Construction of anti-de Sitter-like spacetimes using the metric conformal Einstein field equations: the vacuum case

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October 17, 2018

Abstract

We make use of the metric version of the conformal Einstein field equations to construct anti-de Sitter-like spacetimes by means of a suitably posed initial-boundary value problem. The evolution system associated to this initial-boundary value problem consists of a set of conformal wave equations for a number of conformal fields and the conformal metric. This formulation makes use of generalised wave coordinates and allows the free specification of the Ricci scalar of the conformal metric via a conformal gauge source function. We consider Dirichlet boundary conditions for the evolution equations at the conformal boundary and show that these boundary conditions can, in turn, be constructed from the 3-dimensional Lorentzian metric of the conformal boundary and a linear combination of the incoming and outgoing radiation as measured by certain components of the Weyl tensor. To show that a solution to the conformal evolution equations implies a solution to the Einstein field equations we also provide a discussion of the propagation of the constraints for this initial-boundary value problem. The existence of local solutions to the initial-boundary value problem in a neighbourhood of the corner where the initial hypersurface and the conformal boundary intersect is subject to compatibility conditions between the initial and boundary data. The construction described is amenable to numerical implementation and should allow the systematic exploration of boundary conditions.

1 Introduction

Anti-de Sitter-like spacetimes, i.e. spacetimes satisfying the Einstein field equations with a negative Cosmological constant and admitting a timelike conformal boundary, constitute a basic example of solutions to the Einstein field equations which are not globally hyperbolic —see e.g. [31]. As such, they cannot be constructed solely from data on a spacelike hypersurface and require the prescription of some suitable boundary data. In view of the latter, the methods of conformal geometry provide a natural setting for the discussion of an initial-boundary value problem from which anti-de Sitter-like spacetimes can be constructed in a systematic manner. From the conformal point of view, the timelike conformal boundary of the anti-de Sitter-like spacetime has a finite location (described by the vanishing of the conformal factor) so that analysis of the boundary conditions and its relation to initial data can be carried out with local computations.

A first analysis of the initial–boundary value problem for 4-dimensional vacuum anti-de Sitter-like spacetimes by means of conformal methods has been carried out by Friedrich in [19] —see also [20] for further discussion of the admissible adS-like boundary conditions. This seminal work makes use of a conformal representation of the Einstein field equations known as the extended

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conformal Einstein field equations and a gauge based on the properties of curves with good conformal properties (*conformal geodesics*) to set up an initial-boundary value problem for a first order symmetric hyperbolic system of evolution equations. For this type of evolution equations one can use the theory of maximally dissipative boundary conditions as described in [28, 23] to assert the well-posedness of the problem and to ensure the local existence of solutions in a neighbourhood of the intersection of the initial hypersurface with the conformal boundary (the *corner*). The solutions to these evolution equations can be shown, via a further argument, to constitute a solution to the vacuum Einstein field equations with negative Cosmological constant.

Friedrich's analysis identifies a large class of maximally dissipative boundary conditions involving the outgoing and incoming components of the Weyl tensor —as such, they can be thought of as prescribing the relation between these components. These conditions are given in a very specific gauge and thus it is difficult to assert their physical/geometric meaning. However, it is possible to identify a subclass of boundary conditions which can be recast in a covariant form. More precisely, they can be shown to be equivalent to prescribing the conformal class of the metric on the conformal boundary —see [19], also [31]. The question of recasting the whole class of maximally dissipative boundary conditions obtained by Friedrich in a geometric (i.e. covariant) form remains an interesting open problem.

An alternative construction of anti-de Sitter–like spacetimes, which does not use the conformal Einstein field equations and which also hold for spacetimes of dimension greater than four, can be found in [16]. A discussion of global properties of adS-like spacetimes and the issue of their stability can be found in [2].

Numerical simulations involving anti-de Sitter-like spacetimes is a very active area of current research —see e.g. [6, 5, 15, 14] which kick-started some of the current flurry of interest. In particular, in [6] the evolution of the evolution of the spherically symmetric Einstein-scalar field with relective boundary conditions was considered. Different boundary conditions for numerical evolutions of this system have been considered in [1, 12]. Alternative Cauchy-hyperbolic and characteristic formulations of the spherically symmetric Einstein-scalar field system have been discussed in [30, 29].

Friedrich's results offer a natural and systematic approach to the numerical construction of 4-dimensional vacuum anti-de Sitter-like spacetimes. However, the numerical implementation of these results is not straightforward, among other things, because the equations involved are cast in a form which is not standard for the available numerical codes and moreover, there is very little intuition about the behaviour of the gauges used to formulate the equations. A further difficulty of Friedrich's approach is that it cannot readily be extended to include matter fields —see however [25].

In view of the issues raised in the previous paragraph, it is desirable to have a conformal formulation of the initial-boundary value problem for anti de Sitter-like spacetimes which is closer to the language used in numerical simulations and which exploits familiar gauge conditions. In this article we undertake this task. More precisely, we show that using the better know *metric conformal Einstein field equations* it is possible to construct anti-de Sitter-like spacetimes by formulating an initial-boundary value problem for a system of quasilinear wave equations for the conformal fields governing the geometry of the conformal representation of the anti de Sitter-like spacetimes. The partial differential equations (PDE) theory for this type of systems is available in the literature [10, 13]. Dirichlet boundary data for this system of wave equations can be constructed from the prescription of the 3-dimensional (Lorentzian) metric of the conformal boundary and a relation between the incoming and outgoing components of the Weyl tensor akin to Friedrich's maximally dissipative conditions. Our setting makes use of generalised harmonic coordinates. It also contains a further conformal gauge freedom which can be fixed by specifying the value of the Ricci scalar of the conformal representation. The evolution system to be solved can be thought of as the Einstein field equations coupled to a complicated matter model consisting of several tensorial fields —each of which satisfies its own wave equations. This parallel should ease the numerical implementation of the setting.

In addition to the formulation of an initial-boundary value problem, we also study the relation between the solutions to the conformal Einstein field equations and actual solutions to the Einstein field equations. This analysis requires a discussion similar to that of the *propagation of the* *constraints* in which it is shown that a solution to the evolution equations is, in fact, a solution to the original conformal field equations. For this, one has to construct a subsidiary evolution system for the conformal field equations and show that the boundary data prescription for the evolution equations implies trivial boundary data for the subsidiary equations. Fortunately, most of the lengthy calculations required for this discussion are already available in the literature —see e.g. [27].

Our main result, stating the local existence in time of anti-de Sitter-like spacetimes in a neighbourhood of the corner where the initial hypersurface and the conformal boundary intersect, is provided in Theorem 1.

A feature of our analysis is that it can be extended to include tracefree matter fields. This extension is discussed in a companion article [8].

Outline of the article

This article is structured as follows: in Section 2 we provide an overview of the relevant properties of our main technical tool —the so-called metric conformal Einstein field equations. In particular we discuss the structural properties of the wave equations describing the evolution of the conformal fields. We also discuss the gauge fixing for the evolution equations and the key properties of the subsidiary evolution system responsible of the propagation of the constraints. In Section 3 we discuss relevant properties of the constraint equations implied on spacelike or timelike hypersurfaces by the conformal Einstein field equations —the so-called conformal Einstein constraint equations. These constraint equations are both relevant for the construction of suitable initial and boundary initial data. In Section 4 we describe the general set-up of our construction of anti-de Sitter-like spacetimes. In particular, we analyse the construction of suitable boundary data directly from the knowledge of the metric at the conformal boundary. We also study the properties of the compatibility (corner) conditions between initial and boundary initial data required to ensure the existence of solutions to the initial-boundary value problem for the wave equations describing the evolution of the conformal fields. In Section 5 we analyse the issue of the propagation of the constraints —key to establish the relation between solutions to the conformal wave equations and actual solutions to the Einstein field equations. The propagation of the constraints is established through the analysis of a boundary-initial value problem for the subsidiary evolution system. Finally, in Section 6 we summarise our analysis by stating our main result, Theorem 1. Section 7 provides some concluding remarks to our analysis. In Appendix A we provide a discussion of the integrability conditions associated to the metric conformal field equations. These integrability conditions are fundamental to establish a number of general properties of the conformal Einstein field equations.

Conventions

Throughout, the term *spacetime* will be used to denote a 4-dimensional Lorentzian manifold which not necessarily satisfies the Einstein field equations. Moreover, $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ will denote a vacuum spacetime satisfying the Einstein equations with anti-de Sitter-like cosmological constant λ. The signature of the metric in this article will be $(-,+,+,+)$. It follows that $\lambda < 0$. The lowercase Latin letters a, b, $c, \ldots$ are used as abstract spacetime tensor indices while the indices i, j, $k, \ldots$ are abstract indices on the tensor bundle of hypersurfaces of $\tilde{\mathcal{M}}$. The Greek letters μ, ν, $\lambda, \ldots$ will be used as spacetime coordinate indices while α, β, $\gamma, \ldots$ will serve as indices on a hypersurface. Our conventions for the curvature are

$$\nabla_c \nabla_d u^a - \nabla_d \nabla_c u^a = R^a{}_{bcd} u^b.$$

2 The metric conformal Einstein field equations

The basic tool to be used in this article are the metric conformal field equations. This section reviews the properties of this conformal representation of the Einstein field equations that will be used throughout.

In what follows, let $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ denote a spacetime satisfying the vacuum Einstein field equations

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}, \tag{1}$$

where $\tilde{R}_{ab}$ denotes the Ricci tensor of the metric $\tilde{g}_{ab}$. Further, let $(\mathcal{M}, g_{ab})$ denote a spacetime conformally related to $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ so that, in a slight abuse of notation, we have

$$g_{ab} = \Xi^2 \tilde{g}_{ab},$$

where Ξ is some suitable conformal factor Ξ. The set of points of $\mathcal{M}$ for which Ξ vanishes will be called the *conformal boundary.* We use the notation $\mathscr{I}$ to denote the parts of the conformal boundary which are a hypersurface of $\mathcal{M}$.

2.1 Basic properties

In what follows, let ∇_a denote the Levi-Civita connection of the metric g_{ab}, and let $R^a{}_{bcd}$, R_{ab}, R, $C^a{}_{bcd}$ denote, respectively, the associated Riemann tensor, Ricci tensor, Ricci scalar and (conformally invariant) Weyl tensor. In the discussion of the conformal Einstein field equations it is useful to introduce the *Schouten tensor*, defined as

$$L_{ab} \equiv \frac{1}{2}\left(R_{ab} - \frac{1}{6} R g_{ab}\right).$$

Moreover, let

$$s \equiv \frac{1}{4}\nabla^c \nabla_c \Xi + \frac{1}{24} R \Xi, \qquad d^a{}_{bcd} \equiv \Xi^{-1} C^a{}_{bcd}$$

denote the so-called *Friedrich scalar* and the *rescaled Weyl tensor*, respectively.

In terms of the objects defined in the previous paragraph, the *vacuum metric conformal Einstein field equations* are given by:

$$\nabla_a \nabla_b \Xi = -\Xi L_{ab} + s g_{ab}, \tag{2a}$$
$$\nabla_a s = -L_{ac} \nabla^c \Xi, \tag{2b}$$
$$\nabla_c L_{db} - \nabla_d L_{cd} = \nabla_a \Xi d^a{}_{bcd}, \tag{2c}$$
$$\nabla_a d^a{}_{bcd} = 0, \tag{2d}$$
$$6\Xi s - 3\nabla_c \Xi \nabla^c \Xi = \lambda. \tag{2e}$$

Remark 1. Equations (2a)-(2d) will be read as differential conditions on the fields Ξ, s, L_{ab}, $d^a{}_{bcd}$ while equation (2e) will be regarded as a constraint which is satisfied if it holds at a single point by virtue of the other equations —see Lemma 8.1 in [31].

By a solution to the metric conformal Einstein field equations it is understood a collection of fields

$$(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd})$$

satisfying equations (2a)-(2e). The relation between the metric conformal Einstein field equations and the Einstein field equations is given by the following:

Proposition 1. *Let* $(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd})$ *denote a solution to the metric conformal Einstein field equations* (2a)-(2d) *such that* $\Xi \neq 0$ *on an open set* $\mathcal{U} \subset \mathcal{M}$. *If, in addition, equation* (2e) *is satisfied at a point* $p \in \mathcal{U}$, *then the metric*

$$\tilde{g}_{ab} = \Xi^{-2} g_{ab}$$

is a solution to the Einstein field equations (1) *on* $\mathcal{U}$.

A proof of the above proposition is given in [31] —see Proposition 8.1 in that reference.

We also recall that the causal character of $\mathscr{I}$ is determined by the sign of the Cosmological constant. More precisely, one has that:

Proposition 2. *Suppose that the Friedrich scalar* s *is regular on* $\mathscr{I}$. *Then* $\mathscr{I}$ *is a null, spacelike or timelike hypersurface of* $\mathcal{M}$, *respectively, depending on whether* $\lambda = 0$, $\lambda > 0$ *or* $\lambda < 0$.

This result follows directly from evaluation of equation (2e) on $\mathscr{I}$ and recalling that $\nabla_a \Xi$ is normal to this hypersurface.

2.2 Wave equations for the conformal fields

In [27] it has been shown how the conformal Einstein field equations (2a)-(2d) imply a system of *geometric* wave equations for the components of the fields (Ξ, s, L_{ab}, $d^a{}_{bcd}$). In the subsequent discussion it will be convenient to split the Schouten tensor into a tracefree part and a pure-trace part (the Ricci scalar). Accordingly, one defines the *tracefree Ricci tensor* as

$$\Phi_{ab} \equiv \frac{1}{2}\big(R_{ab} - \frac{1}{4}Rg_{ab}\big),$$

so that the Schouten tensor can be expressed as

$$L_{ab} = \Phi_{ab} + \frac{1}{24}Rg_{ab}. \tag{3}$$

In terms of the above field the main result in [27] can be expressed as:

Proposition 3. *Any solution* (Ξ, s, L_{ab}, $d^a{}_{bcd}$) *to the conformal Einstein field equations* (2a)-(2d) *satisfies the equations*

$$\square\Xi = 4s - \frac{1}{6}\Xi R, \tag{4a}$$

$$\square s = \Xi\Phi_{ab}\Phi^{ab} - \frac{1}{6}sR + \frac{1}{144}\Xi R^2 - \frac{1}{6}\nabla_a R\nabla^a\Xi, \tag{4b}$$

$$\square\Phi_{ab} = 4\Phi_a{}^c\Phi_{bc} - g_{ab}\Phi_{cd}\Phi^{cd} - 2\Xi d_{acbd}\Phi^{cd} + \frac{1}{3}R\Phi_{ab} - \frac{1}{24}g_{ab}\nabla_c\nabla^c R + \frac{1}{6}\nabla_a\nabla_b R, \tag{4c}$$

$$\square d_{abcd} = 2\Xi d_a{}^e{}_d{}^f d_{becf} - 2\Xi d_a{}^e{}_c{}^f d_{bedf} - 2\Xi d_{ab}{}^{ef} d_{cedf} + \frac{1}{2}d_{abcd}R. \tag{4d}$$

Remark 2. The above wave equations are geometric, in the sense that they hold independently of the choice of coordinate system. However, as they stand they are not yet satisfactory second order evolution equations to which one can apply the theory of partial differential equations. For this one has to provide a prescription of the Ricci scalar and introduce suitable coordinates. These issues are discussed in the following subsections.

