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Published in:
Discrete and Continuous Dynamical Systems- Series A

Document Version:
Peer reviewed version

Queen's University Belfast - Research Portal:
Link to publication record in Queen's University Belfast Research Portal

ON AN EXPONENTIAL ATTRACTOR FOR A CLASS OF PDES WITH DEGENERATE DIFFUSION AND CHEMOTAXIS

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(Communicated by Shouchuan Hu)

Abstract. In this article we deal with a class of strongly coupled parabolic systems that encompasses two different effects: degenerate diffusion and chemotaxis. Such classes of equations arise in the mesoscale level modeling of biomass spreading mechanisms via chemotaxis. We show the existence of an exponential attractor and, hence, of a finite-dimensional global attractor under certain 'balance conditions' on the order of the degeneracy and the growth of the chemotactic function.

1. **Introduction.** In this work we continue our studies of the longtime behaviour of a degenerate system modelling a biomass spreading in the presence of chemotaxis which was introduced in [15]:

$$\partial_t M = \nabla \cdot (M^\alpha \nabla M - M^\gamma \nabla \rho) - f(M, \rho) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1a}$$

$$\partial_t \rho = \Delta \rho - g(M, \rho) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1b}$$

$$M = 0, \quad \rho = 1 \quad \text{in } \mathbb{R}^+ \times \partial\Omega, \tag{1c}$$

$$M = M_0, \quad \rho = \rho_0 \quad \text{in } \{0\} \times \Omega, \tag{1d}$$

where Ω is smooth bounded domain in $\mathbb{R}^N$, $N \in \{1, 2, 3\}$, α and γ are two positive constants satisfying certain conditions (we call them 'balance conditions') to be specified below, and $M_0 \in L^\infty(\Omega)$, $\rho_0 \in W^{1,\infty}(\Omega)$ are nonnegative functions.

Equations (1a)-(1b) can model, e.g., the spreading of a bacterial population under the influence of chemotaxis. Chemotaxis systems have been much studied in the recent decades. We refer the interested reader to surveys [3, 22] which cover both modelling and analytical aspects. The available results mostly focus on the uniform boundedness/blow-up for finite/infinite times and convergence of solutions to an equilibrium. It is usually assumed that $\alpha = 0$, $\gamma = 1$, and very specific nonlinearities f and g are chosen. For example, in the case of the classic Keller-Segel model [22] one has $f \equiv 0$, i.e., the absence of proliferation which is not realistic in general. Furthermore, the condition $\alpha = 0$ corresponds to the standard non-degenerate

diffusion. It is has a well known property of the infinite speed of propagation which entails that the population fills the domain instantaneously. Particularly in the case of a bacterial biofilm this falls short to model the experimentally and numerically observed [6] moving fronts. Thus, it was proposed in [15] to consider rather general nonlinearities f and g, thus allowing to model reaction/interaction, and to take $\alpha > 0$. The latter corresponds to the case of a degenerate diffusion (that is, the diffusion coefficient has at least one zero point) of the porous medium type. It is well known (see, e.g., [27]) that such diffusion leads to solutions with a finite speed of propagation.

From the analytical point of view, system (1) is a blend of a porous medium equation with a chemotaxis growth system. The dynamics of a single equation with a porous medium degeneracy has been thoroughly studied in [9, Chapter 4] with the help of exponential attractors (see *Definition 1.1* below). For non-degenerate chemotaxis growth systems under the homogeneous Neumann boundary conditions the existence of attractors and their dimension were studied in [1, 10–12, 16, 25, 26], see also [9, Section 3.6]. For system (1) the well-posedness and the existence of the global attractor were established in [15, 18] and [19–21], respectively, (see also [9, Chapters 9-10] and [28]). The question of finiteness of the attractor dimension has not as yet been studied. We address this issue in the present paper. The answer turns out to be positive under suitable conditions on the problem coefficients. Thus, to the best of our knowledge, system (1) is the second after the porous medium equation class of highly degenerate problems which can exhibit finite-dimensional dynamics.

It is well known that the concept of global attractor has some essential drawbacks. It is in general not stable under perturbations, the speed of convergence to it may be arbitrary slow, and it is usually hard to express it in terms of the parameters of the system. Thus it is often difficult to observe the global attractor in numerical simulations. The notion of exponential attractor (compare *Definition 1.1* below) was first introduced in [7] as an alternative way to capture the dynamics of a dynamical system. It is a finite dimensional positively invariant attracting set that attracts bounded subsets of the phase space with exponential speed. If such a set exists for a dynamical system, it necessarily contains the global attractor of the system, and that global attractor has finite dimension (so it is also a way to show that the global attractor is finite-dimensional). While generally stable and easier to handle, this (eventually bigger) attracting set has it's own faulty: it is not uniquely determined (while the global attractor is).

Unlike the nondegenerate dissipative equations and systems on bounded domains which, as a rule, possess finite-dimensional global and exponential attractors, the dynamics of degenerate problems is much more delicate. The porous medium and p-Laplace equations are two very first examples of autonomous equations which - under rather general conditions- have infinite dimensional attractors, see [13, 14, 17], also [9, Chapters 4-7]. Moreover, the asymptotics of their Kolmogorov ε-entropy turned out to be polynomial. Even the attractors of nondegenerate problems in unbounded domains, which are known to be of infinite dimension, always showed only logarithmic asymptotics of the entropy, see, e.g., [8].

Previous results on the well-posedness and the existence of the global attractor were obtained for (1) under the balance conditions

$$\frac{\alpha}{2} + 1 \leq \gamma < \alpha \tag{2}$$

and the following assumptions on the nonlinearities f and g:

$$|f(M,\rho)| \le F_1(1+|M|^\xi)^{\frac{1}{2}} \quad \text{for all } M,\rho \ge 0 \quad \text{for some } \xi \in [0, \alpha - \gamma + 2), \ F_1 \in \mathbb{R}_0^+, \tag{3}$$

$$f(M,\rho) \ge F_2 M - F_3 \quad \text{for all } M, \rho \ge 0 \quad \text{for some } F_2 \in \mathbb{R}^+, \ F_3 \in \mathbb{R}_0^+, \tag{4}$$

$$g(M,\rho) = G_1\rho + g_2(\rho)M, \ |g_2(\rho)| \le G_2 \quad \text{for all } M,\rho \ge 0 \quad \text{for some } G_1, G_2 \in \mathbb{R}_0^+, \tag{5}$$

$$f(M,\rho) = F_4 M + \widetilde{f}\left(M^{1+\frac{\alpha}{2}}, \rho\right) \quad \text{for all } M, \rho \ge 0 \quad \text{for some } F_4 \in \mathbb{R}, \tag{6}$$

$$\widetilde{f} \in W_{loc}^{1,\infty}(\mathbb{R}^2), \ g_2 \in W_{loc}^{1,\infty}(\mathbb{R}), \ f(0,\rho) = 0 \quad \text{for all } \rho \in \mathbb{R}, \quad g_2(0) \le 0. \tag{7}$$

In this setting, we established in [21] the existence of the weak global attractor in the phase space $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$. Note that in [9, Section 4.4] it was shown that the dimension of the global attractor for the porous medium equation (thus, even without chemotaxis)

$$\partial_t M = \nabla \cdot (M^\alpha \nabla M) - f(M) \tag{8}$$

may be infinite if $-f'(0) > 0$. Observe that this includes the case of the standard logistic growth, i.e., when

$$-f(M) = rM\left(1 - \frac{M}{K}\right) \tag{9}$$

for some growth rate $r > 0$ and carrying capacity $K > 0$. Conditions (2)-(7) therefore cannot guarantee the finite dimension of the global attractor (and hence also the existence of an exponential attractor) for (1) as the following example illustrates.

Example 1. Let

$$-f = -f(M) = M, \qquad g \equiv 0.$$

Observe that this choice of f and g is in line with conditions (2)-(7). Let us assume further that

$$\rho_0 \equiv 1.$$

In this special case equation (1b) together with the corresponding boundary condition can be easily solved explicitly, the solution being $\rho \equiv 1$. Hence, the taxis flux in (1a) completely vanishes on Ω. As a result, M solves the porous medium equation (8) with $-f'(0) = 1 > 0$ thus leading to the infinite dimensional global attractor already for M-component.

We improve conditions (2)-(7) in the following way: we consider now sharper balance conditions than (2), namely

$$1 + \frac{\alpha}{2} < \gamma < \alpha \tag{10}$$

and replace (6) by

$$f(M,\rho) = F_5 M + \widetilde{f}\left(M^\beta, \rho\right), \quad \text{for some } \beta > 1 + \frac{\alpha}{2}, \ F_5 > 0. \tag{11}$$

The following choice of functions f and g satisfies conditions (3)-(5), (7) and (11):

Example 2.

$$f(M,\rho) = -M + \frac{M^\beta}{M^\beta+1}\arctan\rho,$$

$$g(M,\rho) = \rho + M\frac{\rho}{\rho+1}.$$

We recall a definition of the exponential attractor:

Definition 1.1 ([9, Chapter 3, Definition 3.1]). A set $\mathcal{M}$ is an exponential attractor for a semigroup $S(t)$ in a Banach space X if: it

(i) is compact in X;

(ii) is positively invariant, i.e., $S(t)\mathcal{M} \subset \mathcal{M}, \ \forall t \geq 0$;

(iii) attracts bounded sets of initial data exponentially fast in the following sense: there exists a monotonic function Q and a constant $C_{rate} > 0$ (called below attraction parameters) such that

$$\forall B \subset X \text{ bounded, } \operatorname{dist}_X(S(t)B, \mathcal{M}) \leq Q(\|B\|_X)e^{-C_{rate}t}, \ t \geq 0;$$

(iv) has finite fractal dimension.

