Optimal control of a commercial building's thermostatic load for off-peak demand response

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ABSTRACT
This paper studies the optimal control of a commercial building's thermostatic load during off-peak hours as an ancillary service to the power grid. It provides an algorithmic framework that commercial buildings can implement to cost-effectively increase their electricity demand at night while they are unoccupied, instead of using standard inflexible setpoint control. Consequently, there is minimal or no impact on user comfort, while the building manager gains an additional income stream from providing the ancillary service. By introducing a novel benefit-cost ratio of ancillary service payment to night-time price of electricity, we are able to study the building's capability to provide a service that is both useful to the power grid and profitable to the building manager. Numerical results show that there can be an economic incentive to participate even if the payment rate for the ancillary service is less than the price of electricity.

1. Introduction

1.1. *The need for electricity balancing and ancillary services*

A secure and stable power grid requires continuous balance between the electricity supplied and consumed on it. The system operator, an independent entity that is responsible for the security, stability and, in many cases, the transmission system of the power grid (Kirschen and Strbac 2004, p. 3), balances electricity supply and demand by:

- increasing generation or reducing demand when there is a shortfall in supply;
- decreasing generation or increasing demand when there is surplus power.

The latter, which we refer to as *decremental* actions (Szabó and Martyr 2017), are increasingly relevant for power grids with high levels of intermittent generation from renewable energy sources (Rothleder and Loutan 2014). In order to carry out its balancing duties, the system operator procures a variety of *ancillary services* from third-party companies (Kirschen and Strbac 2004, p. 106). There is much interest in

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enabling electricity consumers to provide ancillary services (Hirst and Kirby 1998; Vardakas, Zorba, and Verikoukis 2015; Paterakis, Erdinç, and Catalão 2017), particularly commercial buildings due to the large flexible demand from their heating, ventilation and air conditioning (HVAC) systems (Olivieri et al. 2014; Pavlak, Henze, and Cushing 2014; Lawrence et al. 2016; Kim et al. 2016; De Coninck and Helsen 2016; Blum, Zakula, and Norford 2017; Jensen et al. 2017; Junker et al. 2018). *Replacement reserves*, which are given more time to respond and are used as back up for faster acting services, can be most suitable in this case (Hirst and Kirby 1998, p. 32).

This paper is a quantitative study of the potential for a commercial building with flexible thermostatic load to participate in a decremental replacement reserve (DRR) initiative that is modelled after a real-world example: Demand Turn Up (National Grid UK 2018). The setting of this paper is novel in comparison to previous papers such as De Coninck and Helsen (2016) and Blum, Zakula, and Norford (2017). In the present work the reserve provider bids a schedule of its reserve capability together with a *fixed* utilization payment, rather than a variable payment depending on the quantity utilized. We provide a mathematical and computational framework, inclusive of a novel benefit-cost ratio of utilization payment to electricity price, that enables us to study the commercial building's capability to provide a useful ancillary service to the power grid that is also profitable to the building manager. We also focus on temperature cooling only and note that heating can be treated symmetrically. Data centres, in particular, are an important example since they account for more than 1% of global electricity usage, and their cooling infrastructure typically accounts for about 40% to 50% of their electricity usage (Dayarathna, Wen, and Fan 2016).

1.2. *Buildings as ancillary service providers*

The International Energy Agency (IEA) Energy in Buildings and Communities Programme (EBC) Annex 67, "Energy Flexible Buildings", is a collaboration amongst participants from 16 countries that aims to systematically assess the energy flexibility that buildings can offer to energy systems (Jensen et al. 2017; International Energy Agency (IEA) 2018). One of the key objectives of Annex 67 is the development of a common terminology and definition of this "energy flexibility" which, according to Jensen et al. (2017, p. 28), is the building's ability to "manage its demand and generation according to local climate conditions, user needs, and energy network requirements".

Demand response refers to any programme that motivates changes in an electricity consumer's normal power consumption, typically in response to incentives regarding electricity prices (Vardakas, Zorba, and Verikoukis 2015). It is widely considered as a cost-effective and reliable solution for improving the efficiency, reliability, and safety of the power grid (Vardakas, Zorba, and Verikoukis 2015; Paterakis, Erdinç, and Catalão 2017). There are several examples of initiatives that incentivize electricity consumers to reduce their demand, especially during peak hours, as the references Olivieri et al. (2014); Lawrence et al. (2016); Kim et al. (2016); Jensen et al. (2017); Junker et al. (2018) all show. Demand response schemes that provide incentives for increased electricity demand are much rarer. The Demand Turn Up (DTU) service offered by National Grid UK, the system operator in Great Britain, is one such scheme. This service is meant to incentivize large electricity consumers to increase their demand when there is low overall demand on the network and high output from renewable generation (National Grid UK 2018). It is particularly relevant during the off-peak, night-time hours

of interest to this paper, and below we summarize its key aspects (see National Grid UK (2018) for further details).

Demand turn up: an off-peak demand response scheme

DTU runs during the British Summer Time (BST) period and can be utilized during *availability windows* that last from 23:30 until 08:30 or 09:00 the following day, depending on the time of year, or on weekends and holidays between 13:00 and 16:00. Additionally, there are optional windows that cover all other time periods. DTU providers declare their availability by specifying a schedule for the adjustment in power consumption or generation they can provide, including the payment for utilization of their service. This is primarily done one week in advance of an availability window. National Grid UK then sends a contracted DTU provider instructions for the service according to the capability that was declared. The provider has a deadline for acknowledging receipt of the DTU instruction, then a delivery period for responding as instructed. In 2017, for instance, the average length of time for advanced notice of DTU utilization was 6 hours and 40 minutes. A DTU provider which has declared its availability must be able to deliver the service as instructed or face a penalty.

There two routes to market for DTU candidates:

- Fixed DTU is a medium to long-term procurement process that takes place months in advance of BST.
- Flexible DTU, known as Optional DTU in 2018, is a rolling short-term procurement process that takes place during BST and closer to the period that requires the service.