Remark 3. The wave equations (4a)-(4d) need to be supplemented with an equation for the components of the metric tensor g_{ab}. This equation is given by the definition of the tracefree Ricci tensor, equation (3), rewritten in the form

$$R_{ab} = 2\Phi_{ab} + \frac{1}{4}Rg_{ab}, \tag{5}$$

where R_{ab}, and Φ_{ab} are regarded as independent objects —the former given through the classical expression in terms of second order partial derivatives of the components of the metric tensor while the latter as the field satisfying equations (4a)-(4d).

2.3 Gauge considerations

The conformal Einstein field equations possess both a coordinate and a conformal freedom which can be exploited to cast the geometric wave equations (4a)-(4d) as satisfactory hyperbolic evolution equations.

2.3.1 Conformal gauge source functions

In the following, the Ricci scalar R of the metric g_{ab} will be regarded as a *conformal gauge source* specifying the representative in the conformal class $[\tilde{\boldsymbol{g}}]$ one is working with. Recall that given two conformally related metrics g_{ab} and g'_{ab} such that $g'_{ab} = \vartheta^2 g_{ab}$, the respective Ricci scalars are related to each other via

$$R\vartheta - R'\vartheta^3 = 6\nabla^c\nabla_c\vartheta.$$

If the values of R and R' are prescribed, the above transformation law can be recast as a wave equation for the conformal factor relating the two metrics. Namely, one has that

$$\square\vartheta - \frac{1}{6}R\vartheta = -\frac{1}{6}R'\vartheta^3.$$

Given suitable initial data for this wave equation, it can always be solved locally. Accordingly, it is always possible to find (locally) a conformal rescaling such that the metric g'_{ab} has a prescribed Ricci scalar R'.

Remark 4. Following the previous discussion, in what follows the Ricci scalar of the metric g_{ab} is regarded as a prescribed function $\mathcal{R}(x)$ of the coordinates and one writes

$$R = \mathcal{R}(x).$$

2.3.2 Generalised harmonic coordinates and the reduced Ricci operator

Given general coordinates $x = (x^\mu)$, the components of the Ricci tensor R_{ab} can be explicitly written in terms of the components of the metric tensor g_{ab} and its first and second partial derivatives as

$$R_{\mu\nu} = -\frac{1}{2}g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} + g_{\sigma(\mu}\nabla_{\nu)}\Gamma^\sigma + g_{\lambda\rho}g^{\sigma\tau}\Gamma^\lambda{}_{\sigma\mu}\Gamma^\rho{}_{\tau\nu} + 2\Gamma^\sigma{}_{\lambda\rho}g^{\lambda\tau}g_{\sigma(\mu}\Gamma^\rho{}_{\nu)\tau},$$

with

$$\Gamma^\nu{}_{\mu\lambda} = \frac{1}{2}g^{\nu\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}),$$

and where one has defined the *contracted Christoffel symbols*

$$\Gamma^\nu \equiv g^{\mu\lambda}\Gamma^\nu{}_{\mu\lambda}.$$

A direct computation then gives

$$\square x^\mu = -\Gamma^\mu.$$

In what follows, we introduce *coordinate gauge source functions* $\mathcal{F}^\mu(x)$ to prescribe the value of the contracted Christoffel symbols via the condition

$$\Gamma^\mu = \mathcal{F}^\mu(x),$$

so that the coordinates $x = (x^\mu)$ satisfy the *generalised wave coordinate condition*

$$\square x^\mu = -\mathcal{F}^\mu(x). \tag{6}$$

Associated to the latter coordinate condition one then defines the *reduced Ricci operator* $\mathscr{R}_{\mu\nu}[\boldsymbol{g}]$ as

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] \equiv R_{\mu\nu} - g_{\sigma(\mu}\nabla_{\nu)}\Gamma^\sigma + g_{\sigma(\mu}\nabla_{\nu)}\mathcal{F}^\sigma(x). \tag{7}$$

More explicitly, one has that

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] = -\frac{1}{2}g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} - g_{\sigma(\mu}\nabla_{\nu)}\mathcal{F}^\sigma(x) + g_{\lambda\rho}g^{\sigma\tau}\Gamma^\lambda{}_{\sigma\mu}\Gamma^\rho{}_{\tau\nu} + 2\Gamma^\sigma{}_{\lambda\rho}g^{\lambda\tau}g_{\sigma(\mu}\Gamma^\rho{}_{\nu)\tau}.$$

Thus, by choosing coordinates satisfying the *generalised wave coordinates condition* (6), the *unphysical Einstein equation* (5) takes the form

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] = 2\Phi_{\mu\nu} + \frac{1}{4}\mathcal{R}(x)g_{\mu\nu}. \tag{8}$$

Assuming that the components $\Phi_{\mu\nu}$ are known, the latter is a quasilinear wave equation for the components of the metric tensor.

2.3.3 The reduced wave operator

While equations (4a) and (4b) provide satisfactory wave equations for the scalar fields Ξ and s independently of the choice of coordinates, this is not the case for equations (4c) and (4d). The reason for this is that in these equations the wave operator $\square$ is acting on tensors, and thus, the terms $\square\Phi_{ab}$ and $\square d_{abcd}$, when expressed in a given coordinate system $x = (x^\mu)$, involve derivatives of Christoffel symbols —and consequently, second order derivatives of the metric tensor. This is a problem in situations, like the one considered here, where the metric is an unknown in the problem as the presence of these derivatives in the operator destroys the hyperbolicity of the system.

In what follows, it will be shown how the generalised wave coordinate condition (6) can be used to reduce the geometric wave operator $\square$ to a second order hyperbolic operator. To motivate the procedure consider a covector ω_a with components ω_μ with respect to a coordinate system $x = (x^\mu)$ satisfying condition (6) for some choice of coordinate gauge source functions $\mathcal{F}^\mu(x)$. A direct computation using the expression of the covariant derivative in terms of Christoffel symbols yields

$$\begin{aligned}\square\omega_\lambda &\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\omega_\lambda \\ &= g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda - g^{\mu\nu}\partial_\mu\Gamma^\sigma{}_{\nu\lambda}\omega_\sigma + f_\lambda(g,\partial g,\omega,\partial\omega),\end{aligned}$$

where $f_\lambda(g,\partial g,\omega,\partial\omega)$ denotes an expression depending on the components $g_{\mu\nu}$, ω_μ and their first order partial derivatives. Now, recall the classical expression for the components of the Riemann tensor in terms of the Christoffel symbols and their derivatives,

$$R^\sigma{}_{\mu\lambda\nu} = \partial_\lambda\Gamma^\sigma{}_{\nu\mu} - \partial_\nu\Gamma^\sigma{}_{\lambda\mu} + \Gamma^\sigma{}_{\lambda\tau}\Gamma^\tau{}_{\nu\mu} - \Gamma^\sigma{}_{\nu\tau}\Gamma^\tau{}_{\lambda\mu}$$

so that

$$\begin{aligned}R^\sigma{}_\lambda &= g^{\mu\nu}R^\sigma{}_{\mu\lambda\nu} \\ &= g^{\mu\nu}\partial_\lambda\Gamma^\sigma{}_{\nu\mu} - g^{\mu\nu}\partial_\nu\Gamma^\sigma{}_{\lambda\mu} + g^{\mu\nu}\Gamma^\sigma{}_{\lambda\tau}\Gamma^\tau{}_{\nu\mu} - g^{\mu\nu}\Gamma^\sigma{}_{\nu\tau}\Gamma^\tau{}_{\lambda\mu}.\end{aligned}$$

Making use of this coordinate expression on obtains

$$\begin{aligned}\square\omega_\lambda &= g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda + \big(R^\sigma{}_\lambda - g^{\mu\nu}\partial_\lambda\Gamma^\sigma{}_{\nu\mu}\big)\omega_\sigma + f_\lambda(\boldsymbol{g},\partial\boldsymbol{g},\boldsymbol{\omega},\partial\boldsymbol{\omega}) \\ &= g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda + \big(R^\sigma{}_\lambda - \partial_\lambda\Gamma^\sigma\big)\omega_\sigma + f_\lambda(\boldsymbol{g},\partial\boldsymbol{g},\boldsymbol{\omega},\partial\boldsymbol{\omega}) \\ &= g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda + \big(R_{\tau\lambda} - g_{\sigma\tau}\partial_\lambda\Gamma^\sigma\big)\omega^\tau + f_\lambda(\boldsymbol{g},\partial\boldsymbol{g},\boldsymbol{\omega},\partial\boldsymbol{\omega}),\end{aligned}$$

and finally

$$\square\omega_\lambda = g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda + \big(R_{\tau\lambda} - g_{\sigma\tau}\nabla_\lambda\Gamma^\sigma\big)\omega^\tau + f_\lambda(\boldsymbol{g},\partial\boldsymbol{g},\boldsymbol{\omega},\partial\boldsymbol{\omega}). \tag{9}$$

Making the formal replacements

$$R_{\mu\nu} \mapsto 2\Phi_{\mu\nu} + \frac{1}{4}\mathcal{R}(x)g_{\mu\nu}, \qquad \Gamma^\mu \mapsto \mathcal{F}^\mu(x),$$

in equation (9), one defines the *reduced wave operator* $\blacksquare$, acting on the components ω_μ as

$$\blacksquare\omega_\lambda \equiv g^{\mu\nu}\partial_\mu\partial_\nu\omega_\lambda + \left(2\Phi_{\tau\lambda} + \frac{1}{4}\mathcal{R}(x)g_{\tau\lambda} - g_{\sigma\tau}\nabla_\lambda\mathcal{F}^\sigma(x)\right)\omega^\tau + f_\lambda(\boldsymbol{g},\partial\boldsymbol{g},\boldsymbol{\omega},\partial\boldsymbol{\omega}), \tag{10}$$

where $f_\lambda(\boldsymbol{g},\partial\boldsymbol{g},\boldsymbol{\omega},\partial\boldsymbol{\omega})$ denotes lower order terms whose explicit form will not be required. In fact, from the previous discussion it follows that one can write

$$\blacksquare\omega_\lambda = \square\omega_\lambda + \left((2\Phi_{\tau\lambda} + \frac{1}{4}\mathcal{R}(x)g_{\tau\lambda} - R_{\tau\lambda}) - g_{\sigma\tau}\nabla_\lambda(\mathcal{F}^\sigma(x) - \Gamma^\sigma)\right)\omega^\tau.$$

A similar construction for covariant tensors of arbitrary rank leads to the following:

Definition 1. *The reduced wave operator* $\blacksquare$ *acting on a covariant tensor field* $T_{\lambda\cdots\rho}$ *is defined as*

$$\blacksquare T_{\lambda\cdots\rho} \equiv \square T_{\lambda\cdots\rho} + \left((2\Phi_{\tau\lambda} + \frac{1}{4}\mathcal{R}(x) g_{\tau\lambda} - R_{\tau\lambda}) - g_{\sigma\tau}\nabla_\lambda(\mathcal{F}^\sigma(x) - \Gamma^\sigma) \right) T^\tau{}_{\cdots\rho} + \cdots$$
$$\cdots + \left((2\Phi_{\tau\rho} + \frac{1}{4}\mathcal{R}(x) g_{\tau\rho} - R_{\tau\rho}) - g_{\sigma\tau}\nabla_\rho(\mathcal{F}^\sigma(x) - \Gamma^\sigma) \right) T_{\lambda\cdots}{}^\tau$$

where $\square \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. *The action of* $\blacksquare$ *on a scalar* ϕ *is simply given by*

$$\blacksquare\phi \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\phi.$$

Remark 5. The operator $\blacksquare$ provides a proper second order hyperbolic operator —in contrast to $\square$. Accordingly, when working in generalised harmonic coordinates, all the second order derivatives of the metric tensor can be removed from the principal part of the evolution equations (4c) and (4d).

2.3.4 Summary: gauge reduced evolution equations

The discussion of the previous sections leads us to consider the following *gauge reduced* system of evolution equations for the components of the conformal fields Ξ, s, Φ_{ab}, d_{abcd} and g_{ab} with respect to coordinates $x = (x^\mu)$ satisfying the generalised wave coordinate condition (6):

$$\blacksquare\Xi = 4s - \frac{1}{6}\Xi\mathcal{R}(x), \tag{11a}$$

$$\blacksquare s = \Xi\Phi_{\mu\nu}\Phi^{\mu\nu} - \frac{1}{6}s\mathcal{R}(x) + \frac{1}{144}\Xi\mathcal{R}(x)^2 - \frac{1}{6}\nabla_\mu\mathcal{R}(x)\nabla^\mu\Xi, \tag{11b}$$

$$\begin{aligned}\blacksquare\Phi_{\mu\nu} = {} & 4\Phi_\mu{}^\lambda\Phi_{\nu\lambda} - g_{\mu\nu}\Phi_{\lambda\rho}\Phi^{\lambda\rho} - 2\Xi d_{\mu\lambda\nu\rho}\Phi^{\lambda\rho} \\ & + \frac{1}{3}\mathcal{R}(x)\Phi_{\mu\nu} - \frac{1}{24}g_{\mu\nu}\nabla_\lambda\nabla^\lambda\mathcal{R}(x) + \frac{1}{6}\nabla_\mu\nabla_\nu\mathcal{R}(x),\end{aligned} \tag{11c}$$

$$\blacksquare d_{\mu\nu\lambda\rho} = 2\Xi d_\mu{}^\sigma{}_\rho{}^\tau d_{\nu\sigma\lambda\tau} - 2\Xi d_\mu{}^\sigma{}_\lambda{}^\tau d_{\nu\sigma\rho\tau} - 2\Xi d_{\mu\nu}{}^{\sigma\tau} d_{\lambda\sigma\rho\tau} + \frac{1}{2}d_{\mu\nu\lambda\rho}\mathcal{R}(x), \tag{11d}$$

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] = 2\Phi_{\mu\nu} + \frac{1}{4}\mathcal{R}(x)g_{\mu\nu}. \tag{11e}$$

Remark 6. The reduced system (11a)-(11e) constitutes a system of quasilinear wave equations for the fields Ξ, s, $\Phi_{\mu\nu}$, $d_{\mu\nu\lambda\rho}$ and $g_{\mu\nu}$. More explicitly, one has that

$$\begin{aligned}
g^{\sigma\tau}\partial_\sigma\partial_\tau\Xi &= X\big(\boldsymbol{g}, \partial\boldsymbol{g}, \Xi, s, \mathcal{R}(x)\big),\\
g^{\sigma\tau}\partial_\sigma\partial_\tau s &= S\big(\boldsymbol{g}, \partial\boldsymbol{g}, \Xi, \partial\Xi, s, \boldsymbol{\Phi}, \mathcal{R}(x), \partial\mathcal{R}(x)\big),\\
g^{\sigma\tau}\partial_\sigma\partial_\tau\Phi_{\mu\nu} &= F_{\mu\nu}\big(\boldsymbol{g}, \partial\boldsymbol{g}, \Xi, \boldsymbol{\Phi}, \boldsymbol{d}, \mathcal{R}(x), \partial^2\mathcal{R}(x)\big),\\
g^{\sigma\tau}\partial_\sigma\partial_\tau d_{\mu\nu\lambda\rho} &= D_{\mu\nu\lambda\rho}\big(\boldsymbol{g}, \partial\boldsymbol{g}, \Xi, \boldsymbol{d}, \mathcal{R}(x)\big),\\
g^{\sigma\tau}\partial_\sigma\partial_\tau g_{\mu\nu} &= G_{\mu\nu}\big(\boldsymbol{g}, \partial\boldsymbol{g}, \boldsymbol{\Phi}, \mathcal{R}(x)\big),
\end{aligned}$$

where X, S, $F_{\mu\nu}$, $D_{\mu\nu\lambda\rho}$ and $G_{\mu\nu}$ are polynomial expressions of their arguments. Strictly speaking, the system is a system of wave equations only if $g_{\mu\nu}$ is known to be Lorentzian. This will be case in a perturbative setting or close to an initial hypersurface where initial data can be prescribed to this effect. The local existence theory of initial-boundary value problems for systems of quasilinear differential equations of the above type with Dirichlet boundary data can be found in e.g. [10, 13].