Here $\operatorname{dist}_X(\cdot, \cdot)$ denotes the nonsymmetric Hausdorff distance between subsets of X:

$$\operatorname{dist}_X(A,B) := \sup_{x \in A}\inf_{y \in B}\|x-y\|_X \text{ for all } A, B \subset X.$$

Our main result deals with the existence of exponential attractors for system (1). It reads:

Theorem 1.2 (Exponential attractor for (1)). *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \in \{1,2,3\}$. Let the functions f and g satisfy the assumptions (3)-(5), (7) and (11) and let the given constants α and γ satisfy $1 + \frac{\alpha}{2} < \gamma < \alpha$. Then the initial boundary value problem (1) generates a well defined semigroup $S(t)$, $t \geq 0$, in the (positive cone of the) space $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$. The semigroup $S(t)$ possess an exponential attractor $\mathcal{M}$ (in terms of Definition 1.1) which is a bounded subset of $C^\theta(\overline{\Omega}) \times C^{2+\theta}(\overline{\Omega})$ for some Hölder exponent $\theta \in (0,1)$. The number θ and such parameters of the attractor as: its diameter, fractional dimension, the attraction parameters C_{rate} and Q can be chosen such as to depend only upon the parameters of the problem.*

As a direct corollary of *Theorem 1.2* we have that

Corollary 1 (Finite-dimensional global attractor for (1)). *Under assumptions of Theorem 1.2 the semigroup generated by system (1) possesses the finite-dimensional global attractor $\mathcal{A} \subset \mathcal{M}$. In particular, upper bounds for the attractor diameter and fractal dimension can be chosen to depend upon the parameters of the problem alone.*

There are different constructions [9] of exponential attractors. We use the one based on the so-called smoothing property (see [9] and references therein). One of its simplest abstract versions insures the existence of an exponential attractor for a discrete semigroup S^n, $n \in \mathbb{N}_0$, and takes (see, e.g., [9, Chapter 3]) the form

$$\|S(u_1) - S(u_2)\|_{H_1} \leq K\|u_1 - u_2\|_H \text{ for all } u_1, u_2 \in C. \tag{12}$$

Here H and H_1 are two Banach spaces such that H_1 is compactly embedded in H, S maps between H and H_1, C is a subset of some metric space X and is invariant under S, and the constant $K \geq 0$ is independent of a particular choice of u_1 and

u_2. It is in general not difficult to establish such a property for the semigroup corresponding to a dissipative nondegenerate problem. Moreover, in these cases the ways to choose spaces H and H_1 in an appropriate way are usually in abundance. Very often they are two Hölder spaces or, alternatively, a Lebesgue and a Sobolev space, defined for the whole spatial domain. However, a condition like (12) is in general unattainable for a semigroup of solution operators for a degenerate equation, such as e.g. the porous medium equation. In [9, 17] the smoothing property could be generalised to a form that allows to treat the latter case. It turned out that the underlying spaces, such as H and H_1, cannot be chosen once and for all, but that they need to be changed as one passes from a neighbourhood of one point u_0 in C to another. Thereby, it is necessary to work on functional spaces set up not on the whole spatial domain, but, rather, on some sub- and superlevel sets of u_0. This requires localising techniques. In the present work we use the ideas which were originally developed in [9, 17] for the porous medium equation in order to obtain an exponential attractor for system (1). The presence of a chemotaxis transport term in addition to a degenerate diffusion is a considerable complication. It further reduces the class of norms in which one can estimate the differences of two solutions. For example, while the solution operators of the porous medium equation are Lipschitz continuous both in L^1 (this was essentially used in [9, 17]) and H^{-1}, in our case they are Lipschitz only in H^{-1}. Working in negative Sobolev spaces is more difficult since they are much less suited for the localising techniques.

The rest of the paper is organised as follows. In *Section 2* we fix some notation and then establish some results concerning the regularity and stability of solutions, as well as some properties of an exponentially absorbing set for system (1). In *Section 3* we formulate and prove a smoothing property (*Theorem 3.1* below) for the corresponding semigroup. The proof of *Theorem 1.2* is given in *Section 4*.

2. **Preliminaries.** In this Section we collect some necessary preliminary observations and results.

Basic notation and functional spaces. We denote $\mathbb{R}^+ := (0, \infty)$, $\mathbb{R}_0^+ := [0, \infty)$.

Partial derivatives in the classical or distributional sense with respect to a variable z are denoted by ∂_z. Further, ∇ and Δ stand for the spatial gradient and Laplace operators, respectively.

We assume the reader to be familiar with the standard L^p, Sobolev, and Hölder spaces and their usual properties, as well as with the more general L^p spaces of functions with values in general Banach spaces, and with anisotropic spaces, such as, for any open $O \subset \mathbb{R}^N$ and $0 < t_1 < t$, the parabolic spaces

$$H^1((t_1, t), H_0^1(O), H^{-1}(O)) := \left\{ u \in L^2((t_1, t), H_0^1(O)) \mid \partial_t u \in L^2((t_1, t), H^{-1}(O)) \right\}$$

equipped with the norm

$$\|u\|_{H^1((t_1,t),H_0^1(O),H^{-1}(O))} := \left(\|u\|_{L^2((t_1,t),H_0^1(O))}^2 + \|\partial_t u\|_{L^2((t_1,t),H^{-1}(O))}^2 \right)^{\frac{1}{2}}$$

and

$$W^{(1,2),2}((t_1, t) \times O) := \left\{ u \in L^2((t_1, t), H^2(O)) \mid \partial_t u \in L^2((t_1, t), L^2(O)) \right\}$$

equipped with the norm

$$\|u\|_{W^{(1,2),2}((t_1,t)\times O)} := \left(\|u\|_{L^2((t_1,t),H^2(O))}^2 + \|\partial_t u\|_{L^2((t_1,t),L^2(O))}^2 \right)^{\frac{1}{2}}.$$

As usual, $C^k(\overline{\Omega})$, $k \in \mathbb{N}_0$, denotes the space of k times continuously differentiable functions $u : \overline{\Omega} \to \mathbb{R}$, and $D^\alpha = D^{\alpha_1} \ldots D^{\alpha_n}$, $\alpha_i \in \mathbb{N}_0$, is the corresponding higher order partial derivative of order $|\alpha| = \sum_{i=1}^N \alpha_i$. A norm on $C^k(\overline{\Omega})$ is given by

$$\|u\|_{C^k(\overline{\Omega})} := \sum_{|\alpha| \le k} \max_{x \in \overline{\Omega}} |D^\alpha u(x)|.$$

We recall that a Hölder coefficient for a Hölder exponent $\theta \in (0,1)$ and a real-valued function w defined in a set $A \subset \mathbb{R}^k$, $k \in \mathbb{N}$, is given by

$$|w|_{C^\theta(A)} := \sup_{x,y \in A, \ x \ne y} \frac{|w(x) - w(y)|}{|x-y|^\theta}.$$

This allows to introduce the standard Hölder spaces for $k \in \mathbb{N}_0$:

$$C^{k+\theta}(\overline{\Omega}) := \left\{ u \in C^k(\overline{\Omega}) : \ |D^\alpha u|_{C^\theta(\overline{\Omega})} < \infty \text{ for all } |\alpha| = k \right\}$$

equipped with the norm

$$\|u\|_{C^{k+\theta}(\overline{\Omega})} := \|u\|_{C^k(\overline{\Omega})} + \sum_{|\alpha|=k} |D^\alpha u|_{C^\theta(\overline{\Omega})}.$$

Recall also that due to the Sobolev interpolation inequality for any $\theta \in (0,1)$ there exist numbers $\theta_1, \theta_2 \in (0,1)$ and $C_1 > 0$ such that following interpolation inequalities hold:

$$\|w\|_{L^\infty(\Omega)} \le C_1 \|w\|_{C^\theta(\overline{\Omega})}^{1-\theta_1} \|w\|_{H^{-1}(\Omega)}^{\theta_1} \quad \text{for all } w \in C^\theta(\overline{\Omega}), \tag{13}$$

$$\|v\|_{W^{1,\infty}(\Omega)} \le C_1 \|v\|_{C^{2+\theta}(\overline{\Omega})}^{1-\theta_2} \|v\|_{L^2(\Omega)}^{\theta_2} \quad \text{for all } v \in C^{2+\theta}(\overline{\Omega}). \tag{14}$$

Finally, we make the following two useful conventions. Firstly, for all indices i, C_i denotes a positive constant or, alternatively, a positive continuous function. Secondly, the statement that a constant depends on the parameters of the problem means that it depends upon such parameters as: space dimension N, domain Ω, constants α, β, γ, F_i, G_i, and norms of $\widetilde{f}$ and g_2. This dependence is subsequently **not** indicated in an explicit way.

Sub- and superlevel sets. In what follows we sometimes consider parts of solutions of problem (1) restricted to the sublevel sets $\{M_0 > \delta\}$ for $\delta \in \left(0, \|M_0\|_{L^\infty(\Omega)}\right)$. Observe that if $M_0 \in C^\theta(\overline{\Omega})$, then, in fact, for all $0 < \delta < \frac{1}{2}\|M_0\|_{L^\infty(\Omega)}$

$$d_\mathbb{R}\left(\{M_0 \le \delta\}, \{M_0 \ge 2\delta\}\right) \ge \delta^{\frac{1}{\theta}} |M_0|_{C^\theta(\overline{\Omega})}^{-\frac{1}{\theta}}, \tag{15}$$

where $d_\mathbb{R}$ denotes the standard metric distance between sets in $\mathbb{R}$:

$$d_\mathbb{R}(X,Y) := \inf_{(x,y) \in X \times Y} |x-y|.$$

Thus, we have a control over a lower bound for the distance between sub- and superlevel sets, and that bound depends only upon the quantities which appear on the right-hand side of (15). An important consequence of this observation is the existence for all $0 < \delta_0 < \delta_1 < \frac{1}{2}\|M_0\|_{L^\infty(\Omega)}$ of a smooth cut-off function φ which satisfies the following:

$$\varphi \in C_0^\infty(\Omega), \tag{16a}$$

$$\varphi \in [0,1] \text{ in } \overline{\Omega}, \quad \varphi = 0 \text{ in } \{M_0 < \delta_0\}, \quad \varphi = 1 \text{ in } \{M_0 > \delta_1\}, \tag{16b}$$

$$\left| D^k \varphi(x) \right| \le C_2 \varphi^{1-\omega}(x) \text{ for all } x \in \overline{\Omega} \text{ for all } \omega \in (0,1) \text{ and } k \in \mathbb{N}, \tag{16c}$$

$$C_2 = C_2 \left(|M_0|_{C^\theta(\overline{\Omega})}, \delta_0, \delta_1, \theta, \omega, k \right). \tag{16d}$$

A family of functions with such properties exists for all $M_0 \in C^\theta(\overline{\Omega})$ due to property (15), see Proposition 1.1 of [9].

Regularity of solutions. From now on we assume that assumptions of *Theorem 1.2* are fulfilled. A solution to (1) can be defined as follows:

Definition 2.1 (Weak solution)**.** Let $(M_0, \rho_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)$. We call a pair of functions $M, \rho : \mathbb{R}_0^+ \times \overline{\Omega} \to \mathbb{R}_0^+$ a global weak solution of (1) if for all $0 < T < \infty$ it holds that

(i) $M \in L^\infty((0,T) \times \Omega)$, $M^{\alpha+1} \in L^2\left((0,T); H_0^1(\Omega)\right)$, $\partial_t M \in L^2\left((0,T); H^{-1}(\Omega)\right)$;

(ii) $\rho \in L^\infty((0,T); W^{1,\infty}(\Omega))$, $\partial_t \rho \in L^2\left((0,T); H^{-1}(\Omega)\right)$;

(iii) (M, ρ) satisfies equations (1a)-(1b) in $L^2\left((0,T); H^{-1}(\Omega)\right)$;

(iv) $(M, \rho)(0) = (M_0, \rho_0)$ in $H^{-1}(\Omega) \times L^2(\Omega)$.

