{\mathbf{a}}) * \exp_*(\delta_{\mathbf{b}})$$

It has been shown in [34] that this polynomial can be used to study a wide variety of graph polynomials and that provided that a set of combinatorial objects meet certain requirements, such as forming a minor system, the above definition can be used to obtain a cannonical Tutte polynomial for that set of objects.

Definition 8.2.4. Let $\mathcal{H}$ be a Hopf algebra of a minors system S, and δ_a and δ_b be uniform selectors. Then we say that $\alpha(\mathbf{a}, \mathbf{b})$, as given in Definition 8.2.3, is a *canonical Tutte polynomial* of the minors systems S.

We can now introduce a number of theorems from [34] which we will use in the final chapter.

Theorem 8.2.1. *Let $(\mathcal{H}, m, \eta, \Delta, \epsilon)$ be a combinatorial Hopf algebra of a minors system $\mathcal{S}$. Suppose that the set I indexes the elements of $\mathcal{H}_1$, and that δ_i are the functions defined in definition 2.1. Suppose also that for each j in some indexing set J there is a function $r_j : \mathcal{H} \to \mathbb{Q}$ such that*

$$r_j(S) = r_j(S /\!\!/ e) + m_{ij} \text{ when } \delta_i(S \backslash\!\!\backslash e^c) = 1$$

for all $e \in S$; and such that $r_j(S) = 0$ when $S \in H_0$.

For a set of indeterminates $\{x_j\}_{j \in J}$ define

$$\delta_{\mathbf{a}} = \sum_{i \in I} a_i \delta_i \text{ where } a_i := \prod_{j \in J} x_j^{m_{ij}}$$

Then $\delta_{\mathbf{a}}$ is uniform. Moreover, if $\delta_{\mathbf{b}} = \sum_{i \in I} a_i \delta_i$ with $b_i := \prod_{j \in J} y_j^{m_{ij}}$, then the Tutte polynomial of $\mathcal{H}$ satisfies

$$\alpha(\mathbf{a}, \mathbf{b})(S) = \prod_{j \in J} y_j^{r_j(S)} \sum_{A \subseteq E(S)} \prod_{j \in J} \left(\frac{x_j}{y_j}\right)^{r_j(A)}$$

where $r_j(A) = r_j(S \backslash\!\!\backslash A)$.

Theorem 8.2.2. *Let $\mathcal{H}$be a Hopf algebra of a minors system, and δ_a and δ_b be a uniform selectors. Then the canonical Tutte polynomial $\alpha(\mathbf{a}, \mathbf{b})$ is a* *recursively defined by*

$$\alpha(S) = \begin{cases} \delta_{\mathbf{b}}(S/\!\!/e^c) \cdot \alpha(S\backslash\!\!\backslash e) + \delta_{\mathbf{a}}(S\backslash\!\!\backslash e^c) \cdot \alpha(S/\!\!/e) & \text{if } S \notin S_0 \\ 1 & \text{if } S \in S_0 \end{cases}$$

Definition 8.2.5. Let $\mathcal{H}$ be a Hopf algebra of a minors system. We define a *combinatorial duality* for $\mathcal{H}$ as an involutionary grading preserving algebra morphism $* : \mathcal{H} \to \mathcal{H}$,where we denote $*(S)$ by S^* and call it the dual of S, such that for each $S \in \mathcal{H}$ and each $e \in E(S)$,we have $(S\backslash\!\!\backslash e)^* = S^*/\!\!/e$ and $(S/\!\!/e)^* = S^*\backslash\!\!\backslash e$.

Theorem 8.2.3. *Let $\mathcal{H}$ be a Hopf algebra of a minors system with a combinatorial duality $*$. Let $\delta_a = \sum_{i\in I} a_i\delta_i$ and $\delta_b = \sum_{i\in I} b_i\delta_i$ be selectors for $\mathcal{H}$. Then for all $S \in \mathcal{H}$*

$$\alpha(\mathbf{a}, \mathbf{b})(S) = \alpha(\mathbf{b}^*, \mathbf{a}^*)(S^*)$$

where $\alpha(\mathbf{b}^*, \mathbf{a}^*)(S^*)$ is defined by the selectors $\delta_{\mathbf{a}^*} = \delta_{\mathbf{a}} \circ *$ and $\delta_{\mathbf{b}^*} = \delta_{\mathbf{b}} \circ *$. I.e. $\delta_{\mathbf{a}^*}(S) = \delta_{\mathbf{a}}(S^*)$.

Chapter 9

Hopf Algebras and Delta-Matroid Perspective Polynomials

Contents

In Chapter 8 we showed that we can form Hopf algebras using delta-matroid perspectives and DM-perspectives. We also defined the canonical Tutte polynomial $\alpha(\mathbf{a}, \mathbf{b})$ of a Hopf algebra and stated deletion-contraction, duality and convolution formulas. In this chapter by carefully choosing our starting point we can use these formulas to prove theorems 7.3.1, 7.3.2, 7.3.3, 7.4.1, 7.4.2 and 7.4.3 from chapter 7.

9.1 Canonical Krushkal Polynomial

We will start by looking at the Krushkal Polynomial for a delta-matroid perspective, $K_{(M,D,M')}$. We first need to show that we can obtain $K_{(M,D,M')}$ as a canonical Tutte polynomial.