2.4 The subsidiary evolution equations

In order to analyse the relation between solutions to the system of geometric wave equations (4a)-(4d) and the conformal Einstein field equations (2a)-(2e) one needs to construct a subsidiary system of equations encoding the evolution of these equations. Accordingly, one defines the *zero-quantities*

$$\Upsilon_{ab} \equiv \nabla_a \nabla_b \Xi + \Xi L_{ab} - s g_{ab}, \tag{12a}$$
$$\Theta_a \equiv \nabla_a s + L_{ac} \nabla^c \Xi, \tag{12b}$$
$$\Delta_{cdb} \equiv \nabla_c L_{db} - \nabla_d L_{cd} + \nabla_a \Xi d^a{}_{bcd}, \tag{12c}$$
$$\Lambda_{bcd} \equiv \nabla_a d^a{}_{bcd}. \tag{12d}$$

In terms of the latter, the conformal Einstein field equations (2a)-(2d) can be expressed as

$$\Upsilon_{ab} = 0, \qquad \Theta_a = 0, \qquad \Delta_{cdb} = 0, \qquad \Lambda_{bcd} = 0. \tag{13}$$

A lengthy computation, best done using computer algebra, leads to the following:

Proposition 4. *Assume that the conformal fields Ξ, s, L_{ab} and d_{abcd} satisfy the geometric wave equations* (4a)-(4d). *Then the zero-quantities Θ_a, Υ_{ab}, Δ_{abc} and Λ_{abc} satisfy a system of geometric wave equations of the form*

$$\square \Theta_a = H_a(\boldsymbol{\Theta}, \boldsymbol{\Upsilon}, \boldsymbol{\Delta}, \boldsymbol{\Lambda}), \tag{14a}$$
$$\square \Upsilon_{ab} = H_{ab}(\boldsymbol{\Theta}, \boldsymbol{\Upsilon}, \boldsymbol{\nabla\Upsilon}, \boldsymbol{\Delta}), \tag{14b}$$
$$\square \Delta_{abc} = H_{abc}(\boldsymbol{\Delta}, \boldsymbol{\Lambda}), \tag{14c}$$
$$\square \Lambda_{abc} = L_{abc}(\boldsymbol{\Theta}, \boldsymbol{\Upsilon}, \boldsymbol{\Delta}, \boldsymbol{\Lambda}), \tag{14d}$$

where H_a, H_{ab}, H_{abc} and L_{abc} are homogeneous expressions of their arguments.

The original proof of this result was given in [27]. An alternative derivation, along with several properties of the zero–quantities, can be found in Appendix A.

Remark 7. In practice, the geometric wave equations (14a)-(14d) are replaced by standard wave equations for the components of the zero fields by exchanging the wave operator $\square$ by the reduced wave operator $\blacksquare$.

3 The conformal Einstein constraint equations

In order to formulate an initial-boundary value problem for the wave equations (4a)-(4d) we will need the constraint equations implied by the conformal Einstein field equations (2a)-(2e) on (spacelike and timelike) hypersurfaces of the unphysical spacetime $(\mathcal{M}, g_{ab})$. These equations were first discussed in [17]. A detailed discussion of their derivation and basic properties can be found in [31], Chapter 11.

3.1 The basic expression of the conformal Einstein constraint equations

Let $\mathcal{S}$ denote a (spacelike or timelike) hypersurface of the unphysical spacetime $(\mathcal{M}, g_{ab})$ with unit normal n_a. Furthermore, let

$$\epsilon \equiv n_a n^a,$$

so that $\epsilon = 1$ if $\mathcal{S}$ is timelike and $\epsilon = -1$ if it is spacelike. The projector to $\mathcal{S}$ is defined as

$$h_{ab} \equiv g_{ab} - \epsilon n_a n_b.$$

The extrinsic curvature of $\mathcal{S}$ is defined as

$$K_{ab} = h_a{}^c h_b{}^d \nabla_c n_d.$$

The restriction of the conformal factor Ξ to the hypersurface will be denoted by Ω.

In the following let

$$\Sigma, \quad s, \quad h_{ij} \quad L_i, \quad L_{ij}, \quad d_{ij}, \quad d_{ijk}, \quad d_{ijkl}$$

denote, respectively, the pull-backs of

$$n^a \nabla_a \Xi, \quad s, \quad g_{ab}, \quad n^c h_a{}^d L_{cd}, \quad h_a{}^c h_b{}^d L_{cd},$$
$$n^b n^d h_e{}^a h_f{}^c d_{abcd}, \quad n^b h_e{}^a h_f{}^c h_g{}^d d_{abcd}, \quad h_e{}^a h_f{}^b h_g{}^c h_h{}^d d_{abcd}$$

to $\mathcal{S}$.

Remark 8. In particular, h_{ij} corresponds is the 3-metric induced by g_{ab} on $\mathcal{S}$. Similarly, we will denote by K_{ij} the pull-back of K_{ab} and $K = h^{ij} K_{ij}$. The metric h_{ij} will be Lorentzian or Riemannian depending on whether S is timelike or spacelike.

Remark 9. The fields d_{ij} and d_{ijk} encode, respectively, the *electric* and *magnetic parts* of the rescaled Weyl tensor d_{abcd} with respect to the normal n_a. It can be verified that

$$d_i{}^i = 0, \quad d_{ij} = d_{ji}, \quad d_{ijk} = -d_{ikj}, \quad d_{[ijk]} = 0,$$
$$d_{ijkl} = 2\epsilon(h_{i[l} d_{k]j} + h_{j[k} d_{l]i}).$$

The magnetic part is more commonly encoded in a symmetric traceless tensor of rank 2 defined as

$$d^*_{ij} \equiv \frac{1}{2} \epsilon_j{}^{kl} d_{ikl}$$

where ϵ_{ijk} is the volume form induced on $\mathcal{S}$ by h_{ij}.

In terms of the above fields, a long computation shows that the conformal Einstein field equations (2a)-(2e) imply on the hypersurface $\mathcal{S}$ the *conformal Einstein constraint equations*

$$D_i D_j \Omega = -\epsilon \Sigma K_{ij} - \Omega L_{ij} + s h_{ij}, \tag{15a}$$
$$D_i \Sigma = K_i{}^k D_k \Omega - \Omega L_i, \tag{15b}$$
$$D_i s = -\epsilon L_i \Sigma - L_{ik} D^k \Omega, \tag{15c}$$
$$D_i L_{jk} - D_j L_{ik} = -\epsilon \Sigma d_{kij} + D^l \Omega d_{lkij} - \epsilon (K_{ik} L_j - K_{jk} L_i), \tag{15d}$$
$$D_i L_j - D_j L_i = D^l \Omega d_{lij} + K_i{}^k L_{jk} - K_j{}^k L_{ik}, \tag{15e}$$
$$D^k d_{kij} = \epsilon \big(K^k{}_i d_{jk} - K^k{}_j d_{ik}\big), \tag{15f}$$
$$D^i d_{ij} = K^{ik} d_{ijk}, \tag{15g}$$
$$\lambda = 6\Omega s - 3\epsilon \Sigma^2 - 3 D_k \Omega D^k \Omega. \tag{15h}$$

These equations are supplemented by the Codazzi-Mainardi and Gauss-Codazzi equations which, respectively, take the following form:

$$D_j K_{ki} - D_k K_{ji} = \Omega d_{ijk} + h_{ij} L_k - h_{ik} L_j, \tag{16a}$$
$$l_{ij} = -\epsilon \Omega d_{ij} + L_{ij} + \epsilon \left(K \big(K_{ij} - \frac{1}{4} K h_{ij}\big) - K_{ki} K_j{}^k + \frac{1}{4} K_{kl} K^{kl} h_{\boldsymbol{ij}} \right), \tag{16b}$$

where the Schouten tensor of h_{ij} is defined as

$$l_{ij} \equiv r_{ij} - \frac{1}{4} r h_{ij}.$$

Here, r_{ij} and r are, respectively, the Ricci tensor and scalar of the metric h_{ij}.

3.2 The conformal constraints on the conformal boundary

The conformal Einstein constraint equations simplify considerably when they are evaluated on an hypersurface corresponding to the conformal boundary of a spacetime, in which case Ω vanishes identically. If the conformal boundary is timelike ($\epsilon = 1$) one has the following system:

$$
\begin{aligned}
&s\ell_{ij} \simeq \not{\Sigma}\not{K}_{ij}, && \text{(17a)}\\
&\not{D}_i \not{\Sigma} \simeq 0, && \text{(17b)}\\
&\not{D}_i s \simeq -\not{L}_i \not{\Sigma}, && \text{(17c)}\\
&\not{D}_i \not{L}_{jk} - \not{D}_j L_{ik} \simeq -\not{\Sigma}\not{d}_{kij} + (\not{K}_{jk}\not{L}_i - \not{K}_{ik}\not{L}_j), && \text{(17d)}\\
&\not{D}_i \not{L}_j - \not{D}_j \not{L}_i \simeq \not{K}_i{}^k \not{L}_{jk} - \not{K}_j{}^k \not{L}_{ik}, && \text{(17e)}\\
&\not{D}^k \not{d}_{kij} \simeq \not{K}^k{}_j \not{d}_{ik} - \not{K}^k{}_i \not{d}_{jk}, && \text{(17f)}\\
&\not{D}^i \not{d}_{ij} \simeq \not{K}^{ik} \not{d}_{ijk}, && \text{(17g)}\\
&\lambda \simeq -3\not{\Sigma}^2, && \text{(17h)}\\
&\not{D}_j \not{K}_{ki} - \not{D}_k \not{K}_{ji} \simeq \ell_{ij}\not{L}_k - \ell_{ik}\not{L}_j, && \text{(17i)}\\
&\not{l}_{ij} \simeq \not{L}_{ij} + \not{K}\big(\not{K}_{ij} - \frac{1}{4}\not{K}\ell_{ij}\big) - \not{K}_{ki}\not{K}_j{}^k + \frac{1}{4}\not{K}_{kl}\not{K}^{kl}\ell_{ij}, && \text{(17j)}
\end{aligned}
$$

where $\simeq$ denotes that the equality holds on the conformal boundary and ℓ_{ij} denotes the intrinsic (Lorentzian) 3-metric on $\mathscr{I}$. Moreover, we use the notation $/$ to indicate that the quantities are obtained from a 3+1 split with respect to the (timelike) conformal boundary. In particular, $\not{D}_i$ denotes the Levi-Civita connection of the Lorentzian metric ℓ_{ij}. *This notation will be used in the rest of the article.*

In [18] a procedure to solve the conformal constraints on the conformal boundary has been given. The key observation is to identify the scalar s as gauge dependent quantity and the 3-metric ℓ_{ij} on $\mathscr{I}$ as free data. Instead of directly working with s it is more convenient to consider a scalar $\varkappa$ such that

$$s \simeq \not{\Sigma}\varkappa.$$

One has then that:

Proposition 5. *Given a 3-dimensional Lorentzian metric ℓ_{ij}, a ℓ-divergencefree and tracefree field $\not{d}_{ij}$ and a smooth function $\varkappa$, then the fields*

$$
\begin{aligned}
&\not{\Sigma} \simeq \sqrt{\frac{|\lambda|}{3}}, && \text{(18a)}\\
&s \simeq \not{\Sigma}\varkappa, && \text{(18b)}\\
&\not{K}_{ij} \simeq \varkappa\ell_{ij}, && \text{(18c)}\\
&\not{L}_i \simeq -\not{D}_i\varkappa, && \text{(18d)}\\
&\not{L}_{ij} \simeq \not{l}_{ij} - \frac{1}{2}\varkappa^2\ell_{ij}, && \text{(18e)}\\
&\not{d}_{ijk} \simeq -\not{\Sigma}^{-1} y_{ijk}, && \text{(18f)}
\end{aligned}
$$

where

$$y_{ijk} \equiv \not{D}_j \not{l}_{ki} - \not{D}_k \not{l}_{ji}$$

is the Cotton tensor of ℓ_{ij}, constitute a solution to the conformal constraint equations (17a)-(17j) *with $\epsilon = 1$ and $\Omega = 0$.*

A proof of this result can be found in [31], Section 11.4.4. We will also require the following partial converse the previous result:

Proposition 6. *Assume one has a timelike hypersurface $\mathcal{T}$ of a spacetime $(\mathcal{M}, g_{ab})$ such that conditions* (18c)-(18e) *hold. If, in addition, $\Omega = 0$ on some fiduciary spacelike hypersurface $\mathcal{C}_\star$ of $\mathcal{T}$ then one has that*

$$\Omega = 0 \quad \textit{on} \quad \mathcal{T}.$$

Proof. Assume first that $\varkappa \neq 0$ on $\mathcal{C}_\star$. The substitution of expressions (18c)-(18e) into the general conformal constraint equations (15a) and (15c) yields the relations

$$\not{D}_i \not{D}_j \Omega = -\Omega\big(\not{l}_{ij} - \frac{1}{2}\varkappa^2 \ell_{ij}\big), \tag{19a}$$

$$\varkappa \not{D}_i \Omega = \Omega \not{D}_i \varkappa. \tag{19b}$$

Taking the trace of equation (19a) one obtains the wave equation

$$\square_\ell \Omega = -\frac{1}{4}\big(\not{r} - 6\varkappa^2\big)\Omega \tag{20}$$

on $\mathcal{T}$, where $\square_\ell \equiv \ell^{ij}\not{D}_i\not{D}_j$. We now consider the condition $\Omega = 0$ on $\mathcal{C}_\star$ as initial data for equation (20). We complement this initial condition with $\not{D}_i\Omega = 0$ on $\mathcal{C}_\star$ which follows from condition (19b). It follows from the homogeneity of equation (20) and the uniqueness of solutions to wave equations of this form that $\Omega = 0$ on $\mathcal{T}$.

