

Constants of motion in stationary axisymmetric gravitational fields

C. Markakis^{1,2*}

¹*Department of Physics, University of Wisconsin-Milwaukee, P.O. Box 413, Milwaukee, WI 53201, USA*

²*Theoretical Physics Institute, University of Jena, 07743 Jena, Germany*

5 November 2018

ABSTRACT

The motion of test particles in stationary axisymmetric gravitational fields is generally nonintegrable unless a nontrivial constant of motion, in addition to energy and angular momentum along the symmetry axis, exists. The Carter constant in Kerr-de Sitter spacetime is the only example known to date. Proposed astrophysical tests of the black-hole no-hair theorem have often involved integrable gravitational fields more general than the Kerr family, but the existence of such fields has been a matter of debate. To elucidate this problem, we treat its Newtonian analogue by systematically searching for nontrivial constants of motion polynomial in the momenta and obtain two theorems.

First, solving a set of quadratic integrability conditions, we establish the existence and uniqueness of the family of stationary axisymmetric potentials admitting a quadratic constant. As in Kerr-de Sitter spacetime, the mass moments of this class satisfy a “no-hair” recursion relation $M_{2l+2} = a^2 M_{2l}$, and the constant is Noether-related to a second-order Killing-Stäckel tensor. Second, solving a new set of quartic integrability conditions, we establish nonexistence of quartic constants. Remarkably, a subset of these conditions is satisfied when the mass moments obey a generalized “no-hair” recursion relation $M_{2l+4} = (a^2 + b^2)M_{2l+2} - a^2 b^2 M_{2l}$. The full set of quartic integrability conditions, however, cannot be satisfied nontrivially by any stationary axisymmetric vacuum potential.

Key words: gravitation – black hole physics – celestial mechanics – chaos – stellar dynamics – galaxies: star clusters

1 INTRODUCTION

Since the work of Euler, Lagrange and Jacobi, the motion of a test particle in a Newtonian dipole field¹ has been known to be completely integrable in terms of quadratures. In addition to energy and angular momentum about the symmetry axis, there exists a third nontrivial constant of motion quadratic in the momenta. The constant was first discovered by Euler (1760, 1764) for two-dimensional (meridional) motion, and the problem is known as the *Euler problem* (Lukyanov, Emeljanov, & Shirmin 2005). Lagrange (1766) extended Euler’s solution to three-dimensional motion and made a further generalization by allowing a Hookian (spring) center to be included between the two Newtonian (gravitational) centers. This generalization is known as the *Lagrange*

problem (cf. Lukyanov et al. 2005). The problem was further studied by Jacobi (2009) using separation of variables of the Hamilton-Jacobi equation in prolate spheroidal coordinates.

The literature regarding the problem of two fixed centers is surveyed by Lukyanov et al. (2005) and the orbits are studied in Ó’Mathúna (2008). The quadratic² constant has been studied by several authors using the Hamilton-Jacobi approach (Stäckel 1890; Whittaker 1989; Eddington 1915; Kuzmin 1956; Lynden-Bell 1962; de Zeeuw 1985a,b,c). If one considers the distance between the two centers to be imaginary, then one obtains a potential separable in oblate spheroidal coordinates. This is known in satellite geodesy as the *Vinti potential* and has been used to approximate the gravitational field around the oblate earth (Vinti 1960, 1963, 1969, 1971; Vinti et al. 1998). A further generalization is possible by relaxing equatorial plane symmetry. In the oblate case, this is accomplished by consider-

* E-mail: charalampos.markakis@uni-jena.de

¹ Following terminology in Misner et al. (1973), a Newtonian dipole will refer to a pair of fixed positive-mass centers each of which creates a Newtonian gravitational field

² A quadratic constant will refer to a constant quadratic in the momenta

ing two Newtonian fixed centers with complex-conjugated masses located at constant imaginary distance. The resulting *Darboux–Gredeaks potential* has been used to approximate the gravitational field around other oblate planets (Aksenov, Grebenikov, & Demin 1963, Lukyanov et al. 2005).

Lynden-Bell (2003) has provided a simple and elegant derivation of the quadratic constant in the Euler problem, by noting that the kinetic part of the constant is the dot product of the angular momenta about the two fixed centers of attraction. He also provided a generalization of the constant for a class of potentials that satisfy a certain integrability condition. It will be demonstrated in section 2.6 that this class is essentially that of the Lagrange problem, that is, it amounts to the addition of a Hooke term to the potential.

Israel (1970) and Keres (1967) have demonstrated that the dipole field of the Euler problem can be regarded as the Newtonian analogue of the Kerr solution in general relativity. Following Misner’s suggestion to seek analogues of the quadratic constant in Newtonian dipole fields, Carter discovered his quadratic constant of motion in Kerr spacetime by separation of variables (Carter 1968, 1977; Misner, Thorne, & Wheeler 1973). The analogy has been further elucidated by Lynden-Bell (2003), Flanagan & Hinderer (2007) and Will (2009).

Carter (1968, 2009b, 2010) also discovered a generalization of the Kerr solution with a non-zero cosmological constant, describing a rotating black hole in four dimensional de Sitter (or anti de Sitter) backgrounds. Carter’s quadratic constant of motion exists for this class of spacetimes as well and has been used to solve for the orbits in terms of hypergeometric functions (Kraniotis 2004, 2005, 2011). It will be shown in section 2.6 that the quadratic constant of the Lagrange two-center problem is the Newtonian analogue of the Carter constant in a Kerr-de Sitter spacetime.

The main motivation behind the study of integrability in stationary axisymmetric gravitational fields is to astrophysically test the Kerr solution and the no-hair theorem for black holes. The Kerr solution owes its integrability to the existence of the Carter constant, which is very useful for analyzing the orbit of a small black hole around a massive black hole. Detecting gravitational waves from these extreme mass ratio inspirals is a prime goal of the proposed Laser Interferometer Space Antenna, LISA/eLISA (Jennrich et al. 2011; Amaro et al. 2012a,b). The influence of gravitational radiation reaction on the evolution of the Carter constant has been used to obtain the gravitational waveforms from orbits around Kerr black holes. As demonstrated by Ryan (1995) (see also Sotiriou & Apostolatos 2005), by measuring the mass, spin and at least one more non-trivial moment of the gravitational field of a black hole candidate via gravitational wave observations, one can test the validity of the no-hair theorem. Such tests have been proposed for the supermassive black hole at the centre of the Milky Way, Sgr A*, as reviewed, for example, by Johannsen (2012). Work by Glampedakis & Babak (2006); Gair et al. (2008); Psaltis & Johannsen (2009); Johannsen & Psaltis (2010a,b, 2011); Psaltis & Johannsen (2011, 2012); Collins & Hughes (2004); Hughes (2006); Dubovsky et al. (2007); Vigeland & Hughes (2010); Vigeland (2010);

Vigeland et al. (2011); Johannsen & Psaltis (2013) is a fraction of the vast literature.

In order to provide a systematic framework for testing the Kerr solution, a number of the above articles have introduced “bumpy” black-hole spacetimes, which are stationary, axisymmetric, vacuum spacetimes with arbitrary multipole moments (deviating from those of Kerr), in terms of which the stellar orbits and their associated observables are parametrized. This parametrization allows one to astrophysically test the relationship between the multipole moments, and thus probe the validity of the Kerr solution in general relativity. Because the arbitrariness of the moments leads to loss of the Carter constant, some authors have sought other stationary axisymmetric vacuum spacetimes that may be integrable, i.e. that admit a generalized Carter constant (Brink 2008a,b, 2010a,b, 2011). However, there exist conflicting claims in the literature about whether such spacetimes actually exist (Mirshekari & Will 2010; Kruglikov & Matveev 2011; Lukes-Gerakopoulos 2012).

The subject of the present article is a systematic approach towards resolving such conflicts and elucidating the relation between the no-hair theorem and integrability of motion in the Newtonian regime. We use a direct approach (Hietarinta 1987) to systematically search for a nontrivial constant polynomial in the momenta. This approach consists of directly solving the Killing equations and certain integrability conditions. In section 2 we show that the constant in the Euler and Lagrange two-center problems is the *unique quadratic constant* for motion around a stationary axisymmetric massive object with equatorial reflection symmetry. Although this result is implicit in other work (cf. Kalnins et al. 2009, 2010; Kalnins 2012) no explicit proof of uniqueness³ existed in the literature prior to that of Will (2009). The merits of the systematic procedure outlined here are that it provides a complete proof (as it involves no assumptions regarding separability or the form of the quadratic constant) and that it is generalizable to more complicated systems with higher-order Killing tensors, as illustrated in subsequent sections. In section 2.7 the quadratic constant is shown to be Noether-related. In section 3 we consider the next natural generalization, a constant *quartic* in the momenta. Using the direct approach, we show that *no such constant exists* for stationary axisymmetric vacuum potentials in Newtonian gravity.

Extrapolating these uniqueness and non-existence theorems into the relativistic regime provides highly suggestive evidence in favour of the conjecture that a stationary axisymmetric vacuum spacetime is integrable if and only if it belongs to the Kerr (or Kerr-de Sitter) class. Besides the theoretical implications, this is important from an observational/astronomical point of view. For example, this conjecture is a working assumption of a number of articles discussing astrophysical tests of the Kerr black hole solution (Apostolatos et al. 2009; Lukes-Gerakopoulos et al. 2010; Lukes et al. 2010; Contopoulos et al. 2011). Conclu-

³ The advantage of the derivation in Will (2009), based on a multipole expansion, is that it establishes uniqueness among all stationary axisymmetric potentials, but this uniqueness is restricted to invariants constructed from a certain combination of the linear or angular momentum and position vectors.

sions, and a discussion of this matter, are given in section 4.

2 INVARIANTS QUADRATIC IN THE MOMENTA

2.1 Killing equation and integrability conditions

The motion of a test particle in a Newtonian gravitational field Φ is independent of the particle mass, so for simplicity we can set the latter equal to unity. This motion is described by an action functional

$$S[x, p] = \int_{t_1}^{t_2} dt [p_i \dot{x}^i - H(x, p)] \quad (1)$$

where position x^i and momentum p_i are treated as independent variables,

$$H(x, p) = \frac{1}{2} g^{ij}(x) p_i p_j + \Phi(x) \quad (2)$$

is the Hamiltonian, g^{ij} is the inverse of the Euclidian metric g_{ij} in \mathbb{E}^3 and summation over repeated indices is implied. Indices are raised and lowered with this metric throughout the paper. Unless otherwise noted, we will be using Cartesian coordinates, so that $g_{ij} = \delta_{ij}$ where δ_{ij} is the Kronecker delta. The Cartesian components of the canonical momentum $p_i = g_{ij} \dot{x}^j$ will then coincide with those of the kinetic momentum \dot{x}^i .