Lemma 9.1.1. *Let $D = (M, D, M')$ be a delta-matroid perspective where $D = (E, \rho)$ is a delta-matroid and $M = (E, r)$ and $M' = (E, r')$ are matroids. If*

$$S_1 = (U_{1,1}, D_c, U_{1,1}) \qquad S_2 = (U_{0,1}, D_0, U_{0,1}) \qquad S_3 = (U_{1,1}, D_c, U_{0,1})$$
$$S_4 = (U_{1,1}, D_n, U_{0,1}) \qquad S_5 = (U_{1,1}, D_0, U_{0,1})$$

and

$$r_1(A) = r'(A) \qquad r_2(A) = |A| - r(A)$$
$$r_3(A) = \rho(A) - r'(A) \qquad r_4(A) = r(A) - \rho(A)$$

then

$$r_j(S) = r_j(S/e) + m_{ij} \text{ when } \delta_i(S \backslash e^c) = 1,$$

where the m_{ij} are constants such that

$$\begin{array}{llll}
m_{11} = 1 & m_{12} = 0 & m_{13} = 0 & m_{14} = 0 \\
m_{21} = 0 & m_{22} = 1 & m_{23} = 0 & m_{24} = 0 \\
m_{31} = 0 & m_{32} = 0 & m_{33} = 1 & m_{34} = 0 \\
m_{41} = 0 & m_{42} = 0 & m_{43} = \frac{1}{2} & m_{44} = \frac{1}{2} \\
m_{51} = 0 & m_{52} = 0 & m_{53} = 0 & m_{54} = 1.
\end{array}$$

Proof. We have $r_1(S) = r'(S)$ and we know from Lemma 4.1.3 that

$$r'(S) = \begin{cases} r'(S/e) & \text{if } e \text{ is a loop of M'}, \\ r'(S/e) + 1 & \text{otherwise} \end{cases}$$

and we know that e is a loop in M' if and only if $M' \backslash e^c = U_{0,1}$ using Lemma

4.1.4 therefore we have

$$r_1(S) = \begin{cases} r_1(S/e) + 1 & \text{if } S\backslash e^c = S_1, \\ r_1(S/e) & \text{otherwise.} \end{cases}$$

Hence

$$m_{11} = 1 \text{ and } m_{21} = m_{31} = m_{41} = m_{51} = 0.$$

Next we have $r_2(S) = |S| - r(S)$, we know that $|S| = |S/e| + 1$ for all e and that e is a loop in M if $M\backslash e^c = U_{0,1}$ and so we have

$$r_2(S) = \begin{cases} r_2(S/e) + 1 & \text{if } S\backslash e^c = S_2, \\ r_1(S/e) & \text{otherwise.} \end{cases}$$

Hence

$$m_{22} = 1 \text{ and } m_{12} = m_{32} = m_{42} = m_{52} = 0.$$

Then we have $r_3(E) = \rho(E) - r'(E)$, and we know from Lemma 4.2.5. that

$$\rho(E) = \begin{cases} \rho(E/e) + 1 & \text{if } e \text{ is not a ribbon loop,} \\ \rho(E/e) & \text{if } e \text{ is an orientable ribbon loop,} \\ \rho(E/e) + \frac{1}{2} & \text{if } e \text{ is a non-orientable(S ribbon loop.} \end{cases}$$

We also know from Lemma 4.2.7 that e is not a ribbon loop (is an orientable ribbon loop, is a non orientable ribbon loop) if and only if $D\backslash e^c$ is isomorphic to D_c (D_0, D_n respectively).

Therefore

$$r_3(S) = \begin{cases} r_3(S/e) + 1 & \text{if } S\backslash e^c = S_3, \\ r_3(S/e) + \frac{1}{2} & \text{if } S\backslash e^c = S_4, \\ r_3(S/e) & \text{otherwise.} \end{cases}$$

Hence

$$m_{33} = 1, m_{13} = m_{23} = m_{53} = 0 \text{ and } m_{43} = \frac{1}{2}.$$

Finally we have that $r_3(S) = r(S) - \rho(S)$. Therefore

$$r_4(S) = \begin{cases} r_4(S/e) + 1 & \text{if } S\backslash e^c = S_5, \\ r_4(S/e) + \frac{1}{2} & \text{if } S\backslash e^c = S_4, \\ r_4(S/e) & \text{otherwise.} \end{cases}$$