To deal with the case $\varkappa = 0$ we observe that it is always possible to carry out a rescaling $\Xi \mapsto \Xi' \equiv \vartheta\Xi$ of the spacetime conformal factor Ξ with $\vartheta \simeq 1$ and $\mathbf{d}\vartheta \neq 0$ such that if $s \not\simeq 0$ on $\mathscr{I}$ then $s' \simeq 0$ —see [31] Section 11.4.4, page 268. Thus, if $\varkappa \not\simeq 0$ initially, then using the above rescaling and taking into account relation (18b) for s', it follows that $\varkappa' \simeq 0$. The rescaling $\Xi \mapsto \Xi' \equiv \vartheta\Xi$ does not change the value of Ξ on $\mathcal{T}$ —accordingly one has that $\Omega = 0$ on $\mathcal{T}$ even if $\varkappa = 0$. $\square$

3.3 Solutions to the conformal constraints on a spacelike hypersurface

In addition to analysing the conformal constraint equations on a timelike hypersurface corresponding to the conformal boundary of the spacetime, we will also need to consider solutions to the constraints (15a)-(16b) on spacelike hypersurfaces. These solutions provide part of the initial data for the wave equations (11a)-(11d).

The conformal constraint equations (15a)-(16b) with $\epsilon = -1$ can be combined to obtain the *conformal Hamiltonian* and *momentum constraints*

$$\lambda = -\frac{1}{2}\Omega^2 K_{ij}K^{ij} + \frac{1}{2}\Omega^2 K^2 + \frac{1}{2}\Omega^2 r - 2\Omega K\Sigma + 3\Sigma^2 - 3D_i\Omega D^i\Omega + 2\Omega D_i D^i\Omega, \tag{21a}$$

$$\Omega D^j K_{ij} - \Omega D_i K = 2K_{ij}D^j\Omega - 2D_i\Sigma. \tag{21b}$$

For a solution of the above equations it will be understood a collection of fields $(\Omega, h_{ij}, K_{ij}, \Sigma)$ satisfying them. The collection $(\Omega, h_{ij}, K_{ij}, \Sigma)$ constitutes the *basic data* from which the rest of the initial data set for the conformal wave equations (11a)-(11e) can be computed. Indeed, a calculation shows that:

$$s = \frac{1}{3}\bigg(\Delta\Omega + \frac{1}{4}\Omega\big(r - K_{ij}K^{ij} + K^2\big) - \Sigma K\bigg), \tag{22a}$$

$$L_{ij} = \frac{1}{\Omega}\bigg(- D_i D_j\Omega + \Sigma K_{ij} + s h_{ij}\bigg), \tag{22b}$$

$$L_i = \frac{1}{\Omega}\big(K_i{}^k D_k\Omega - D_i\Sigma\big), \tag{22c}$$

$$d_{ij} = \frac{1}{\Omega}\bigg(- L_{ij} + l_{ij} + \big(K\big(K_{ij} - \frac{1}{4}Kh_{ij}\big) - K_{ki}K_j{}^k + \frac{1}{4}K_{kl}K^{kl}h_{ij}\big)\bigg), \tag{22d}$$

$$d_{ijk} = \frac{1}{\Omega}\big(D_j K_{ki} - D_k K_{ji} + h_{ik}L_j - h_{ij}L_k\big). \tag{22e}$$

Observe that the above expressions are formally singular at the points where $\Omega = 0$. This observation leads to the following:

Definition 2 (anti-de Sitter-like initial data). *For an anti-de Sitter initial data set it is understood a 3-manifold $\mathcal{S}_\star$ with boundary $\partial\mathcal{S}_\star \approx \mathbb{S}^2$ together with a collection of smooth fields $(\Omega, h_{ij}, K_{ij}, \Sigma)$ such that:*

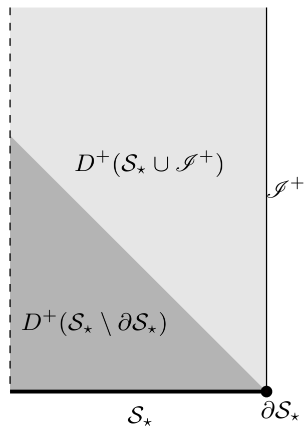

Figure 1: Penrose diagram of the set-up for the construction of anti-de Sitter-like spacetimes as described in the main text. Initial data prescribed on $\mathcal{S}_\star \setminus \partial\mathcal{S}_\star$ allows to recover the dark shaded region $D^+(\mathcal{S}_\star \setminus \partial\mathcal{S}_\star)$. In order to recover $D^+(\mathcal{S}_\star \cup \mathscr{I}^+)$ it is necessary to prescribe boundary data on $\mathscr{I}^+$. Notice that $D^+(\mathcal{S}_\star \cup \mathscr{I}^+) = J^+(\mathcal{S}_\star)$.

(i) $\Omega > 0$ *on* $\operatorname{int}\mathcal{S}_\star$*;*

(ii) $\Omega = 0$ *and* $|\mathrm{d}\Omega|^2 = \Sigma^2 - \frac{1}{3}\lambda > 0$ *on* $\partial\mathcal{S}_\star$*;*

(iii) the fields s*,* L_{ij}*,* L_i*,* d_{ij} *and* d_{ijk} *computed from relations* (22a)-(22e) *extend smoothly to* $\partial\mathcal{S}_\star$*.*

Remark 10. Anti-de Sitter-like initial data sets are closely related to so-called hyperboloidal data sets for Minkowski-like spacetimes —see [24]. By means of this correspondence it is possible to adapt the existence results for hyperboloidal initial data sets in [4, 3] to the anti-de Sitter-like setting. In particular, this shows the existence of a large class of time symmetric data —i.e. data for which $K_{ij} = 0$.

Remark 11. The fields given by equations (22a)–(22e) represent part of the initial data required to evolve the system of wave equations (11a)–(11e). A calculation shows that the remaining component, $n^a n^b L_{ab}$, can be computed directly from L_{ij} and the gauge function $\mathcal{R}(x)$. On the other hand, the normal derivatives of the fields s, L_{ab} and $d^a{}_{bcd}$ on $\mathcal{S}_\star$ can be computed via the system (2a)–(2d) along with the contracted Bianchi identity. Furthermore, notice that this construction guarantees that the zero–quantities trivially vanish on $\mathcal{S}_\star$.

4 General set-up

In this section we discuss in detail the gauge fixing and the boundary data prescription for an initial-boundary problem for the conformal Einstein field equations which, in turn, gives rise to anti-de Sitter-like spacetimes.

In what follows, let $(\mathcal{M}, g_{ab}, \Xi)$ denote a conformal extension of an anti-de Sitter-like spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ with $g_{ab} = \Xi^2 \tilde{g}_{ab}$. It will be assumed that the spacetime is causal (i.e. it contains no closed timelike curves) and that it contains a smooth, oriented and compact spacelike hypersurface $\mathcal{S}_\star$ with boundary $\partial\mathcal{S}_\star$ which intersects the conformal boundary $\mathscr{I}$ in such a way that $\mathcal{S}_\star \cap \mathscr{I} = \partial\mathcal{S}_\star$. It is convenient to define $\tilde{\mathcal{S}}_\star \equiv \mathcal{S}_\star \setminus \partial\mathcal{S}_\star$. The portion of $\mathscr{I}$ in the future of $\mathcal{S}_\star$ will be denoted by $\mathscr{I}^+$. Furthermore, it will be assumed that the causal future $J^+(\mathcal{S}_\star)$ coincides with the future domain of dependence $D^+(\mathcal{S}_\star \cup \mathscr{I}^+)$ and that $\mathcal{S}_\star \cup \mathscr{I}^+ \approx [0,1) \times \mathcal{S}_\star$ so that, in particular, $\mathscr{I}^+ \approx [0,1) \times \partial\mathcal{S}_\star$.

4.1 Coordinates

Close to the conformal boundary $\mathscr{I}$ we will make use of adapted coordinates $x = (x^\mu)$ such that in terms of these coordinates

$$\mathscr{I} = \{x \in \mathbb{R}^3 \mid x^1 = 0\}.$$

The coordinate x^0 is chosen so that the initial hypersurface $\mathcal{S}_\star$ corresponds to the condition $x^0 = 0$. Accordingly, the corner $\partial\mathcal{S}_\star$ is described by the conditions $x^0 = 0$ and $x^1 = 0$.

The coordinates $x = (x^\mu)$ are propagated off the initial hypersurface $\mathcal{S}_\star$ through the generalised wave coordinate condition

$$\Box x^\mu = -\mathcal{F}^\mu(x). \tag{23}$$

The value of the coordinates on $\mathcal{S}_\star$ provides the initial data for the equation (23). The initial value of the normal derivatives to $\mathcal{S}_\star$ is obtained from the requirement that (x^μ) are independent —that is, the coordinate differentials $\mathbf{d}x^\mu$ must be linearly independent.

4.2 Boundary conditions for the conformal evolution equations

In this subsection we discuss the boundary conditions to be imposed on the various conformal fields. In [19] it has been shown that it is possible to formulate an initial boundary-initial value problem for anti-de Sitter-like spacetimes in which the conformal class of the metric on the conformal boundary is specified freely. In the following, we investigate whether it is possible to make a similar prescription in our scheme. More precisely, we would like to specify *Dirichlet boundary data* for the wave equations (11a)-(11e) —that is, one would like to specify the values of the scalar fields Ξ, s and the components of the tensors $g_{\mu\nu}$, $\Phi_{\mu\nu}$ and $d_{\mu\nu\lambda\rho}$ on $\mathscr{I}$.

4.2.1 Boundary data for the conformal factor

The evolution of the conformal factor Ξ is described by the wave equation (11a). For this equation one naturally prescribes Dirichlet boundary conditions such that

$$\Xi \simeq 0.$$

In other words, one has that $\Xi = O(x^1)$ close to $\mathscr{I}$. On $\mathcal{S}_\star$ one wants to identify Ξ with some 3-dimensional conformal factor Ω such that $\Omega = 0$, $\mathbf{d}\Omega \neq 0$ at $\partial\mathcal{S}_\star$, consistent with Definition 2.

4.2.2 The Friedrich scalar

The evolution of the Friedrich scalar s is governed by the wave equation (11b). In the context of the conformal constraint equations on the conformal boundary, the Friedrich scalar s is a gauge dependent quantity which contains information about the manner the conformal boundary embeds in the spacetime. Following Proposition 5 we set

$$s \simeq \varkappa(x)\not{\Sigma}, \qquad \not{\Sigma} = \sqrt{\frac{|\lambda|}{3}}, \qquad \not{K}_{ij} \simeq \varkappa(x)\ell_{ij}, \tag{24}$$

where $\varkappa(x)$ is an arbitrary scalar field. This specification of s is independent of the choice of the gauge source function $\mathcal{R}(x)$ associated to the Ricci scalar —see the discussion in Remark 4. In particular, it is possible, say, to have two related conformal representations of the same physical solution with the same spacetime Ricci scalar, one with a conformal boundary which is extrinsically curved and the other extrinsically flat.

Remark 12. Observe that the particular choice $\varkappa(x) = 0$ renders a conformal boundary which is *extrinsically flat* with respect to the ambient spacetime —see equation (18c).

4.2.3 Boundary data for the components of the conformal metric

In the following it is convenient to make use of the 3+1 decomposition of the metric g_{ab} with respect to the unit normal to the conformal boundary —namely

$$\boldsymbol{g} = \not{\alpha}^2 \mathbf{d}x^1 \otimes \mathbf{d}x^1 + \ell_{\gamma\delta}\big(\not{\beta}^\gamma \mathbf{d}x^1 + \mathbf{d}x^\gamma\big) \otimes \big(\not{\beta}^\delta \mathbf{d}x^1 + \mathbf{d}x^\delta\big), \qquad \gamma,\, \delta = 0,\, 2,\, 3.$$

In particular, $(\ell_{\gamma\delta})$ denote the components of the intrinsic metric ℓ_{ij} of the conformal boundary and $\not{\alpha}$ and $\not{\beta}^\gamma$ are, respectively, the lapse and shift. As $\mathscr{I}$ is timelike, then ℓ_{ij} is a 3-dimensional Lorentzian metric of signature $(-++)$. Accordingly, the components $(g_{\mu\nu})$ are given by

$$(g_{\mu\nu}) = \begin{pmatrix} \not{\alpha}^2 + \not{\beta}_\gamma \not{\beta}^\gamma & \not{\beta}_\gamma \\ \not{\beta}_\delta & \ell_{\gamma\delta} \end{pmatrix}, \tag{25}$$

so that for the components of the contravariant metric one has

$$(g^{\mu\nu}) = \begin{pmatrix} \not{\alpha}^{-2} & -\not{\alpha}^{-2}\not{\beta}^\gamma \\ -\not{\alpha}^{-2}\not{\beta}^\delta & \ell^{\gamma\delta} + \not{\alpha}^{-2}\not{\beta}^\gamma\not{\beta}^\delta \end{pmatrix}.$$

Remark 13. In the following we regard the components $(\ell_{\alpha\beta})$ as our basic boundary data.

Without loss of generality, we adopt a *Gaussian gauge* at the conformal boundary so that

$$\not{\alpha} \simeq 1, \qquad \not{\beta}^\gamma \simeq 0, \tag{26}$$

and the metric g_{ab} takes the form

$$\boldsymbol{g} \simeq \mathbf{d}x^1 \otimes \mathbf{d}x^1 + \ell_{\alpha\beta}\mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta.$$

Remark 14. The prescription of the gauge conditions at the conformal boundary (26) is independent of the generalised harmonic condition (23) and, thus, consistent with each other. Indeed, a calculation shows that for a metric in the form given by (25) one has that

$$\Gamma^1 = \frac{1}{\not{\alpha}^3}\big(\partial_1\not{\alpha} - \not{\beta}^\gamma\partial_\gamma\not{\alpha} + \not{\alpha}^2\not{K}\big), \tag{27a}$$

$$\Gamma^\delta = \gamma^\delta - \frac{\not{\beta}^\delta}{\not{\alpha}^3}\big(\partial_1\not{\alpha} - \not{\beta}^\gamma\partial_\gamma\not{\alpha} + \not{\alpha}^2\not{K}\big) + \frac{1}{\not{\alpha}^2}\big(\partial_1\not{\beta}^\delta - \not{\beta}^\gamma\partial_\gamma\not{\beta}^\delta + \not{\alpha}\partial^\delta\not{\alpha}\big), \tag{27b}$$

and $\gamma^\delta \equiv \ell^{\eta\theta}\gamma^\delta{}_{\eta\theta}$ denote the 3-dimensional contracted Christoffel symbols. Thus, the generalised harmonic condition (23) only prescribes the propagation of the gauge fields $\not{\alpha}$ and $\not{\beta}^\gamma$ off the conformal boundary and do not constraint the components of the 3-metric ℓ_{ij}. Observe that $\not{\alpha}$ and $\not{\beta}^\gamma$ depend on the choice of $\varkappa(x)$ as $\not{K} = 3\varkappa(x)$ as a consequence of equation (24).