Spatial translations and rotations in \mathbb{E}^3 are generated by the vector fields

$$\mathbf{X}_k = \mathbf{e}_k, \quad \mathbf{R}_i = \varepsilon^k_{ji} x^j \mathbf{e}_k \quad (3)$$

where $\mathbf{e}_k = \{\hat{x}, \hat{y}, \hat{z}\}$ are Cartesian basis vectors and ε_{ijk} is the Levi-Civita tensor. We are interested in orbits of a test particle in the vicinity of a *stationary* and *axisymmetric* massive object, an object whose gravitational potential satisfies

$$\partial_t \Phi = 0 \quad (4)$$

$$\mathcal{L}_\varphi \Phi = \varphi^k \partial_k \Phi = (x \partial_y - y \partial_x) \Phi = 0 \quad (5)$$

Here, \mathcal{L}_φ denotes the Lie-derivative along the Killing vector field

$$\varphi \equiv \mathbf{R}_z = x \hat{y} - y \hat{x} \quad (6)$$

that generates rotations about the z -axis (the axis of symmetry). By virtue of Noether's theorem, invariance with respect to time translations and azimuthal rotations implies respectively conservation of the Hamiltonian H and the component

$$L_z = \varphi^i p_i = R_z^i p_i = x p_y - y p_x \quad (7)$$

of angular momentum. Rotations about the x and y axes, generated by the vector fields \mathbf{R}_x and \mathbf{R}_y are *not* considered symmetries of the problem, so the components $L_x = R_x^i p_i$ and $L_y = R_y^i p_i$ of angular momentum are *not* generally conserved. For example, the potential around a rotating massive object (such as a star or planet) in hydrostationary equilibrium, may be expected to be axisymmetric, but not spherically symmetric, due to the rotationally induced deformation. Without further symmetries, H and L_z are the only independent integrals of motion and, since the motion of

test particles is three-dimensional, the problem is in general non-integrable. Nevertheless, one may seek special types of stationary axisymmetric potentials that admit a third nontrivial constant of motion and are therefore integrable.

If one seeks a constant of motion linear in the momenta, then one is quickly led to rotational or translational symmetries as the only choices, which have been already exhausted as explained above. The next natural step is to seek potentials admitting a nontrivial constant of motion *quadratic* in the momenta,

$$I(x, p) = K^{ij}(x) p_i p_j + K(x), \quad (8)$$

where the symmetric tensor K^{ij} and the scalar K are functions of position. The above quantity is conserved iff it commutes with the Hamiltonian, in the sense of a vanishing Poisson bracket:

$$\frac{dI}{dt} = \{I, H\} \equiv \frac{\partial I}{\partial x^k} \frac{\partial H}{\partial p_k} - \frac{\partial I}{\partial p_k} \frac{\partial H}{\partial x^k} = 0. \quad (9)$$

Substituting the Hamiltonian (2) and the ansatz (8) into the above Poisson bracket yields

$$\{I, H\} = \partial^k K^{ij} p_i p_j p_k + (\partial^j K - 2K^{jk} \partial_k \Phi) p_j \quad (10)$$

where $\partial_k = \frac{\partial}{\partial x^k}$. In order that this Poisson bracket vanish for all orbits, the following necessary and sufficient conditions (Boccaletti & Pucacco 2003) must be satisfied:

$$\partial^{(k} K^{ij)} = 0 \quad (11)$$

$$\partial^j K = 2K^{jk} \partial_k \Phi, \quad (12)$$

where parentheses denote symmetrization over the enclosed indices; that is, $A^{(ij)} = \frac{1}{2}(A^{ij} + A^{ji})$. Eq. (11) is a Killing equation in Euclidian space; a solution K^{ij} to the above equations will be referred to as a Killing-Stäckel tensor (Carter 1977). From Eq. (12), we obtain a necessary, but not sufficient, integrability condition

$$\partial^{[i} (K^{j]k} \partial_k \Phi) = \frac{1}{2} \partial^i \partial^j K = 0 \quad (13)$$

where square brackets denote antisymmetrization over the enclosed indices, that is $A^{[ij]} = \frac{1}{2}(A^{ij} - A^{ji})$.

Our assumptions of stationarity and axisymmetry already guarantee the existence of two independent solutions to the above set of equations. First, the metric itself is already a Killing-Stäckel tensor, because equations (11)–(13) are satisfied by $K^{ij} = \frac{1}{2} g^{ij} = \frac{1}{2} \delta^{ij}$, $K = \Phi$, and the associated conserved quantity is simply the Hamiltonian (2). Secondly, the above equations are also satisfied by the reducible Killing-Stäckel tensor $K^{ij} = \varphi^i \varphi^j$, with $K = 0$, implying conservation of the quantity L_z^2 . This second case is reducible to the linear invariant (7) associated with the axial Killing vector φ^i . We now explore the possibility of a third independent solution that can render the problem integrable.

The conditions (11)–(13) suggest a systematic method (Hietarinta 1987) for obtaining integrable potentials and the associated invariants of motion:

(i) The tensor K^{ij} may be computed by solving Eq. (11) subject to the symmetries of the problem.

(ii) With this tensor known, the integrability condition (13) provides a key restriction on the Newtonian potential Φ . One may solve this condition (e.g. via a multipole method) to obtain a family of integrable potentials.

4 C. Markakis

(iii) Given a family of potentials Φ that satisfy this integrability condition, one may obtain the scalar function K by integrating the components of Eq. (12).

With this method as the basis of our analysis, we proceed to carry out the prescribed steps in more detail.

2.2 Rank-two Killing-Stäckel tensors

One may straightforwardly solve Eq. (11) by noticing that the solution must be polynomial in the Cartesian coordinates. This is easily seen in one or two dimensions (c.f. Appendix A) and can be generalized to arbitrary dimensions and tensor rank (Horwood 2008). In three dimensions, Eq. (11) constitutes an overdetermined system of ten equations for the six independent components of the symmetric tensor K^{ij} . As shown by Horwood (2008), the most general solution to this system is a sum of symmetrized products of the translational and rotational vectors (3):

$$\mathbf{K} = K^{ij} \mathbf{X}_i \otimes \mathbf{X}_j = \mathcal{A}^{ij} \mathbf{X}_i \otimes \mathbf{X}_j + 2\mathcal{B}^{ij} \mathbf{X}_i \otimes \mathbf{R}_j + \mathcal{C}^{ij} \mathbf{R}_i \otimes \mathbf{R}_j \quad (14)$$

The components K^{ij} of the tensor \mathbf{K} are polynomial of second order in the Cartesian coordinates x^i and can be written as

$$K^{ij} = \mathcal{A}^{ij} + 2\mathcal{B}^{k(i} \varepsilon^{j)kl} x^l + \mathcal{C}^{mn} \varepsilon^i{}_{km} \varepsilon^j{}_{ln} x^k x^l \quad (15)$$

where $\mathcal{A}^{ij} = \mathcal{A}^{(ij)}$, $\mathcal{C}^{ij} = \mathcal{C}^{(ij)}$ are *symmetric* 3×3 constant matrices and \mathcal{B}^{ij} is a *nonsymmetric* 3×3 constant matrix. While a superficial counting results in 21 independent coefficients, the actual number of independent coefficients is 20. This is because the main diagonal elements of \mathcal{B}^{ij} appear in K^{ij} only in the combinations $\mathcal{B}^{xx} - \mathcal{B}^{yy}$, $\mathcal{B}^{yy} - \mathcal{B}^{zz}$, $\mathcal{B}^{zz} - \mathcal{B}^{xx}$ and the sum of these three terms is zero.

2.3 Isometries

One may considerably reduce the number of unknown coefficients by imposing known symmetries of the action functional S on the orbital invariant I .

(i) *Stationarity* corresponds to invariance of S under the group action of $(\mathbb{R}, +)$ which represents time translations $t \rightarrow t + \delta t$. This symmetry has already been taken into account, since all quantities have no explicit dependence on time t and an additive term $\partial I / \partial t$ has been set to zero in Eq. (9).

(ii) *Axisymmetry* corresponds to invariance of S under the group action of $\text{SO}(2)$ which represents infinitesimal rotations $\{x, y, z\} \rightarrow \{x + y \delta\varphi, y - x \delta\varphi, z\}$ about the z -axis. This symmetry is generated by the Killing vector (6) and may be imposed by requiring

$$\left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + p_x \frac{\partial}{\partial p_y} - p_y \frac{\partial}{\partial p_x} \right) I = 0 \quad (16)$$

With I given by Eq. (8), the above condition can be shown to be equivalent to the requirement that the functions K^{ij} and K remain unchanged under rotations about the z -axis:

$$\mathcal{L}_\varphi K^{ij} = \varphi^k \partial_k K^{ij} - K^{kj} \partial_k \varphi^i - K^{ki} \partial_k \varphi^j = 0 \quad (17)$$

$$\mathcal{L}_\varphi K = \varphi^k \partial_k K = 0 \quad (18)$$

Evidently, the symmetries (16), (17), (18) are inherited from

the Hamiltonian, which is seen by the fact that these equations are also satisfied by replacing I, K^{ij}, K with $H, \frac{1}{2}\gamma^{ij}, \Phi$ in the above three equations respectively. Substituting the general solution (15) into Eq. (17) yields the axisymmetry constraints

$$\mathcal{A}^{xy} = \mathcal{A}^{xz} = \mathcal{A}^{yz} = \mathcal{A}^{xx} - \mathcal{A}^{yy} = 0 \quad (19)$$

$$\begin{aligned} \mathcal{B}^{xz} = \mathcal{B}^{zx} = \mathcal{B}^{yz} = \mathcal{B}^{zy} \\ = \mathcal{B}^{xx} - \mathcal{B}^{yy} = \mathcal{B}^{xy} + \mathcal{B}^{yx} = 0 \end{aligned} \quad (20)$$

$$\mathcal{C}^{xy} = \mathcal{C}^{xz} = \mathcal{C}^{yz} = \mathcal{C}^{xx} - \mathcal{C}^{yy} = 0 \quad (21)$$