Hence

$$m_{54} = 1, m_{14} = m_{34} = m_{24} = 0 \text{ and } m_{44} = \frac{1}{2}.$$

□

This means that we can apply Theorem 8.2.1 and so we have the following theorem

Theorem 9.1.1. *Let* $\mathcal{H} = (\mathbf{D}, m, \eta, \Delta, \epsilon)$ *be the Hopf algebra of* $\mathbf{D}$ *and let* $S = (M, D, M')$ *be a delta-matroid perspective. Then if*

$$S_1 = (U_{1,1}, D_c, U_{1,1}) \quad S_2 = (U_{0,1}, D_0, U_{0,1}) \quad S_3 = (U_{1,1}, D_c, U_{0,1})$$
$$S_4 = (U_{1,1}, D_n, U_{0,1}) \quad S_5 = (U_{1,1}, D_0, U_{0,1})$$

and

$$r_1(A) = r'(A) \qquad r_2(A) = |A| - r(A)$$
$$r_3(A) = \rho(A) - r'(A) \qquad r_4(A) = r(A) - \rho(A)$$

and

$$\mathbf{a} = (1, y, a, (ab)^{\frac{1}{2}}, b) \text{ and } \mathbf{b} = (x, 1, 1, 1, 1),$$

we have

$$\alpha(\mathbf{a}, \mathbf{b})(S) = K_{(M,D,M')}(x, y, a, b).$$

Proof. If we set x_i and y_i as follows

$$x_1 = 1 \qquad x_2 = y \qquad x_3 = a \qquad x_4 = b$$
$$y_1 = x \qquad y_2 = 1 \qquad y_3 = 1 \qquad y_4 = 1$$

and use the m_{ij} from Lemma 9.1.1 we get the a_i and b_i required and if we substitute them into the formulae for α from Theorem 8.2.1 we get

$$\begin{aligned}\alpha(\mathbf{a},\mathbf{b})(S) &= \prod_{j\in J} y_j^{r_j(S)} \sum_{A\subseteq E(S)} \prod_{j\in J}\left(\frac{x_j}{y_j}\right)^{r_j(A)} \\ &= x^{r'(G)} \sum_{A\subseteq E(S)} x^{-r'(A)} y^{|A|-r(A)} a^{\rho(A)-r'(A)} b^{r(A)-\rho(A)} \\ &= K_{(M,D,M')}(x,y,a,b)\end{aligned}$$

□

We can now use this relationship along with Theorem 8.2.2 to prove Theorem 7.3.1 and obtain a deletion-contraction formula for $K_{(M,D,M')}$.

Theorem 9.1.2. *Let $D = (M, D, M')$ be a delta-matroid perspective where $D = (E, \rho)$ is a delta-matroid and $M = (E, r)$ and $M' = (E, r')$ are matroids. Then*

$$K_{(M,D,M')}(x,y,a,b) = \begin{cases} K_{(M\backslash e,D\backslash e,M'\backslash e)} + K_{(M/e,D/e,M'/e)} & \text{if } e \text{ is not a loop or a coloop in } M', \\ K_{(M\backslash e,D\backslash e,M'\backslash e)} + yK_{(M/e,D/e,M'/e)} & \text{if } e \text{ is a loop in } M, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)} + aK_{(M/e,D/e,M'/e)} & \text{if } e \text{ is a loop in } M' \text{ and } e \text{ is not a ribbon loop in } D, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)} + (ab)^{\frac{1}{2}}K_{(M/e,D/e,M'/e)} & \text{if } e \text{ is not a loop in } M \text{ and } e \text{ is a non-orientable ribbon loop in } D, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)} + bK_{(M/e,D/e,M'/e)} & \text{if } e \text{ is not a loop in } M \text{ and } e \text{ is an orientable ribbon loop in } D, \\ xK_{(M\backslash e,D\backslash e,M'\backslash e)} + K_{(M/e,D/e,M'/e)} & \text{if } e \text{ is a coloop in } M'. \end{cases}$$

Proof. We know from Theorem 8.2.2 that

$$\alpha(S) = \delta_{\mathbf{b}}(S/e^c) \cdot \alpha(S\backslash e) + \delta_{\mathbf{a}}(S\backslash e^c) \cdot \alpha(S/e)$$

and we have shown in Theorem 9.1.1 that

$$\alpha(S) = K_{(M,D,M')}.$$

Therefore we have

$$K_{(M,D,M')} = \delta_{\mathbf{b}}(S/e^c) \cdot K_{(M\backslash e,D\backslash e,M'\backslash e)} + \delta_{\mathbf{a}}(S\backslash e^c) \cdot K_{(M/e,D/e,M'/e)}.$$

Recall that for all i $\delta_{\mathbf{a}}(S_i) = a_i$ and $\delta_{\mathbf{b}} = b_i$. Therefore since S/e^c and $S\backslash e^c$ must equal one of $S_1, \dots S_5$ and

$$\mathbf{a} = (1, y, a, (ab)^{\frac{1}{2}}, b) \text{ and } \mathbf{b} = (x, 1, 1, 1, 1),$$