4.2.4 Boundary data for the components of the Schouten tensor

Given the 3-metric ℓ_{ij} of the conformal boundary, one can compute the tangential components $(\not{L}_{\alpha\beta})$ and tangential-normal components $(\not{L}_\alpha)$ of the spacetime Schouten tensor at the conformal boundary using formulae (18d) and (18e). One has then that

$$\not{L}_\alpha \simeq -\not{D}_\alpha\varkappa(x), \qquad \not{L}_{\alpha\beta} \simeq \not{l}_{\alpha\beta} - \frac{1}{2}\varkappa^2(x)\ell_{\alpha\beta}, \tag{28}$$

where $\varkappa(x)$ is the arbitrary scalar field determining the extrinsic curvature of the conformal boundary according to equation (18c) and $\not{l}_{\alpha\beta}$ denotes the components of the Schouten tensor $\not{l}_{ij}$ of the metric ℓ_{ij}. To compute the normal-normal component $\not{L}_{11}$ we notice that

$$g^{\mu\nu}\not{L}_{\mu\nu} = \frac{1}{6}R.$$

Thus, one has that

$$
\begin{aligned}
\not{L}_{11} &\simeq \frac{1}{6}\mathcal{R}(x) - \ell^{\alpha\beta}\not{Y}_{\alpha\beta} + \frac{1}{2}\varkappa^2(x)\ell_{\alpha\beta}\ell^{\alpha\beta} \\
&\simeq \frac{1}{6}\mathcal{R}(x) - \frac{1}{4}r + \frac{3}{2}\varkappa^2(x)
\end{aligned}
\tag{29}
$$

where it is recalled that $\mathcal{R}(x)$ denotes the conformal gauge source function introduced in Remark 4.

4.2.5 The rescaled Weyl tensor

The boundary data for the magnetic part of the rescaled Weyl tensor is directly computed from the metric ℓ_{ij} using the formula

$$
\not{d}_{ijk} \simeq -\sqrt{\frac{3}{|\lambda|}}y_{ijk}, \tag{30}
$$

where y_{ijk} denotes the Cotton tensor of ℓ_{ij}—see equation (18f) in Proposition 5.

The computation of the boundary data for the electric part requires more work. From the discussion in Section 3.2 it follows that the electric part of the rescaled Weyl tensor satisfies on $\mathscr{I}$ the equation

$$
\not{D}^i \not{d}_{ij} \simeq 0. \tag{31}
$$

We now consider a $2+1$ decomposition of this equation on $\mathscr{I}$. To this end let $\partial\mathcal{S}_t$, $t \in [0,\infty)$ with $\partial\mathcal{S}_0 = \partial\mathcal{S}_\star$ denote a foliation of the conformal boundary and let ν_i denote the normal to this foliation. The projector $s_i{}^j$ onto the leaves $\partial\mathcal{S}_t$ is given by

$$
s_{ij} = \ell_{ij} + \nu_i\nu_j.
$$

The covariant derivative $\not{D}_i$ can be decomposed, in turn, as

$$
\not{D}_i = -\nu_i\delta + \delta_i
$$

where δ is the covariant directional derivative in the direction of ν_i and δ_i is the Levi-Civita covariant derivative associated to the 2-dimensional metric s_{ij}. The normal ν_i induces the decomposition

$$
\not{d}_{ij} = w_{ij} - \nu_i w_j - \nu_j w_i + \nu_i\nu_j w, \qquad w_{ij} = w_{(ij)},
$$

of the electric part of the rescaled Weyl tensor, where

$$
w_{ij} \equiv s_i{}^k s_j{}^l \not{d}_{kl}, \qquad w_i \equiv s_i{}^k \nu^l \not{d}_{kl}, \qquad w \equiv \nu^i\nu^j \not{d}_{ij}.
$$

Using the above expressions, and observing that $w = w_i{}^i$, one obtains the following decomposition of equation (31):

$$
\delta w - \delta^i w_i = -\frac{3}{2}kw - k^{ij}w_{\{ij\}}, \tag{32a}
$$

$$
2\delta w_i - \delta_i w = -2kw_i - 2k_i{}^j w_j + 2\delta^j w_{\{ij\}}, \tag{32b}
$$

where the 2-dimensional extrinsic curvature of the leaves of the foliation $\partial\mathcal{S}_t$, k_{ij}, and the acceleration, a_i, are defined via the relation

$$
\not{D}_i\nu_j = k_{ij} + \nu_i a_j, \qquad k \equiv s^{ij}k_{ij},
$$

and $w_{\{ij\}} \equiv w_{ij} - \frac{1}{2}s_{ij}w$ is the s-tracefree part of w_{ij}.

Remark 15. Expressing equations (32a)-(32b) in terms of coordinates $(x^{\mathcal{A}}) = (t, x^A)$ adapted to the foliation $\partial\mathcal{S}_t$, one finds that the former imply a first order symmetric hyperbolic system for w and the two non-trivial independent components $w_{\mathcal{A}}$ of w_i provided that the components $w_{\{AB\}}$ are known. Thus, the components $w_{\{AB\}}$ of the electric part of the rescaled Weyl tensor constitute an independent piece of boundary data that supplements the prescription of the Lorentzian 3-metric ℓ_{ij}.

Remark 16. The restriction to $\mathscr{I}$ of the generalised wave coordinate conditions (23) allows to specify, via the relation (27b), a natural choice for the lapse and shift (and thus a choice of the foliation of $\partial\mathcal{S}_t$) for which equations (32a)-(32b) are to be solved.

The discussion of the previous paragraphs leads to the following:

Lemma 1. *Let on $\mathscr{I}$ be given:*

(i) a smooth 3-dimensional Lorentzian metric ℓ_{ij};

(ii) a prescription of coordinate gauge source functions $\mathcal{F}^\mu(x)$ and the intrinsic gauge function $\varkappa(x)$;

(iii) a smooth symmetric tensor $w_{\{ij\}}$ which is spatial with respect to the foliation induced on $\mathscr{I}$ by the functions $\mathcal{F}^\mu(x)$ and tracefree with respect to the metric induced on the leaves of the foliation;

(iv) a smooth choice of fields w and w_i on a fiduciary hypersurface $\partial\mathcal{S}_\star$ of $\mathscr{I}$.

Then, there exists a $t_\bullet > 0$ such that on $\mathscr{I}_{t_\bullet} \approx [0, t_\bullet) \times \partial\mathcal{S}_\star$ there exists unique fields w and w_i which together with the prescribed choice of $w_{\{ij\}}$ satisfy the constraint (31).

Proof. The proof of this result follows from the discussion in the previous paragraphs and the theory of local existence of first order symmetric hyperbolic systems. □

Remark 17. The *free data* $w_{\{ij\}}$ can be related to the notion of incoming and outgoing radiation. In order make this evident, let $(\boldsymbol{l}, \boldsymbol{l}', \boldsymbol{m}, \bar{\boldsymbol{m}})$ be a Newman–Penrose tetrad satisfying the following normalisation relations in accordance with our conventions:

$$l_a l'^a = -1, \quad m_a \bar{m}^a = 1,$$

while all the remaining contractions vanish. The normal unit vectors are expressed, respectively, as $n_a = \frac{1}{\sqrt{2}}(l_a + l'_a)$ and $\not{n}_a = \frac{1}{\sqrt{2}}(l_a - l'_a)$. Using this, the different metrics take the following form:

$$g_{ab} = -2l_{(a}l'_{b)} + 2m_{(a}\bar{m}_{b)}, \quad \ell_{ab} = 2m_{(a}\bar{m}_{b)} - l_{(a}l'_{b)} - \tfrac{1}{2}(l_a l_b + l'_a l'_b), \quad s_{ab} = 2m_{(a}\bar{m}_{b)}.$$

Making use of this and observing that $\omega_{ab} = \not{n}^q \not{n}^s s_a{}^p s_b{}^r d_{dpqrs}$, an expansion of the Weyl tensor in terms of the tetrad defined above leads, after a straightforward calculation, to:

$$\omega_{\{ab\}} = \frac{1}{2}\bigg((\psi_0 + \psi_4^*)\bar{m}_a\bar{m}_b + (\psi_0^* + \psi_4)m_a m_b\bigg),$$

where $\psi_0 \equiv d_{pqrs}l^p m^q l^r m^s$ and $\psi_4 \equiv d_{pqrs}l'^p \bar{m}^q l'^r \bar{m}^s$ —see e.g. [9]. This shows that ψ_0 and ψ_4 constitute part of the basic data one must provide on $\mathscr{I}$.

4.2.6 Summary

The analysis of this section can be summarised as follows:

Proposition 7. *Let on $\mathscr{I}$ be given a smooth Lorentzian metric ℓ_{ij} and a smooth tensor field $w_{\{ij\}}$ as in Lemma 1. Moreover, let the fields*

$$\not{\Sigma}, \qquad s, \qquad \not{K}_{ij}, \qquad \not{L}_i, \qquad \not{L}_{ij}, \qquad \not{d}_{ijk}$$

be constructed according to formulae (24), (28) *and* (30). *Finally, let Θ_a, Υ_a, Δ_{abc} and Λ_{abc} be the zero-quantities defined by relations* (12a)-(12d). *One has then that*

$$\begin{gathered}\ell_b{}^a\Theta_a \simeq 0,\\ \ell_c{}^a\ell_d{}^b\Upsilon_{ab} \simeq 0, \quad \not{n}^a\ell_c{}^b\Upsilon_{ab} \simeq 0,\\ \ell_e{}^c\ell_f{}^d\ell_g{}^b\Delta_{cdb} \simeq 0, \quad \not{n}^b\ell_e{}^c\ell_f{}^d\Delta_{cdb} \simeq 0,\\ \not{n}^b\ell_e{}^c\ell_f{}^d\Lambda_{bcd} \simeq 0, \quad \not{n}^b\not{n}^d\ell_e{}^c\Lambda_{bcd} \simeq 0,\end{gathered}$$

at least on $\mathscr{I}_{t_\bullet} \approx [0, t_\bullet) \times \partial\mathcal{S}_\star$, where $\not{n}^a$ and $\ell_a{}^b$ denote, respectively, the normal and projector of the conformal boundary $\mathscr{I}$.

4.3 Corner conditions

In the previous sections we have discussed the problem of the determination of initial and boundary data. In particular, it is clear that once boundary data have been provided on $\mathscr{I}$, time derivatives of the various conformal fields can be directly calculated. However, these data do not necessarily match smoothly with the ones corresponding to $\mathcal{S}_\star$ at the corner. The purpose of this section is to analyse the compatibility conditions, at different orders, arising from the conformal Einstein field equations and the wave equations —these conditions are commonly known as *corner conditions*. In the following, the subscript $_\odot$ will stand for a quantity evaluated at $\partial\mathcal{S}_\star$.

4.3.1 Conditions for the metric

In terms of the adapted coordinates previously introduced, the corner $\partial\mathcal{S}_\star$ is defined by the conditions $x^0 = 0$ and $x^1 = 0$. Exploiting the gauge freedom, we adopt local Gaussian coordinates both on $\mathcal{S}_\star$ and $\mathscr{I}$. Denoting as $h_{\gamma\delta}$ and $\ell_{\mathcal{AB}}$ the intrinsic 3–metrics corresponding to these hypersurfaces, respectively, this condition implies that the spacetime metric at $\partial\mathcal{S}_\star$ can be written in the two following ways:

$$
\begin{aligned}
\boldsymbol{g} &= -\mathbf{d}x^0 \otimes \mathbf{d}x^0 + h_{\gamma\delta}\mathbf{d}x^\gamma \otimes \mathbf{d}x^\delta, \qquad (\gamma,\delta = 1,2,3),\\
\boldsymbol{g} &= \mathbf{d}x^1 \otimes \mathbf{d}x^1 + \ell_{\mathcal{AB}}\mathbf{d}x^{\mathcal{A}} \otimes \mathbf{d}x^{\mathcal{B}}, \qquad (\mathcal{A},\mathcal{B} = 0,2,3).
\end{aligned}
$$

Hereafter, the previous convention for the indices will be used. Additionally, uppercase indices A, $B,\ldots$ will stand for the coordinates x^2 and x^3 (which we will refer to as *angular*) of the sections of $\mathscr{I}$.

Zero order conditions. Comparing the two last expressions for the metric, one readily finds that

$$
(\ell_{00})_\odot = -1, \quad (h_{11})_\odot = 1, \quad (\ell_{AB})_\odot = (h_{AB})_\odot, \tag{33}
$$

while the remaining components vanish at $\partial\mathcal{S}_\star$.

First order conditions. In Gaussian coordinates, we can express the normal derivatives of the metric in terms of the corresponding extrinsic curvature. Explicitly, one has:

$$
K_{\gamma\delta}|_{\mathcal{S}_\star} = \frac{1}{2}\partial_0 h_{\gamma\delta}|_{\mathcal{S}_\star} = \Gamma^0{}_{\gamma\delta}|_{\mathcal{S}_\star}, \tag{34a}
$$

$$
\not{K}_{\mathcal{AB}} \simeq \frac{1}{2}\partial_1 \ell_{\mathcal{AB}} \simeq -\Gamma^1{}_{\mathcal{AB}}. \tag{34b}
$$

As $K_{\gamma\delta}$ is part of the initial data, this establishes a corner condition for $\partial_0 h_{\gamma\delta}$; in particular, the angular components must satisfy the condition $(\partial_0 h_{AB})_\odot = (\partial_0 \ell_{AB})_\odot$.

Recall that in Gaussian coordinates the propagation of the timelike vector $(\partial_0)^a$ along itself implies that $\Gamma^\mu_{00}|_{\mathcal{S}_\star} = 0$; similarly, for the normal to $\mathscr{I}$ one has that $\Gamma^\mu_{11} \simeq 0$. The previous conditions on the Christoffel symbols, along with equations (34a) and (34b), imply that K_{11} and $\not{K}_{00}$ vanish at the corner. Furthermore, the traces of the extrinsic curvature can be related to the gauge functions $\mathcal{F}^\mu(x)$ as follows:

$$
K_\odot = (h^{AB}K_{AB})_\odot = \mathcal{F}^0(x)_\odot, \qquad \not{K}_\odot = (\ell^{AB}\not{K}_{AB})_\odot = -\mathcal{F}^1(x)_\odot.
$$

Finally, given that ∇ is a Levi–Civita connection and the acceleration is zero, our coordinate choice determines the remaining partial derivatives: $(\partial_0 g_{0\mu})_\odot = -(\Gamma^0_{0\mu})_\odot = 0$.