In addition to the utilization payment, successful Fixed DTU candidates receive a guaranteed payment for their availability according to the price tendered. Although this availability payment is appealing, Fixed DTU candidates have their utilization payment capped at tender. This can make Flexible DTU preferable since it gives candidates more flexibility to adjust their declaration in response to weather and market conditions.

While our subsequent analysis only accounts for the utilization payment, we acknowledge that nearly all of the contracts in 2017 were for Fixed DTU. Inclusion of the availability payment merely strengthens our case for profitable participation in DTU. It is also important to note that the availability payment for Fixed DTU providers is typically small in relation to the utilization payment. For instance, the average utilization payment for 2017 was 67.50 £/MWh or, equivalently, 6.75 p/kWh, whilst the average accepted availability payment was 1.51 £/MW/h or, equivalently, 0.151 p/kW/h (National Grid UK 2018). Besides cost, the magnitude and sustainability of a response are important in the assessment of an offer to provide DTU. The current entry threshold is 1 MW, which can be aggregated from sources of at least 0.1 MW (equivalently, 100 kW). Regarding sustainability, in 2017 the average duration of a single DTU request was 3 hours and 34 minutes (National Grid UK 2018). Therefore, both power and energy are important for the DTU service.

Settlement is the process of compensation for successful provision of the DTU service. A provider has two options for settlement, forecast or baseline, and its choice is fixed for the contract's duration. Both options produce a *reference schedule* to which the actual metered electricity consumption or generation is compared, and the difference is settled against the DTU service instruction that was sent. The forecast method uses the provider's prediction for electricity consumption or generation during that ser-

vice period, whereas the baseline method uses the average metered output from recent entries for that day and time in which the provider did not render a DTU service.

Quantifying a building's potential to provide an ancillary service

As mentioned previously, commercial buildings have significant potential to provide ancillary services to the system operator. According to Blum, Zakula, and Norford (2017, p. 1266), *capacity* and *performance* are the two main components of this ancillary service provision, each of which has a magnitude and cost. Capacity refers to the capability of the building's HVAC system to provide the ancillary service, whereas performance refers to the work that the HVAC system does to provide the ancillary service in response to the system operator's instructions. Similarly, Jensen et al. (2017) report three general properties of energy flexibility in buildings:

- the time over which energy and power can be altered
- the amount of energy or power that can altered
- the cost associated with this energy or power alteration.

Several papers have argued that buildings can be incentivized to participate in ancillary services markets, despite high energy prices or less efficient operating conditions, provided they are adequately compensated (Olivieri et al. 2014; Pavlak, Henze, and Cushing 2014; Lawrence et al. 2016; De Coninck and Helsen 2016; Blum, Zakula, and Norford 2017).

Blum, Zakula, and Norford (2017) propose a methodology for quantifying the opportunity costs arising from the provision of ancillary services by buildings' HVAC systems. The authors identify sources of these opportunity costs, and develop a method of accounting for them through time that is consistent with current practice for generators. This is done by recognizing the impacts of ancillary service provision on daily energy efficiency and costs. De Coninck and Helsen (2016) previously addressed a similar problem. However, unlike Blum, Zakula, and Norford (2017) they focused on a building's capability to alter its total energy use over a period of time, and the methodology put forward was not intended for real-time dynamic operations (De Coninck and Helsen 2016, p. 654). Both papers use optimal control to determine the building's capability to provide a given level of reserve and the associated opportunity cost. In that setting, an opportunity cost curve can be constructed by varying the level of reserve, and this cost curve can be used in the ancillary services market for the purpose of dispatch by the system operator, or for bidding purposes by the building manager.

1.3.   *Aim of this work*

In this paper we study the potential for a commercial building to participate in an ancillary service scheme such as Demand Turn Up by controlling the electricity it consumes for temperature cooling at night, which is the longest possible period for providing this service. Unlike the setting studied in De Coninck and Helsen (2016) and Blum, Zakula, and Norford (2017), the reserve provider bids a schedule of its reserve capability (in kW) together with a fixed utilization payment (per kWh), rather than a variable payment depending on the quantity utilized. Our main contribution is an analysis of the building manager's incentives in this novel setting by using the *benefit-cost ratio* of utilization payment to night-time price of electricity. By varying this ratio, we can see how it affects the magnitude of reserve power offered for the ancillary service.

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We approach the overall problem of economically providing the ancillary service by breaking it up into three smaller problems:

(1) *Reference:* determine an optimal reference control schedule to use for settlement of the service.
(2) *Capability:* using a prospective control schedule, determine an optimal offer of reserve relative to the reference schedule obtained from the previous step.
(3) *Delivery:* determine an optimal control schedule for delivering given service instructions that agree with the capability identified in the previous step.

We formulate each of these problems as a constrained optimal control problem (Clarke 2013), and use the control parametrization method (Teo and Goh 1991) to obtain approximate numerical solutions for different scenarios. Optimal control is one of several mathematical techniques that can be used to optimize the provision of an ancillary service from a commercial building (Wang and Ma 2008; Olivieri et al. 2014; Deng et al. 2015; Jensen et al. 2017). Moreover, it can be an effective solution for control of the building's thermostatic load (Vardakas, Zorba, and Verikoukis 2015, p. 158).

Our methodology is suitable for assessing the building's ability to participate in any demand response scheme, for either incremental or decremental reserve, where the utilization payment is fixed and the reserve provider bids capacity curves. We are able to identify the incentives that drive the optimal actions, leading to recommendations that are intuitive and implementable using a variety of control architectures. Consistent with previous studies (Olivieri et al. 2014; Pavlak, Henze, and Cushing 2014; Lawrence et al. 2016; De Coninck and Helsen 2016; Blum, Zakula, and Norford 2017), we find that, besides the dynamics and constraints for the internal temperature, the level of participation in the ancillary service depends on how well the building manager is compensated relative to the additional cost incurred.