we have

$$K_{(M,D,M')}(x,y,a,b) = \begin{cases} K_{(M\backslash e,D\backslash e,M'\backslash e)}+K_{(M/e,D/e,M'/e)} & \text{if } S/e^c \neq S_1 \text{ and } S\backslash e^c = S_1, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)}+yK_{(M/e,D/e,M'/e)} & \text{if } S/e^c \neq S_1 \text{ and } S\backslash e^c = S_2, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)}+aK_{(M/e,D/e,M'/e)} & \text{if } S/e^c \neq S_1 \text{ and } S\backslash e^c = S_3, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)}+(ab)^{\frac{1}{2}}K_{(M/e,D/e,M'/e)} & \text{if } S/e^c \neq S_1 \text{ and } S\backslash e^c = S_4, \\ K_{(M\backslash e,D\backslash e,M'\backslash e)}+bK_{(M/e,D/e,M'/e)} & \text{if } S/e^c \neq S_1 \text{ and } S\backslash e^c = S_5, \\ xK_{(M\backslash e,D\backslash e,M'\backslash e)}+K_{(M/e,D/e,M'/e)} & \text{if } S/e^c = S_1 \text{ and } S\backslash e^c = S_1, \\ xK_{(M\backslash e,D\backslash e,M'\backslash e)}+yK_{(M/e,D/e,M'/e)} & \text{if } S/e^c = S_1 \text{ and } S\backslash e^c = S_2, \\ xK_{(M\backslash e,D\backslash e,M'\backslash e)}+aK_{(M/e,D/e,M'/e)} & \text{if } S/e^c = S_1 \text{ and } S\backslash e^c = S_3, \\ xK_{(M\backslash e,D\backslash e,M'\backslash e)}+(ab)^{\frac{1}{2}}K_{(M/e,D/e,M'/e)} & \text{if } S/e^c = S_1 \text{ and } S\backslash e^c = S_4, \\ xK_{(M\backslash e,D\backslash e,M'\backslash e)}+bK_{(M/e,D/e,M'/e)} & \text{if } S/e^c = S_1 \text{ and } S\backslash e^c = S_5. \end{cases}$$

We can now reduce this as follows. If $M'/e^c = U_{1,1}$ then we know from Lemma 4.1.4 that e must be a coloop in M'. Therefore $M'\backslash e^c = U_{1,1}$, since we also know from Lemma 4.1.4 that $M'\backslash e^c = U_{0,1}$ if and only if e is a loop in M'. Hence if $S/e^c = S_1$ then we must have $S\backslash e^c = S_1$ and so we can delete all the cases where $S/e^c = S_1$ and $S\backslash e^c \neq S_1$. This leaves us with

$$K_{(M,D,M')} = xK_{(M\backslash e,D\backslash e,M'\backslash e)} + K_{(M/e,D/e,M'/e)}$$

if e is a coloop in M'.

If e is not a coloop in M' then we know that $S/e^c \neq S_1$ and therefore we only need to look at $S\backslash e^c$. If $S\backslash e^c = S_1$ then $M'\backslash e^c = U_{1,1}$ and so e is not a loop in M' hence we have

$$K_{(M,D,M')} = K_{(M\backslash e,D\backslash e,M'\backslash e)} + K_{(M/e,D/e,M'/e)}$$

if e is not a loop or a coloop in M'.

If $S\backslash e^c = S_2$ then $M\backslash e^c = U_{0,1}$ so e is a loop in M hence

$$K_{(M,D,M')} = K_{(M\backslash e,D\backslash e,M'\backslash e)} + yK_{(M/e,D/e,M'/e)}$$

if e is a loop in M.

If $S\backslash e^c$ is one of S_3, S_4, S_5 then e is a loop in M' and not a loop in M so we need to consider what e is in D. If $S\backslash e^c = S_3$ then $D\backslash e^c = D_c$ so e is not a ribbon loop. Hence

$$K_{(M,D,M')} = K_{(M\backslash e,D\backslash e,M'\backslash e)} + aK_{(M/e,D/e,M'/e)}$$

if e is a loop in $M'$and e is not a ribbon loop in D.

If $S\backslash e^c = S_4$ then $D\backslash e^c = D_n$ so e is a non orientable ribbon loop hence

$$K_{(M,D,M')} = K_{(M\backslash e,D\backslash e,M'\backslash e)} + (ab)^{\frac{1}{2}}K_{(M/e,D/e,M'/e)}$$

if e is not a loop in M and e is a non-orientable ribbon loop in D.

Finally if $S\backslash e^c = S_5$ then $D\backslash e^c = D_0$ so e is a orientable ribbon loop hence

$$K_{(M,D,M')} = K_{(M\backslash e,D\backslash e,M'\backslash e)} + bK_{(M/e,D/e,M'/e)}$$

if e is not a loop in M and e is a orientable ribbon loop in D.

Therefore we have the required result. □

We can also use the results from Chapter 8 to prove Theorem 7.3.2 which shows the relationship of the Krushkal polynomial with its dual.

Theorem 9.1.3. *Let $S = (M, D, M')$ be a delta-matroid perspective where*

$D = (E, \mathcal{F})$ is a delta-matroid and $M = (E, r)$ and $M' = (E, r')$ are matroids. Then

$$K_{(M,D,M')}(x, y, a, b) = b^{\rho_{D^*}(E)-r'(E)} a^{r(E)-\rho_{D^*}(E)} K_{((M')^*,D^*,M^*)}(y, x, b^{-1}, a^{-1}),$$

and

$$K_{((M')^*,D^*,M^*)}(x, y, a, b) = b^{\rho_D(E)-r'(E)} a^{r(E)-\rho_D(E)} K_{(M,D,M')}(y, x, b^{-1}, a^{-1}).$$

Proof. We have from Theorem 8.2.3 that

$$\alpha(\mathbf{a}, \mathbf{b})(S) = \alpha(\mathbf{b}^*, \mathbf{a}^*)(S^*)$$

where $\alpha(\mathbf{b}^*, \mathbf{a}^*)(S^*)$ is defined by the selectors $\delta_{\mathbf{a}^*} = \delta_{\mathbf{a}} \circ *$ and $\delta_{\mathbf{b}^*} = \delta_{\mathbf{b}} \circ *$.