It was proved in [15, 18] (see also [28, Section 3.2]) that for all $(M_0, \rho_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ the initial boundary value problem (1) possess a unique solution with regularity as stated in *Definition 2.1* and, moreover, this solution is uniformly bounded in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$. In general, a solution of a degenerate equation like (1a) is only weak and not classical [27]. Still, it is well-understood [2, 4, 5, 23, 29] that under reasonable conditions on the equation coefficients bounded weak solutions are Hölder continuous. In our case the following regularity result holds:

Lemma 2.2 (Regularity and positivity)**.** *Let* $(M_0, \rho_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ *with* $\|(M_0, \rho_0)\|_{L^\infty(\Omega) \times W^{1,\infty}(\Omega)} \le R$ *for some* $R > 0$ *and let* $(M, \rho) : [0,T] \times \overline{\Omega} \to \mathbb{R}_0^+ \times \mathbb{R}_0^+$ *be the corresponding weak solution to* (1)*. Then:*

1. *(Hölder regularity) There exists a number* $\theta = \theta(R) \in (0,1)$ *such that* (M, ρ) *belongs to* $C^{\frac{\theta}{2}, \theta}(\mathbb{R}^+ \times \overline{\Omega}) \times C^{1+\frac{\theta}{2}, 2+\theta}(\mathbb{R}^+ \times \overline{\Omega})$*, and for all* $0 < \tau < T$ *it holds that*

$$\|(M, \rho)\|_{C^{\frac{\theta}{2}, \theta}([\tau, T] \times \overline{\Omega}) \times C^{1+\frac{\theta}{2}, 2+\theta}([\tau, T] \times \overline{\Omega})} \le C_3\left(\tau, T, R\right). \tag{17}$$

2. *(Preservation of positivity)* $\mathbb{R}^+ \times \{M_0 > 0\} \subset \{M > 0\}$*, and for all* $\delta \in \left(0, \|M_0\|_{L^\infty(\Omega)}\right)$ *and* $T > 0$ *it holds that*

$$\inf \{M(s,x)|\ t \in [0,T],\ x \in \{M_0 > \delta\}\} \ge C_4(\delta, T, R). \tag{18}$$

3. *(Regularity on sublevel sets)* $M \in C^{1,2}\left(\mathbb{R}^+ \times \overline{\{M_0 > \delta\}}\right)$*, and for all* $\delta \in \left(0, \|M_0\|_{L^\infty(\Omega)}\right)$ *and* $0 < \tau < T$ *it holds that*

$$\|M\|_{C^{1,2}\left([\tau, T] \times \overline{\{M_0 > \delta\}}\right)} \le C_5\left(\delta, \tau, T, R\right). \tag{19}$$

Proof. **1.** Observe that equation (1a) can be written in the following form:

$$\partial_t M = \nabla \cdot A(t, x, M, \nabla M) + b(t, x),$$

where we introduced

$$A(t, x, M, p) := M^\alpha p - M^\gamma \nabla \rho(t, x),$$
$$b(t, x) := f(M, \rho)(t, x).$$

Clearly, functions A and b satisfy the following conditions:

$$|A(t, x, M, p)| \le M^\alpha |p| + \|M^\gamma \nabla \rho\|_{(L^\infty(\mathbb{R}^+ \times \Omega))^n}, \tag{20}$$
$$|b(t, x)| \le \|f(M, \rho)\|_{L^\infty(\mathbb{R}^+ \times \Omega)}. \tag{21}$$

Moreover, with the help of the Hölder inequality we deduce that

$$
\begin{aligned}
A(t,x,M,p) \cdot p =& M^\alpha |p|^2 + M^\gamma \nabla \rho(t,x) \cdot p \\
=& M^\alpha |p|^2 + M^{\frac{\alpha}{2}} p \cdot M^{\gamma - \frac{\alpha}{2}} \nabla \rho(t,x) \\
\geq& M^\alpha |p|^2 - M^{\frac{\alpha}{2}} |p| \cdot M^{\gamma - \frac{\alpha}{2}} |\nabla \rho(t,x)| \\
\geq& \frac{1}{2} M^\alpha |p|^2 - \frac{1}{2} \left\| M^{\gamma - \frac{\alpha}{2}} \nabla \rho \right\|^2_{(L^\infty(\mathbb{R}^+ \times \Omega))^n} ,
\end{aligned}
\tag{22}
$$

Due to (20)-(22), $\alpha > 0$, $\gamma \geq \frac{\alpha}{2}$, and the fact that (M, ρ) is uniformly bounded in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ by a constant which depends only upon the parameters of the problem and R, we can apply Theorems 2.I and 3.I from [23] on inner and boundary regularity for degenerate parabolic PDEs. These results imply the existence of a number $\theta = \theta(R) \in (0,1)$ such that $M \in C^{\frac{\theta}{2}, \theta}([\tau, T] \times \overline{\Omega})$ for all $0 < \tau < T$, and

$$
\|M\|_{C^{\frac{\theta}{2}, \theta}([\tau, T] \times \overline{\Omega})} \leq C_6(\tau, T, R).
\tag{23}
$$

Consequently, equation (1b) together with the boundary condition $\rho \equiv 1$ can be seen as a linear parabolic equation for ρ with Hölder continuous coefficients. Thus, standard Schauder estimates entail that ρ is a classical solution to (1b) and satisfies

$$
\|\rho\|_{C^{1 + \frac{\theta}{2}, 2 + \theta}([\tau, T] \times \overline{\Omega})} \leq C_7(\tau, T, R).
\tag{24}
$$

Combining (23)-(24), we obtain (17).

2. We start by proving the quantitative estimate (18). For this purpose we make use of the classical idea of propagation of L^p bounds. Since we aim at an estimate from below, we estimate M^{-1} from above. Of course, this can only be done in those areas where the M-component is strictly bounded from below by a positive constant. For that reason we use a cutoff function from (16) in order to eliminate the part of Ω where M_0 is small. More precisely, let us multiply equation (1a) by $-a \varphi_\delta^a M^{-a-1}$ for $a \geq 2\alpha$ and φ_δ as in (16) while we choose $\delta_0 := \delta$, $\delta_1 := 2\delta$, so that, in particular, $\varphi_\delta = 0$ in $\{|M_0| \leq \delta\}$. Integrating (formally) over Ω and using integration by parts where necessary we obtain that

$$
\begin{aligned}
&\frac{d}{dt} \left\| \varphi_\delta^a M^{-a} \right\|_{L^1(\Omega)} \\
=& -4 \frac{(a+1)a}{(a-\alpha)^2} \left\| \varphi_\delta^{\frac{a}{2}} \nabla M^{-\frac{a-\alpha}{2}} \right\|^2_{(L^2(\Omega))^n} \\
& -2a \frac{a}{a-\alpha} \left(\varphi_\delta^{\frac{a}{2}} \nabla M^{-\frac{a-\alpha}{2}}, \left(\varphi_\delta M^{-1} \right)^{\frac{a}{2}-1} M^{\frac{\alpha-2}{2}} \nabla \varphi_\delta \right)_{(L^2(\Omega))^n} \\
& +2a \frac{a+1}{a-\alpha} \left(\varphi_\delta^{\frac{a}{2}} \nabla M^{-\frac{a-\alpha}{2}}, \left(\varphi_\delta M^{-1} \right)^{\frac{a}{2}} M^{\gamma - 1 - \frac{\alpha}{2}} \nabla \rho \right)_{(L^2(\Omega))^n} \\
& -a^2 \left(\left(\varphi_\delta M^{-1} \right)^{a-1} M^{\gamma - 2}, \nabla \varphi_\delta \cdot \nabla \rho \right)_{(L^2(\Omega))^n} + a \left(\varphi_\delta^a M^{-a}, \frac{f(M, \rho)}{M} \right)_{L^2(\Omega)}.
\end{aligned}
\tag{25}
$$

Using the Young and Hölder inequalities, the assumptions on α, γ, and f, and the properties of φ_δ we estimate the right-hand side of (25) on $[0, T]$ as follows:

$$
\begin{aligned}
&\frac{d}{dt} \left\| \varphi_\delta^a M^{-a} \right\|_{L^1(\Omega)} \\
\leq& -C_8 \left\| \varphi_\delta^{\frac{a}{2}} \nabla M^{-\frac{a-\alpha}{2}} \right\|^2_{(L^2(\Omega))^n} \\
& + C_9(\delta, T, R) a \left\| \varphi_\delta^{\frac{a}{2}} \nabla M^{-\frac{a-\alpha}{2}} \right\|_{(L^2(\Omega))^n} \left\| \varphi_\delta^a M^{-a} \right\|^{\frac{a-2}{2a}}_{L^1(\Omega)}
\end{aligned}
$$

$$+ C_{10}(\delta,T,R)a \left\|\varphi_\delta^{\frac{a}{2}}\nabla M^{-\frac{a-\alpha}{2}}\right\|_{(L^2(\Omega))^n} \left\|\varphi_\delta^a M^{-a}\right\|_{L^1(\Omega)}^{\frac{1}{2}}$$

$$+ C_{11}(\delta,T,R)a^2 \left\|\varphi_\delta^a M^{-a}\right\|_{L^1(\Omega)}^{\frac{a-1}{a}} + C_{12}(\delta,T,R)a \left\|\varphi_\delta^a M^{-a}\right\|_{L^1(\Omega)}$$

$$\leq - C_{13}(\delta,T,R) \left\|\varphi_\delta^{\frac{a}{2}}\nabla M^{-\frac{a-\alpha}{2}}\right\|_{(L^2(\Omega))^n}^2 + C_{14}(\delta,T,R)a^2 \left(\left\|\varphi_\delta^a M^{-a}\right\|_{L^1(\Omega)} + 1\right)$$

$$\leq - \frac{1}{2}C_{13}(\delta,T,R) \left\|\nabla\left(\varphi_\delta^{\frac{a}{2}} M^{-\frac{a-\alpha}{2}}\right)\right\|_{(L^2(\Omega))^n}^2$$

$$+ C_{13}(\delta,T,R)\frac{a^2}{4} \left\|(\varphi_\delta M^{-1})^{\frac{a}{2}-1} M^{\frac{\alpha-2}{2}}\nabla\varphi\right\|_{(L^2(\Omega))^n}^2$$

$$+ C_{14}(\delta,T,R)a^2 \left(\left\|\varphi_\delta^a M^{-a}\right\|_{L^1(\Omega)} + 1\right)$$

$$\leq - C_{15}(\delta,T,R) \left\|\nabla\left(\varphi_\delta^{\frac{a}{2}} M^{-\frac{a-\alpha}{2}}\right)\right\|_{(L^2(\Omega))^n}^2 + C_{16}(\delta,T,R)a^2 \left\|\varphi_\delta^a M^{-a}\right\|_{L^1(\Omega)}^{\frac{a-2}{a}}$$

$$+ C_{14}(\delta,T,R)a^2 \left(\left\|\varphi_\delta^a M^{-a}\right\|_{L^1(\Omega)} + 1\right)$$

$$\leq - C_{15}(\delta,T,R) \left\|\nabla\left(\varphi_\delta^{\frac{a}{2}} M^{-\frac{a-\alpha}{2}}\right)\right\|_{(L^2(\Omega))^n}^2$$

$$+ C_{17}(\delta,T,R)a^2 \left(\left\|\varphi_\delta^a M^{-a}\right\|_{L^1(\Omega)} + 1\right). \tag{26}$$