(iii) Further simplification is possible by assuming *equatorial plane reflection symmetry*. This corresponds to invariance of S under the discrete group action of \mathbb{Z}_2 which represents reflections $\{x, y, z, p_x, p_y, p_z\} \rightarrow \{x, y, -z, p_x, p_y, -p_z\}$ about the equatorial plane. (This assumption is not necessary for integrability, c.f. Lynden-Bell (2003), but we retain it for simplicity). Imposing this symmetry on the invariant (8) leads to the constraints

$$\mathcal{A}^{xz} = \mathcal{A}^{yz} = 0 \quad (22)$$

$$\mathcal{B}^{xx} = \mathcal{B}^{yy} = \mathcal{B}^{zz} \quad (23)$$

$$\mathcal{B}^{xy} = \mathcal{B}^{yx} = 0 \quad (24)$$

$$\mathcal{C}^{xz} = \mathcal{C}^{yz} = 0 \quad (25)$$

Note that the constraints (19)–(21) and (22)–(25) are independent, since axisymmetry and reflection symmetry are separate assumptions. Imposing these two sets of constraints on the tensor (15) yields

$$\begin{aligned} \mathbf{K} &= \mathcal{A}^{yy} (\mathbf{X}_x \otimes \mathbf{X}_x + \mathbf{X}_y \otimes \mathbf{X}_y) + \mathcal{A}^{zz} \mathbf{X}_z \otimes \mathbf{X}_z \\ &+ \mathcal{C}^{yy} (\mathbf{R}_x \otimes \mathbf{R}_x + \mathbf{R}_y \otimes \mathbf{R}_y) + \mathcal{C}^{zz} \mathbf{R}_z \otimes \mathbf{R}_z \\ &= \mathcal{C}^{yy} \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} \\ &+ (\mathcal{A}^{zz} - \mathcal{A}^{yy}) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{A}^{yy} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &+ (\mathcal{C}^{zz} - \mathcal{C}^{yy}) \begin{pmatrix} y^2 & -xy & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (26)$$

This is the most general solution to Eq. (11) consistent with the symmetry $(\mathbb{R}, +) \times \text{SO}(2) \times \mathbb{Z}_2$. For clarity, we set $\kappa \equiv \mathcal{C}^{yy}$, $\lambda \equiv \mathcal{A}^{zz}$, $\mu \equiv \mathcal{C}^{zz} - \mathcal{C}^{yy}$ and $a \equiv \sqrt{(\mathcal{A}^{zz} - \mathcal{A}^{yy}) / \mathcal{C}^{yy}}$. The parameter a is allowed to be real or imaginary and will be shown to depend on the gravitational source. (This reparametrization entails no loss of generality, as the sign of $(\mathcal{A}^{zz} - \mathcal{A}^{yy}) / \mathcal{C}^{yy}$ is unrestricted. We have also excluded the possibility of vanishing \mathcal{C}^{yy} with nonvanishing $(\mathcal{A}^{zz} - \mathcal{A}^{yy})$. This case would lead to a Killing-Stäckel tensor $(\mathcal{A}^{zz} - \mathcal{A}^{yy}) \delta_z^i \delta_z^j$, reducible to a Killing vector δ_z^i that generates translations along the z -axis. But this is *not* a symmetry of the problem by assumption, i.e. we have excluded cylindrical symmetry.) The above solution may then be written as

$$K^{ij} = \kappa A^{ij} + \lambda g^{ij} + \mu \varphi^i \varphi^j \quad (27)$$

where $g_{ij} = \delta_{ij}$ is the Euclidian metric, φ is the axial Killing vector (6),

$$\begin{aligned} A^{ij} &= \delta^{kl} R_k^i R_l^j + a^i a^j - a^2 g^{ij} \\ &= (r^2 g^{ij} - x^i x^j) - (a^2 g^{ij} - a^i a^j) \\ &= g^{mn} \varepsilon^i_{km} \varepsilon^j_{ln} (x^k x^l - a^k a^l) \\ &= g^{mn} \varepsilon^i_{km} \varepsilon^j_{ln} (x^k + a^k)(x^l - a^l) \end{aligned} \quad (28)$$

is a new nontrivial Killing-Stäckel tensor, $r = \sqrt{x^k x_k}$ is a radial coordinate and $a^i = a \delta^i_z$ are the components of the vector $\mathbf{a} = a \hat{z}$. With κ, λ, μ regarded as arbitrary coefficients, the general solution (27) is a linear combination of three independent solutions: the known Killing-Stäckel tensors $\gamma^{ij} = \delta^{ij}$ and $\varphi^i \varphi^j$, associated with stationarity and axisymmetry, and a third independent solution A^{ij} associated with a Noether symmetry to be discussed later. The above expression gives the *most general rank-two Killing tensor* in stationary-axisymmetric vacuum potentials. This completes step (i) of the prescribed procedure.

2.4 Integrability condition: no-hair relation

Although the solution (28) satisfies Eq. (11) and the symmetries of the problem, it does not lead to a conserved quantity unless the condition (12) is satisfied. Our next step is thus to use the integrability condition (13) of Eq. (12) to obtain a family of potentials Φ for which A^{ij} leads to a conserved quantity. The Newtonian gravitational field around an axisymmetric object is completely characterized by a set of mass multipole moments $\{M_L\}$, by means of the expansion

$$\Phi = - \sum_{L=0}^{\infty} \frac{M_L}{r^{L+1}} P_L(z/r) \quad (29)$$

where P_L are the Legendre polynomials. This expansion is consistent with a stationary axisymmetric vacuum potential that vanishes at infinity.

If one also requires equatorial plane reflection symmetry, then the potential Φ must be an even function of z , so the odd moments M_1, M_3, \dots must vanish. Then, substituting $K^{ij} = A^{ij}$ and Φ , as given by eqs. (28) and (29), into the integrability condition (13), we find by straightforward algebra that the latter is satisfied *if and only if* the even multipole moments satisfy the recursion relation:

$$M_{L+2} = a^2 M_L \quad (L = 0, 2, \dots) \quad (30)$$

or, equivalently,

$$M_L = a^L M_0 \quad (L = 0, 2, \dots). \quad (31)$$

where a^L denotes the L -th power of a in the above two equations. This relation is analogous to the “no-hair” relation for Kerr black holes (Israel 1970; Hansen 1974; Will 2009). All nonzero multipoles are determined by the mass M_0 and the parameter a . If a is real (imaginary) then the quadrupole moment $M_2 = a^2 M_0$ is positive (negative) and the expansion (29) describes the field of a prolate (oblate) object. Summing the Legendre series (29) using Eq. (31) yields (Trahanas 2004; Will 2009)

$$\Phi = - \frac{M_0/2}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{M_0/2}{\sqrt{x^2 + y^2 + (z+a)^2}} \quad (32)$$

In the prolate case, the above potential is that of Euler’s

three body problem: a test mass moving in the gravitational field of two point sources, each of mass $M_0/2$, fixed at positions $\pm a \hat{z}$.

In the oblate case, the replacement $a \rightarrow i a$ (where $i^2 = -1$) allows one to write the above potential as

$$\Phi = - \frac{M_0}{\rho + a^2 z^2 / \rho^3} \quad (33)$$

where ρ is an ellipsoidal coordinate defined by

$$\frac{x^2 + y^2}{\rho^2 + a^2} + \frac{z^2}{\rho^2} = 1 \quad (34)$$

Vinti (1960, 1963, 1969, 1971); Vinti et al. (1998) has shown that the above potential is the most general solution to the three dimensional Laplace equation that separates the Hamilton-Jacobi equation in oblate spheroidal coordinates. (A similar statement can be made for the potential (32) in prolate spheroidal coordinates). Vinti’s potential has been frequently used in satellite geodesy to approximate the gravitational field around the oblate earth.

We have so far shown that *the solutions (28) and (32) are the unique stationary, axisymmetric, equatorially symmetric, vacuum solutions to eqs. (11) and (13)*. (Note that we have excluded the possibility of cylindrical symmetry as we are interested in the gravitational field of massive objects with compact support near the origin.) This completes step (ii) of our procedure.

2.5 Existence and uniqueness of the quadratic invariant

The last step towards obtaining an invariant is to obtain the scalar contribution K to the invariant (8). This is done by solving the condition (12) with $K^{ij} = A^{ij}$ and Φ given respectively by eqs. (28) and (29). The solution $K = A$ is obtained by integrating the x component of Eq. (12) with respect to x , or the y component with respect to y . Up to some additive constant, we find

$$\begin{aligned} A &= 2 \int dx A^{ix} \partial_i \Phi = 2 \int dy A^{iy} \partial_i \Phi \\ &= \frac{M_0 a (z+a)}{\sqrt{x^2 + y^2 + (z+a)^2}} - \frac{M_0 a (z-a)}{\sqrt{x^2 + y^2 + (z-a)^2}} \end{aligned} \quad (35)$$

Finally, substituting Eq. (28) into (8) and using the Lagrange identity, the quadratic invariant is written

$$\begin{aligned} I &= A^{ij} p_i p_j + A \\ &= r^2 |\mathbf{p}|^2 - |\mathbf{x} \cdot \mathbf{p}|^2 - (a^2 |\mathbf{p}|^2 - |\mathbf{a} \cdot \mathbf{p}|^2) + A \\ &= |\mathbf{x} \times \mathbf{p}|^2 - |\mathbf{a} \times \mathbf{p}|^2 + A \\ &= [(\mathbf{x} + \mathbf{a}) \times \mathbf{p}] \cdot [(\mathbf{x} - \mathbf{a}) \times \mathbf{p}] + A \end{aligned} \quad (36)$$

with A given by Eq. (35). The above expression holds for the prolate case, $a = |a|$. In the oblate case, $a = i|a|$, the above expression still gives a true result. In the spherically symmetric limit, $a \rightarrow 0$, the above invariant reduces to $I \rightarrow |\mathbf{x} \times \mathbf{p}|^2$ which is a natural consequence of conserved net angular momentum. The first integrals H, L_z, I are independent and in involution (that is, $\{H, L_z\} = 0, \{H, I\} = 0$ and $\{I, L_z\} = 0$). Thus, the system is Liouville-integrable.

We have shown that the potential (33) is the unique stationary, axisymmetric, equatorially symmetric, vacuum potential admitting a nontrivial constant of motion quadratic

in the momenta, Eq. (36). The potential (33) can be considered the Newtonian analogue (Keres 1967; Israel 1970; Lynden-Bell 2003; Will 2009) of the Kerr solution in general relativity. The analogy is manifest when the Kerr metric is written in Kerr-Schild coordinates (Kerr & Schild 1965). The associated constant of motion is traditionally obtained by separating the Hamilton-Jacobi equation in Boyer-Lindquist coordinates (Carter 1968; Misner et al. 1973). It was in this way that Carter originally discovered his constant, following Misner's suggestion to seek analogies to the constant (36) in Newtonian dipole fields. This analogy persists in the presence of a cosmological constant as demonstrated in the following section.