We also have $\mathbf{a} = (1, y, a, (ab)^{\frac{1}{2}}, b)$ and $\mathbf{b} = (x, 1, 1, 1, 1)$ and since

$$(S_1)^* = S_2,\ (S_2)^* = S_1,\ (S_3)^* = S_5,\ (S_4)^* = S_4 \text{ and } (S_5)^* = S_3$$

then we have

$$\begin{aligned}
\delta_{\mathbf{a}^*}(S_1) &= \delta_{\mathbf{a}}(S_2) \quad \text{and } \delta_{\mathbf{b}^*}(S_1) = \delta_{\mathbf{b}}(S_2)\\
\delta_{\mathbf{a}^*}(S_2) &= \delta_{\mathbf{a}}(S_1) \quad \text{and } \delta_{\mathbf{b}^*}(S_2) = \delta_{\mathbf{b}}(S_1)\\
\delta_{\mathbf{a}^*}(S_3) &= \delta_{\mathbf{a}}(S_5) \quad \text{and } \delta_{\mathbf{b}^*}(S_3) = \delta_{\mathbf{b}}(S_5)\\
\delta_{\mathbf{a}^*}(S_4) &= \delta_{\mathbf{a}}(S_4) \quad \text{and } \delta_{\mathbf{b}^*}(S_4) = \delta_{\mathbf{b}}(S_4)\\
\delta_{\mathbf{a}^*}(S_5) &= \delta_{\mathbf{a}}(S_3) \quad \text{and } \delta_{\mathbf{b}^*}(S_5) = \delta_{\mathbf{b}}(S_3).
\end{aligned}$$

Hence

$$\mathbf{a}^* = (y, 1, b, (ab)^{\frac{1}{2}}, a) \text{ and } \mathbf{b}^* = (1, x, 1, 1, 1).$$

Now $S^* = ((M')^*, D^*, M^*)$ so if we set

$$r_1(A) = r_{M^*}(A) \qquad r_2(A) = |A| - r_{(M')^*}(A)$$
$$r_3(A) = \rho_{D^*}(A) - r_{M^*}(A) \qquad r_4(A) = r_{(M')^*}(A) - \rho_{D^*}(A)$$

and

$$\begin{array}{llll} m_{11} = 1 & m_{12} = 0 & m_{13} = 0 & m_{14} = 0 \\ m_{21} = 0 & m_{22} = 1 & m_{23} = 0 & m_{24} = 0 \\ m_{31} = 0 & m_{32} = 0 & m_{33} = 1 & m_{34} = 0 \\ m_{41} = 0 & m_{42} = 0 & m_{43} = \frac{1}{2} & m_{44} = \frac{1}{2} \\ m_{51} = 0 & m_{52} = 0 & m_{53} = 0 & m_{54} = 1 \end{array}$$

and we have that $a_i := \prod_{j \in J} s_j^{m_{ij}}$ and $b_i := \prod_{j \in J} t_j^{m_{ij}}$ then in order to get the required values for $\mathbf{a}^*$ and $\mathbf{b}^*$ we set

$$\begin{array}{llll} s_1 = y & s_2 = 1 & s_3 = b & s_4 = a \\ t_1 = 1 & t_2 = x & t_3 = 1 & t_4 = 1. \end{array}$$

Therefore by Theorem 8.2.1 we have

$$\begin{aligned} \alpha(\mathbf{b}^*, \mathbf{a}^*)(S^*) &= \prod_{j \in J} s_j^{r_j(S^*)} \sum_{A \subseteq E(S^*)} \prod_{j \in J} \left(\frac{t_j}{s_j}\right)^{r_j(A)} \\ &= b^{\rho_{D^*}(E) - r_{M^*}(E)} a^{r_{(M')^*}(E) - \rho_{D^*}(E)} \\ &\cdot \sum_{A \subseteq E(S^*)} y^{r_{M^*}(E) - r_{M^*}(A)} x^{|A| - r_{(M')^*}(A)} b^{-\left(\rho_{D^*}(A) - r_{M^*}(A)\right)} a^{-\left(r_{(M')^*}(A) - \rho_{D^*}(A)\right)} \\ &= b^{\rho_{D^*}(E) - r_{M^*}(E)} a^{r_{(M')^*}(E) - \rho_{D^*}(E)} K_{((M')^*, D^*, M^*)}(y, x, b^{-1}, a^{-1}). \end{aligned}$$

hence

$$\begin{aligned} K_{(M,D,M')}(x, y, a, b) &= \alpha(\mathbf{a}, \mathbf{b})(S) \\ &= \alpha(\mathbf{b}^*, \mathbf{a}^*)(S^*) \\ &= b^{\rho_{D^*}(E) - r_{M^*}(E)} a^{r_{(M')^*}(E) - \rho_{D^*}(E)} K_{((M')^*, D^*, M^*)}(y, x, b^{-1}, a^{-1}) \end{aligned}$$