The first consequence of (26) is due to the Gronwall lemma:

$$\sup_{t\in[0,T]} \left\|\varphi_\delta^a M^{-a}(t)\right\|_{L^1(\Omega)} + 1 \leq e^{TC_{17}(\delta,T,R)a^2} \left(\left\|\varphi_\delta^a M_0^{-a}\right\|_{L^1(\Omega)} + 1\right)$$

$$\leq e^{TC_{17}(\delta,T,R)a^2} \left(\delta^{-a} + 1\right)$$

$$=: C_{18}(a,\delta,T,R). \tag{27}$$

This shows the a priori boundedness of $\left\|\varphi_\delta M^{-1}\right\|_{L^a(\Omega)}$ for all $a \in [2\alpha,\infty)$. To get an estimate for $\left\|\varphi_\delta M^{-1}\right\|_{L^\infty(\Omega)}$ observe that due to the interpolation inequality for Lebesgue spaces

$$\left\|\varphi_\delta^a M^{-a}\right\|_{L^1(\Omega)} = \left\|\varphi_\delta^{\frac{a}{2}} M^{-\frac{a-\alpha}{2}} M^{-\frac{\alpha}{2}}\right\|_{L^2(\Omega)}^2$$

$$\leq \left\|M^{-1}\right\|_{L^{3\alpha}(\{M_0\geq\delta\})}^\alpha \left\|\varphi_\delta^{\frac{a}{2}} M^{-\frac{a-\alpha}{2}}\right\|_{L^3(\Omega)}^2$$

$$\leq \left\|\varphi_{\frac{\delta}{2}} M^{-1}\right\|_{L^{3\alpha}(\Omega)}^\alpha \left\|\varphi_\delta^{\frac{a}{2}} M^{-\frac{a-\alpha}{2}}\right\|_{L^3(\Omega)}^2$$

$$\leq C_{18}^\alpha\left(3\alpha,\frac{\delta}{2},T,R\right) \left\|\varphi_\delta^{\frac{a}{2}} M^{-\frac{a-\alpha}{2}}\right\|_{L^3(\Omega)}^2. \tag{28}$$

Combining (26) with (28), the Young inequality, and the Sobolev interpolation inequality

$$\|u\|_{L^3(\Omega)} \leq C_{19}\|u\|_{H_0^1(\Omega)}^{\frac{4N}{6+3N}}\|u\|_{L^1(\Omega)}^{\frac{6-N}{6+3N}}$$

for $u := \varphi_\delta^{\frac{a}{2}} M^{-\frac{a-\alpha}{2}}$ yields

$$\frac{d}{dt}\left\|\varphi_\delta^a M^{-a}\right\|_{L^1(\Omega)} \leq - C_{20}(\delta,T,R) \left\|\nabla\left(\varphi_\delta^{\frac{a}{2}} M^{-\frac{a-\alpha}{2}}\right)\right\|_{(L^2(\Omega))^n}^2$$

$$+ C_{21}(\delta, T, R)a^2 \left\| \nabla \left(\varphi_\delta^{\frac{a}{2}} M^{-\frac{a-\alpha}{2}} \right) \right\|_{(L^2(\Omega))^n}^{2\frac{4N}{6+3N}} \left\| \varphi_\delta^{\frac{a}{2}} M^{-\frac{a-\alpha}{2}} \right\|_{L^1(\Omega)}^{2\frac{6-N}{6+3N}}$$

$$+ C_{17}(\delta, T, R)a^2$$

$$\leq C_{22}(\delta, T, R)a^{\frac{12+6N}{6-N}} \left\| \varphi_\delta^{\frac{a}{2}} M^{-\frac{a-\alpha}{2}} \right\|_{L^1(\Omega)}^2 + C_{17}(\delta, T, R)a^2$$

$$\leq C_{23}(\delta, T, R)a^{\frac{12+6N}{6-N}} \left\| \varphi_\delta^{\frac{a}{2}} M^{-\frac{a}{2}} \right\|_{L^1(\Omega)}^2 + C_{17}(\delta, T, R)a^2$$

$$\leq C_{24}(\delta, T, R)a^{\frac{12+6N}{6-N}} \left(\left\| \varphi_\delta^{\frac{a}{2}} M^{-\frac{a}{2}} \right\|_{L^1(\Omega)}^2 + 1 \right). \tag{29}$$

Integrating (29) over $[0, T]$ and taking maximum on both sides of (30) we obtain that

$$\max_{t \in [0,T]} \left(\left\| \varphi_\delta^a M^{-a}(t) \right\|_{L^1(\Omega)} + 1 \right)$$

$$\leq \left(\left\| \varphi_\delta^a M_0^{-a} \right\|_{L^1(\Omega)} + 1 \right) + C_{24}(\delta, T, R)a^{\frac{12+6N}{6-N}} \int_0^T \left\| \varphi_\delta^{\frac{a}{2}} M^{-\frac{a}{2}} \right\|_{L^1(\Omega)}^2 + 1 \, ds$$

$$\leq \left(\delta^{-a}|\Omega| + 1 \right) + C_{25}(\delta, T, R)a^{\frac{12+6N}{6-N}} \max_{t \in [0,T]} \left(\left\| \varphi_\delta^{\frac{a}{2}} M^{-\frac{a}{2}} \right\|_{L^1(\Omega)} + 1 \right)^2. \tag{30}$$

Here we used the properties of the cutoff function φ_δ. Since for $a \geq 2\alpha > 1$ it holds that $\delta^{-a}|\Omega| + 1 \leq C_{26}{}^a(\delta)$, estimate (30) leads to a recursive inequality

$$A_a \leq C_{27}(\delta, T, R)a^{C_{28}} A_{\frac{a}{2}}^2 \tag{31}$$

for

$$A_a := \frac{\max\limits_{s \in [0,t]} \left(\left\| \varphi_\delta^a M^{-a}(t) \right\|_{L^1(\Omega)} + 1 \right)}{C_{26}^a(\delta)} + 1.$$

A standard induction argument together with estimate (27) for $a := 2\alpha$ implies that

$$A_{2^{n+1}\alpha}^{\frac{1}{2^n}} \leq \left(C_{27}(\delta, T, R)\alpha^{C_{28}} \right)^{\sum_{k=0}^{n-1} 2^{k-n}} 2^{C_{28} \sum_{k=0}^{n-1}(n+1-k)2^{k-n}} A_{2\alpha}$$

$$\underset{n \to \infty}{\to} C_{27}(\delta, T, R)\alpha^{C_{28}} 2^{3C_{28}} A_{2\alpha}$$

$$\leq C_{29}(\delta, T, R). \tag{32}$$

Thanks to (32) we obtain that

$$\max_{s \in [0,t]} \left\| M^{-1}(s) \right\|_{L^\infty(\{M_0 \geq \delta\})} \leq \max_{s \in [0,t]} \left\| \varphi_\delta M^{-1}(s) \right\|_{L^\infty(\Omega)}$$

$$= \max_{s \in [0,t]} \lim_{n \to \infty} \left\| \varphi_\delta M^{-1}(t) \right\|_{L^{2^{n+1}\alpha}(\Omega)}$$

$$\leq \limsup_{n \to \infty} C_{26}(\delta) A_{2^{n+1}\alpha}^{\frac{1}{2^{n+1}\alpha}}$$

$$\leq C_{26}(\delta) C_{29}^{\frac{1}{2\alpha}}(\delta, T, R)$$

$$=: C_{30}(\delta, T, R).$$

Consequently, it holds that

$$\inf \left\{ M(s, x) \mid t \in [0, T], \, x \in \{M_0 > \delta\} \right\} \geq \left(C_{30}\left(\delta/2, T, R \right) \right)^{-1} =: C_4(\delta, T, R) > 0$$

for all $\delta \in \left(0, \|M_0\|_{L^\infty(\Omega)} \right)$ and $T > 0$, which proves (18).

The above calculations -and thus also the quantitative estimate (18)- are only valid if $M(t, \cdot)$ is uniformly bounded from below on $\{M_0 > \delta\}$ for all $t \in [0, T]$. In order to justify the latter we apply a standard approximation argument. In [15, 18] (see also [28, Section 3.2]) we proved that on the one hand a weak solution is unique, and on the other hand it can be obtained as a limit of approximations which solve a nondegenerate system such as, e.g., for $n \in \mathbb{N}$ the system

$$
\begin{aligned}
\partial_t M_n =& \nabla \cdot \left(\left(M_n + \frac{1}{n} \right)^\alpha \nabla M_n \right) - \nabla \cdot \left(\left(M_n + \frac{1}{n} \right)^\gamma \nabla \rho_n \right) \\
& - f\left(M_n, \rho_n \right) && \text{in } (0, T) \times \Omega, \\
\partial_t \rho_n =& \Delta \rho_n - g(M_n, \rho_n) && \text{in } (0, T) \times \Omega
\end{aligned}
$$

equipped with the same initial and boundary conditions as for the original system (1). Since $M_n \geq 0$ solves a nondegenerate equation, its strict positivity is guaranteed. Moreover, it is not difficult to see that family M_n, $n \in \mathbb{N}$, satisfies a positive bound similar to (18) with some constant which is independent of n. Consequently, the limit function M is indeed uniformly bounded from below by a positive constant and satisfies (18).

3. Let $M := u^{\frac{1}{\alpha+1}}$. Under this change of variables equation (1a) takes the form

$$
\frac{1}{\alpha+1} u^{\frac{1}{\alpha+1}-1} \partial_t u = \frac{1}{\alpha+1} \Delta u - \frac{\gamma}{\alpha+1} u^{\frac{\gamma}{\alpha+1}-1} \nabla \rho \cdot \nabla u - u^{\frac{\gamma}{\alpha+1}} \Delta \rho - f\left(u^{\frac{1}{\alpha+1}}, \rho \right).
\tag{33}
$$

Dividing (33) by $\frac{1}{\alpha+1} u^{\frac{1}{\alpha+1}-1}$, we obtain that

$$
\partial_t u = u^{\frac{\alpha}{\alpha+1}} \Delta u - \gamma u^{\frac{\gamma-1}{\alpha+1}} \nabla \rho \cdot \nabla u - (\alpha+1) u^{\frac{\alpha+\gamma}{\alpha+1}} \Delta \rho - (\alpha+1) u^{\frac{\alpha}{\alpha+1}} f\left(u^{\frac{1}{\alpha+1}}, \rho \right),
\tag{34}
$$

i.e.,

$$
\partial_t u = a_0 \Delta u + a_1 \cdot \nabla u + a_3,
\tag{35}
$$

with

$$
\begin{aligned}
a_0 &:= u^{\frac{\alpha}{\alpha+1}}, \\
a_1 &:= -\gamma u^{\frac{\gamma-1}{\alpha+1}} \nabla \rho, \\
a_2 &:= -(\alpha+1) u^{\frac{\alpha+\gamma}{\alpha+1}} \Delta \rho - (\alpha+1) u^{\frac{\alpha}{\alpha+1}} f\left(u^{\frac{1}{\alpha+1}}, \rho \right).
\end{aligned}
$$

Due to the results of parts 1. and 2. of this Lemma and assumptions on α, γ, and f, we have for all $0 < \tau < T$ and $\delta \in (0, \|M_0\|_{L^\infty(\Omega)})$ that in $[\tau, T] \times \{M_0 > \delta\} = [\tau, T] \times \{u_0 > \delta^{\alpha+1}\}$ equation (35) is a nondegenerate linear parabolic PDE with Hölder continuous coefficients. Standard result [24, Chapter 10, Theorem 10.1] on *interior* regularity in Hölder spaces yields that

$$
\|u\|_{C^{1,2}\left([\tau,T] \times \overline{\{u_0 > \delta^{\alpha+1}\}}\right)} \leq C_{31}\left(\delta, \tau, T, R \right).
\tag{36}
$$

Since the map $u \mapsto u^{\frac{1}{\alpha+1}}$ is smooth in $\mathbb{R}^+$, (19) is a consequence of (36). *Lemma 2.2* is proved. $\square$