Another interesting property is the following: One may interchange the roles of the metric g_{ij} and potential Φ with those of the Killing-Stäckel tensor A_{ij} and scalar field A respectively. Then the momentum map is generated by $I = A^{ij}p_i p_j + A$ (which plays the role of the Hamiltonian) and the quadratic constant of motion is given by $H = \frac{1}{2}g^{ij}p_i p_j + \Phi$. This duality also exists in the relativistic case of the Kerr spacetime (Carter 2009a).

2.6 Cosmological constant: analogy with a Kerr-de Sitter spacetime

One may generalize the result of the previous section in various directions, by relaxing certain assumptions. The assumption that the motion is in vacuum may be relaxed by adding terms with a non-vanishing Laplacian to the multipole expansion (29) of the gravitational potential. For example, one may attempt to add a spherically symmetric power-law potential of the form r^l . Doing so, and repeating the previous steps, leads to the same expression (28) for the Killing tensor A^{ij} as before. Direct substitution shows that the integrability condition (13) is satisfied *if and only if* the moments $\{M_L\}$ satisfy the no-hair relation (30) and the only power law term that can be added is proportional to r^2 . This leads to a potential

$$\Phi = -\frac{M_0/2}{\sqrt{x^2 + y^2 + (z+a)^2}} - \frac{M_0/2}{\sqrt{x^2 + y^2 + (z-a)^2}} - \Lambda(x^2 + y^2 + z^2) \quad (37)$$

In the prolate case, with a real, the above potential is that of two force centers, with Newton's inverse square law of gravitational attraction supplemented by a linear cosmological constant (or Hooke) term. (Note that the last term in the above equation may be regarded as the contribution of a spring connecting the test particle to the origin or as the contribution of two springs connecting the test particle to the two centers at $\pm a \hat{z}$). The associated constant of motion is given by Eq. (36) with

$$A = \frac{M_0 a(z+a)}{\sqrt{x^2 + y^2 + (z+a)^2}} - \frac{M_0 a(z-a)}{\sqrt{x^2 + y^2 + (z-a)^2}} + 2\Lambda a^2(x^2 + y^2) \quad (38)$$

In the oblate case, with a imaginary, the last term may again be interpreted as the contribution of a cosmological constant. With the substitution $a \rightarrow ia$, the potential (37) can be naturally regarded as the Newtonian analogue of a Kerr-de Sitter spacetime and the constant (38) as the analogue of the Carter constant in that spacetime (Carter 1968).

For this spacetime the metric has a tt component given by $g_{tt} = -(1 + 2\Phi)$ in Kerr-Schild coordinates, with Φ given by Eq. (37) and a replaced by ia .

Lynden-Bell (2003) provided a generalization of the quadratic constant (36) and the two-center potential (32). His result holds for potentials satisfying a certain integrability condition, which takes the form of a wave equation, $(\partial^2/\partial r_1^2 - \partial^2/\partial r_2^2)(r_1 r_2 \Phi) = 0$ in his two-center coordinates $\mathbf{r}_{1,2} = \mathbf{r} \pm \mathbf{a}$ (Lynden-Bell 2003, section 3). Assuming equatorial symmetry and substituting the multipole expansion (29) into this integrability condition, we recover the no-hair relation (29) and the potential (32). If we attempt to add power law terms of the form r_1^l, r_2^l to the potential Φ we find that the only possibility is $r_1^2 + r_2^2 = 2(r^2 + a^2)$, giving rise to the potential (37) up to a constant. We infer that Lynden-Bell's generalization accounts for the presence of a cosmological constant term, which gives rise to the Newtonian analogue of the Kerr-de Sitter spacetime discussed above.

The direct approach employed here is more algorithmic and computationally intensive compared to other approaches (Lynden-Bell 2003; Will 2009). But the advantages of the direct approach are that it establishes uniqueness and that it can be straightforwardly generalized to higher order or relativistic invariants. We have shown that the unique stationary, axisymmetric, equatorially symmetric potential admitting an invariant quadratic in the momenta is given by Eq. (37). The assumption of equatorial symmetry may be relaxed (Lynden-Bell 2003) by dropping conditions (22)-(25), but we do not pursue this here.

2.7 Generalized Noether symmetry

The generalized Noether theorem may be stated as follows (Ioannou & Apostolatos 2004). Consider the ϵ -family of infinitesimal transformations

$$x^i \rightarrow x_\epsilon^i = x^i + \epsilon K^i(x, \dot{x}, t) \quad (39)$$

which depend on position *and velocity*, for a small parameter ϵ . If this family of transformations leaves the action $S = \int L dt$ invariant, or, equivalently, changes the Lagrangian L by a total time derivative of some scalar $K(x, t)$,

$$L \rightarrow L_\epsilon = L - \epsilon \frac{dK}{dt} \quad (40)$$

then the quantity

$$I = \frac{\partial L}{\partial \dot{x}^i} K^i + K \quad (41)$$

is a constant of motion. Since the family (39) of transformations is velocity-dependent, it is not generally considered a family of diffeomorphisms. Nevertheless, it is a generalized symmetry of the action and Noether-related to an invariant of the form (41).

Conversely, if the quantity I is a constant of motion, then the ϵ -family of transformations generated by $K^i(x, \dot{x}, t)$, obtained by solving the linear system (Ioannou & Apostolatos 2004)

$$\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} K^i = \frac{\partial I}{\partial \dot{x}^j}, \quad (42)$$

is a generalized symmetry of the action.

The motion of a test particle in a Newtonian gravitational field can be obtained from the Lagrangian $L(x, \dot{x}) = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j + \Phi$. If the motion admits a quadratic invariant $I(x, \dot{x}) = K_{ij}\dot{x}^i\dot{x}^j + K$, then the inverse Noether theorem (42) implies that the ϵ -family of transformations (39) generated by $K^i = K^{ij}\dot{x}_j$ is a generalized symmetry of the action. We infer, for the problem of the previous section with K^{ij} given by Eq. (27), that the action (1) is invariant under three families of infinitesimal transformations:

(i) The Killing-Stäckel tensor $\varphi^i\varphi^j$ is Noether-related to the family of transformations

$$x_\epsilon^i = x^i + \epsilon\varphi^i\varphi^j\dot{x}_j = x^i + \epsilon L_z\varphi^i$$

The tensor $\varphi^i\varphi^j$ is of course reducible to the axial Killing vector φ^i , related to the family of diffeomorphisms $x_\epsilon^i = x^i + \epsilon\varphi^i$. These represent azimuthal rotations and give rise to conservation of angular momentum (7).

(ii) The metric g^{ij} is a Killing-Stäckel tensor and is Noether-related to the family of transformations

$$x_\epsilon^i = x^i + \epsilon g^{ij}\dot{x}_j = x^i + \epsilon\dot{x}^i$$

or, equivalently,

$$x_\epsilon^i(t) = x^i(t) + \epsilon\dot{x}^i(t) = x^i(t + \epsilon)$$

The metric tensor g^{ij} is therefore Noether-related to invariance with respect to time translations $t \rightarrow t + \epsilon$ and gives rise to conservation of energy (or the Hamiltonian) given by Eq. (2).

(iii) The irreducible Killing-Stäckel tensor A^{ij} given by eq. (28) is Noether-related to the family of transformations

$$x_\epsilon^i = x^i + \epsilon A^{ij}\dot{x}_j$$

or, equivalently,

$$\begin{aligned} \mathbf{x}_\epsilon &= \mathbf{x} + \epsilon [(\mathbf{x} \times \dot{\mathbf{x}}) \times \mathbf{x} - (\mathbf{a} \times \dot{\mathbf{x}}) \times \mathbf{a}] \\ &= \mathbf{x} + \epsilon [(\mathbf{x} - \mathbf{a}) \times \dot{\mathbf{x}}] \times (\mathbf{x} + \mathbf{a}) \end{aligned}$$

This a-posteriori knowledge of the symmetry transformation allows a fast “derivation” of the quadratic invariant via the Noether procedure: varying the action of the two-center problem with respect to the above family of transformations yields no change, while the Lagrangian changes by $-\epsilon dA/dt$ with A given by eq. (38). Then, Eq. (41) leads immediately to the quadratic invariant (36).

As mentioned above, these transformations are a symmetry of the action, giving rise to the constant of motion (35), but are not diffeomorphisms since they depend on position and velocity. Nevertheless, writing $\epsilon\dot{x}_j(t) = x^j(t + \epsilon) - x^j(t)$ they can be expressed as transformations in position and time:

$$x_\epsilon^i(t) = x^i(t) - A^{ij}x_j(t) + A^{ij}x_j(t + \epsilon)$$

where A^{ij} is a function of $x^i(t)$.

The Noether theorem and its inverse, expressed by Eqs. (39)-(42), also hold for a relativistic Lagrangian and can be used to show that the four constants of geodesic motion in a Kerr (or Kerr-de Sitter) spacetime are also Noether-related to symmetries of the action. Axisymmetry is related to conservation of angular momentum about the symmetry axis as in case (i) above. Case (ii) discussed above has two analogues in general relativity: Stationarity (invariance of the

Lagrangian under time translations along the integral curves of a timelike Killing vector) is related to conservation of energy or Hamiltonian. Metric affinity (invariance of the action under proper time translations) is related to conservation of the magnitude of four-velocity (or the super-Hamiltonian) and is associated with the four-metric being a Killing tensor. Finally, a family of transformations analogous to those of case (iii) is related to the Carter constant of motion (cf. Padmanabhan 2010, page 381).