On the one hand, our numerical results for the optimal reference and delivery control schedules are intuitive and show that these schedules both minimize the overall cost of consuming power to satisfy the corresponding operational constraints. On the other hand, our numerical results for the optimal prospective control schedule, which is used to determine the optimal level of reserve to bid, are more nuanced as they show that there is an economic incentive to participate in the ancillary service even when the utilization payment R is less than the night-time price of electricity P. The optimal prospective control schedule's structure depends crucially on the benefit-cost ratio $\frac{R}{P}$.

- When $\frac{R}{P} < 1$ the optimal prospective control schedule has a complicated structure with time-shifts in consumption causing intermittent periods of constant reserve which are too short to be useful in practice. Nevertheless, this schedule ends with a sufficiently long interval of constant reserve which can be offered for the ancillary service.
- When $\frac{R}{P} = 1$ the optimal prospective control schedule also has a complicated structure which lacks sufficiently long periods of constant reserve. Nevertheless, it ends with a relatively long interval where the level of reserve steadily increases. The totality of this additional energy usage can be offered for the ancillary service, if allowed.
- When $\frac{R}{P} > 1$ the optimal prospective control schedule has the least complicated structure and sustains constant reserve for long periods throughout the night-time period. It can consist entirely of two contiguous intervals of constant reserve, at two different power levels, which is consistent with current practice (National Grid UK 2018).

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In the first two cases, the strikingly complicated optimal prospective control schedules result from the need to use highly variable power consumption to maximize profit when there is insufficient compensation for deviating from the reference schedule. The extremity of this behaviour is also a consequence of the continuous-time setting of our optimal control problem. While the optimal prospective control schedule in this case may not be entirely practicable for an ancillary service such as DTU, a suboptimal one can be derived from it which is both practicable and profitable to the building manager. It is also possible that an ensemble of buildings can be used to obtain a less variable and practicable optimal prospective control schedule in aggregate.

In the following section we present our mathematical framework for optimizing the reference, capability and delivery control schedules for the reserve service. This framework uses the building's internal temperature as a controlled variable. In principle, any model that describes the temperature dynamics using ordinary differential equations can be used, and for our study a simple linear model suffices. For realistic applications, the temperature relaxation behaviour of a given building is measured and used as input to the optimal control scheme. However, modelling the fine details of temperature evolution and building characteristics is beyond the scope of this paper. In Section 2.1 we present the model for temperature dynamics, which is used to obtain the numerical solutions to the optimization problems presented in Section 3. The paper concludes with a summary of the main results and practical recommendations in Section 4.

2. Optimal control problems for off-peak demand response from thermostatic load

The *control horizon* is a period of time during which the building's internal temperature is controlled for the ancillary service. In this paper, the control horizon is included within the night-time availability window for DTU that lasts for 9 or 9.5 hours starting from 23:30. Let $T > 0$ denote the control horizon's length in minutes and $x = (x(t))_{0 \leq t \leq T}$ denote the building's internal temperature in °C during this time.

2.1. *Internal temperature modelling*

We assume that the internal temperature x evolves according to the following linear dynamics (Ihara and Schweppe 1981; Tindemans, Trovato, and Strbac 2015):

$$\dot{x}(t) = -\frac{1}{\tau}\left[x(t) - X_{off} + (X_{off} - X_{on})u(t)\right],$$
$$x(0) \in [X_{on}, X_{off}], \tag{1}$$

where $\dot{x}$ is the time derivative of x and,

- X_{on} and X_{off} are the asymptotic temperatures reached when the cooling equipment operates in the "on" and "off" states respectively;
- $\tau > 0$ is the thermal time constant;
- $u(t) \in [0, 1]$, the normalized cooling control power, is the fraction of actual power $C(t)$ consumed at time t,

$$u(t) = \frac{C(t)}{C_{max}},$$

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where C_{max} (kW) is the maximum power consumption of the cooling equipment.

Definition 2.1. Let $\mathcal{U}$ denote the set of normalized control schedules $\big(u(t)\big)_{0 \leq t \leq T}$ where $u \colon [0, T] \to [0, 1]$.

Suppose $u \in \mathcal{U}$ is a *step function* of the form,

$$u(t) = \sum_{k=1}^{n_p} u_k \mathbf{1}_{[t_{k-1}, t_k)}(t), \quad t \in [0, T], \tag{2}$$

where $n_p > 1$ is an integer, $\{t_k\}_{k=0}^{n_p}$ is a sequence of time points $0 = t_0 < \ldots < t_{n_p} = T$ that partition the control horizon $[0, T]$ into n_p contiguous subintervals, and $\mathbf{1}_{[t_{k-1}, t_k)}$ is the indicator function of the set $[t_{k-1}, t_k)$. There is a unique solution to the differential equation (1) corresponding to such a schedule u, which is also a continuous function of time (see Theorem 6.5 of Adkins and Davidson (2012), for example). Moreover, since equation (1) is a first-order linear ordinary differential equation with constant coefficients, Adkins and Davidson (2012, p. 47), for example, shows that it can be solved explicitly on each subinterval $[t_{k-1}, t_k)$ to get,

$$x(t) = e^{-\frac{t-t_{k-1}}{\tau}} x(t_{k-1}) + \left(1 - e^{-\frac{t-t_{k-1}}{\tau}}\right) \big(X_{off} + (X_{on} - X_{off})u_k\big),$$
$$t_{k-1} \leq t < t_k, \quad 1 \leq k \leq n_p. \tag{3}$$

This solution can be verified by showing that it solves the differential equation (1) with control (2).