Stability. As was mentioned earlier, the initial boundary value problem (1) is well-posed (in the usual Hadamard sense). In particular, the following stability result was proved in [15, 18]:

Lemma 2.3 (Lipschitz property, [15,18])**.** *There exists a function $L_0 : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $L_0(0, \cdot) \equiv 1$, which is continuous, increasing in each variable, depends only upon the parameters of the problem, and such that the following Lipschitz-type estimate holds:*

$$\max_{s \in [0,t]} \|(M_1 - M_2, \rho_1 - \rho_2)(s)\|_{H^{-1}(\Omega) \times L^2(\Omega)}$$

$$+ \left(\int_0^t \left(M_1^{\alpha+1} - M_2^{\alpha+1}, M_1 - M_2 \right)_{L^2(\Omega)} ds \right)^{\frac{1}{2}} + \|\rho_1 - \rho_2\|_{L^2(0,t;H_0^1(\Omega))}$$

$$\leq L_0(t,R) \|(M_{10} - M_{20}, \rho_{10} - \rho_{20})\|_{H^{-1}(\Omega) \times L^2(\Omega)}. \tag{37}$$

It is possible to estimate the difference $M_1 - M_2$ in $L^\infty(\Omega)$ by replacing (37) with a Hölder property:

Lemma 2.4 (Hölder property)**.** *There exists a constant $\theta_\infty \in (0,1)$ and a function $L_1 : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}^+$ which is continuous, increasing in each variable, depends only upon the parameters of the problem, and such that for all $(M_{10}, \rho_{10}), (M_{20}, \rho_{20}) \in C^\theta(\overline{\Omega}) \times C^{2+\theta}(\overline{\Omega})$ with $\|(M_0, \rho_0)\|_{C^\theta(\overline{\Omega}) \times C^{2+\theta}(\overline{\Omega})} \leq R$ for some $R \geq 0$ it holds for all $t > 0$ that*

$$\max_{s \in [0,t]} \|(M_1 - M_2, \rho_1 - \rho_2)(s)\|_{L^\infty(\Omega)}$$

$$\leq L_1(t,R) \|(M_{10} - M_{20}, \rho_{10} - \rho_{20})\|_{H^{-1}(\Omega) \times L^2(\Omega)}^{\theta_\infty}. \tag{38}$$

Proof. Combining (37) with the interpolation inequalities (13)-(14) applied to $w = (M_1 - M_2)(s)$ and $v = (\rho_1 - \rho_2)(s)$, we deduce that

$$\max_{s \in [0,t]} \|(M_1 - M_2, \rho_1 - \rho_2)(s)\|_{L^\infty(\Omega) \times W^{1,\infty}(\Omega)}$$

$$\leq C_{32} L_0^{\theta_1}(t,R)(2R)^{1-\theta_1} \|M_{10} - M_{20}\|_{H^{-1}(\Omega)}^{\theta_1}$$

$$+ C_{32} L_0^{\theta_2}(t,R)(2R)^{1-\theta_2} \|\rho_{10} - \rho_{20}\|_{L^2(\Omega)}^{\theta_2}$$

$$\leq L_1(t,R) \|(M_{10} - M_{20}, \rho_{10} - \rho_{20})\|_{H^{-1}(\Omega) \times L^2(\Omega)}^{\theta_\infty},$$

where $\theta_\infty := \min\{\theta_1, \theta_2\}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Absorbing set. It was proved in [21] that the initial boundary value problem (1) generates a well defined semigroup $S(t)$, $t \geq 0$, in the phase space $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ which possesses a bounded exponentially absorbing positively invariant set $\mathcal{B}_0$. In particular, the following dissipative estimate holds [21]:

$$\|(M,\rho)(t)\|_{L^\infty(\Omega) \times W^{1,\infty}(\Omega)} \leq C_\infty \|(M_0, \rho_0)\|_{L^\infty(\Omega) \times W^{1,\infty}(\Omega)}^{r_\infty} e^{-\omega_\infty t}$$

$$+ D_\infty \text{ for all } t \geq 0, \tag{39}$$

where the positive constants $C_\infty, r_\infty, \omega_\infty, D_\infty$ depend only upon the parameters of the problem. Due to *Lemma 2.2*, the absorbing set can actually be chosen in a nicer space:

Lemma 2.5 (Absorbing set)**.** *The semigroup $S(t)$ possesses an exponentially absorbing positively invariant set $\mathcal{B} \subset C^\theta(\overline{\Omega}) \times C^{2+\theta}(\overline{\Omega})$ ($\theta \in (0,1)$ as in Lemma 2.2)*

such that for some $R > 0$ which depends only upon the parameters of the problem it holds for all $(M_0, \rho_0) \in \mathcal{B}$ that

$$\|(M_0, \rho_0)\|_{C^\theta(\overline{\Omega}) \times C^{2+\theta}(\overline{\Omega})} \leq R. \tag{40}$$

Moreover, the solutions $(M, \rho) := S(\cdot)(M_0, \rho_0)$ belong to
$\left(C^{\frac{\theta}{2}, \theta}([0, \infty) \times \overline{\Omega}) \cap C^{1,2}(\mathbb{R}^+ \times \{M_0 > 0\}) \right) \times C^{1 + \frac{\theta}{2}, 2 + \theta}([0, \infty) \times \overline{\Omega})$ *and satisfy for all $\delta \in \left(0, \|M_0\|_{L^\infty(\Omega)}\right)$ and $t > 0$ the inequalities*

$$\|(M, \rho)\|_{C^{\frac{\theta}{2}, \theta}([0,t] \times \overline{\Omega}) \times C^{1 + \frac{\theta}{2}, 2 + \theta}([0,t] \times \overline{\Omega})} \leq C_{33}(t), \tag{41}$$

$$\inf \left\{ M(s, x) \mid s \in [0, t], \, x \in \{M_0 > \delta\} \right\} \geq C_{34}(\delta, t), \tag{42}$$

$$\|M\|_{C^{1,2}\left([0,t] \times \overline{\{M_0 > \delta\}}\right)} \leq C_{35}(\delta, t). \tag{43}$$

Proof. Set $\mathcal{B} := S(1)\mathcal{B}_0$. Due to a standard argument, $\mathcal{B}$ remains an exponentially absorbing positively invariant set for $S(t)$. On the other hand, *Lemma 2.2* ensures (40)-(43) (choose $\tau := 1$ and $T := t + 1$). $\square$

3. **A smoothing property.** The aim of this section is to prove that the semigroup $S(t)$ generated by system (1) is asymptotically smooth. Recall that due to the general theory presented in [9] (see also references therein) every asymptotically smooth semigroup possesses an exponential attractor. More notation is needed first. Let

$$u := (M, \rho),$$
$$X := H^{-1}(\Omega) \times L^2(\Omega). \tag{44}$$

Following [17], we introduce for any $u_0 := (M_0, \rho_0) \in \mathcal{B}$ some suitable u_0-dependent spaces and an operator. In our case a possible choice is as follows: for $\delta \in (0, \|M_0\|_{L^\infty(\Omega)})$ and $0 < t_1 < t$ set

$$Y_{u_0}^{(\delta)} := L^2((t_1, t) \times \{M_0 > \delta\}) \times \left(L^2((t_1, t), H^1(\{M_0 > \delta\})) \cap L^2((t_1, t) \times \Omega) \right),$$

$$Z_{u_0}^{(\delta)} := W^{(1,2),2}((t_1, t) \times \{M_0 > \delta\})$$
$$\times \left(W^{(1,2),2}((t_1, t) \times \{M_0 > \delta\}) \cap H^1([0, t], H_0^1(\Omega), H^{-1}(\Omega)) \right),$$

$$K_{u_0}^{(\delta)} : \mathcal{B} \to Z_{u_0}^{(\delta)}, \, K_{u_0}^{(\delta)}(u_{10})(s) := (S(s)u_{10})|_{\{M_0 > \delta\}} \text{ for all } s \in [0, t], \, u_{10} \in \mathcal{B}.$$

Observe that since M_0 is a continuous function, the level sets $\{M_0 > \delta\}$ are open. Hence, the spaces $Y_{u_0}^{(\delta)}$ and $Z_{u_0}^{(\delta)}$ are well defined.

Now we can formulate a smoothing property for our case:

Theorem 3.1 (Smoothing property). *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \in \{1, 2, 3\}$. Let the functions f and g satisfy assumptions (4)-(7) and (11) and let the given constants α and γ satisfy $\frac{\alpha}{2} + 1 < \gamma < \alpha$. Then there exist some constants $A_1, A_2, \delta, \varepsilon, T > 0$ depending only on the parameters of the problem and such that the following smoothing property holds for the operator $S(T)$:*

$$\|(S(T))(u_{10}) - (S(T))(u_{20})\|_X \leq \frac{1}{2}\|u_{10} - u_{20}\|_X$$
$$+ A_1 \left\| K_{u_0}^{\left(\frac{\delta}{2}\right)}(u_{10}) - K_{u_0}^{\left(\frac{\delta}{2}\right)}(u_{20}) \right\|_{Y_{u_0}^{(\delta)}}, \tag{45a}$$

$$\left\| K_{u_0}^{\left(\frac{\delta}{2}\right)}(u_{10}) - K_{u_0}^{\left(\frac{\delta}{2}\right)}(u_{20}) \right\|_{Z_{u_0}^{\left(\frac{\delta}{2}\right)}} \leq A_2 \|u_{10} - u_{20}\|_X \tag{45b}$$

$$for\ \|u_0 - u_{10}\|_X,\ \|u_0 - u_{20}\|_X \leq \varepsilon,\ u_0, u_{10}, u_{20} \in \mathcal{B}, \tag{45c}$$

where $\mathcal{B}$ is the absorbing set from Lemma 2.5.

Due to the presence of the taxis term it seems impossible to handle the difference $M_1 - M_2$ in L^1, as is done for the porous medium equation in [9, Chapter 4]. This is the reason why we use the H^{-1}-norm instead. It offers another convenient choice for a degenerate equation. We first recall some useful and well known facts about the gradient operator (∇), its adjoint (∇^*), and pseudo-inverse (∇^+):

$$H_0^1(\Omega) \underset{\nabla^+}{\overset{\nabla}{\rightleftharpoons}} (L^2(\Omega))^N \underset{\nabla^{+*}}{\overset{\nabla^*}{\rightleftharpoons}} H^{-1}(\Omega),$$

$$\nabla^* = -\nabla\cdot, \quad \nabla^+ = \nabla^{-1}\Pi_{\nabla(H_0^1(\Omega))},$$

$$\nabla^+\nabla = id, \tag{46}$$

$$(-\Delta)^{-1} = \nabla^+\nabla^{+*} \tag{47}$$

$$\nabla(-\Delta)^{-1} = \nabla^{+*}, \tag{48}$$

$$\|\nabla^{+*}u^*\|_{(L^2(\Omega))^N} = \|u^*\|_{H^{-1}(\Omega)}. \tag{49}$$

Here $\Pi_{\nabla(H_0^1(\Omega))}$ denotes the orthogonal projection on $\nabla(H_0^1(\Omega))$ which is a closed subspace of $(L^2(\Omega))^N$.