3 INVARIANTS QUARTIC IN THE MOMENTA

3.1 Killing equation and integrability conditions

In the limit $a \rightarrow 0$, the rank-2 Killing tensor (28) of the 2-centre potential reduces to a combination $R_x^i R_x^j + R_y^i R_y^j + R_z^i R_z^j$ of the axial Killing vectors R_x^i, R_y^i, R_z^i of the spherically symmetric 1-centre potential. Intuitively, one might then expect a hierarchy, whereby a 4-centre potential (with source centres at $\pm a, \pm b$) admits a rank-4 Killing tensor (reducible to a combination of rank-2 tensors in the 2-centre limit $a \rightarrow b$), an 8-centre potential admits a rank-8 Killing tensor, and so forth. In light of this, we consider the next natural generalization of the previous section, that is, invariants quartic in the momenta

$$I(x, p) = K^{ijkl}(x)p_i p_j p_k p_l + K^{ij}(x)p_i p_j + K(x) \quad (43)$$

associated with a rank-four Killing-Stäckel tensor K^{ijkl} . The above quantity is conserved iff it commutes with the Hamiltonian, in the sense of a vanishing Poisson bracket, Eq. (9). Evaluating the bracket with I given by Eq. (43) and H given by Eq. (2) yields

$$\begin{aligned} \{I, H\} &= \partial^m K^{ijkl} p_i p_j p_k p_l p_m \\ &+ (\partial^k K^{ij} - 4K^{ijkl} \partial_l \Phi) p_i p_j p_k \\ &+ (\partial^k K - 2K^{ik} \partial_i \Phi) p_k \end{aligned} \quad (44)$$

Demanding strong integrability, that is, requiring that this Poisson bracket vanish for all orbits, we have

$$\partial^{(m} K^{ijkl)} = 0 \quad (45)$$

$$\partial^{(k} K^{ij)} = 4K^{ijkl} \partial_l \Phi \quad (46)$$

$$\partial^k K = 2K^{ik} \partial_i \Phi \quad (47)$$

As in the previous section, a solution to Eq. (47) exists only if the following integrability condition is satisfied:

$$\partial^{[i} (K^{j]k} \partial_k \Phi) = 0 \quad (48)$$

which is identical to condition (13). The above set of equations is employed in Hietarinta (1987) to find quartic invariants for various two-dimensional systems.

Eq. (46) may be regarded as an inhomogeneous Killing equation and is also subject to an integrability condition. The general solution to this equation and its integrability condition are obtained in Appendix A. We find that a solution to Eq. (46) in \mathbb{E}^2 exists only if the following integrability condition is satisfied:

$$\begin{aligned} \partial_{zzz} (K^{yyyy} \partial_i \Phi) - 3\partial_{zzy} (K^{yyzi} \partial_i \Phi) = \\ \partial_{yyy} (K^{zzzi} \partial_i \Phi) - 3\partial_{yyz} (K^{yzzz} \partial_i \Phi) \end{aligned} \quad (49)$$

8 C. Markakis

where i is summed over the y and z components and $\partial_{ij\dots k}$ is an abbreviation for $\partial_i\partial_j\dots\partial_k$. Extending this integrability condition to \mathbb{E}^n is straightforward and, as shown in Appendix A, given by

$$\begin{aligned} & \partial_{nml}f_{ijk} - \partial_{nmi}f_{ljk} - \partial_{njl}f_{imk} + \partial_{nji}f_{lmk} \\ & - \partial_{kml}f_{ijn} + \partial_{kmi}f_{ljn} + \partial_{kjl}f_{imn} - \partial_{kji}f_{lmn} = 0 \end{aligned} \quad (50)$$

where $f^{ijk} = 4K^{ijkl}\partial_l\Phi$. Eq. (49) follows from (50) by setting $n = m = i = z$ and $i = j = k = y$. Eq. (49) will suffice for our present purposes, as we shall restrict attention to two-dimensional motion in section 3.3 and beyond. If, in addition to H and L_z , a third constant of motion exists for all initial conditions in three dimensions, then this constant ought to also be conserved in the special case of orbits with $L_z = 0$ that lie on a meridional plane. That is, three dimensional motion is integrable only if meridional motion is integrable. We may thus set $x = 0$ and study motion restricted on the meridional (y - z) plane first. If such motion is found to be integrable, generalization to three dimensions is straightforward by virtue of axisymmetry.

Eqs. (45)–(49) suggest a systematic method for obtaining potentials admitting quartic invariants:

(i) The tensor K^{ijkl} may be computed by solving Eq. (45) subject to the symmetries of the problem.

(ii) With this tensor known, the integrability condition (49) provides a restriction on the Newtonian potential Φ . Solving this condition yields a family of possibly integrable potentials.

(iii) Given a family of potentials Φ which satisfy the condition (49), one may obtain the tensor K^{ij} by solving the inhomogenous Killing equation (46).

(iv) With K^{ijkl} and K^{ij} known, Eq. (49) provides a further restriction on the potential Φ .

(v) If a family of potentials Φ are found that satisfy the above integrability conditions, then one may obtain the scalar function K by integrating the components of Eq. (47).

We now proceed to carry out the above steps in more detail.

3.2 Rank-four Killing-Stäckel tensors

In \mathbb{E}^3 , Eq. (45) constitutes an overdetermined system of 21 equations for the 15 unknown independent components of the symmetric tensor K^{ijkl} . Similarly, in \mathbb{E}^2 , Eq. (45) is a system of 6 equations for 5 unknowns. The system may be solved by noticing that each component of K^{ijkl} must be a fourth order polynomial in the Cartesian coordinates x^i . Horwood (2008) considered the equation $\partial^{(j}K^{i_1\dots i_N)} = 0$ for a tensor $K^{i_1\dots i_N}$ of arbitrary valence N and showed that the general solution is given by

$$\begin{aligned} \mathbf{K} = & \sum_{L=0}^N \binom{N}{L} C_L^{i_1\dots i_L i_{L+1}\dots i_N} \mathbf{X}_{i_1} \otimes \dots \\ & \dots \otimes \mathbf{X}_{i_L} \otimes \mathbf{R}_{i_{L+1}} \otimes \dots \otimes \mathbf{R}_{i_N} \end{aligned} \quad (51)$$

where the objects $C_L^{i_1\dots i_N}$, labelled by $L = 0, \dots, N$, are constant and subject to the symmetries $C_L^{i_1\dots i_L j_{L+1}\dots j_N} = C_L^{(i_1\dots i_L)(j_{L+1}\dots j_N)}$. The above expression, for $N = 4$ and

with $\mathbf{X}_i, \mathbf{R}_j$ given by Eq. (3), provides the general solution to Eq. (45). Imposing known isometries on the quartic invariant (43) constrains the coefficients $C_L^{i_1\dots i_N}$

Stationarity has already been imposed (as the system is autonomous). For three dimensional motion, axisymmetry may be imposed via the requirement that the above Killing-Stäckel tensor is Lie-derived by the axial vector $\boldsymbol{\varphi} = \mathbf{R}_z$, that is

$$\begin{aligned} \mathcal{L}_{\boldsymbol{\varphi}} K^{ijkl} = & \varphi^m \partial_m K^{ijkl} - K^{mjkl} \partial_m \varphi^i - K^{mikl} \partial_m \varphi^j \\ & - K^{mijl} \partial_m \varphi^k - K^{mijk} \partial_m \varphi^l = 0 \end{aligned} \quad (52)$$

(For purely meridional motion, enforcing the above condition does not change the relevant coefficients of K^{ijkl} projected to the meridional plane.) In addition, we also require \mathbb{Z}_2 reflection symmetry about the equatorial plane, that is, the replacement $\{z, p_z\} \rightarrow \{-z, -p_z\}$ leaves the quantity $K^{ijkl} p_i p_j p_k p_l$ unchanged. Since the latter quantity is a fourth order polynomial in the (Cartesian) position and momentum variables, imposing the above isometries is straightforward via algebraic manipulation software such as Mathematica. Imposing the above symmetries significantly reduces the number of independent polynomial coefficients. Then, after dropping trivial terms such as $g^{(ij}g^{kl)}$, $g^{(ij}\varphi^k\varphi^l)$, $\varphi^i\varphi^j\varphi^k\varphi^l$ which correspond to reducible Killing-Stäckel tensors, we find that the most general nontrivial solution to (45) subject to the symmetry $(\mathbb{R}, +) \times \text{SO}(2) \times \mathbb{Z}_2$ can be written in the remarkably simple form

$$K^{ijkl} = \kappa A^{(ij} B^{kl)} + \mu g^{(ij} C^{kl)} \quad (53)$$

where A^{ij} is given by Eq. (28) and B^{ij}, C^{ij} are given by the same equation with the replacements $\mathbf{a} \rightarrow \mathbf{b}$, $\mathbf{a} \rightarrow \mathbf{c}$ respectively:

$$B^{ij} = g^{mn} \varepsilon^i_{km} \varepsilon^j_{ln} (x^k + b^k)(x^l - b^l) \quad (54)$$

$$C^{ij} = g^{mn} \varepsilon^i_{km} \varepsilon^j_{ln} (x^k + c^k)(x^l - c^l) \quad (55)$$

where $\mathbf{b} = b \hat{\mathbf{z}}$, $\mathbf{c} = c \hat{\mathbf{z}}$ and a, b, c are arbitrary parameters. It will be shown in Sec. 3.4 that $\mu = 0$, leaving the first term $A^{(ij} B^{kl)}$ in Eq. (53) as the only possibility for an irreducible Killing tensor of rank-4. Note that the above expressions were obtained in two dimensions. In three dimensions, additional reducible terms constructed from combinations of the axial Killing vector φ^i and other Killing tensors (φ^j or g^{kl}) will be present, which vanish for meridional motion. These reducible terms can be dropped, by virtue of energy and angular momentum conservation. The combination $A^{(ij} B^{kl)}$ is thus the only nontrivial possibility in both two and three dimensions. This completes step (i) of our prescribed procedure.

3.3 First integrability condition: no-hair relation

With the tensor K^{ijkl} known, the next step consists of finding the class of gravitational potentials Φ that satisfy the integrability condition (49). A stationary axisymmetric vacuum potential that vanishes at infinity is characterized by the multipole expansion (29). Assuming equatorial plane reflection symmetry, the odd moments M_L ($L = 1, 3, \dots$) vanish. Restricting attention to purely meridional motion, with K^{ijkl} and Φ given by eqs. (53) and (29), we find with straightforward algebra that the integrability condition (49)

is satisfied if and only if the even multipole moments satisfy the two-step recursion relation

$$\kappa M_{L+4} = \left[\kappa(a^2 + b^2) - \frac{\mu}{8} \right] M_{L+2} - \left(\kappa a^2 b^2 - \frac{\mu}{8} c^2 \right) M_L \quad (56)$$

If $\kappa = 0$ and $\mu \neq 0$, the above relation reduces to the “no-hair” relation (30) and Eq. (53) gives a reducible rank-4 Killing tensor, constructed from the rank-2 Killing tensor (29) and the metric g^{ij} .