Similarly if we started with $K_{((M')^*,D^*,M^*)}$ and applied the same process we would obtain

$$K_{((M')^*,D^*,M^*)}(x,y,a,b) = b^{\rho_D(E)-r_{M'}(E)}a^{r_{(M)}(E)-\rho_D(E)}K_{(M,D,M')}(y,x,b^{-1},a^{-1}).$$

□

9.1.1 Convolution Formulas

Theorem 8.2.2 gives us the following

$$\alpha(\mathbf{a},\mathbf{b})(S) = \sum_{A\subseteq E} \alpha(\mathbf{a},\mathbf{c})(S\backslash\!\backslash A^c)\cdot\alpha(-\mathbf{c},\mathbf{b})(S/\!\!/A).$$

We use this to obtain a convolution formula for K. However if we try to simply solve for $\mathbf{c}$ we come into some problems as to get $\alpha(\mathbf{a},\mathbf{c})$ and $\alpha(-\mathbf{c},\mathbf{b})$ to be Krushkal polynomials we need

$$(c_1,c_2,c_3,c_4,c_5) = (x,1,1,1,1)$$

and

$$(-c_1,-c_2,-c_3,-c_4,-c_5) = (1,y,a,(ab)^{\frac{1}{2}},b)$$

and this means that $c_4 = 1$ which means that $-c_4 = -1$ which means that $(ab)^{\frac{1}{2}} = -1$ which is problematic. One way to solve this problem is to substitute ab^2 for b so this gives us $\mathbf{a} = (1,y,a,ab,ab^2)$. We can then prove Theorem 7.3.3.

Theorem. *Let $M = (E,r)$ and $M' = (E,r')$ be matroids and $D = (E,\rho)$ be a delta-matroid. Then*

$$K_{(M,D,M')}(x,y,a,ab^2) = \sum_{A\subseteq E} K_{(M,D,M')\backslash A^c}(-1,y,a,ab^2)\cdot K_{(M,D,M')/A}(x,-1,-1,-1)$$

Proof. Let $\mathbf{a} = (1,y,a,ab,ab^2)$ $\mathbf{b} = (x,1,1,1,1)$ and $\mathbf{c} = (-1,1,1,1,1)$ then

$$\alpha(\mathbf{a},\mathbf{c}) = K_{(M,D,M')}(-1,y,a,ab^2)$$

and

$$\alpha(-\mathbf{c}, \mathbf{b}) = K_{(M,D,M')}(x, -1, -1, -1)$$

hence by Theorem 8.2.2

$$K_{(M,D,M')}(x, y, a, b) = \sum_{A \subseteq E} K_{(M,D,M')\backslash A^c}(-1, y, a, ab^2) \cdot K_{(M,D,M')/A}(x, -1, -1, -1).$$

□

9.2 The Bollobás-Riordan Polynomial

We can use DM-perspectives in a similar way to deduce some results about the Bollobás-Riordan Polynomial. We start by showing that we can recover the BR-polynomial from the canonical Tutte polynomial in the same way as we did for the Krushkal polynomial.

Lemma 9.2.1. *Let $D = (E, \mathcal{F})$ be a delta-matroid, $M = (E, r)$ be a matroid and let $S = (D, M)$ be a DM perspective. If*

$$T_1 = (D_c, U_{1,1}) \qquad T_2 = (D_0, U_{0,1})$$
$$T_3 = (D_c, U_{0,1}) \qquad T_4 = (D_n, U_{0,1})$$

and

$$r_1(A) = r(A) \qquad r_2(A) = |A| - \rho(A)$$
$$r_3(A) = \rho(A) - r(A)$$

then

$$r_j(S) = r_j(S/e) + m_{ij} \text{ when } \delta_i(S \backslash e^c) = 1$$

where the m_{ij} are given by

$$\begin{array}{lll} m_{11} = 1 & m_{12} = 0 & m_{13} = 0 \\ m_{21} = 0 & m_{22} = 1 & m_{23} = 0 \\ m_{31} = 0 & m_{32} = 0 & m_{33} = 1 \\ m_{41} = 0 & m_{42} = \dfrac{1}{2} & m_{43} = \dfrac{1}{2} \end{array}$$

Proof. For $S \backslash e^c$ to be a matroid perspective it must equal T_i for some i so we only need to consider the evaluation of each r_i in these cases. We have that $r_1(S) = r(S)$ and we know from Lemma 4.1.3 that

$$r(S) = \begin{cases} r(S/e) & \text{if } e \text{ is a loop,} \\ r(S/e) + 1 & \text{otherwise,} \end{cases}$$

and e is a loop in M if $M \backslash e^c = U_{0,1}$. Therefore we have

$$r_1(S) = \begin{cases} r_1(S/e) + 1 & \text{if } S \backslash e^c = S_1, \\ r_1(S/e) & \text{otherwise.} \end{cases}$$

Hence

$$m_{11} = 1 \text{ and } m_{21} = m_{31} = m_{41} = 0.$$

Then we have $r_2(S) = |S| - \rho(S)$, $|S| = |S/e| + 1$ for all e and we know from Lemma 4.2.5 that

$$\rho(S) = \begin{cases} \rho(S/e) + 1 & \text{if } e \text{ is not a ribbon loop,} \\ \rho(S/e) & \text{if } e \text{ is an orientable ribbon loop,} \\ \rho(S/e) + \frac{1}{2} & \text{if } e \text{ is a non-orientable ribbon loop.} \end{cases}$$