Constraints for the internal temperature. We suppose that the internal temperature must be kept between lower and upper limits X_{min} and X_{max} overnight,

$$X_{min} \leq x(t) \leq X_{max}, \quad t \in [0, T], \tag{4}$$

where $X_{on} < X_{min} < X_{max} < X_{off}$. A *pre-cooling operational strategy* refers to the act of increasing cooling power and using the building's thermal inertia to reduce the need for cooling power at later periods (Reddy, Norford, and Kempton 1991; Roth, Dieckmann, and Brodrick 2009). We include pre-cooling in our framework by imposing a constraint on the final temperature value $x(T)$ as follows,

$$X_{min} \leq x(T) \leq \hat{X}, \tag{5}$$

where $\hat{X}$ in $[X_{min}, X_{max}]$ is set by the building manager. Maximum pre-cooling is achieved by setting $\hat{X} = X_{min}$.

2.2. *Three optimal control problems for off-peak decremental replacement reserve provision*

In this section we discuss our mathematical framework for analysing the economic provision of off-peak decremental replacement reserve which determines,

(1) the optimal reference control schedule C_{ref} to use for settlement of this ancillary service,

(2) an optimal offer of instantaneous reserve capability C_{cap} from a prospective schedule C_{pro} as follows,

$$C_{cap}(t) = C_{pro}(t) - C_{ref}(t), \quad t \in [0, T]. \tag{6}$$

(3) the optimal delivery control schedule C_{del} that fulfils reserve service instructions that are compatible with the declared reserve capability C_{cap}.

If the right-hand side of (6) is negative then the building is unable to deliver *instantaneous* decremental reserve at that time. Negative instantaneous reserve exemplifies a possible consequence of demand response, which some authors call the *payback effect*, where operational constraints enforce a period of recovery subsequent to demand response provision (Tindemans, Trovato, and Strbac 2015; Jensen et al. 2017). It is important to note that despite occurrences of negative instantaneous reserve, the building can still deliver *total* decremental reserve over the control horizon, provided

$$\int_0^T C_{cap}(t)dt > 0.$$

Problem 1. Optimal reference power consumption

Suppose there is no request for decremental reserve during $[0, T]$ and the building manager implements the reference schedule C_{ref}. Letting P (p/kWh), where "p" stands for pence, denote the positive and constant night-time price of electricity, the total cost to the building manager is,

$$\frac{1}{60} \int_0^T P C_{ref}(t)dt, \tag{7}$$

where we divide by 60 since T is given in minutes. In this case it is reasonable to assume that the building manager aims for energy efficiency, so that C_{ref} minimizes (7) subject to the internal temperature constraints. This reasoning is consistent with related studies such as De Coninck and Helsen (2016); Blum, Zakula, and Norford (2017) and forms the basis for the following optimal control problem.

$$\text{minimize} \int_0^T \left[u_{ref}(t) + \alpha_{ref} \big(u_{ref}(t) \big)^2 \right] dt \ \text{ over } \ u_{ref} \in \mathcal{U} \text{ subject to:}$$

$$\begin{aligned}
(i) \quad & \dot{x}(t) = f(t, x(t), u_{ref}(t)) \text{ given by (1),} \\
(ii) \quad & X_{min} \leq x(t) \leq X_{max}, \quad t \in [0, T], \\
(iii) \quad & x(0) \in [X_{min}, X_{max}] \text{ and } x(T) \in [X_{min}, \hat{X}],
\end{aligned} \tag{8}$$

where u_{ref} is the normalized reference control schedule, and $\alpha_{ref} > 0$ is a constant that weighs the importance of the *regularization term* $\big(u_{ref}(t) \big)^2$. This regularizer is used in the control problem (8) to disfavour solutions where u_{ref} alternates rapidly between its minimum and maximum possible values. In the absence of this regularizer, theory states that this unwanted behaviour can be optimal in (8) since the control variable u_{ref} then appears linearly in both the cost criterion and state dynamics (Maurer 1977).

Problem 2. Optimal prospective power consumption

Let R (p/kWh) denote the positive utilization payment received as a reward for the electricity consumed in excess of the reference level C_{ref}. When decremental reserve is being delivered according to a prospective schedule C_{pro}, the *instantaneous net cost* is,

$$PC_{pro}(t) - R\big(C_{pro}(t) - C_{ref}(t)\big)^+,$$

where $y^+ = \max(y, 0)$. The *total net cost* is therefore,

$$\frac{1}{60} \int_0^T \big[PC_{pro}(t) - R\big(C_{pro}(t) - C_{ref}(t)\big)^+\big]dt. \tag{9}$$

Assuming that the system operator will utilize all of the decremental reserve provided by the prospective schedule, it is reasonable to choose C_{pro} so that it minimizes the total net cost (9) subject to the internal temperature constraints. We therefore formulate an optimal control problem to achieve this. Given the normalized reference control schedule u_{ref}, night-time electricity price P, and utilization payment R,

$$\text{minimize } J(u_{pro}\,;\, u_{ref}, P, R) \text{ over } u_{pro} \in \mathcal{U} \text{ subject to:}$$
$$(i) \quad \dot{x}(t) = f(t, x(t), u_{pro}(t)) \text{ given by (1),}$$
$$(ii) \quad X_{min} \leq x(t) \leq X_{max}, \;\; t \in [0, T],$$
$$(iii) \quad x(0) \in [X_{min}, X_{max}] \text{ and } x(T) \in [X_{min}, \hat{X}], \tag{10}$$
$$(iv) \quad \int_0^T \big[u_{pro}(t) - u_{ref}(t) - \tfrac{R}{P}\big(u_{pro}(t) - u_{ref}(t)\big)^+\big]dt \leq 0,$$

where u_{pro} is the normalized prospective control schedule, the cost criterion $J(u_{pro}\,;\, u_{ref}, P, R)$ is given by,

$$\begin{aligned}
J(u_{pro}\,;\, u_{ref}, P, R) &= \int_0^T \big[u_{pro}(t) - \tfrac{R}{P}\big(u_{pro}(t) - u_{ref}(t)\big)^+\big]dt \\
&\quad + \alpha_{pro} \int_0^T \big(u_{pro}(t)\big)^2 dt,
\end{aligned} \tag{11}$$

and, similar to (8) above, $\alpha_{pro} > 0$ is a constant used to weigh the importance of a quadratic regularizer. The effect this parameter can have on the results is illustrated in Section 3 below. Constraint (10)-(iv) ensures that the building manager is not worse off financially by following u_{pro} instead of u_{ref}, and is always satisfied when $R \geq P$. If "0" on the right-hand side of (10)–(iv) is replaced by "$-\gamma$" where $\gamma \geq 0$, then a solution to (10) guarantees the building manager a minimum total net profit of $\gamma\big(\frac{C_{max}P}{60}\big)$ pence relative to the reference schedule's cost.