If $\kappa \neq 0$ and $\mu = 0$, the above relation simplifies to

$$M_{L+4} = (a^2 + b^2)M_{L+2} - a^2 b^2 M_L \quad (57)$$

which generalizes the “no-hair” relation (30). One may obtain all higher moments recursively from the first two non-vanishing moments. This yields

$$M_L = \frac{(a^2 b^L - a^L b^2)M_0 + (a^L - b^L)M_2}{a^2 - b^2} \quad (58)$$

for L even (and $M_L = 0$ for L odd). In the limit $b \rightarrow 0$, one recovers the no-hair relations (30) and (31) (except for $L = 0$). The limit $b \rightarrow a$ is slightly more subtle. Although the $b \rightarrow a$ limit of eq. (57) is satisfied by (30) the converse is not necessarily true, because Eq. (57) does not fix the relation between the first two moments. Applying Eq. (57) recursively after taking the limit $b \rightarrow a$ (or taking the same limit of Eq. (58) and applying the l’Hôpital rule) yields

$$\lim_{b \rightarrow a} M_L = a^L \left[M_0 + \frac{L}{2} (a^{-2} M_2 - M_0) \right] \quad (59)$$

If the mass and quadrupole moment are related by $M_2 = a^2 M_0$ then one recovers Eq. (31), but this need not be the case and M_2 is generally considered independent of M_0 . If $a \neq b$, then the gravitational field depends on four independent parameters (M_0, M_2, a, b) and the reparametrization $m_a = (M_2 - b^2 M_0)/(a^2 - b^2)$, $m_b = (M_2 - a^2 M_0)/(b^2 - a^2)$ allows us to write Eq. (58) in the suggestive form

$$M_L = a^L m_a + b^L m_b \quad (60)$$

Then, summing the Legendre series (29) yields the potential

$$\Phi = - \frac{m_a/2}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{m_a/2}{\sqrt{x^2 + y^2 + (z+a)^2}} - \frac{m_b/2}{\sqrt{x^2 + y^2 + (z-b)^2}} - \frac{m_b/2}{\sqrt{x^2 + y^2 + (z+b)^2}} \quad (61)$$

created by four fixed point sources with mass $m_a/2$ at positions $\pm a \hat{z}$ and mass $m_b/2$ at positions $\pm b \hat{z}$.

If $\kappa \neq 0$ and $\mu \neq 0$, one may introduce new parameters α, β such that $\kappa(a^2 + b^2) - \frac{\mu}{8} = \kappa(\alpha^2 + \beta^2)$ and $\kappa a^2 b^2 - \frac{\mu}{8} c^2 = \kappa \alpha^2 \beta^2$. Eq. (56) then takes the form

$$M_{L+4} = (\alpha^2 + \beta^2)M_{L+2} - \alpha^2 \beta^2 M_L \quad (62)$$

which is analogous to Eq. (57). One can then proceed as in the previous case and obtain the multipole moments

$$M_L = \alpha^L m_\alpha + \beta^L m_\beta \quad (63)$$

of a potential analogous to (61), with a, b replaced by α, β .

Our solution procedure guarantees that Eq. (60) (or its analogue for nonzero μ , Eq. (63)) gives *the unique vacuum potential with the symmetry* $(\mathbb{R}, +) \times \text{SO}(2) \times \mathbb{Z}_2$ *compatible with the integrability condition* (49). This completes step (ii) of the prescribed procedure.

3.4 Second integrability condition: nonexistence of quartic invariants

With Φ obtained from Eq. (61) ($\kappa \neq 0$ and $\mu = 0$) or (63) ($\kappa \neq 0$ and $\mu \neq 0$), one may use eqs. (A6)-(A9) to solve the inhomogeneous Killing equation (46).

In the case $\kappa \neq 0$, $\mu = 0$, we find

$$K^{ij} = 2BA^{ij} + 2AB^{ij} + (b^2 - a^2)(\tilde{A} - \tilde{B})(z\delta_y^i \delta_y^j - y\delta_y^i \delta_z^j) + \nu D^{ij} \quad (64)$$

where ν is a constant, A is given by Eq. (35) with the replacement $M_0 \rightarrow 2m_a$, \tilde{A} is given by

$$\tilde{A} = \frac{m_a(z+a)}{\sqrt{y^2 + (z+a)^2}} + \frac{m_a(z-a)}{\sqrt{y^2 + (z-a)^2}} \quad (65)$$

and B, \tilde{B} are given by Eqs. (35), (65) with the replacements $a \rightarrow b$ and $m_a \rightarrow m_b$. The additive contribution C^{ij} to Eq. (64) is a solution to the homogenous Killing equation (11), which is polynomial of second order in the cartesian coordinates. Since the homogeneous solution must obey the symmetries of the problem, it must have the form of Eq. (28), with a replaced by some other parameter c , that is

$$D^{ij} = g^{mn} \varepsilon^i_{km} \varepsilon^j_{ln} (x^k + d^k)(x^l - d^l) \quad (66)$$

where $\mathbf{d} = d \hat{z}$.

In the case $\kappa \neq 0$, $\mu \neq 0$, one may proceed in a similar way. However, upon substitution of the resulting expressions into (A8), (A9), we find that the two relations yield different results, unless $\mu = 0$. Thus, since K^{ij} must be symmetric, we are left with $\kappa \neq 0$, $\mu = 0$, and Eq. (57), as the only viable possibility. This completes step (iii) of our prescribed procedure.

The next step is to solve the second integrability condition (48) for a potential with moments given by Eq. (57). However, substituting Eqs. (61) and (64) into the condition (48), we find that the latter *cannot be satisfied* except in the limit $b \rightarrow a$, whence the quartic invariant is reducible to the quadratic invariant of the previous section.

We infer that *there exists no stationary, axisymmetric, equatorially symmetric vacuum Newtonian potential that admits an independent nontrivial invariant quartic in the momenta*. That is, the only quartic invariants are trivial products of lower-order invariants.

The proof of this nonexistence result used the fact that strong integrability requires existence of a constant of motion for all values of energy E and angular momentum L_z , including the case of purely meridional orbits with $L_z = 0$. Failure to satisfy Eq. (49) means that there exists no independent constant for purely meridional motion; therefore there exists no strong integral for three-dimensional motion. One could alternatively use the integrability condition (50) to derive the same nonexistence result directly in \mathbb{E}^3 .

As mentioned earlier, this section was motivated by the intuitive expectation that 4-centre potentials may admit rank-4 Killing tensors, reducible to a combination of rank-2 tensors in the appropriate limit. Remarkably, this expectation was partially fulfilled, in the sense that the most general rank-4 tensor consistent with stationarity, axisymmetry and equatorial symmetry, given by Eq. (53), is precisely such a combination and that its integrability conditions (49) lead uniquely to 4-centre potentials, given by Eq. (61). Nevertheless, since the remaining integrability conditions (48) are not

satisfied, this tensor does not give rise to a quartic invariant. Simple considerations based on Poisson brackets show that integrability is a non-linear property, in the sense that a linear superposition of integrable Newtonian potentials need not be also integrable. In view of this, the integrability of 2-centre potentials (and their relativistic analogues) is exceptional, but the non-integrability of 4-centre potentials is not surprising.

The nonexistence of a quartic invariant does not nevertheless preclude the possibility that the potential (61) admits some other constant of motion with different dependence in the momenta. A superficial study of Poincaré maps of three dimensional orbits in the potential (61) may show that most orbits appear to be regular. This could lead to the (false) impression that the system is integrable. However, a thorough scan of initial conditions reveals the existence of *Birkhoff chains* and in some cases *ergodic motion* surrounds the main island of stability on the Poincaré maps, confirming the non-integrability of 4-centre potentials. In a relativistic context, a similar behaviour has been observed for orbits in certain stationary axisymmetric spacetimes (Apostolatos et al. 2009; Lukes-Gerakopoulos et al. 2010; Contopoulos et al. 2011).

4 SUMMARY AND CONCLUSIONS

The aim of this work was to use a direct approach in order to study polynomial constants of motion in stationary axisymmetric gravitational fields in a Newtonian context and to gain insight into analogous problems in relativistic gravity. The results established via this direct search include:

- (i) the uniqueness of the constant (35)-(36) of the Euler problem: Eq. (32) gives the only stationary, axisymmetric, equatorially symmetric Newtonian vacuum potential admitting a Killing-Stäckel tensor of rank two;
- (ii) the uniqueness of the constant (36)-(38) of the Lagrange problem: Eq. (37) gives the only stationary, axisymmetric, equatorially symmetric Newtonian potential admitting a Killing-Stäckel tensor of rank two;
- (iii) the relation of the quadratic constant in the Lagrange problem to that of Lynden-Bell (2003), and the analogy with the Carter constant in a Kerr-de Sitter spacetime;
- (iv) the integrability conditions (49), (50) which do not appear to have been implemented previously in direct searches for quartic invariants (c.f. Hietarinta 1987).
- (v) the non-existence of stationary, axisymmetric, equatorially symmetric, vacuum Newtonian potentials admitting a Killing-Stäckel tensor of rank four.

Note that the assumptions of vacuum and equatorial symmetry may be relaxed without loss of integrability in the two-center problem (Lynden-Bell 2003). Note also that the integrability condition (49) and its generalization to arbitrary dimension, Eq. (50), can be quite useful for other direct searches of quartic invariants in physical systems.

Electromagnetic or gravitational-wave observations of orbital motion around a massive black hole at the galactic centre can probe its gravitational field and test the validity of the Kerr solution and the no-hair theorem. To this end, several authors have explored the possibility of spacetime mapping by modelling the central ob-

ject as a bumpy black hole (cf. the review by Johannsen 2012 and references therein). To this end, Gair et al. (2008) and Brink (2008a, 2010a,b) suggested the possibility of using integrable stationary axisymmetric spacetimes of the Manko-Novikov type. However, a conjecture on existence of a quartic constant of motion (Brink 2011) was later disproven by (Kruglikov & Matveev 2011) and (Lukes-Gerakopoulos 2012) for the case of the Zipoy-Voorhees metric. Mirshekari & Will (2010) provide a non-existence proof for the Bach-Weyl spacetime. Although non-existence results have been obtained for particular stationary axisymmetric spacetimes, no such result has been established for stationary axisymmetric spacetimes with arbitrary multipole moments.

The present work provides a non-existence proof of quartic invariants for generic stationary axisymmetric vacuum gravitational fields in the Newtonian regime. Extending the present analysis to the relativistic regime is a nontrivial task. However, the methods employed here provide useful intuition for treating the analogous problem in relativistic gravity or post-Newtonian approximations to it. In particular, if a stationary axisymmetric system is nonintegrable in the Newtonian limit, it is unlikely to be integrable in the relativistic regime (although the converse is not true). This is consistent with Kruglikov & Matveev (2011) and Lukes-Gerakopoulos (2012) and provides evidence in favour of the conjecture that a stationary, axisymmetric, equatorially symmetric, vacuum (modulo a cosmological constant) spacetime is integrable if and only if it belongs to the Kerr (or Kerr-de Sitter) class. Therefore, attempts to seek stationary axisymmetric and equatorially symmetric vacuum solutions in general relativity (other than Kerr) that admit irreducible polynomial invariants are unlikely to yield positive results.