We also know from Lemma 4.2.7 that e is not a ribbon loop (is an orientable ribbon loop, is a non orientable ribbon loop) if and only if $D \backslash e^c$ is isomorphic to D_c (D_0, D_n respectively).

$$r_2(S) = \begin{cases} r_2(S/e) + 1 & \text{if } S\backslash e^c = T_2, \\ r_2(S/e) + \frac{1}{2} & \text{if } S\backslash e^c = T_4, \\ r_2(S/e) & \text{otherwise.} \end{cases}$$

Hence

$$m_{22} = 1, m_{42} = \frac{1}{2} \text{ and } m_{12} = m_{32} = 0.$$

Finally we have $r_3(S) = \rho(S) - r(S)$. Therefore

$$r_3(S) = \begin{cases} r_3(S/e) + 1 & \text{if } S\backslash e^c = T_3, \\ r_3(S/e) + \frac{1}{2} & \text{if } S\backslash e^c = T_4, \\ r_3(S/e) & \text{otherwise.} \end{cases}$$

Hence

$$m_{33} = 1, m_{13} = m_{23} = 0 \text{ and } m_{43} = \frac{1}{2}.$$

□

Therefore we can apply Theorem 8.2.1 in the setting described above to show that the canonical Tutte polynomial of a DM-perspective is $R_{(D,M)}(x, y, z)$.

Theorem 9.2.1. *Let $\mathcal{H} = (\mathbf{DM}, m, \eta, \Delta, \epsilon)$ be the Hopf algebra for* **DM** *and let $T = (D, M)$ be a delta-matroid perspective. Then if*

$$T_1 = (D_c, U_{1,1}) \qquad T_2 = (D_0, U_{0,1})$$
$$T_3 = (D_c, U_{0,1}) \qquad T_4 = (D_n, U_{0,1})$$

and

$$r_1(A) = r(A) \qquad r_2(A) = |A| - \rho(A)$$
$$r_3(A) = \rho(A) - r(A)$$

and

$$\mathbf{a} = (1, y, z, (yz)^{\frac{1}{2}})$$

and

$$\mathbf{b} = (x, 1, 1, 1)$$

then

$$\alpha(\mathbf{a}, \mathbf{b})(T) = R_{(D,M)}(x, y, z).$$

Proof. If we set

$$x_1 = 1, x_2 = y, x_3 = z, y_1 = x \text{ and } y_2 = y_3 = 1,$$

then Theorem 8.2.1 gives

$$\begin{aligned}\alpha(\mathbf{a}, \mathbf{b})(S) &= x^{r_1(S)} \sum_{A \subseteq E} x^{-r_1(A)} y^{r_2(A)} z^{r_3(A)} \\ &= R_{(D,M')}(x, y, z)\end{aligned}$$

and we achieve the desired result. □

We can also use the same method that we used for the Krushkal polynomial to prove the deletion-contraction relationship for $R_{(D,M)}(x, y, z)$ stated in Theorem 7.4.1.

Theorem 9.2.2. *Let (D, M) be a DM perspective where $D = (E, \rho)$ is a* *delta-matroid and $M = (E, r)$ is a matroid. Then*

$$R_{(D,M)}(x,y,z) = \begin{cases} R_{(D\backslash e,M\backslash e)} + R_{(D/e,M/e)} & \text{if } e \text{ is not a loop or a coloop in } M, \\ R_{(D\backslash e,M\backslash e)} + yR_{(D/e,M/e)} & \text{if } e \text{ is a orientable ribbon loop in } D, \\ R_{(D\backslash e,M\backslash e)} + zR_{(D/e,M/e)} & \text{if } e \text{ is a loop in } M \text{ and } e \text{ is not a ribbon loop in } D, \\ R_{(D\backslash e,M\backslash e)} + (yz)^{\frac{1}{2}}R_{(D/e,M/e)} & \text{if } e \text{ is a non-orientable ribbon loop in } D, \\ xR_{(D\backslash e,M\backslash e)} + R_{(D/e,M/e)} & \text{if } e \text{ is a coloop in } M. \end{cases} \tag{9.1}$$

Proof. We know from Theorem 8.2.2 that

$$\alpha(S) = \delta_{\mathbf{b}}(S/e^c) \cdot \alpha(S\backslash e) + \delta_{\mathbf{a}}(S\backslash e^c) \cdot \alpha(S/e)$$

and we have shown above that

$$\alpha(S) = R_{(D,M)}$$

so we have

$$R_{(D,M)} = \delta_{\mathbf{b}}(S/e^c) \cdot R_{(D\backslash e,M\backslash e)} + \delta_{\mathbf{a}}(S\backslash e^c) \cdot R_{(D/e,M/e)}$$