Proof of Theorem 3.1. Let $(M_0, \rho_0), (M_{10}, \rho_{10}), (M_{20}, \rho_{20}) \in \mathcal{B}$ be any points such that

$$\|(M_0 - M_{10}, \rho_0 - \rho_{10})\|_{H^{-1}(\Omega) \times L^2(\Omega)},\ \|(M_0 - M_{20}, \rho_0 - \rho_{20})\|_{H^{-1}(\Omega) \times L^2(\Omega)} \leq \varepsilon. \tag{50}$$

for some $\varepsilon > 0$ which we will fix later on. Due to the interpolation inequality (13) we obtain that

$$\|M_0 - M_{10}\|_{L^\infty(\Omega)},\ \|M_0 - M_{20}\|_{L^\infty(\Omega)} \leq C_{36}\varepsilon^{\theta_1}. \tag{51}$$

Let $(M, \rho), (M_1, \rho_1), (M_2, \rho_2)$ be the corresponding solutions to system (1). Subtracting equation (1a) for solutions (M_1, ρ_1) and (M_2, ρ_2) we obtain with (11) that

$$\partial_t(M_1 - M_2) = \frac{1}{\alpha+1}\Delta\left(M_1^{\alpha+1} - M_2^{\alpha+1}\right) - \nabla \cdot \left((M_1^\gamma - M_2^\gamma)\nabla\rho_2\right)$$
$$- \nabla \cdot (M_1^\gamma\nabla(\rho_1 - \rho_2))$$
$$- F_5(M_1 - M_2) - \left(\widetilde{f}\left(M_1^\beta, \rho_1\right) - \widetilde{f}\left(M_2^\beta, \rho_2\right)\right). \tag{52}$$

Further, in order to shorten the notation we introduce the quantities

$$W := M_1 - M_2, \quad W_0 := W(0), \quad v := \rho_1 - \rho_2, \quad v_0 := v(0),$$
$$U_{\delta,\varepsilon,T} := \max\left\{M_1(s,x), M_2(s,x)|\ t \in [0,T],\ x \in \{M_0 \leq \delta\}\right\}.$$

Our first goal is to progress towards the 'contractive' part of estimate (45a). In order to achieve this we need to obtain some kind of dissipativity estimate for $\|M\|_{H^{-1}(\Omega)}$ with perturbation terms which do not contain norms of M-component on level sets $\{M_0 \leq \delta\}$. Multiplying (52) by $(-\Delta)^{-1}W$ and integrating over Ω, we arrive at

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla^{+*}W\right\|_{(L^2(\Omega))^n}^2 = -\frac{1}{\alpha+1}\left(M_1^{\alpha+1} - M_2^{\alpha+1}, M_1 - M_2\right)_{L^2(\Omega)}$$

$$+ \left((M_1^\gamma - M_2^\gamma)\nabla\rho_2, \nabla^{+*}W \right)_{(L^2(\Omega))^n}$$

$$+ \left(M_1^\gamma \nabla v, \nabla^{+*}W \right)_{(L^2(\Omega))^n} - F_5 \left\| \nabla^{+*}W \right\|^2_{(L^2(\Omega))^n}$$

$$- \left(\widetilde{f}\left(M_1^\beta, \rho_1\right)_{L^2(\Omega)} - \widetilde{f}\left(M_2^\beta, \rho_2\right), \nabla^+\nabla^{+*}W \right)_{L^2(\Omega)}.$$

$$(53)$$

Here we used the definition of the adjoint and properties (47)-(48). Using the inequalities

$$\left(M_1^{\alpha+1} - M_2^{\alpha+1} \right)(M_1 - M_2) \geq \frac{\alpha+1}{\left(1+\frac{\alpha}{2}\right)^2}\left(M_1^{1+\frac{\alpha}{2}} - M_2^{1+\frac{\alpha}{2}} \right)^2, \qquad (54)$$

$$\frac{c}{b}\max\left\{M_1, M_2\right\}^{\frac{c-b}{b}}\left| M_1^b - M_2^b \right| \geq \left| M_1^c - M_2^c \right| \text{ for all } c \geq b > 0, \qquad (55)$$

and assumptions $\gamma, \beta > 1 + \frac{\alpha}{2}$, we can estimate the terms on the right-hand side of (53) in the following way:

$$- \left(M_1^{\alpha+1} - M_2^{\alpha+1}, M_1 - M_2 \right)_{L^2(\Omega)} \leq -C_{37}\left\| M_1^{1+\frac{\alpha}{2}} - M_2^{1+\frac{\alpha}{2}} \right\|^2_{L^2(\Omega)}, \qquad (56)$$

$$\left| \left((M_1^\gamma - M_2^\gamma)\nabla\rho_2, \nabla^{+*}W \right)_{(L^2(\Omega))^n} \right|$$

$$\leq C_{38}\left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n}\left(\left\| M_1^\gamma - M_2^\gamma \right\|_{L^2(\{M_0 \leq \delta\})} + \left\| M_1^\gamma - M_2^\gamma \right\|_{L^2(\{M_0 > \delta\})} \right)$$

$$\leq C_{39}\left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n}\left(U_{\delta,\varepsilon,T}^{\frac{2\gamma-2-\alpha}{\alpha+2}}\left\| M_1^{1+\frac{\alpha}{2}} - M_2^{1+\frac{\alpha}{2}} \right\|_{L^2(\Omega)} + \left\| W \right\|_{L^2(\{M_0 > \delta\})} \right),$$

$$(57)$$

$$\left| \left(M_1^\gamma \nabla v, \nabla^{+*}W \right)_{(L^2(\Omega))^n} \right| \leq C_{40}\left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n}$$

$$\cdot \left(\left\| M_1^\gamma \nabla v \right\|_{L^2(\{M_0 \leq \delta\})} + \left\| M_1^\gamma \nabla v \right\|_{(L^2(\{M_0 > \delta\}))^n} \right)$$

$$\leq C_{41}\left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n}$$

$$\cdot \left(U_{\delta,\varepsilon,T}^\gamma\left\| \nabla v \right\|_{(L^2(\Omega))^n} + \left\| \nabla v \right\|_{(L^2(\{M_0 > \delta\}))^n} \right), \quad (58)$$

$$\left| \left(\widetilde{f}\left(M_1^\beta, \rho_1\right) - \widetilde{f}\left(M_2^\beta, \rho_2\right), \nabla^+\nabla^{+*}W \right)_{L^2(\Omega)} \right|$$

$$\leq C_{42}\left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n}\left(\left\| M_1^\beta - M_2^\beta \right\|_{L^2(\Omega)} + \left\| v \right\|_{L^2(\Omega)} \right)$$

$$\leq C_{42}\left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n}$$

$$\cdot \left(\left\| M_1^\beta - M_2^\beta \right\|_{L^2(\{M_0 \leq \delta\})} + \left\| M_1^\beta - M_2^\beta \right\|_{L^2(\{M_0 > \delta\})} + \left\| v \right\|_{L^2(\Omega)} \right)$$

$$\leq C_{43}\left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n}$$

$$\cdot \left(U_{\delta,\varepsilon,T}^{\frac{2\beta-2-\alpha}{\alpha+2}}\left\| M_1^{1+\frac{\alpha}{2}} - M_2^{1+\frac{\alpha}{2}} \right\|_{L^2(\Omega)} + \left\| W \right\|_{L^2(\{M_0 > \delta\})} + \left\| v \right\|_{L^2(\Omega)} \right). \qquad (59)$$

Combining (53) and (56)-(59) with the Young inequality, we conclude that

$$\frac{1}{2}\frac{d}{dt}\left\| \nabla^{+*}W \right\|^2_{(L^2(\Omega))^n}$$

$$\leq \left(-\frac{F_5}{2} + C_{44} \left(U_{\delta,\varepsilon,T}^{2\frac{2\gamma-2-\alpha}{\alpha+2}} + U_{\delta,\varepsilon,T}^{2\frac{2\beta-2-\alpha}{\alpha+2}} \right) \right) \left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n}^2$$

$$+ C_{45} U_{\delta,\varepsilon,T}^{2\gamma} \left\| \nabla v \right\|_{(L^2(\Omega))^n}^2$$

$$+ C_{46} \left(\|W\|_{L^2(\{M_0>\delta\})}^2 + \|\nabla v\|_{(L^2(\{M_0>\delta\}))^n}^2 + \|v\|_{L^2(\Omega)}^2 \right)$$

$$\leq \left(-\frac{F_5}{2} + C_{47} U_{\delta,\varepsilon,T}^{\kappa} \right) \left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n}^2 + C_{45} U_{\delta,\varepsilon,T}^{\kappa} \left\| \nabla v \right\|_{(L^2(\Omega))^n}^2$$

$$+ C_{48} \left(\|W\|_{L^2(\{M_0>\delta\})}^2 + \|\nabla v\|_{(L^2(\{M_0>\delta\}))^n}^2 + \|v\|_{(L^2(\Omega))^n}^2 \right), \tag{60}$$

where

$$\kappa := 2 \min \left\{ \frac{2\min\{\gamma,\beta\}-2-\alpha}{\alpha+2}, \gamma \right\} = 2 \frac{2\min\{\gamma,\beta\}-2-\alpha}{\alpha+2} > 0 \tag{61}$$

due to the assumptions on α, β, and γ. We emphasise at this point that our new and sharper assumptions $\gamma, \beta > 1 + \frac{\alpha}{2}$ allow not only to absorb the L^2 norms of the differences of some powers of M_1 and M_2 coming from the taxis and reaction terms, but also to obtain some dissipativity with respect to $\left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n}^2$ in (60). Indeed, as we will see later, $U_{\delta,\varepsilon,T}$ can be made arbitrary small by choosing ε and δ sufficiently small. Thanks to $\kappa > 0$ this leads to a negative coefficient $\left(-\frac{F_5}{2} + C_{47} U_{\delta,\varepsilon,T}^{\kappa} \right)$. With γ or β equal to $1 + \frac{\alpha}{2}$ we would have $\kappa = 0$ instead and thus a potentially positive coefficient $\left(-\frac{F_5}{2} + C_{47} \right)$ which only guaranties an estimate such as (37).