This can lead to further insights on how to parametrize the departure of a spacetime geometry from that of a Kerr spacetime and the relation of this departure to integrability or ergodicity. In particular, if the nonexistence conjecture is true, at least two conclusions can be drawn. First, ergodic geodesics exist if and only if the spacetime geometry deviates from Kerr. This is in agreement with numerical evidence in Apostolatos et al. (2009); Lukes-Gerakopoulos et al. (2010); Lukes et al. (2010); Contopoulos et al. (2011), although a general proof is still lacking. Second, approaches towards modelling bumpy black holes as integrable, such as those attempted by Gair et al. (2008); Brink (2008a, 2010a,b) are less likely to be successful than approaches that do not require integrability, such as those based on canonical perturbation theory (Vigeland 2010; Vigeland & Hughes 2010).

As mentioned earlier, the uniqueness and non-existence proofs for Newtonian gravity are strongly suggestive of similar behaviour in General Relativity. A rotating black hole's mass moments are identical to those of two Newtonian centers fixed at imaginary distance from each other. Thus, the Euler (and Lagrange) problems can qualitatively capture many features (such as integrability of motion, cosmological constant, resonant frequencies, separability of wave equations and other non-gravitomagnetic phenomena) of their relativistic counter-parts. Newtonian systems can therefore serve as simple toy-models whose study as surrogates of Kerr and non-Kerr spacetimes offers valuable insights and opens interesting questions. For example, in Newtonian gravity,

the quadratic constant is known to extend beyond equatorial plane symmetry (Aksenov, Grebenikov, & Demin 1963; Lukyanov et al. 2005; Lynden-Bell 2003). Since the Kerr family is equatorially symmetric, this raises the question of whether other stationary axisymmetric solutions to the vacuum Einstein equations exist that still admit a quadratic constant of motion but are not equatorially symmetric.

ACKNOWLEDGMENTS

The author thanks T. Apostolatos, J. Brink, B. Carter, É. É. Flanagan, J. L. Friedman, K. Glampedakis and G. Pappas and the anonymous referees for fruitful suggestions and comments. The author also thanks N. K. Johnson-McDaniel for corrections to the manuscript and for generalizing Eq. (49) to Eq. (50) and G. Lukes-Gerakopoulos for useful comments and for providing Poincaré maps of the four-center potential. This work was supported in part by the Greek State Scholarships Foundation, by NSF Grant PHY1001515 and by DFG grant SFB/Transregio 7 “Gravitational Wave Astronomy”.

REFERENCES

Aksenov E. P., Grebenikov E. A., Demin V. G., 1963, *Soviet Astronomy*, 7, 491
 Amaro P. et al., 2012a, arXiv:1202.0839, 20
 Amaro P. et al., 2012b, arXiv:1201.3621
 Apostolatos T., Lukes-Gerakopoulos G., Contopoulos G., 2009, *Physical Review Letters*, 103, 111101
 Boccaletti D., Pucacco G., 2003, *Theory of Orbits: Volume 1: Integrable Systems and Non-perturbative Methods*. Springer, p. 406
 Brink J., 2008a, *Physical Review D*, 78, 102001
 Brink J., 2008b, *Physical Review D*, 78, 102002
 Brink J., 2010a, *Physical Review D*, 81, 022001
 Brink J., 2010b, *Physical Review D*, 81, 022002
 Brink J., 2011, *Physical Review D*, 84, 104015
 Carter B., 1968, *Communications in Mathematical Physics* (1965-1997), 10, 280
 Carter B., 1977, *Physical Review D*, 16, 3395
 Carter B., 2009a, Personal communication
 Carter B., 2009b, *General Relativity and Gravitation*, 41, 2873
 Carter B., 2010, *General Relativity and Gravitation*, 42, 653
 Collins N., Hughes S., 2004, *Physical Review D*, 69, 124022
 Contopoulos G., Lukes-Gerakopoulos G., Apostolatos T. A., 2011, *International Journal of Bifurcation and Chaos*, 21, 2261
 de Zeeuw T., 1985a, *Monthly Notices of the Royal Astronomical Society*, 216, 599
 de Zeeuw T., 1985b, *Monthly Notices of the Royal Astronomical Society*, 216, 273
 de Zeeuw T., 1985c, *Monthly Notices of the Royal Astronomical Society*, 215, 731
 Dubovsky S., Tinyakov P., Zaldarriaga M., 2007, *Journal of High Energy Physics*, 2007, 083
 Eddington A. S., 1915, *Monthly Notices of the Royal Astronomical Society*, 76, 37
 Euler L., 1760, *Mem. Berlin*, 228

Euler L., 1764, *Novi Commet. Acad. Scient. Imperial. Petropolit.*, 10, 207
 Flanagan E., Hinderer T., 2007, *Physical Review D*, 75, 124007
 Gair J., Li C., Mandel I., 2008, *Physical Review D*, 77
 Glampedakis K., Babak S., 2006, *Classical and Quantum Gravity*, 23, 4167
 Hansen R. O., 1974, *Journal of Mathematical Physics*, 15, 46
 Hietarinta J., 1987, *Physics Reports*, 147, 87
 Horwood J., 2008, *Journal of Geometry and Physics*, 58, 487
 Hughes S. A., 2006in , *AIP*, pp. 233–240
 Ioannou P. J., Apostolatos T. A., 2004, *Elements of Theoretical Mechanics* (in Greek), 1st edn. Leader Books, Athens, p. 421
 Israel W., 1970, *Physical Review D*, 2, 641
 Jacobi C. G. J., 2009, *Jacobi’s Lectures on Dynamics*, 2nd edn., Clebsch A., ed. Hindustan Book Agency, New Delhi, p. 339
 Jennrich O. et al., 2011, *NGO assessment study report* (Yellow Book). Tech. rep.
 Johannsen T., 2012, *Advances in Astronomy*, 2012
 Johannsen T., Psaltis D., 2010a, *The Astrophysical Journal*, 716, 187
 Johannsen T., Psaltis D., 2010b
 Johannsen T., Psaltis D., 2011, *Physical Review D*, 83, 124015
 Johannsen T., Psaltis D., 2013, *The Astrophysical Journal*, 773, 57
 Kalnins E. G., 2012, *Symmetry, Integrability and Geometry: Methods and Applications*, 8, 034
 Kalnins E. G., Kress J. M., Miller W., 2010, *Journal of Physics A: Mathematical and Theoretical*, 43, 265205
 Kalnins E. G., Kress J. M., Miller Jr W., 2009
 Keres H., 1967, *Soviet Journal of Experimental and Theoretical Physics*, 25, 504
 Kerr R. P., Schild A., 1965, *Proc. Symp. Appl. Math.*, 17
 Kraniotis G. V., 2004, *Classical and Quantum Gravity*, 21, 4743
 Kraniotis G. V., 2005, *Classical and Quantum Gravity*, 22, 4391
 Kraniotis G. V., 2011, *Classical and Quantum Gravity*, 28, 085021
 Kruglikov B. S., Matveev V. S., 2011, arXiv:1111.4690
 Kuzmin G. G., 1956, *Astr. Zh.*, 33
 Lagrange J.-L., 1766, *Oeuvres de Lagrange*, 2, 67
 Lukes G., Apostolatos T. A., Contopoulos G., 2010, *Physical Review D*, 81, 25
 Lukes-Gerakopoulos G., 2012, *Physical Review D*, 86, 044013
 Lukes-Gerakopoulos G., Apostolatos T. A., Contopoulos G., 2010, *Physical Review D*, 81, 124005
 Lukyanov L. G., Emeljanov N. V., Shirmin G. I., 2005, *Cosmic Research*, 43, 186
 Lynden-Bell D., 1962, *Monthly Notices of the Royal Astronomical Society*, 124, 95
 Lynden-Bell D., 2003, *Monthly Notices of the Royal Astronomical Society*, 338, 208
 Mirshekari S., Will C. M., 2010, *Classical and Quantum Gravity*, 27, 235021
 Misner C. W., Thorne K. S., Wheeler J. A., 1973, *Gravitation*. W. H. Freeman, p. 1215

- Ó'Mathúna D., 2008, Integrable Systems in Celestial Mechanics. Birkhäuser Boston, p. 244
- Padmanabhan T., 2010, Gravitation: Foundations and Frontiers. Cambridge University Press, p. 728
- Psaltis D., Johannsen T., 2009, 4
- Psaltis D., Johannsen T., 2011, Journal of Physics: Conference Series, 283, 012030
- Psaltis D., Johannsen T., 2012, The Astrophysical Journal, 745, 1
- Ryan F., 1995, Physical Review D, 52, 5707
- Sotiriou T. P., Apostolatos T. A., 2005, Phys. Rev. D, 71, 044005
- Stäckel P., 1890, Math. Ann., 35
- Trahanas S., 2004, Partial Differential Equations (in Greek). Crete University Press, Heraklion, Crete
- Vigeland S., 2010, Physical Review D, 82, 104041
- Vigeland S., Yunes N., Stein L., 2011, Physical Review D, 83, 104027
- Vigeland S. J., Hughes S. A., 2010, Physical Review D, 81, 024030
- Vinti J. P., 1960, The Astronomical Journal, 65, 353
- Vinti J. P., 1963, in The Use of Artificial Satellites for Geodesy, Veis G., ed., Interscience Publishers, Amsterdam, p. 12
- Vinti J. P., 1969, The Astronomical Journal, 74, 25
- Vinti J. P., 1971, Celestial Mechanics, 4, 348
- Vinti J. P., Der G. J., Bonavito N. L., 1998, Orbital and celestial mechanics. American Institute of Aeronautics and Astronautics, p. 409
- Whittaker E. T., 1989, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. Cambridge University Press, p. 480
- Will C., 2009, Physical Review Letters, 102

APPENDIX A: INTEGRABILITY CONDITIONS AND INHOMOGENEOUS KILLING EQUATIONS

We consider the inhomogeneous Killing equation

$$\partial_{(k}K_{ij)} = f_{ijk} \quad (\text{A1})$$

where $f^{ijk} = 4K^{ijkl}\partial_l\Phi$. In \mathbb{E}^2 , the cartesian components of the above equation read

$$\partial_y K_{yy} = f_{yyy} \quad (\text{A2})$$

$$\partial_z K_{yy} + 2\partial_y K_{yz} = 3f_{yyz} \quad (\text{A3})$$

$$\partial_y K_{zz} + 2\partial_z K_{zy} = 3f_{yzz} \quad (\text{A4})$$

$$\partial_z K_{zz} = f_{zzz} \quad (\text{A5})$$

The above system is overdetermined and may be solved as follows: Integrating Eq. (A2) with respect to y and Eq. (A5) with respect to z yields

$$K_{yy} = Z(z) + \int dy f_{yyy} \quad (\text{A6})$$

$$K_{zz} = Y(y) + \int dz f_{zzz} \quad (\text{A7})$$

where $Z(z)$ and $Y(y)$ are scalar functions of their arguments. Then, integrating Eq. (A3) with respect to y and Eq. (A4)

with respect to z yields

$$K_{yz} = \zeta(z) + \frac{1}{2} \int dy (3f_{yyz} - \partial_z K_{yy}) \quad (\text{A8})$$

$$K_{zy} = \psi(y) + \frac{1}{2} \int dz (3f_{yzz} - \partial_y K_{zz}) \quad (\text{A9})$$

where $\zeta(z)$ and $\psi(y)$ are scalar functions of their arguments. Eqs. (A6)-(A9) provide the solution to the inhomogeneous Killing equation (A1).