Problem 3. Optimal reserve service delivery

Reserve service instructions are described by a control schedule C_{ask} that indicates how much additional power the building should consume relative to the reference level C_{ref}. The schedule C_{ask} is a non-negative function of time that should not exceed the

declared reserve capability C_{cap} (cf. (6)) whenever the latter is positive,

$$0 \le C_{ask}(t) \le \max(C_{cap}(t), 0), \quad t \in [0, T].$$

We suppose furthermore that C_{ask} is of the form,

$$C_{ask}(t) = \sum_{i=1}^{N} C_{i,ask} \mathbf{1}_{[s_i, e_i)}(t),$$

where $\{(C_{i,ask}, s_i, e_i)\}_{i=1,\dots,N}$ with $N \ge 1$ is a sequence of instructions, each consisting of a *delivery amount* $C_{i,ask}$ (kW), *delivery start time* s_i (min), and *delivery end time* e_i (min), satisfying

$$(i) \quad 0 \le s_i < e_i \le T \text{ for } i \in \{1, \dots, N\},$$
$$(ii) \quad e_i = s_{i+1} \text{ for } N \ge 2 \text{ and } i \in \{1, \dots, N-1\}.$$

Therefore, at any time $t \in [s_i, e_i)$ the building's power consumption must be at least $C_{i,ask}$ (kW) more than the reference level $C_{ref}(t)$. This leads to the following constraint on any delivery schedule C_{del} that satisfies the reserve service instructions,

$$C_{del}(t) \ge C_{ref}(t) + C_{i,ask}, \quad s_i \le t < e_i, \quad i = 1, \dots, N. \tag{12}$$

For computations and illustrations, it is convenient to represent the constraint (12) by its equivalent form,

$$C_{del}(t) \ge C_{min}(t), \quad t \in [0, T],$$

where C_{min} is the schedule of minimal instructed power consumption,

$$C_{min}(t) = \sum_{i=1}^{N} \left[C_{ref}(t) + C_{i,ask} \right] \mathbf{1}_{[s_i, e_i)}(t),$$

and we used the non-negativity of C_{del}.

We assume that the building's instantaneous consumption can be less than the reference level outside of the reserve service instructions' times, allowing it to recover from providing the service as needed. Since the building manager is only compensated for the additional demand as instructed, it is reasonable to require that the building uses no more power than that needed to satisfy the reserve instructions and internal temperature constraints. Using the normalized schedules u_{del}, u_{ref} and u_{ask}, we formulate the reserve service delivery problem as follows.

$$\text{minimize} \int_0^T \left[u_{del}(t) + \alpha_{del} \left(u_{del}(t) \right)^2 \right] dt \text{ over } u_{del} \in \mathcal{U} \text{ subject to:}$$
$$(i) \quad \dot{x}(t) = f(t, x(t), u_{del}(t)) \text{ given by (1)},$$
$$(ii) \quad X_{min} \le x(t) \le X_{max}, \quad t \in [0, T],$$
$$(iii) \quad x(0) \in [X_{min}, X_{max}] \text{ and } x(T) \in [X_{min}, \hat{X}], \tag{13}$$
$$(iv) \quad u_{del}(t) \ge u_{ref}(t) + u_{i,ask}, \quad s_i \le t < e_i, \quad i = 1, \dots, N,$$

where, similar to (8) above, $\alpha_{del} > 0$ is a constant used to weigh the importance of the quadratic regularizer.

3. Numerical simulations

Theory, for example Cesari (1983); Clarke (2013), guarantees the existence of a solution to each of the problems (8), (10), and (13). In this section we present results of the associated numerical solutions, which we obtained using the control parametrization method (Teo and Goh 1991) outlined in Appendix A. Table 1 lists values for some of the parameters used to generate the numerical results.

**Table 1.** Parameters for numerical simulations.

Parameter	Value	Units
T	360	min
τ	120	min
X_{off}	35	°C
X_{on}	10	°C
$x(0)$	27	°C
X_{min}	18	°C
X_{max}	27	°C

The hypothetical building for the experiments has computing equipment that generates a significant amount of electricity demand for temperature cooling. We do not investigate specifics of the building here, but have chosen the parameters in Table 1 to produce sensible temperature values according to the dynamics (1). We note that the chosen internal temperature range agrees with the recommended range in ASHRAE Standard 90.4-2016 for data centres (ASHRAE 2016). The control horizon's length, $T = 360$, corresponds to the night-time period 00:00 to 06:00. Although papers similar to ours such as Blum, Zakula, and Norford (2017); De Coninck and Helsen (2016) tend to have longer control horizons, say twenty-four hours, the short horizon we consider accurately reflects the costs and benefits accrued from providing a night-time ancillary service such as DTU. Moreover, the operational constraints for temperature regulation do not permit a large range of time for intra-day electricity demand shifting. Nevertheless, our analysis can be made consistent with another one that occurs over a longer control horizon by appropriately specifying the terminal temperature constraint (5).