Second order conditions. Second order conditions can be extracted in a straightforward way from the wave equation for the metric, equation (11e) —namely

$$
g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} = 2\bigg(g_{\lambda\rho}g^{\sigma\tau}\Gamma^\lambda{}_{\sigma\mu}\Gamma^\rho{}_{\tau\nu} + 2\Gamma^\sigma{}_{\lambda\rho}g^{\lambda\tau}g_{\sigma(\mu}\Gamma^\rho{}_{\nu)\tau} - g_{\sigma(\mu}\nabla_{\nu)}\mathcal{F}^\sigma(x) - 2\Phi_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}(x)\bigg).
$$

Using the conditions discussed above for the first order derivatives, the wave equation for the components $g_{\mu\nu}$ can be written schematically as:

$$
(\partial_0^2 \ell_{\mu\nu})_\odot = (\partial_1^2 h_{\mu\nu})_\odot + (h^{CD}\partial_C\partial_D h_{\mu\nu})_\odot + f_{\mu\nu}(\boldsymbol{g}, \boldsymbol{K}, \not{\boldsymbol{K}}, \boldsymbol{\mathcal{F}}(x), \boldsymbol{\Phi}, \mathcal{R}(x))_\odot.
$$

Apart from the components of the Schouten tensor encoded into $\Phi_{\mu\nu}$ (to be discussed below), the second order condition can be expressed in terms of the initial data, lower order corner conditions and gauge functions at the corner. Further application of ∂_0 enables to obtain higher order conditions.

4.3.2 Conditions for the conformal factor

As, by definition, $\Xi = 0$ on the conformal boundary, then all its intrinsic derivatives of any order will vanish. In particular, $\partial\mathcal{S}_\star$ automatically inherits these conditions. Regarding the normal derivative, solution (18a) gives its value on $\mathscr{I}$. Accordingly, one has that

$$(\not\Sigma)_\odot = \sqrt{-\frac{\lambda}{3}}.$$

When smoothness is imposed, higher order partial derivatives both on $\mathcal{S}_\star$ as well as on $\mathscr{I}$ are forced to coincide at $\partial\mathcal{S}_\star$.

4.3.3 Conditions for the Friedrich scalar

Zero order condition. As discussed previously, the Friedrich scalar s is determined on the conformal boundary by the gauge function $\varkappa(x)$. Nevertheless, when the 00 component of equation (2a) is evaluated at the corner, our choice of Gaussian coordinates imply that,

$$s_\odot = 0.$$

First order conditions. Equation (17c) —or alternatively (2b)— determines the intrinsic derivatives of s on the boundary. In particular, the time derivative takes the following form at the corner:

$$(\partial_0 s)_\odot = -\not\Sigma(L_{01})_\odot.$$

This expression is equivalent to the one given in (28) for the tangential-normal components of L_{ab} on $\mathscr{I}$.

Second order conditions. The second order condition for s can be extracted from the wave equation (11b) expressed in Gaussian coordinates. The evaluation of this equation at the corner yields:

$$(\partial_0^2 s)_\odot = (\partial_1^2 s)_\odot + (h^{AB}\partial_A\partial_B s)_\odot - (\mathcal{F}^\mu(x)\partial_\mu s + \frac{1}{6}(s\mathcal{R}(x) + \not\Sigma\partial_1\mathcal{R}(x))_\odot.$$

Here, the spatial derivatives of s can be computed from the restriction of the initial data to $\partial\mathcal{S}_\star$ while $\partial_0 s$ corresponds to the first order condition. The functions $\mathcal{F}^\mu(x)$ and $\mathcal{R}(x)$ are gauge-dependent prescribed quantities. Furthermore, we observe that $\partial_0^2 s$ is written in terms of the first order derivatives, indicating then a recursive procedure to find higher order conditions — computed by further application of ∂_0 to equation (11b).

4.3.4 Conditions for the Schouten tensor

Next, we will show how the constraint equations impose restrictions on the components of L_{ab}, which along with the gauge quantity $\mathcal{R}(x)$ determines the tracefree tensor Φ_{ab} on $\mathscr{I}$.

Zero order corner conditions. The value of components $L_{\alpha\beta}$ and $L_{0\alpha}$ at the corner can be obtained from the initial data (22b) and (22c) taking the limit $\Omega \to 0$. Imposing smoothness, they must match the boundary data given by equations (28) and (29) at $\partial\mathcal{S}_\star$. The same is imposed for component L_{00}.

First order corner conditions. First time derivatives of the components $L_{\alpha\beta}$ and $L_{0\alpha}$ can be obtained via equation (2c). More explicitly one has:

$$\begin{aligned}
(\partial_0 L_{\alpha\beta})_\odot &= \not\Sigma(d^1{}_{\beta 0\alpha})_\odot + f_{\alpha\beta}(\boldsymbol{L}, \boldsymbol{h}, \boldsymbol{K}, \not{\boldsymbol{K}})_\odot, \\
(\partial_0 L_{\alpha_0})_\odot &= \not\Sigma(d^1{}_{00\alpha})_\odot + f_\alpha(\boldsymbol{L}, \boldsymbol{h}, \boldsymbol{K}, \not{\boldsymbol{K}})_\odot.
\end{aligned}$$

As it will be seen below, the components of the Weyl tensor appearing here, are part of the data satisfying zero-order conditions, so they must be consistent with the last equations. On the other hand, a condition for $(\partial_0 L_{00})_\odot$ can be obtained via the contracted Bianchi identity.

Second order corner conditions. Second order time derivatives of L_{ab} are to be obtained by evaluating the wave equation (11c) at $\partial\mathcal{S}_\star$. For $L_{\alpha\beta}$ one has:

$$(\partial_0^2 L_{\alpha\beta})_\odot = (\partial_1^2 L_{\alpha\beta})_\odot + (h^{CD}\partial_C\partial_D L_{\alpha\beta})_\odot + f_{\alpha\beta}(\boldsymbol{h}, \boldsymbol{L}, \boldsymbol{K}, \not{\boldsymbol{K}}, \partial\mathcal{F}(x), \mathcal{R}(x))_\odot.$$

Similar expressions can be obtained for the rest of the components.

4.3.5 Conditions for the Weyl tensor

Information about the Weyl tensor is encoded in the electric and magnetic parts. These are given on $\mathcal{S}_\star$ by equations (22d) and (22e), and has been discussed in section 4.2.5 for $\mathscr{I}$. As these data have been obtained using different projections, their components must be carefully matched. One can check that they share the components d_{0101}, d_{010A}, d_{01A1}, d_{01AB} and d_{0A1B} so, when matched, they represent the zero-order conditions.

First order corner conditions. Given the structure of equation (2d), only certain conditions can be extracted from it. Ultimately, when it is evaluated at the corner it takes the form:

$$(\partial_0 d^0{}_{\lambda\mu\nu})_\odot = f_{\lambda\mu\nu}(\boldsymbol{K}, \not{\boldsymbol{K}}, \boldsymbol{d})_\odot.$$

Second order corner conditions. Second order time derivatives of the rescaled Weyl tensor are given by the wave equation (11d). As Ξ vanishes at the corner, the equation is significantly simplified. Expanding the reduced wave operator $\blacksquare$ it takes the schematic form

$$(\partial_0^2 d_{\lambda\mu\nu\sigma})_\odot = (\partial_1^2 d_{\lambda\mu\nu\sigma})_\odot + (\partial_A\partial_B d_{\mu\nu\lambda\sigma})_\odot + f_{\lambda\mu\nu\sigma}(\boldsymbol{g}, \boldsymbol{K}, \not{\boldsymbol{K}}, \boldsymbol{d})_\odot.$$

4.3.6 Concluding remarks regarding the corner conditions

The discussion in the previous paragraphs provides a recursive procedure to compute the corner conditions to any required order. Given this procedure, its natural to ask whether there exist any examples of pairs of initial data and boundary conditions which satisfy the corner conditions to *any arbitrary order*. The difficulties in implementing corner conditions to any arbitrary order have been discussed in [20]. A way of satisfying corner conditions to an arbitrary order is to make use of the gluing constructions for asymptotically hyperbolic initial data sets in [11]. Given an asymptotically hyperbolic initial data set satisfying certain smallness conditions, these constructions allow to deform the data by a deformation which is supported arbitrarily far in the asymptotic region, to ones which are exactly Schwarzschild-anti de Sitter in the asymptotic region. This class of data is naturally supplemented by Schwarzschild-anti de Sitter boundary initial data —and thus it trivially satisfies the corner conditions to any order. The resulting spacetime has, accordingly, a very special behaviour near the corner. In particular, the metric ℓ_{ij} must be conformally flat near the corner. It is of interest to analyse whether it is possible to construct a more general class of initial–boundary data for adS-like spacetimes satisfying the corner conditions at any order.

5 Propagation of the constraints

The purpose of this section is to analyse the propagation of the gauge conditions and to discuss the relation of the evolution system (11a)-(11e) to the Einstein field equations.

5.1 Boundary conditions for the subsidiary equations

The purpose of this section is to show that the boundary conditions for the conformal wave equations (11a)-(11e) discussed in the previous section imply trivial (i.e. vanishing) Dirichlet boundary conditions for the subsidiary wave equations (14a)-(14d).

5.1.1 Transport equations for the subsidiary fields

Proposition 7 shows that as a consequence of our Dirichlet boundary data prescription, the components of the zero fields Θ_a, Υ_a, Δ_{abc} and Λ_{abc} which only involve derivatives intrinsic to $\mathscr{I}$ vanish. In order to show that the remaining components also vanish, it is necessary to construct suitable transport equations for the zero-quantities on the conformal boundary. As it will be seen, the equations for Υ_{ab} and Θ_a can be constructed in a straightforward manner, whereas Δ_{abc} and Λ_{abc} require a more detailed treatment. The integrability conditions for the zero–quantities (56a)–(57b) —see Appendix A.2— will prove to be key to obtain these equations. In what follows let τ^i denote a timelike vector on $\mathscr{I}$ with pushforward to the spacetime $(\mathcal{M}, g_{ab})$ given by τ^a and let $\mathcal{P} \equiv \tau^a \nabla_a$. Notice then that $\cancel{n}^a \tau_a = 0$

Transport equations for Θ_a and Υ_{ab}. First, consider the expression $2\tau^a \nabla_{[a} \Upsilon_{b]c}$. On the one hand, a calculation shows that

$$2\tau^a \nabla_{[a} \Upsilon_{b]c} = \mathcal{P}\Upsilon_{bc} + \Upsilon_{ac}\chi_b{}^a - \nabla_b(\tau^a \Upsilon_{ac}) \simeq \mathcal{P}\Upsilon_{bc} + \Upsilon_{ac}\chi_b{}^a,$$

where $\chi_{ab} \equiv \nabla_a \tau_b$ and the second equality follows from the fact that $\ell_a{}^c \ell_b{}^d \Upsilon_{cd} \simeq 0$ and $\ell_a{}^c n^b \Upsilon_{bc} \simeq 0$, which are a consequence of the validity of constraints (17a) and (17b) on $\mathscr{I}$. On the other hand, using the integrability condition (56a) one obtains the following transport equation:

$$\mathcal{P}\Upsilon_{bc} \simeq 2\tau^a g_{c[a}\Theta_{b]} - \Upsilon_{ac}\chi_b{}^a, \tag{35}$$

which crucially is homogeneous in the zero-quantities.

Now, for Θ_a, consider the expression $2\tau^a \nabla_{[a}\Theta_{b]}$. Expanding as in the case for Υ_{ab} one finds that

$$2\tau^a \nabla_{[a}\Theta_{b]} \simeq \mathcal{P}\Theta_b - \nabla_b\big(\tau^a \Theta_a\big) + \Theta_a \chi_b{}^a = \mathcal{P}\Theta_b + \Theta_a \chi_b{}^a,$$

where it has been used that $\ell_b{}^a \Theta_a \simeq 0$ —as this is equivalent to satisfy the constraint (17c)— so that $\tau^a \Theta_a \simeq 0$. Using the integrability condition (56b), the following homogeneous transport equation is directly obtained:

$$\mathcal{P}\Theta_b \simeq \tau^a \Delta_{abc} \nabla^c \Xi - \tau^a L^c{}_{[a}\Upsilon_{b]c} - \Theta_a \chi_b{}^a. \tag{36}$$

Transport equations for Δ_{abc} and Λ_{abc}. For the zero-quantity Δ_{abc} consider $3\tau^e \nabla_{[e}\Delta_{ab]c}$. A direct calculation shows that

$$3\tau^e \nabla_{[e}\Delta_{ab]c} = \mathcal{P}\Delta_{abc} - 2\chi_{[a}{}^e \Delta_{b]ec} + 2\nabla_{[a}(\tau^e \Delta_{b]ec}). \tag{37}$$

As before, one needs to show that the last term in the previous expression vanishes on the boundary. For this purpose a decomposition with respect to $\ell_a{}^b$ can be performed. Observing that the components $\ell_a{}^d \ell_b{}^e \ell_c{}^f \Delta_{def} \equiv \Delta^{(3)}_{abc}$ and $\ell_a{}^c \ell_b{}^d n^e \Delta_{cde}$ vanish by virtue of the constraints (17d) and (17e), a calculation leads to

$$\tau^b \Delta_{abc} \simeq \tau^b \ell_b{}^e \ell_c{}^f n_a n^d \Delta_{def} + \tau^b \ell_b{}^f n_a n_c n^d n^e \Delta_{dfe} \equiv \tau^b \Delta_{bc} n_a + \tau^b \Delta_b n_a n_c.$$

In view of this, further homogeneous transport equations for Δ_{ab} and Δ_a on $\mathscr{I}$ are required. Regarding Δ_{ab}, after performing suitable projections in equation (37) and eliminating the normal derivatives via the relevant constraints, its right–hand side takes the following form on the conformal boundary:

$$\mathcal{P}\Delta_{ab} - \cancel{D}_a(\tau^e \Delta_{eb}) + f(\boldsymbol{\chi}, \cancel{\boldsymbol{K}}, \boldsymbol{\Delta}),$$

where the function f is homogeneous in Δ_{abc}. Then, using the integrability condition (57a), a homogeneous transport equation for Δ_{ab} on the boundary is obtained. In the case of Δ_a, a completely analogous procedure leads directly to a similar transport equation.