Because K_{ij} is a symmetric tensor, the above two expressions must be equal. Acting with $\partial_{yyy} = \frac{\partial^4}{\partial y^2 \partial z^2}$ on Eqs. (A8), (A9), demanding that the two expressions be equal and using Eqs. (A6), (A7), yields the integrability condition

$$\partial_{zzz} f_{yyy} - 3\partial_{zz} f_{yyz} = \partial_{yyy} f_{zzz} - 3\partial_{yy} f_{yzz} \quad (\text{A10})$$

which must necessarily be satisfied by f_{ijk} in order for Eq. (A1) to have a solution. Note that the unknown functions Z, Y, ζ, ψ do not appear in the above condition. In the homogeneous case, these functions can be easily shown to be quadratic polynomials in their arguments by setting $f_{ijk} = 0$, demanding that expressions (A8) and (A9) be equal and separating variables.

The integrability condition (A10) may be generalized to \mathbb{E}^n as follows. Eq. (A1) can be expanded out as

$$\partial_i K_{jk} + \partial_j K_{ik} + \partial_k K_{ij} = 3f_{ijk} \quad (\text{A11})$$

The first term vanishes if we apply ∂_l and antisymmetrize over l and i . The second term vanishes if we subsequently apply ∂_m and antisymmetrize over m and j . Finally, the third term vanishes if we apply ∂_n and antisymmetrize over n and k . This yields the integrability condition

$$\begin{aligned} \partial_{nml} f_{ijk} - \partial_{nmi} f_{ljk} - \partial_{njl} f_{imk} + \partial_{nji} f_{lmk} \\ - \partial_{kml} f_{ijn} + \partial_{kmi} f_{ljn} + \partial_{kjl} f_{imn} - \partial_{kji} f_{lmn} = 0 \end{aligned} \quad (\text{A12})$$

which generalizes Eq. (A10) to arbitrary dimension.

APPENDIX B: KILLING-STÄCKEL TENSORS OF RANK-FOUR WITH REFLECTION SYMMETRY IN TWO DIMENSIONS

As explained in Sec. 3.2, we are interested in solving Eq. (45) in \mathbb{E}^2 . The most general solution, according to Eq. (51) can be written as

$$\begin{aligned} \mathbf{K} = \mathcal{A}^{ijkl} \mathbf{X}_i \otimes \mathbf{X}_j \otimes \mathbf{X}_k \otimes \mathbf{X}_l + \mathcal{B}^{ijkl} \mathbf{X}_i \otimes \mathbf{X}_j \otimes \mathbf{X}_k \otimes \mathbf{R}_l \\ + \mathcal{C}^{ijkl} \mathbf{X}_i \otimes \mathbf{X}_j \otimes \mathbf{R}_k \otimes \mathbf{R}_l + \mathcal{D}^{ijkl} \mathbf{X}_i \otimes \mathbf{R}_j \otimes \mathbf{R}_k \otimes \mathbf{R}_l \\ + \mathcal{E}^{ijkl} \mathbf{R}_i \otimes \mathbf{R}_j \otimes \mathbf{R}_k \otimes \mathbf{R}_l \end{aligned} \quad (\text{B1})$$

where the generators of translations and rotations are given by Eq. (3) and the constant coefficients possess the symmetries $\mathcal{A}^{ijkl} = \mathcal{A}^{(ijkl)}$, $\mathcal{B}^{ijkl} = \mathcal{B}^{(ijk)l}$, $\mathcal{C}^{ijkl} = \mathcal{C}^{(ij)(kl)}$, $\mathcal{D}^{ijkl} = \mathcal{D}^{i(jkl)}$ and $\mathcal{E}^{ijkl} = \mathcal{E}^{(ijkl)}$. Here, we consider a meridional (yz) slice of the respective three-dimensional system. Indices are thus summed over values such that only $\mathbf{e}_y, \mathbf{e}_z$ and their linear combination $\mathbf{R}_x = y\mathbf{e}_z - z\mathbf{e}_y$ appear in \mathbf{K} ; all other components vanish in the meridional plane. The number of independent coefficients is reduced upon imposing symmetries.

First, we impose \mathbb{Z}_2 reflection symmetry about the equatorial (xy) plane by requiring that \mathbf{K} be invariant under the replacement $\{z, \mathbf{e}_z\} \rightarrow \{-z, -\mathbf{e}_z\}$. (This amounts

to invariance of $K^{ijkl}p_i p_j p_k p_l$ under $\{z, p_z\} \rightarrow \{-z, -p_z\}$.) This yields the constraints

$$\mathcal{A}^{yyyyz} = \mathcal{A}^{yzzz} = 0 \quad (\text{B2a})$$

$$\mathcal{B}^{yyzx} = \mathcal{B}^{zzzx} = 0 \quad (\text{B2b})$$

$$\mathcal{C}^{yzzx} = 0 \quad (\text{B2c})$$

$$\mathcal{D}^{zzxx} = 0 \quad (\text{B2d})$$

Second, recall that the respective three-dimensional system is axisymmetric about the z -axis. Considering a meridional (yz) slice of this system and making a rotation of 2π radians about the z -axis amounts to a reflection about the xz plane. Thus, the respective two-dimensional system inherits a \mathbb{Z}_2 reflection symmetry about the xz plane. Imposing invariance with respect to the replacement $\{y, e_y\} \rightarrow \{-y, -e_y\}$ yields the constraints

$$\mathcal{A}^{yyyyz} = \mathcal{A}^{yzzz} = 0 \quad (\text{B3a})$$

$$\mathcal{B}^{yyyx} = \mathcal{B}^{yzzx} = 0 \quad (\text{B3b})$$

$$\mathcal{C}^{yzzx} = 0 \quad (\text{B3c})$$

$$\mathcal{D}^{yxxx} = 0 \quad (\text{B3d})$$

Imposing both sets of constraints (B2),(B3) to the tensor (B1) yields the solution

$$\begin{aligned} \mathbf{K} = & \mathcal{A}^{yzzz} (\mathbf{X}_y \otimes \mathbf{X}_y \otimes \mathbf{X}_z \otimes \mathbf{X}_z + \mathbf{X}_y \otimes \mathbf{X}_z \otimes \mathbf{X}_y \otimes \mathbf{X}_z \\ & + \mathbf{X}_y \otimes \mathbf{X}_z \otimes \mathbf{X}_z \otimes \mathbf{X}_y + \mathbf{X}_z \otimes \mathbf{X}_y \otimes \mathbf{X}_z \otimes \mathbf{X}_y \\ & + \mathbf{X}_z \otimes \mathbf{X}_y \otimes \mathbf{X}_y \otimes \mathbf{X}_z + \mathbf{X}_z \otimes \mathbf{X}_z \otimes \mathbf{X}_y \otimes \mathbf{X}_y) \\ & + \mathcal{A}^{yyyy} \mathbf{X}_y \otimes \mathbf{X}_y \otimes \mathbf{X}_y \otimes \mathbf{X}_y + \mathcal{A}^{zzzz} \mathbf{X}_z \otimes \mathbf{X}_z \otimes \mathbf{X}_z \otimes \mathbf{X}_z \\ & + \mathcal{C}^{yyxx} \mathbf{X}_y \otimes \mathbf{X}_y \otimes \mathbf{R}_x \otimes \mathbf{R}_x + \mathcal{C}^{zzxx} \mathbf{X}_z \otimes \mathbf{X}_z \otimes \mathbf{R}_x \otimes \mathbf{R}_x \\ & + \mathcal{E}^{xxxx} \mathbf{R}_x \otimes \mathbf{R}_x \otimes \mathbf{R}_x \otimes \mathbf{R}_x \end{aligned} \quad (\text{B4})$$

The above expression gives the most general rank-4 Killing-Stäckel tensor in \mathbb{E}^2 consistent with the symmetry $(\mathbb{R}, +) \times \mathbb{Z}_2 \times \mathbb{Z}_2$. For clarity, let us set define new parameters $\kappa, \lambda, \mu, a, b, c$ such that $\mathcal{E}^{xxxx} = \kappa$, $\mathcal{A}^{zzzz} = \lambda$, $\mathcal{C}^{zzxx} = \mu$, $\mathcal{A}^{yzzz} = (2\lambda - \mu c^2)/6$, $\mathcal{A}^{yyyy} = \lambda - \mu c^2 + \kappa a^2 b^2$, $\mathcal{C}^{yyxx} = \mu - \kappa(a^2 + b^2)$. This reparametrization involves no loss of generality, since a, b, c are allowed to be real or imaginary. Then, the solution (B4) takes the simple form

$$K^{ijkl} = \kappa A^{(ij} B^{kl)} + \lambda g^{(ij} g^{kl)} + \mu g^{(ij} C^{kl)} \quad (\text{B5})$$

where the tensors A^{ij}, B^{ij}, C^{ij} are given by Eqs. (28) and (55). When contracted with the momenta, the term $g^{(ij} g^{kl)}$ in the above expression is the quartic part of the squared energy, which is conserved; this term is thus dropped from Eq. (53).

This paper has been typeset from a $\text{T}_{\text{E}}\text{X}/\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$ file prepared by the author.