3.1. *Optimal reference power consumption*

In this section we highlight important characteristics of the control and temperature trajectories corresponding to an optimal reference control schedule u_{ref}. These trajectories, displayed in Figure 1, are intuitive and show that the optimal control uses minimal energy to satisfy the temperature constraints. In particular,

(1) Starting at the maximum allowed level X_{max}, the temperature is kept constant at this level until a time $\bar{t}_1 \in (0, T)$.
(2) From time $\bar{t}_1$ until $\bar{t}_2 \in (\bar{t}_1, T)$, the power consumption is increased quickly at a constant rate until it reaches the maximum level $u_{ref} = 1$. Furthermore, the
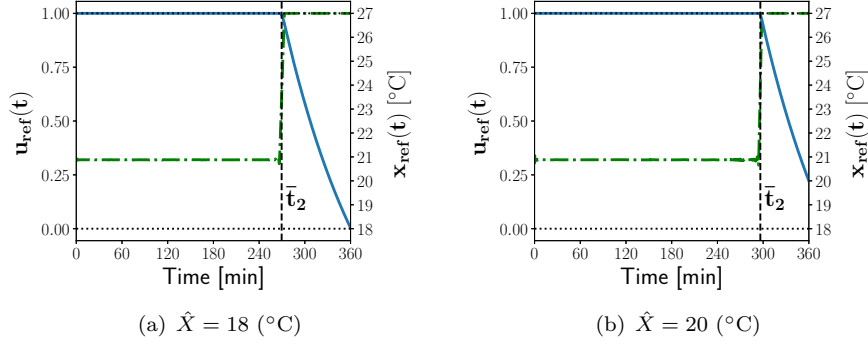
Figure 1. Optimized reference control schedule u_{ref} (dash-dotted line) and internal temperature x_{ref} (solid line) corresponding to two different values for the pre-cooling temperature $\hat{X}$. The schedule u_{ref} keeps the temperature constant at the maximum level for some time, and then quickly ramps up to full power around time $\bar{t}_2$ (vertical dashed line) so that the temperature hits $\hat{X}$ exactly at the terminal time. The initial temperature is $x(0) = 27$ and the regularizer weight is $\alpha_{ref} = 0.01$. Control and temperature constraints are shown using dotted horizontal lines.

difference between $\bar{t}_1$ and $\bar{t}_2$ is negligible.

(3) Maximum power is used from time $\bar{t}_2$ until T to steer the temperature to $\hat{X}$, the upper limit of allowed values at time T.

Consequently, if the optimal reference schedule u_{ref} is used then the building is unable to provide decremental reserve after $\bar{t}_2$ as u_{ref} already consumes maximum power. Note that these optimal characteristics are sensible for temperature dynamics other than the linear one (1).

The linear model allows for easy approximation of the terms described above. For example, if the temperature x is constant over an interval $[\bar{t}_0, \bar{t}_1]$ with $0 \leq \bar{t}_0 < \bar{t}_1 \leq T$, then setting $\dot{x}(t) = 0$ in (1) shows that any control $\bar{u} \in \mathcal{U}$ that achieves this condition satisfies,

$$-\frac{1}{\tau}\left[x(\bar{t}_0) - X_{off} + (X_{off} - X_{on})\bar{u}(t)\right] = 0, \quad t \in [\bar{t}_0, \bar{t}_1],$$

and, after a rearrangement of the terms,

$$\bar{u}(t) = \frac{X_{off} - x(\bar{t}_0)}{X_{off} - X_{on}}, \quad t \in [\bar{t}_0, \bar{t}_1]. \tag{14}$$

Using (14) with $\bar{t}_0 = 0$ and $x(0) = X_{max}$, we see that the control u_{ref} approximately satisfies,

$$u_{ref}(t) = \frac{X_{off} - X_{max}}{X_{off} - X_{on}}, \quad t \in [0, \bar{t}_1]. \tag{15}$$

Using the explicit representation (3) for the temperature schedule corresponding to a step control, the time $\bar{t}_2$ in the description approximately satisfies,

$$\hat{X} = \left(e^{-\frac{T - \bar{t}_2}{\tau}}\right)X_{max} + \left(1 - e^{-\frac{T - \bar{t}_2}{\tau}}\right)X_{on},$$

which we solve to get,

$$\bar{t}_2 = T - \tau \log \left(\frac{X_{max} - X_{on}}{\hat{X} - X_{on}} \right). \tag{16}$$

Since the difference between $\bar{t}_1$ and $\bar{t}_2$ can be made negligible, we only need to calculate $\bar{t}_2$ in practice. Nevertheless, the time $\bar{t}_1$ can be determined by calculating how long it takes to increase power from the normalized level $u_{ref}(\bar{t}_1) = \frac{X_{off} - X_{max}}{X_{off} - X_{on}}$ at $\bar{t}_1$ to the maximum level $u_{ref}(\bar{t}_2) = 1$ at $\bar{t}_2$ for a given rate of increase. Using (16) we see that the duration of unavailability, $T - \bar{t}_2$, is proportional to the building's thermal time constant τ, and also increases with the level of pre-cooling at time T, which is controlled by the parameter $\hat{X}$.

3.2. *Optimal reserve level capability*

This section illustrates the numerical results for the reserve capability problem (10). For the simulations we set the utilization payment R at either 75%, 100% or 125% of the night-time electricity price P. Figure 2 shows the prospective and reserve power schedules corresponding to these three different values of $\frac{R}{P}$, whilst Table 2 further describes the observed reserve power schedules. Figure 2 also shows the normalized net profit relative to the reference schedule for the three cases, which is defined as the negative of constraint (10)-(iv),

$$NNP(u_{pro}; u_{ref}, R, P) = \int_0^T \left[u_{ref}(t) - u_{pro}(t) + \frac{R}{P} \big(u_{pro}(t) - u_{ref}(t) \big)^+ \right] dt.$$

Multiplying this value by $\left(\frac{C_{max} P}{60} \right)$ gives the total net profit relative to the reference schedule in pence. The results in this section demonstrate how crucial the benefit-cost ratio $\frac{R}{P}$ is in incentivizing practicable participation in the reserve service.