Leaving (60) for while we will now establish an bound for $\|v\|$. Once again, we need to take care so as not to include any $L^p(\{M_0 \leq \delta\})$-norms of W in our estimates. Subtracting equation (1b) for (M_1, ρ_1) and (M_2, ρ_2) we obtain that

$$\partial_t v = \Delta v - G_1 v - (g_2(\rho_1) - g_2(\rho_2))M_1 - g_2(\rho_2)W. \tag{62}$$

Multiplying (62) by v, integrating over Ω, and using (46), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2(\Omega)}^2 = -\|\nabla v\|_{(L^2(\Omega))^n}^2 - G_1\|v\|_{L^2(\Omega)}^2 - (g_2(\rho_1) - g_2(\rho_2), M_1 v)_{L^2(\Omega)}$$

$$- (W, g_2(\rho_2)v)_{L^2(\Omega)}$$

$$= -\|\nabla v\|_{(L^2(\Omega))^n}^2 - G_1\|v\|_{L^2(\Omega)}^2 - (g_2(\rho_1) - g_2(\rho_2), M_1 v)_{L^2(\Omega)}$$

$$- \left(\nabla^{+*}W, g_2(\rho_2)\nabla v + v\frac{dg_2}{d\rho}(\rho_2)\nabla\rho_2 \right)_{(L^2(\Omega))^n}. \tag{63}$$

Using the assumptions on g_2 and the Young inequality we conclude from (63) that

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2(\Omega)}^2 \leq -\|\nabla v\|_{(L^2(\Omega))^n}^2 - G_1\|v\|_{L^2(\Omega)}^2 + C_{49}\left(M_1, v^2\right)_{L^2(\Omega)}$$

$$+ \left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n} \left(\|\nabla v\|_{(L^2(\Omega))^n} + \|v\|_{L^2(\Omega)} \right)$$

$$\leq C_{50}\left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n}^2 - \frac{1}{2}\|\nabla v\|_{(L^2(\Omega))^n}^2 + C_{51}\|v\|_{L^2(\Omega)}^2. \tag{64}$$

Let us now multiply the inequality (64) by the constant $C_{52} := \frac{F_5}{4C_{50}}$ and add it to (60). This yields

$$\frac{1}{2}\frac{d}{dt}\left(\left\| \nabla^{+*}W \right\|_{(L^2(\Omega))^n}^2 + C_{52}\|v\|_{L^2(\Omega)}^2 \right)$$

$$\leq \left(-\frac{F_5}{4} + C_{47} U_{\delta,\varepsilon,T}^\kappa \right) \left\| \nabla^{+*} W \right\|_{(L^2(\Omega))^n}^2 + \left(-C_{53} + C_{45} U_{\delta,\varepsilon,T}^\kappa \right) \left\| \nabla v \right\|_{(L^2(\Omega))^n}^2$$

$$+ C_{54} \|v\|_{L^2(\Omega)}^2 + C_{48} \left(\|W\|_{L^2(\{M_0 > \delta\})}^2 + \|\nabla v\|_{(L^2(\{M_0 > \delta\}))^n}^2 \right). \tag{65}$$

Observe that due to (38) and (50) it follows for all $\delta \in (0, \|M_0\|_{L^\infty(\Omega)})$ that

$$U_{\delta,\varepsilon,T} \leq \delta + L(T,R)\varepsilon^{\theta_\infty}. \tag{66}$$

Combining (65)-(66) and recalling that $\kappa > 0$ due to (61), we conclude that if δ and ε are chosen in such a way that

$$\delta + L(T,R)\varepsilon^{\theta_\infty} \leq \min\left\{ \frac{F_5}{8C_{47}}, \frac{C_{53}}{2C_{45}} \right\}^{\frac{1}{\kappa}}, \tag{67}$$

then

$$\frac{1}{2}\frac{d}{dt}\left(\left\| \nabla^{+*} W \right\|_{(L^2(\Omega))^n}^2 + C_{52} \|v\|_{L^2(\Omega)}^2 \right)$$

$$\leq -\frac{F_5}{8} \left\| \nabla^{+*} W \right\|_{(L^2(\Omega))^n}^2 - \frac{C_{53}}{2} \left\| \nabla v \right\|_{(L^2(\Omega))^n}^2 + C_{54} \|v\|_{L^2(\Omega)}^2$$

$$+ C_{48} \left(\|W\|_{L^2(\{M_0 > \delta\})}^2 + \|\nabla v\|_{(L^2(\{M_0 > \delta\}))^n}^2 \right)$$

$$\leq -C_{55} \left(\left\| \nabla^{+*} W \right\|_{(L^2(\Omega))^n}^2 + C_{52} \|v\|_{L^2(\Omega)}^2 \right)$$

$$+ C_{56} \left(\|W\|_{L^2(\{M_0 > \delta\})}^2 + \|\nabla v\|_{(L^2(\{M_0 > \delta\}))^n}^2 + \|v\|_{L^2(\Omega)}^2 \right). \tag{68}$$

Using the Gronwall lemma, the Lipschitz property (37), and property (49) we obtain with (68) that for all $0 < t_1 < T$

$$\|W(T)\|_{H^{-1}(\Omega)}^2 + C_{52} \|v(T)\|_{L^2(\Omega)}^2$$

$$\leq e^{-2C_{55}(T-t_1)} \left(\|W(t_1)\|_{H^{-1}(\Omega)}^2 + C_{52} \|v(t_1)\|_{L^2(\Omega)}^2 \right)$$

$$+ 2C_{56} \int_{t_1}^T e^{2C_{55}(s-T)} \left(\|W\|_{L^2(\{M_0 > \delta\})}^2 + \|\nabla v\|_{(L^2(\{M_0 > \delta\}))^n}^2 + \|v\|_{L^2(\Omega)}^2 \right) ds$$

$$\leq C_{57} L_0^2(t_1,R) e^{-2C_{55}(T-t_1)} \left(\|W_0\|_{H^{-1}(\Omega)}^2 + C_{52} \|v_0\|_{L^2(\Omega)}^2 \right)$$

$$+ 2C_{56} \int_{t_1}^T \|W\|_{L^2(\{M_0 > \delta\})}^2 + \|\nabla v\|_{(L^2(\{M_0 > \delta\}))^n}^2 + \|v\|_{L^2(\Omega)}^2 \, ds, \tag{69}$$

which finally leads to the estimate

$$\|(W(T), v(T))\|_X \leq C_{58} L_0(t_1,R) e^{-C_{55}(T-t_1)} \|(W_0, v_0)\|_X + C_{59} \|(W, v)\|_{Y_{u_0}^{(\delta)}}$$

$$\leq \frac{1}{2}\|(W_0, v_0)\|_X + C_{59} \|(W, v)\|_{Y_{u_0}^{(\delta)}} \tag{70}$$

if t_1 and T are such that

$$C_{58} L_0(t_1,R) e^{-C_{55}(T-t_1)} \leq \frac{1}{2}. \tag{71}$$

Next, we study the pair (W, v) on the sets $\{M_0 > \delta\}$. On these sets the equation for M is non-degenerate, which allows to use standard estimates for uniformly parabolic PDEs and thus obtain better regularity. Starting once again with equations (52) and (62), we now rewrite them in the following way:

$$\partial_t W = M_1^\alpha \Delta W + b_1 \cdot \nabla W + b_2 W + b_3 \Delta v + b_4 \cdot \nabla v + b_5 v, \tag{72a}$$

$$\partial_t v = \Delta v + b_6 v + b_7 W, \tag{72b}$$

where

$$
\begin{aligned}
b_1 :=& \alpha M_1^{\alpha-1} \nabla(M_1 + M_2) - \gamma M_2^{\gamma-1} \nabla \rho_1, \\
b_2 :=& \frac{M_1^{\alpha} - M_2^{\alpha}}{M_1 - M_2} \Delta M_2 + \alpha \frac{M_1^{\alpha-1} - M_2^{\alpha-1}}{M_1 - M_2} |\nabla M_2|^2 - \frac{M_1^{\gamma} - M_2^{\gamma}}{M_1 - M_2} \Delta \rho_1 \\
& - \gamma \frac{M_1^{\gamma-1} - M_2^{\gamma-1}}{M_1 - M_2} \nabla M_1 \cdot \nabla \rho_1 - \frac{f(M_1, \rho_1) - f(M_2, \rho_1)}{M_1 - M_2}, \\
b_3 :=& - M_2^{\gamma}, \\
b_4 :=& - \gamma M_2^{\gamma-1} \nabla M_2, \\
b_5 :=& - \frac{f(M_2, \rho_1) - f(M_2, \rho_2)}{\rho_1 - \rho_2}, \\
b_6 :=& - \frac{g(M_1, \rho_1) - g(M_1, \rho_2)}{\rho_1 - \rho_2}, \\
b_7 :=& - \frac{g(M_1, \rho_2) - g(M_2, \rho_2)}{M_1 - M_2}.
\end{aligned}
$$

Observe that due to (51) we have for all $\delta_0 \in \left(0, \|M_0\|_{L^\infty(\Omega)}\right)$ that

$$
\begin{aligned}
\inf \{M_{10}(x), M_{20}(x) | \, x \in \{M_0 > \delta_0\}\} \geq & \delta_0 - C_{36} \varepsilon^{\theta_1} \\
\geq & \frac{\delta_0}{2}, \tag{73}
\end{aligned}
$$

so that, due to (42),

$$\inf \{M(t, x) | \, s \in [0, t], \, x \in \{M_0 > \delta_0\}\} \geq C_{34} \left(\frac{\delta_0}{2}, t\right), \tag{74}$$

if

$$C_{36} \varepsilon^{\theta_1} \leq \frac{\delta_0}{2}. \tag{75}$$

Thus, for such δ_0 and ε system (72a) is a nondegenerate linear parabolic system w.r.t. (W, v). Moreover, coefficients b_i are compositions of continuous functions with M_k and ρ_k, $k = 1, 2$, and their partial derivatives up to the second order. *Lemma 2.5* implies that b_i's all belong to $L^\infty((0, T) \times \{M_0 > \delta_0\})$. Altogether, standard results on interior regularity in Sobolev spaces (see, e.g., Theorems 9.1 and 10.1, and the remark on local estimates in Sobolev spaces at the end of §10 in [24, Chapter IV]) together with estimates from *Lemma 2.5* imply that for all $0 < t_0 < t_1 < T$ and $\delta_0 < \delta_1 < \delta_2$ it holds

$$
\begin{aligned}
&\|W\|_{W^{(1,2),2}((t_1,T) \times \{M_0 > \delta_2\})} \\
\leq & C_{60}(\delta_1, \delta_2, t_1, T) \left(\|W\|_{L^2((t_0,T) \times \{M_0 > \delta_1\})} + \|v\|_{W^{(1,2),2}((t_0,T) \times \{M_0 > \delta_1\})}\right), \tag{76}
\end{aligned}
$$

$$
\begin{aligned}
&\|v\|_{W^{(1,2),2}((t_0,T) \times \{M_0 > \delta_1\})} \\
\leq & C_{61}(\delta, \delta_1, t_0, T) \left(\|v\|_{L^2((0,T) \times \{M_0 > \delta_0\})} + \|W\|_{L^2((0,T) \times \{M_0 > \delta_0\})}\right). \tag{77}
\end{aligned}
$$

Plugging (77) into (76), we obtain that

$$\|W\|_{W^{(1,2),2}((t_1,T) \times \{M_0 > \delta_2\})} \leq C_{62}(\delta, \delta_1, \delta_2, t_0, t_1, t) \|(W, v)\|_{L^2([0,t] \times \{M_0 > \delta_0\})}. \tag{78}$$

Thus, choosing $\delta_0 := \frac{\delta}{4}$, $\delta_1 := \frac{3\delta}{8}$, $\delta_2 := \frac{\delta}{2}$ in (78) and $\delta_0 := \frac{\delta}{4}$, $\delta_1 := \frac{\delta}{2}$, $t_0 := t_1$ in (77) yields

$$\|(W, v)\|_{(W^{(1,2),2}((t_1,T)\times\{M_0>\frac{\delta}{2}\}))^2} \leq C_{63}(\delta, t_1, T)\|(W, v)\|_{(L^2((0,T)\times\{M_0>\frac{\delta}{4}\}))^2}. \quad (79)$$