Finally, for Λ_{abc} consider the analogous expression $3\tau^a \nabla_{[e}\Lambda_{|a|bc]}$:

$$3\tau^d \nabla_{[d}\Lambda_{|a|bc]} = \mathcal{P}\Lambda_{abc} - 2\chi_{[b}{}^d \Lambda_{|a|c]d} + 2\nabla_{[b}(\tau^d \Lambda_{|a|c]d}). \tag{38}$$

Performing a decomposition for $\tau^d\Lambda_{dbc}$ and observing that the components $\ell_a{}^e\ell_b{}^f n^d\Lambda_{def}$ and $\ell_a{}^f n^d n^e\Lambda_{def}$ vanish on $\mathscr{I}$ due to constraints (17f) and (17g), a calculation yields:

$$\tau^c\Lambda_{abc} \simeq \tau^c\ell_a{}^d\ell_b{}^e\ell_c{}^f\Delta_{def} + \tau^c\ell_a{}^d\ell_c{}^e n^f\Delta_{dfe}n_b \equiv \tau^c\Lambda^{(3)}_{abc} + \tau^c\Lambda_{ac}n_b.$$

As in the analysis of Δ_{abc}, this means that suitable transport equations must be constructed for $\Lambda^{(3)}_{abc}$ and Λ_{ab}. When all the indices in equation (38) are projected with the metric $\ell_a{}^b$, its right–hand side takes the form:

$$\mathcal{P}\Lambda^{(3)}{}_{abc} + \not{D}_b(\tau^d\Lambda^{(3)}_{acd}) + f(\boldsymbol{\chi}, \not{\boldsymbol{K}}, \boldsymbol{\Lambda}).$$

Now, the left–hand side of equation (38) can be expressed via the integrability condition (57b). Projecting in the same way as before, and after a long calculation, one finds it can be expressed schematically as $f(\boldsymbol{\chi}, \not{\boldsymbol{K}}, \boldsymbol{\Lambda}, \not{\boldsymbol{D}}\boldsymbol{\Lambda}^{(3)})$, being this function homogeneous in Δ_{abc}. Thus, this implies a transport equation for $\Lambda^{(3)}_{abc}$. Regarding Λ_{ab}, an analogous transport equation is found via suitable contractions as well as exploiting the constraints for Λ_{abc}.

Remark 18. The main observation following the previous calculations is that one has homogeneous propagation equations intrinsic to $\mathscr{I}$ for all the components of the zero-quantities which do not directly vanish by virtue of Proposition 7. Thus, if one can ensure that these intrinsic propagation equations have vanishing initial data at the corner, their solutions have to vanish along the conformal boundary as well —accordingly, the full set of zero-quantities associated to the conformal field equations will vanish on $\mathscr{I}$.

5.1.2 The propagation argument

Once we have obtained the relevant transport equations we are in position to state the following lemma:

Lemma 2. *Consider vanishing initial data for the zero-quantities Υ_{ab}, Θ_a, Δ_{abc} and Λ_{abc} at $\partial\mathcal{S}_\star$ and assume that the conformal constraints* (17a)–(17g) *are satisfied. Then, all the components of the zero-quantities vanish on the conformal boundary.*

Remark 19. A similar approach can be employed to prove that the normal derivatives of the zero–quantities vanish on $\mathcal{S}_\star$ via projecting the integrability conditions (56a)–(57b) with respect to n^a. The result readily follows from the fact that all the components of the zero–quantities vanish on $\mathcal{S}_\star$ —see Remark 11. Thus, one has vanishing initial data for the wave equations (14a)–(14d).

5.2 Propagation of the gauge

The discussion of the propagation of the zero-quantities associated to the conformal Einstein field equations needs to be supplemented with a discussion of the propagation of the gauge. The strategy in this regard is similar to that used in the analysis of the propagation of the constraints —i.e. one introduces a set of zero-quantities associated to the gauge and purports constructing a suitable system of subsidiary homogeneous evolution equations.

5.2.1 Basic relations

In what follows it is convenient to define

$$Q \equiv R - \mathcal{R}(x), \tag{39a}$$
$$Q^\mu \equiv \Gamma^\mu - \mathcal{F}^\mu(x), \tag{39b}$$
$$Q_{\mu\nu} \equiv R_{\mu\nu} - 2\Phi_{\mu\nu} - \frac{1}{4}\mathcal{R}(x)g_{\mu\nu}. \tag{39c}$$

Remark 20. The zero-quantity Q encodes the relation between the Ricci scalar of the unphysical spacetime and the conformal gauge source function. The zero-quantity Q^μ corresponds to the relation between the contracted Christoffel symbols and the coordinate gauge source function giving rise to the generalised wave coordinates. Finally, Q_μ is associated to the relation between the Ricci tensor and the reduced Ricci tensor —compare with equation (8).

Remark 21. In what follows we regard the field $g_{\mu\nu}$ as the components of a metric tensor g_{ab} in the coordinates $x = (x^\mu)$. Let $R_{\mu\nu}$ denote the components of the Ricci tensor, R_{ab}, of g_{ab} in the coordinates (x^μ) and let R be the associated Ricci scalar. The objective of the subsequent analysis is to investigate under what circumstances one has that R coincides with $\mathcal{R}(x)$, $R_{\mu\nu}$ coincides with $\mathscr{R}_{\mu\nu}$ and $\Phi_{\mu\nu}$ are the components of the symmetric tracefree part of $R_{\mu\nu}$ so that one can write

$$R_{\mu\nu} = 2\Phi_{\mu\nu} + \frac{1}{4}\mathcal{R}(x)g_{\mu\nu}.$$

This is equivalent to showing that

$$Q = 0, \qquad Q^\mu = 0, \qquad Q^{\mu\nu} = 0.$$

The definitions of Q^μ and $Q_{\mu\nu}$ allows one to rewrite the reduced Ricci operator, equation (7), and the reduced wave operator acting on $\Phi_{\mu\nu}$, equation (10), as

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] = R_{\mu\nu} - \nabla_{(\mu}Q_{\nu)}, \tag{40a}$$

$$\blacksquare\Phi_{\mu\nu} = \Box\Phi_{\mu\nu} - (Q_{\mu\sigma} - \nabla_\mu Q_\sigma)\Phi^\sigma{}_\nu - (Q_{\nu\sigma} - \nabla_\nu Q_\sigma)\Phi^\sigma{}_\mu. \tag{40b}$$

5.2.2 The subsidiary gauge evolution system

In the calculations of this section we make the following assumption:

Assumption 1. Let $g_{\mu\nu}$ and $\Phi_{\mu\nu}$, with $\Phi_\mu{}^\mu = g^{\mu\nu}\Phi_{\mu\nu} = 0$, be smooth solutions to the equations

$$\mathscr{R}_{\mu\nu} = 2\Phi_{\mu\nu} + \frac{1}{4}\mathcal{R}(x)g_{\mu\nu}, \tag{41a}$$

$$\blacksquare\Phi_{\mu\nu} = 4\Phi_\mu{}^\lambda\Phi_{\nu\lambda} - \Phi_{\lambda\rho}\Phi^{\lambda\rho}g_{\mu\nu} + \frac{1}{3}\mathcal{R}(x)\Phi_{\mu\nu} + \frac{1}{6}\nabla_\mu\nabla_\nu\mathcal{R}(x) - \Box\mathcal{R}(x)g_{\mu\nu}, \tag{41b}$$

for some smooth choice of the gauge source functions $\mathcal{F}^\mu(x)$ and $\mathcal{R}(x)$.

Combining equation (41a) with identity (40a) one finds the relation

$$R_{\mu\nu} = 2\Phi_{\mu\nu} + \frac{1}{4}\mathcal{R}(x)g_{\mu\nu} + \nabla_{(\mu}Q_{\nu)}, \tag{42}$$

in which the reduced Ricci operator has been eliminated. The latter implies, in turn, that

$$R = \mathcal{R}(x) + \nabla^\mu Q_\mu,$$

so that, in fact

$$Q = \nabla^\mu Q_\mu. \tag{43}$$

Also, it follows from its definition that

$$Q = Q_\mu{}^\mu.$$

Moreover, substituting equation (42) into the definition of $Q_{\mu\nu}$, equation (39c), one obtains the relation

$$Q_{\mu\nu} = \nabla_{(\mu}Q_{\nu)}. \tag{44}$$

Taking the divergence of this last identity, commuting covariant derivatives and using expression (42) to eliminate the components of the Ricci tensor which appears after commuting derivatives one obtains

$$\nabla^\mu Q_{\mu\nu} = \frac{1}{8}Q_\nu\mathcal{R} + Q^\mu\Phi_{\mu\nu} + \frac{1}{4}Q^\mu\nabla_\mu Q_\nu + \frac{1}{2}\Box Q_\nu + \frac{1}{2}\nabla_\nu Q + \frac{1}{4}Q^\mu\nabla_\nu Q_\mu. \tag{45}$$

Remark 22. Equations (43) and (44) show that the zero–quantities Q, Q^μ and $Q_{\mu\nu}$ are not independent of each other. In what follows we will regard Q^μ as the fundamental zero quantity. Clearly, if $Q^\mu = 0$ then necessarily $Q = 0$ and $Q_{\mu\nu} = 0$.

The construction of a suitable system of subsidiary equations for the fields Q, Q_μ and $Q_{\mu\nu}$ makes use of the properties of the *Bach tensor* B_{ab} —see Appendix A.4. From the definition of the Bach tensor given in equation (58) one can find an expression for B_{ab} which is homogeneous in the fields Q, Q_μ and $Q_{\mu\nu}$:

$$
\begin{aligned}
B_{\mu\nu} = & -\frac{5}{12}Q\Phi_{\mu\nu} - \Phi_\nu{}^\lambda Q_{\mu\lambda} - \Phi_\mu{}^\lambda Q_{\nu\lambda} + \frac{1}{24}Q^2 g_{\mu\nu} - \frac{1}{48}Q\mathcal{R}(x)g_{\mu\nu} - \frac{5}{48}Q\nabla_\mu Q_\nu + \frac{1}{48}\mathcal{R}(x)\nabla_\mu Q_\nu \\
& +2\Phi_{\nu\lambda}\nabla_\mu Q^\lambda - \frac{1}{4}\nabla_\mu\nabla_\lambda\nabla^\lambda Q_\nu - \frac{1}{16}Q\nabla_\nu Q_\mu + \frac{1}{16}\mathcal{R}(x)\nabla_\nu Q_\mu + \frac{3}{16}\nabla_\mu Q^\lambda \nabla_\nu Q_\lambda \\
& +\frac{7}{4}\Phi_{\mu\lambda}\nabla_\nu Q^\lambda + \frac{1}{6}\nabla_\nu\nabla_\mu Q + \frac{1}{4}\nabla_\lambda\nabla_\mu\nabla^\lambda Q_\nu + \frac{1}{12}g_{\mu\nu}\nabla_\lambda\nabla^\lambda Q - \frac{1}{4}\nabla_\lambda\nabla^\lambda\nabla_\mu Q_\nu \\
& -\frac{1}{4}\nabla_\lambda\nabla^\lambda\nabla_\nu Q_\mu + \frac{3}{4}\Phi_{\nu\lambda}\nabla^\lambda Q_\mu + \frac{1}{8}\nabla_\nu Q_\lambda \nabla^\lambda Q_\mu + \frac{1}{16}\nabla_\lambda Q_\nu \nabla^\lambda Q_\mu + \frac{1}{2}\Phi_{\mu\lambda}\nabla^\lambda Q_\nu \\
& +\frac{1}{8}\nabla_\mu Q_\lambda \nabla^\lambda Q_\nu - \frac{1}{2}\Xi d_{\mu\lambda\nu\rho}\nabla^\rho Q^\lambda - \frac{1}{4}\Xi d_{\mu\rho\nu\lambda}\nabla^\rho Q^\lambda - \frac{3}{4}\Phi_{\lambda\rho}g_{\mu\nu}\nabla^\rho Q^\lambda \\
& -\frac{1}{16}g_{\mu\nu}\nabla_\lambda Q_\rho \nabla^\rho Q^\lambda - \frac{1}{16}g_{\mu\nu}\nabla_\rho Q_\lambda \nabla^\rho Q^\lambda.
\end{aligned}
$$

From the previous identity one finds, after some manipulations, that

$$
\nabla^\mu B_{\mu\nu} = -\frac{1}{4}\square^2 Q_\nu + H_\nu(\nabla\square\boldsymbol{Q}, \nabla Q, \nabla\boldsymbol{Q}, \boldsymbol{Q}, Q).
$$

In view of the above, it is convenient to define the auxiliary tensor field

$$
M_a \equiv \square Q_a.
$$

A further calculation then shows that

$$
\square Q = H(\nabla\boldsymbol{M}, \nabla Q, \nabla\boldsymbol{Q}, Q).
$$

Recalling that the Bach tensor is divergence free, i.e. $\nabla^a B_{ab} = 0$, it follows from the discussion in the previous paragraph that the fields M_a, Q_a and Q satisfy an homogeneous system of wave equations of the form

$$
\begin{aligned}
&\square M_\mu = 4H_\mu(\nabla\boldsymbol{M}, \nabla Q, \nabla\boldsymbol{Q}, \boldsymbol{Q}, Q), && \text{(46a)}\\
&\square Q_\mu = M_\mu, && \text{(46b)}\\
&\square Q = H(\nabla\boldsymbol{M}, \nabla Q, \nabla\boldsymbol{Q}, Q). && \text{(46c)}
\end{aligned}
$$

In the following, the above system will be known as *gauge subsidiary evolution system*. Given the homogeneous nature of (46a)-(46c), if the system is supplemented with vanishing boundary and initial conditions, one necessarily has the unique solution

$$
M_\mu = 0, \qquad Q_\mu = 0, \qquad Q = 0, \qquad \text{in a neighbourhood of } \partial\mathcal{S}_\star.
$$

The latter, in turn, implies that

$$
Q_{\mu\nu} = 0 \qquad \text{in a neighbourhood of } \partial\mathcal{S}_\star.
$$

Remark 23. If this is the case, then, at least in a neighbourhood of $\partial\mathcal{S}_\star$ one has that

$$
R = \mathcal{R}(x), \qquad \Gamma^\mu = \mathcal{F}^\mu(x)
$$

and the tensor Φ_{ab} coincides with one half of the tracefree part of the Ricci tensor, R_{ab}, of the metric g_{ab}.

5.2.3 Initial and Boundary conditions for the subsidiary gauge evolution system

In this section we analyse the trivial initial conditions

$$M_\mu = 0, \qquad Q_\mu = 0, \qquad Q = 0, \qquad \nabla_\mu M_\nu = 0, \qquad \nabla_\mu Q_\nu = 0, \qquad \nabla_\mu Q = 0 \qquad \text{on} \quad \mathcal{S}_\star$$

and the trivial boundary conditions

$$M_\mu = 0, \qquad Q_\mu = 0, \qquad Q = 0 \qquad \text{on} \quad \mathscr{I}$$

and consider the conditions under which they can be enforced.