Case 1: $\frac{R}{P} = \frac{3}{4}$

In Figure 2(a) the benefit-cost ratio satisfies $\frac{R}{P} < 1$, and the optimal control tends to use short bursts of pre-cooling to minimize the cost of providing reserve power. Consequently, there are many intervals during which the building is either unavailable for decremental reserve, or available for short periods at the maximum level. The longest period of sustained maximal reserve power occurs just before time $\bar{t}_2$ when the reference schedule's power consumption is at its highest. The normalized level of sustained reserve power during this time is approximately,

$$u_{cap}(t) = 1 - \frac{X_{off} - X_{max}}{X_{off} - X_{on}}$$
$$= \frac{X_{max} - X_{on}}{X_{off} - X_{on}}, \quad t \in [\hat{t}, \bar{t}_2],$$

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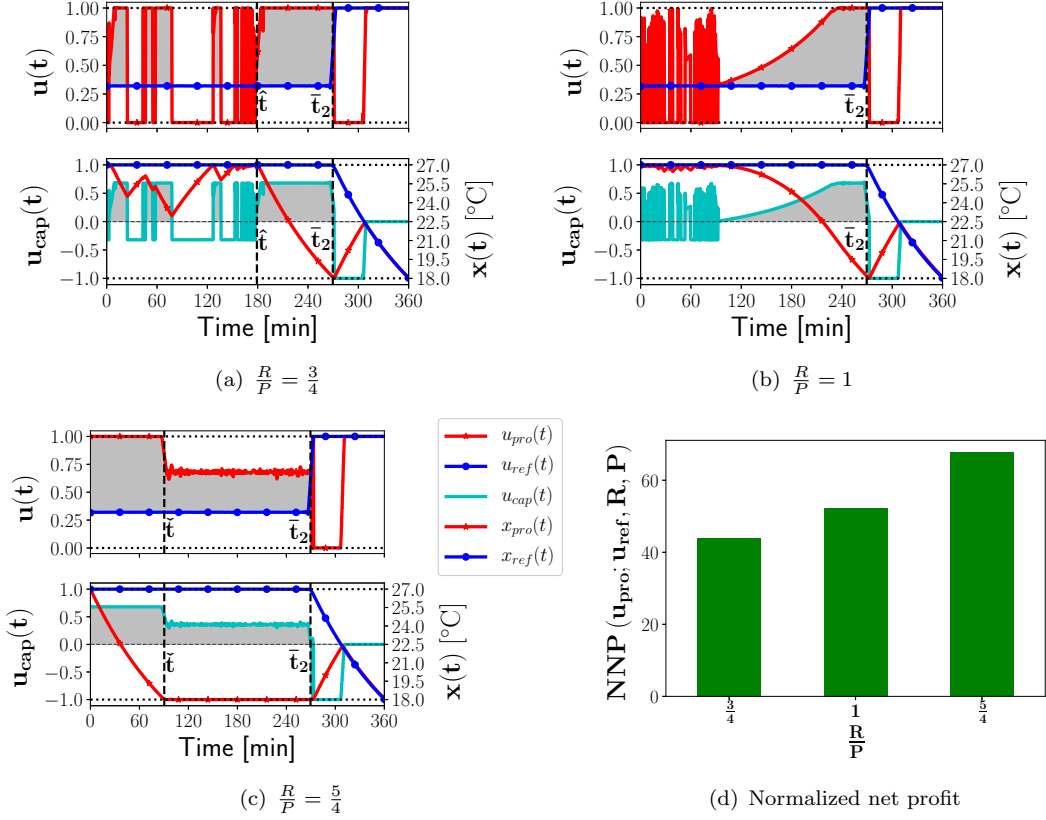
Figure 2. Optimized reserve power u_{cap} and prospective power consumption and internal temperature, u_{pro} and x_{pro}, corresponding to three different values for the benefit-cost ratio $\frac{R}{P}$. Solid lines with star (respectively, circle) markers indicate the prospective (respectively, reference) schedules. The shaded region highlights where there is a positive value of instantaneous reserve power. Also shown in each case is the approximate time $\bar{t}_2$ from which the reference schedule is at maximum power. In (a) $\frac{R}{P} = \frac{3}{4}$ and the building tends to be available for decremental reserve at maximum level on short intervals. The longest duration of reserve occurs on an interval $[\hat{t}, \bar{t}_2]$ where the temperature is cooled from the maximum allowed value to the minimum one. In (b) $\frac{R}{P} = 1$ and there is an initial period of highly variable decremental reserve, followed by a longer period in which decremental reserve is stable but mainly below the maximum level. In (c) $\frac{R}{P} = \frac{5}{4}$ and the building is mostly available for decremental reserve. Maximum power is applied until the time $\hat{t}$ that the temperature first hits the minimum allowed value, then near constant power is applied to keep the temperature at this level until $\bar{t}_2$. Figure (d) shows that, as expected, the normalized net profit increases with $\frac{R}{P}$. Parameter values are $x(0) = 27$ for the initial temperature, $\hat{X} = 18$ for the pre-cooling value, and $\alpha_{ref} = \alpha_{pro} = 0.01$ for the control regularizer weights. Temperature and control constraints are shown using dotted horizontal lines.

where, similar to (16) above for the reference schedule, $\hat{t}$ is given by,

$$
\begin{aligned}
\hat{t} &= \bar{t}_2 - \tau \log \left(\frac{X_{max} - X_{on}}{X_{min} - X_{on}} \right) \\
&= T - \tau \log \left(\frac{(X_{max} - X_{on})^2}{(\hat{X} - X_{on})(X_{min} - X_{on})} \right).
\end{aligned} \tag{17}
$$

Using the expression for $\bar{t}_2$ in (16) and setting $\hat{X} = X_{min}$ in (17) shows that for Figure 2(a),

$$
\bar{t}_2 - \hat{t} = T - \bar{t}_2,
$$

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Table 2. Summary description of the reserve power schedules shown in Figure 2 based on the values used for the parameters. In all cases, the building is unable to provide reserve for a period of 90.5 minutes from $\bar{t}_2 = 269.5$ to $T = 360$ since the reference power is at maximum level $u_{ref}(t) = 1$ during this time.