Combining (79) with (37), (74), and the inequalities (54) and

$$|M_1 - M_2| \leq \frac{2}{\alpha+2}\inf\{M_1, M_2\}^{-\frac{\alpha}{2}}\left|M_1^{1+\frac{\alpha}{2}} - M_2^{1+\frac{\alpha}{2}}\right|, \quad (80)$$

we thus arrive at the estimate

$$\|(W, v)\|_{(W^{(1,2),2}((t_1,T)\times\{M_0>\frac{\delta}{2}\}))^2}$$

$$\leq C_{64}(\delta, t_1, T)\left(\int_0^T \left(M_1^{\alpha+1} - M_2^{\alpha+1}, M_1 - M_2\right) ds\right)^{\frac{1}{2}} + C_{63}(\delta, t_1, T)\|v\|_{L^2((0,T)\times\Omega)}$$

$$\leq C_{65}(\delta, t_1, T)\|(W_0, v_0)\|_{H^{-1}(\Omega)\times L^2(\Omega)}. \quad (81)$$

Next, we recall that due to (37) it holds that

$$\|v\|_{L^2(0,T;H_0^1(\Omega))} \leq L_0(T, R)\|(W_0, v_0)\|_{H^{-1}(\Omega)\times L^2(\Omega)}. \quad (82)$$

Going back to (62), we compute that

$$\|\partial_t v\|_{H^{-1}(\Omega)}$$
$$=\|\Delta v - G_1 v - (g_2(\rho_1) - g_2(\rho_2))M_1 - g_2(\rho_2)W\|_{H^{-1}(\Omega)}$$
$$\leq\|v\|_{H_0^1(\Omega)} + G_1\|v\|_{H^{-1}(\Omega)} + C_{66}\|M_1\|_{L^\infty(\Omega)}\|g_2(\rho_1) - g_2(\rho_2)\|_{L^2(\Omega)}$$
$$\quad + C_{67}\|g_2(\rho_2)\|_{W^{1,\infty}(\Omega)}\|W\|_{H^{-1}(\Omega)}$$
$$\leq C_{68}(R)\left(\|v\|_{H_0^1(\Omega)} + \|W\|_{H^{-1}(\Omega)}\right). \quad (83)$$

Integrating (83) over $(0, t)$ and combining with (37) and (82), we finally obtain that

$$\|v\|_{H^1((0,T),H_0^1(\Omega),H^{-1}(\Omega))} \leq C_{69}(R, T)\|(W_0, v_0)\|_{H^{-1}(\Omega)\times L^2(\Omega)}. \quad (84)$$

With (70), (81), (84) we have the conditions of the smoothing property (45), it only remains to choose the parameters in such a way that conditions (67), (71), and (75) (recall that $\delta_0 = \frac{\delta}{4}$) are satisfied, i.e., if

$$\delta + L(T, R)\varepsilon^{\theta_1} \leq \min\left\{\frac{F_5}{8C_{47}}, \frac{C_{53}}{2C_{45}}\right\}^{\frac{1}{\kappa}}, \quad (85)$$

$$C_{15}L_0(t_1, R)e^{-C_{55}(T-t_1)} \leq \frac{1}{2}, \quad (86)$$

$$C_{36}\varepsilon^{\theta_1} \leq \frac{\delta}{8}. \quad (87)$$

Clearly, the exist such t_1, T, δ, and ε, that conditions (85)-(87) are satisfied. Indeed, for any $t_1 > 0$ one can choose T large enough so as to fulfil (86). Then, choosing $\delta := 8C_{36}\varepsilon^{\theta_1}$ in order to comply with (87), it remains to choose ε so small as to meet (85). *Theorem 3.1* is proved.

$\square$

4. **Proof of Theorem 1.2.** We are finally ready for the

Proof of Theorem 1.2. Our proof goes through the following steps. First, we prove the existence of an exponential attractor $\mathcal{M}$ in $H^{-1}(\Omega) \times L^2(\Omega)$-metric. This we achieve with the help of the smoothing property (45). Finally, we use the Sobolev interpolation inequality in order to show that $\mathcal{M}$ is at the same time an exponential attractor in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$-metric.

Due to *Theorem 3.1* the exists a number $T > 0$ such that for $S(T)$ the smoothing property (45) holds. The existence of an exponential attractor for the *discrete* semigroup $S(nT), n \in \mathbb{N}$, in the set $\mathcal{B} \subset X$ is a consequence of Remark 4.3 of [9], we only need to verify that $Z_{u_0}^{\left(\frac{\delta}{2}\right)}$ is uniformly (w.r.t. $u_0 \in \mathcal{B}$) compactly embedded in $Y_{u_0}^{(\delta)}$. Due to Lions-Aubin lemma, we have that

$$H^1\left((t_1, T), H_0^1(\Omega), H^{-1}(\Omega)\right) \subset\subset L^2\left((t_1, T), L^2(\Omega)\right). \tag{88}$$

Therefore, we only need to study the (obviously continuous) canonical embedding

$$i_{u_0} : W^{(1,2),2}\left((t_1, t) \times \left\{M_0 > \frac{\delta}{2}\right\}\right) \to L^2((t_1, t), H^1(\{M_0 > \delta\})),$$

$$i_{u_0}u := u|_{(t_1,t) \times \{M_0 > \delta\}}.$$

We proceed similar to [17, Proposition A.5], where the case of a Hölder space embedded in the space of continuous functions on a smaller domain was considered. Let us define for each $u_0 \in \mathcal{B}$ an extension operator

$$p_{u_0} : W^{(1,2),2}\left((t_1, t) \times \left\{M_0 > \frac{\delta}{2}\right\}\right) \to W^{(1,2),2}\left((t_1, t) \times \Omega\right),$$

$$p_{u_0}u := \begin{cases} \varphi_{M_0}u & \text{in } \left\{M_0 > \frac{\delta}{2}\right\}, \\ 0 & \text{in } \left\{M_0 \leq \frac{\delta}{2}\right\}. \end{cases}$$

Here φ_{M_0} is any cutoff function which satisfies (16) for $\delta_0 := \frac{3\delta}{4}$ and $\delta_1 := \delta$. Since φ_{M_0} is a test function and compactly supported in $\left\{M_0 > \frac{\delta}{2}\right\}$, it follows that $\{p_{u_0}\}_{u_0 \in \mathcal{B}}$ is a family of well defined continuous linear operators. Moreover, even though these operators are defined on different spaces, their norms are uniformly bounded:

$$\|p_{u_0}\| \leq A_p \quad \text{for all } u_0 \in \mathcal{B} \tag{89}$$

for some constant $A_p > 0$. This is a consequence of property (16c). Note also that our choice of cutoff function guaranties that

$$p_{u_0}u = u \quad \text{in } (t_1, t) \times \{M_0 > \delta\}. \tag{90}$$

Next, we define a restriction operator

$$c_{u_0} : L^2((t_1, t), H^1(\Omega)) \to L^2((t_1, t), H^1(\{M_0 > \delta\})), \quad c_{u_0}u = u|_{(t_1,t) \times \{M_0 > \delta\}}.$$

In this case, the value ranges depend upon u_0, but, clearly,

$$\|c_{u_0}\| \leq 1 \quad \text{for all } u_0 \in \mathcal{B}. \tag{91}$$

Finally, we recall that due to the Lions-Aubin lemma the canonical embedding

$$j : W^{(1,2),2}\left((t_1, t) \times \Omega\right) \to L^2((t_1, t), H^1(\Omega)), \quad ju = u$$

is compact. Observe that due to (90)

$$i_{u_0} = c_{u_0}jp_{u_0}. \tag{92}$$

Using (89) and (91), we compute that

$$
\begin{aligned}
& N_r \left(i_{u_0} \left(B \left(0, 1; W^{(1,2),2} \left((t_1, t) \times \left\{ M_0 > \frac{\delta}{2} \right\} \right) \right) \right); \right. \\
& \left. L^2((t_1, t), H^1(\{M_0 > \delta\}))) \right. \\
=& N_r \left(c_{u_0} j p_{u_0} \left(B \left(0, 1; W^{(1,2),2} \left((t_1, t) \times \left\{ M_0 > \frac{\delta}{2} \right\} \right) \right) \right); \right. \\
& \left. L^2((t_1, t), H^1(\{M_0 > \delta\}))) \right. \\
\leq& N_r \left(j p_{u_0} \left(B \left(0, 1; W^{(1,2),2} \left((t_1, t) \times \left\{ M_0 > \frac{\delta}{2} \right\} \right) \right) \right); L^2((t_1, t), H^1(\Omega)) \right) \\
\leq& N_r \left(j \left(B \left(0, A_p; W^{(1,2),2} \left((t_1, t) \times \Omega \right) \right) \right); L^2((t_1, t), H^1(\Omega)) \right), \quad (93)
\end{aligned}
$$

where $B(0, 1; V)$ denotes the unit ball in a normed space V, and $N_r(C; V)$ denotes the minimum number of balls of radius $r > 0$ needed in order to cover a compact set $C \subset V$. Since the bound on the right-hand side of (93) is independent of u_0, the embedding family $\{i_{u_0}\}$ is indeed uniformly compact. Due to the above observation this carries over to the embedding $Z_{u_0}^{\left(\frac{\delta}{2}\right)} \subset\subset Y_{u_0}^{(\delta)}$. Therefore, with Remark 4.3 from [9] we conclude that there exists an exponential attractor $\mathcal{M}_T$ for the semigroup $S(nT), n \in \mathbb{N}$, in $\mathcal{B}$ (equipped with the $H^{-1}(\Omega) \times L^2(\Omega)$-topology) and its dimension and attraction parameters depend only upon the parameters of the problem. As usual (see, e.g., [9, Remark 3.2]), the required exponential attractor $\mathcal{M} \subset \mathcal{B}$ for the continuous-time semigroup $S(t), t \geq 0$ can be defined via

$$
\mathcal{M} := \bigcup_{t \in [0,T]} S(t) \mathcal{M}_T \subset \mathcal{B}.
$$

For this construction to work, it suffices (compare [9, Remark 3.2]) to check that the map $(t, u_0) \mapsto S(t) u_0$ is, say, Hölder continuous on $[0, T] \times \mathcal{B}$. The Hölder continuity w.r.t. t is a consequence of the regularity result (41) and the Sobolev embedding theorem. The Lipschitz continuity with respect to u_0 is given by the Lipschitz property (37). In both cases such parameters as the Hölder/Lipschitz constants and the Hölder exponent can be chosen to depend upon the parameters of the problem. Consequently, $\mathcal{M}$ is indeed an exponential attractor for $S(t)$ in $\mathcal{B}$ equipped with the $H^{-1}(\Omega) \times L^2(\Omega)$-topology and its dimension and attraction parameters depend only upon the parameters of the problem.

Finally, we observe that due to the interpolation inequalities (13)-(14) the canonical embedding of $\mathcal{B}$ equipped with (the norm-induced) $H^{-1}(\Omega) \times L^2(\Omega)$-metric and $\mathcal{B}$ equipped with (the norm-induced) $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$-metric is Hölder continuous. Consequently, $\mathcal{M}$ is an exponential attractor for $S(t)$ in $\mathcal{B}$ equipped with $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$-metric and, once again, its dimension and attraction parameters depend only upon the parameters of the problem. Combining this with the fact that $\mathcal{B}$ is an exponentially absorbing set in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ and its diameter and absorption parameters depend only upon the parameters of the problem, we conclude that $\mathcal{M}$ is an exponential attractor for $S(t)$ in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ and its dimension and attraction parameters depend only upon the parameters of the problem, as required. *Theorem 1.2* is thus proved. $\square$

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