In order to study the consequences of these vanishing initial-boundary conditions, it is convenient to decompose Q_μ in terms of its intrinsic and normal components. In the case of $\mathscr{I}$, the projections $\hat{q}_\mu \equiv \ell_\mu{}^\nu Q_\nu$ and $\hat{q} \equiv \not{n}^\nu Q_\nu$ are naturally introduced. The fundamental zero-quantity is then written as

$$Q_\mu \simeq \hat{q}_\mu + \hat{q}\not{n}_\mu.$$

Adopting a Gaussian gauge as in Section 4, the conditions $\hat{q}_\mu \simeq 0$ and $\hat{q} \simeq 0$ imply a system of equations for the normal derivatives of the lapse and shift —see equations (27a) and (27b). Namely, one has that

$$\partial_1 \not{\alpha} \simeq 3\varkappa + \mathcal{F}^1, \qquad \partial_1 \not{\beta}^\delta \simeq F^\delta - \gamma^\delta. \tag{47}$$

Additionally, when the conditions $Q \simeq 0$ and $M_\mu \simeq 0$ are imposed, the following relations are found:

$$\not{D}\hat{q} \simeq 0, \qquad \not{D}^2\hat{q} \simeq 0, \qquad \not{D}^2\hat{q}_\mu + \varkappa\not{D}\hat{q}_\mu \simeq 0. \tag{48}$$

These can be read as higher order differential equations for the normal derivatives of $\not{\alpha}$ and $\not{\beta}$.

Following the same approach, in the case of $\mathcal{S}_\star$ one defines the projections $q_\mu \equiv h_\mu{}^\nu Q_\nu$ and $q \equiv n^\nu Q_\nu$, so one has

$$Q_\mu = q_\mu - q n_\mu.$$

Setting $Q_\mu = 0$, an analogous decomposition of the metric implies a pair of evolution equations for α and β in terms of the gauge source functions. When the remaining vanishing initial data are analysed, a series of straightforward calculations leads to the following conditions on q and q_μ:

$$q = 0, \quad q_\mu = 0, \quad D^{(n)}q = 0, \quad D^{(n)}q_\mu = 0, \quad n = 1, 2, 3. \tag{49}$$

Therefore, more restrictions in the form of higher order constraints for the lapse and shift functions are imposed.

Remark 24. Even though the condition $Q = 0$ on $\mathcal{S}_\star$ and $\mathscr{I}$ implies, respectively, that $Dq = 0$ and $\not{D}\hat{q} \simeq 0$, this can be equivalently stated as imposing that the function $\mathcal{R}(x)$ coincides with the Ricci scalar of the metric g_{ab}.

The discussion of the section can be summarised in the following lemma:

Lemma 3. *Let Q, Q_μ and $Q_{\mu\nu}$ be defined as in* (39a)–(39c)*. If conditions* (47) *and* (48) *are satisfied on $\mathscr{I}$, and* (49) *is satisfied on $\mathcal{S}_\star$, then Q, Q_μ and $Q_{\mu\nu}$ vanish identically in a neighbourhood of $\partial\mathcal{S}_\star$.*

6 The local existence result

We are now in the position of formulating the main result of this article: a local in time existence result for the conformal Einstein field equations in a neighbourhood of the corner $\partial\mathcal{S}_\star$. This result can, in turn, be patched together with the domain of dependence of open subsets of $\mathcal{S}_\star$ away from $\partial\mathcal{S}_\star$ to obtain a solution on a slab around $\mathcal{S}_\star$ —see e.g. [31], Section 12.3.

One has the following:

Theorem 1. *Let $\mathcal{S}_\star$ be a 3-dimensional spacelike hypersurface with boundary $\partial\mathcal{S}_\star$ and smooth anti-de Sitter-like initial data defined on it. Consider the cylinder $[0,\tau_\bullet)\times\partial\mathcal{S}_\star$, for some $\tau_\bullet>0$, endowed with a smooth 3-dimensional Lorentzian metric ℓ_{ij} and let ψ_0, ψ_4 be two complex-valued scalar functions. Assume that the data on $\mathcal{S}_\star$ and the cylinder satisfy the corner conditions at $\partial\mathcal{S}_\star$. Then, there exists a smooth solution to the Einstein field equations with $\lambda<0$ in a neighbourhood of $\mathcal{S}_\star$.*

Proof. Consider initial data on $\mathcal{S}_\star$ given as in Definition 2. Given a 3–dimensional Lorentzian metric ℓ_{ij} on the cylinder $[0,\tau_\bullet)\times\partial\mathcal{S}_\star$, the data given by (24), (28) and (30) can be computed. On the other hand, $\not{d}_{ij}$ is determined via the system (32a)–(32b), which requires the specification of ψ_0 and ψ_4 along with initial values for w and w_i. Notice that the latter ones are prescribed by the initial data at $\partial\mathcal{S}_\star$. If these two sets of initial and boundary data satisfy the corner conditions at $\partial\mathcal{S}_\star$ then the theory of initial-boundary value problems, as given in e.g. [10, 13], guarantees the existence of a unique solution to the system of wave equations (11a)–(11e) in a neighbourhood of $\partial\mathcal{S}_\star$.

Given the boundary data described above, Proposition 7 and Lemma 2 imply that all the components of the zero-quantities vanish on the cylinder. On the other hand, from Remark 11 and Remark 19 we have that the initial data on $\mathcal{S}_\star$ yield vanishing data for the zero–quantities and their first-order derivatives on this hypersurface. Thus, Proposition 4 implies that a solution to the system (11a)–(11e) guarantees the existence and uniqueness of a vanishing solution of equations (14a)–(14d). From the definition of the zero–quantities it follows then that the conformal Einstein field equations (2a)–(2e) are satisfied in a neighbourhood of $\partial\mathcal{S}_\star$.

Finally, having a solution to the conformal Einstein field equations, Proposition 1 implies that the metric $\tilde{g}_{ab}=\Xi^{-2}g_{ab}$ is a solution to the Einstein field equations (1) with $\lambda<0$ for $\Xi\neq 0$. □

Remark 25. A more precise statement about the regularity of the initial data and boundary conditions needs to be expressed in terms of suitable Sobolev spaces and goes beyond the scope of this article. Here, for the sake of simplicity of the presentation we have opted for phrase these conditions in terms of the word *smooth*.

7 Conclusions

The construction carried out in this work can be implemented to numerical codes in a systematic way. Moreover, it represents a step forward with respect to the work in [19] as the wave equations to be solved are manifestly hyperbolic. Furthermore, the free boundary data for the Weyl tensor are explicitly related to the incoming and outgoing radiation. This may make possible to study more general boundary conditions, relevant for a better understanding of the instability of anti-de Sitter-like spacetimes. Nevertheless, as the geometric character of these data is broken by the performed decompositions, further work must be done in order to obtain a completely covariant formulation.

The local existence result presented assumes a vanishing matter–energy component. However, under the methods of conformal geometry, it is not clear how completely general scenarios can be studied —see [21] for a discussion of a particular case of the Einstein-massive scalar field case which is particularly amenable to the use of conformal methods and [22] for a discussion about dust models coupled to Einstein equations with $\lambda>0$. Despite this fact, if a *tracefree* matter field is considered, it is possible to establish a well–posed problem and analyse it in a similar fashion to the one described here. In this context, work is currently under progress to investigate a possible extension of the result to anti-de Sitter-like spacetimes for this class of energy–momentum tensors. Moreover, this naturally leads to several particular cases of interest, namely, Maxwell, Yang–Mills and scalar fields.

Acknowledgements

JAVK thanks the hospitality of Erwin Schrödinger Institute for Mathematics and Physics of the University of Vienna during the programme *Geometry and Relativity* which took place between the months of July and September of 2017. DAC thanks support granted by CONACyT (480147). The calculations described in this article have been carried out in the suite `xAct` for `Mathematica` —see [26].

A Zero-quantities and integrability conditions

The zero-quantities (12a)–(12d) defined in Section 2.4 possess a set of *integrability conditions* which allow us to understand the interrelations between the various equations. These conditions arise in a natural way when appropriate antisymmetrisations of the derivatives of the zero-quantities are taken into consideration.

A.1 Basic properties

First, the zero–quantities possess the following symmetries:

$$\Upsilon_{ab} = \Upsilon_{(ab)}, \quad \Delta_{abc} = \Delta_{[ab]c}, \quad \Delta_{[abc]} = 0, \quad \Lambda_{abc} = \Lambda_{a[bc]}, \quad \Lambda_{[abc]} = 0. \tag{50}$$

Directly from their definitions and using wave equations (4a)–(4c) some useful identities are found, namely:

$$\begin{gathered} \Upsilon_a{}^a = 0, \quad \nabla_b \Upsilon_a{}^b = 3\Theta_a, \quad \nabla_a \Theta^a = \Upsilon^{ab} L_{ab}, \quad \Delta_a{}^b{}_b = 0, \quad \Lambda^b{}_{ab} = 0, \\ \nabla_c \Delta_a{}^c{}_b = \Lambda_{abc} \nabla^c \Xi, \quad \nabla_c \Delta_{ab}{}^c = -\Lambda_{cab} \nabla^c \Xi, \quad \nabla_c \Lambda^c{}_{ab} = 0. \end{gathered} \tag{51}$$

Remarkably, the second Bianchi identity implies the following relation:

$$\Delta_{abc} = -\Xi \Lambda_{cab}. \tag{52}$$

Given the structure of Λ_{abc} as given in (12d), it will be more convenient to define the following auxiliary zero–quantity:

$$\Lambda_{abcde} \equiv 3\nabla_{[a} d_{bc]de} = \Lambda_{d[ab} g_{c]e} - \Lambda_{e[ab} g_{c]d}. \tag{53}$$

As noticed in [7], this new object arises in a natural way when (2d) is suitably contracted with the 4–volume form ϵ_{abce}. Also, observe that Λ_{abcde} has only two independent divergences. From its definition, and assuming the validity of the wave equation (4d), it follows that

$$\nabla^c \Lambda_{abcde} = 2\nabla_{[a} \Lambda_{b]de}, \tag{54a}$$
$$\nabla^e \Lambda_{abcde} = 3\nabla_{[a} \Lambda_{|d|bc]}. \tag{54b}$$

Moreover, by contracting equation (54a), an additional identity involving the other independent divergence of Λ_{abc} is directly found:

$$\nabla_c \Lambda_{ab}{}^c = \nabla_c \Lambda_{ba}{}^c. \tag{55}$$

Finally, one can observe that the expressions for $\Upsilon_a{}^a$, $\nabla_a \Theta^a$, $\nabla_c \Delta_a{}^c{}_b$ and $\nabla^c \Lambda_{abcde}$ given above represent the geometric wave equations (4a)-(4d), respectively.

A.2 Integrability conditions

Making use of the identities presented in previous section, direct computations yield the following integrability conditions for Υ_{ab} and Θ_a:

$$2\nabla_{[a} \Upsilon_{b]c} = 2g_{c[a} \Theta_{b]} + \Xi \Delta_{abc}, \tag{56a}$$
$$2\nabla_{[a} \Theta_{b]} = -2L_{[a}{}^c \Upsilon_{b]c} + \Delta_{abc} \nabla^c \Xi. \tag{56b}$$

Regarding the other two zero–quantities, equation (54b) represents an integrability condition for Λ_{abc}. The latter, along with identity (52), implies an analogous equation for Δ_{abc}:

$$3\nabla_{[a}\Delta_{bc]d} = -3\Lambda_{d[ab}\nabla_{c]}\Xi - \Xi\nabla_e\Lambda_{bcad}{}^e, \tag{57a}$$
$$3\nabla_{[a}\Lambda_{|d|bc]} = \nabla^e\Lambda_{abcde}. \tag{57b}$$

Equations (56a)–(57b) are the integrability conditions for the zero-quantities associated to vacuum conformal Einstein field equations:

Remark 26. Notice that equation (54a) is also an integrability condition for Λ_{abc}. In this sense, the wave equations obtained form (56a)–(57b) are in general not unique.

A.3 The subsidiary equations

With the previous equations at hand, and assuming that the fields Ξ, s, L_{ab} and d_{abcd} satisfy the wave equations (4a)–(4d), suitable evolution (i.e. wave) equations for the zero-quantities can be obtained by direct application of a covariant derivative. Commuting derivatives, and using the identities exposed in A.1, straightforward but lengthy calculations yield:

$$\nabla_d\nabla^d Z_{ab} = -\Xi Z^{dc}d_{adbc} + \frac{1}{6}Z_{ab}R + Z_b{}^d L_{ad} + 3Z_a{}^d L_{bd} - 2Z^{dc}\ L_{dc}g_{ab} + \nabla_a Z_b + 3\nabla_b Z_a - \Xi\ \Lambda_{dab}\nabla^d\Xi,$$
$$\nabla_b\nabla^b Z_a = 6L_{ab}Z^b - Z^{bc}\Delta_{abc} + 2\Xi L^{bc}\Delta_{abc} + Z^{bc}\nabla_a L_{bc} - \tfrac{1}{6}Z_{ab}\nabla^b R + Z^{bc}\nabla_c L_{ab},$$
$$\begin{aligned}\nabla_d\nabla^d\Delta_{abc} = {}& -4\Lambda_{cab}s - \Xi^2\Lambda_c{}^{de}d_{adbe} + 2\Xi^2\Lambda^d{}_{[a}{}^e d_{b]dce} + \frac{1}{6}\Xi\Lambda_{cab}R - 4\Xi\Lambda^d{}_{ab}L_{cd} + 4\Xi\Lambda^d{}_{c[a}L_{b]d}\\ & -2\Xi^2\Lambda^{def}g_{c[a}d_{b]edf} - 4\Xi g_{c[a}\Lambda^d{}_{b]}{}^e L_{de} - 2\nabla_d\Lambda_{cab}\nabla^d\Xi,\end{aligned}$$
$$\nabla_e\nabla^e\Lambda_{abc} = \Xi\Lambda_a{}^{ef}d_{becf} - 2\Xi\Lambda^e{}_{[b}{}^f d_{c]eaf} + 4\Lambda^e{}_{bc}L_{ae} - 4\Lambda^e{}_{a[b}L_{c]e} + 2\Xi\Lambda^{efd}g_{a[b}d_{c]fed} + 4g_{a[b}\Lambda^e{}_{c]}{}^f L_{ef}.$$

These equations are clearly homogeneous in the zero-quantities and their first derivatives.

A.4 The Bach tensor

The Bach tensor in 4-dimensions is defined as

$$B_{ab} \equiv \nabla^c\nabla_a L_{bc} - \nabla^c\nabla_c L_{ab} - C_{acbd}L^{cd}. \tag{58}$$

It can be written in terms of zero-quantities as

$$B_{ab} = \nabla^c\Delta_{acb} - \Lambda_{abc}\nabla^c\Xi - d_{acbd}Z^{cd}. \tag{59}$$

Thus, for any solution to the conformal Einstein field equations one has that $B_{ab} = 0$. Finally, it is observed that

$$\nabla^a B_{ab} = 0$$

independently of whether the conformal Einstein field equations hold or not.

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