Hence $\delta_{\mathbf{a}}(S_i) = a_i$ and $\delta_{\mathbf{b}} = b_i$ Therefore since S/e^c and $S\backslash e^c$ must equal one of $T_1, \ldots, T_5$ and

$$\mathbf{a} = (1, y, z, (yz)^{\frac{1}{2}}) \text{ and } \mathbf{b} = (x, 1, 1, 1)$$

we have

$$R_{(D,M)}(x,y,z) = \begin{cases} R_{(D\backslash e,M\backslash e)} + R_{(D/e,M/e)} & \text{if } S/e^c \neq T_1 \text{ and } S\backslash e^c = T_1, \\ R_{(D\backslash e,M\backslash e)} + yR_{(D/e,M/e)} & \text{if } S/e^c \neq T_1 \text{ and } S\backslash e^c = T_2, \\ R_{(D\backslash e,M\backslash e)} + zR_{(D/e,M/e)} & \text{if } S/e^c \neq T_1 \text{ and } S\backslash e^c = T_3, \\ R_{(D\backslash e,M\backslash e)} + (yz)^{\frac{1}{2}}R_{(D/e,M/e)} & \text{if } S/e^c \neq T_1 \text{ and } S\backslash e^c = T_4, \\ xR_{(D\backslash e,M\backslash e)} + R_{(D/e,M/e)} & \text{if } S/e^c = T_1 \text{ and } S\backslash e^c = T_1, \\ xR_{(D\backslash e,M\backslash e)} + yR_{(D/e,M/e)} & \text{if } S/e^c = T_1 \text{ and } S\backslash e^c = T_2, \\ xR_{(D\backslash e,M\backslash e)} + zR_{(D/e,M/e)} & \text{if } S/e^c = T_1 \text{ and } S\backslash e^c = T_3, \\ xR_{(D\backslash e,M\backslash e)} + (yz)^{\frac{1}{2}}R_{(D/e,M/e)} & \text{if } S/e^c = T_1 \text{ and } S\backslash e^c = T_4. \end{cases} \tag{9.2}$$

We can now reduce this as follows. If $S/e^c = T_1$ then $M/e^c = U_{1,1}$. So e must be a coloop in M, hence $M\backslash e^c = U_{1,1}$, since $M\backslash e^c = U_{0,1}$ if and only if e is a loop in M. Hence if $S/e^c = T_1$ then we must have $M\backslash e^c = U_{1,1}$ which only occurs when $S\backslash e^c = T_1$ and so we can delete all the cases where $S/e^c = S_1$ and $S\backslash e^c \neq S_1$. This leaves us with

$$R_{(D,M)} = xR_{(D\backslash e,M\backslash e)} + R_{(D/e,M/e)}$$

if e is a coloop in M.

If e is not a coloop in M then we know that $S/e^c \neq T_1$ and therefore we only need to look at $S\backslash e^c$.

If $S\backslash e^c = T_1$ then $M\backslash e^c = U_{1,1}$ and so e is not a loop in M hence we have

$$R_{(D,M)} = R_{(D\backslash e,M\backslash e)} + R_{(D/e,M/e)}$$

if e s not a loop or a coloop in M. If $S\backslash e^c = T_2, T_3$ or T_4, then e is a loop in M so we need to consider what e is in D. If $S\backslash e^c = T_2$ then $D\backslash e^c = D_0$, so e is a orientable ribbon loop hence

$$R_{(D,M)} = R_{(D\backslash e,M\backslash e)} + yR_{(D/e,M/e)}$$

if e is a orientable ribbon loop in D.

If $S\backslash e^c = T_3$ then $D\backslash e^c = D_c$ so e is not a ribbon loop hence

$$R_{(D,M)} = R_{(D\backslash e,M\backslash e)} + zR_{(D/e,M/e)}$$

if e is a loop in M and e is not a ribbon loop in D.

If $S\backslash e^c = T_4$ then $D\backslash e^c = D_n$. So e is a non orientable ribbon loop hence

$$R_{(D,M)} = R_{(D\backslash e,M\backslash e)} + (yz)^{\frac{1}{2}})R_{(D/e,M/e)}$$

if e is a non orientable ribbon loop in D. Therefore we have the required result. Also note that we have covered all possibilities as if e is a ribbon loop in D then it must be a loop in M. □

Similarly we can now also prove Theorem 7.4.2 which gives the convolution formula for $R_{(D,M)}$.

Theorem 9.2.3. *Let (D, M) be a DM-perspective where $M = (E, r)$ is a matroid and $D = (E, \mathcal{F})$ a delta-matroid. Then*

$$R_{(D,M)}(x, y, yz^2) = \sum_{A\subseteq E} R_{(D,M'\backslash A^c}(-1, y, yz^2) \cdot R_{(D,M)/A}(x, -1, -1)$$

Proof. Let $\mathbf{a} = (1, y, yz^2, yz)$ $\mathbf{b} = (x, 1, 1, 1)$ and $\mathbf{c} = (-1, 1, 1, 1)$ then

$$\alpha(\mathbf{a}, \mathbf{c}) = R_{(D,M)}(-1, y, yz^2)$$

and

$$\alpha(-\mathbf{c}, \mathbf{b}) = R_{(D,M)}(x, -1, -1)$$

hence by Theorem 8.2.2

$$R_{(D,M)}(x, y, yz^2) = \sum_{A\subseteq E} R_{(D,M)\backslash A^c}(-1, y, yz^2) \cdot R_{(D,M)/A}(x, -1, -1,).$$

□

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