$\frac{R}{P}$	Summary
$\frac{3}{4}$	There are few intervals of sustained decremental reserve, the longest of which is a period of 90.5 minutes from $\hat{t} = 179$ to $\bar{t}_2 = 269.5$ with an approximate normalized reserve power value $u_{cap}(t) = 0.68$.
1	There is an initial period of highly variable decremental reserve, followed by a longer period of stable decremental reserve. During the latter period, reserve power u_{cap} starts at a value near 0 and increases slowly to the maximum possible value 0.68.
$\frac{5}{4}$	There are two contiguous periods of sustained decremental reserve. These periods are bordered by the time $\check{t} = 90.5$ that the temperature first hits the minimum allowed value $X_{min} = 18$ after starting from $x(0) = 27$ and being cooled using maximum power. From $\check{t}$ to $\bar{t}_2$, which is a period of 179 minutes, reserve power hovers around the value 0.36 while the temperature is steady at its minimum allowed level. Decremental reserve is provided uninterruptedly for 4.5 hours in this case, which is almost an hour longer than the average duration of a single DTU request in 2017 (National Grid UK 2018).

and u_{pro} essentially time-shifts the final period of maximum consumption that occurred under the reference schedule u_{ref}.

These numerical results indicate that even if the benefit-cost ratio $\frac{R}{P}$ is less than 1, it is still possible to obtain a prospective control schedule that is profitable and suitable for a decremental reserve service such as DTU, at least on the interval $[\hat{t}, \bar{t}_2]$. It is important to note, however, that the net cost of providing maximum reserve power increases as R decreases. In the presence of thermal losses, this means there may no longer be any profitable solutions to (10) of this form if R is too low. In this case it may be beneficial for profitability, and also practicality, to aggregate the decremental reserve from multiple buildings.

Case 2: $R = P$

In Figure 2(b) the benefit-cost ratio satisfies $\frac{R}{P} = 1$ and the prospective schedule u_{pro} exhibits complex bang-bang behaviour early within the control horizon. This complex behaviour can be explained by noticing that when $\frac{R}{P} = 1$ the instantaneous cost in (11) satisfies,

$$u_{pro}(t) - \big(u_{pro}(t) - u_{ref}(t)\big)^+ + \alpha_{pro}\big(u_{pro}(t)\big)^2 = \min\big(u_{pro}(t), u_{ref}(t)\big) + \alpha_{pro}\big(u_{pro}(t)\big)^2. \tag{18}$$

The term $\min\big(u_{pro}(t), u_{ref}(t)\big)$ can dominate (18) when α_{pro} is low, incentivizing u_{pro} to use bang-bang control behaviour in order to be more energy efficient than the reference schedule u_{ref}, which already uses minimal energy to satisfy the temperature constraints. This line of reasoning also helps explain the complexity of u_{pro} in the case $\frac{R}{P} < 1$. Figure 3 shows furthermore that the fluctuations of u_{pro} increase in magnitude as α_{pro} decreases. The prospective schedule u_{pro} shown in Figure 2(b) may not be
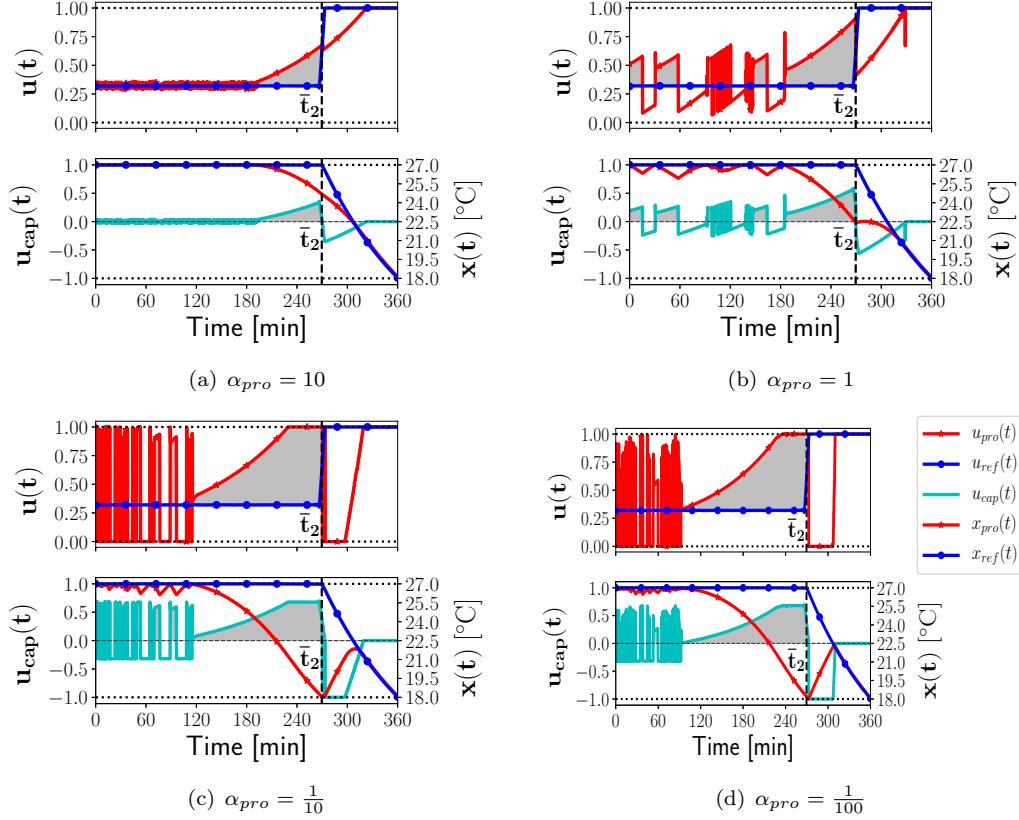
Figure 3. Optimized reserve power u_{cap} and prospective power consumption and internal temperature, u_{pro} and x_{pro}, corresponding to the benefit-cost ratio $\frac{R}{P} = 1$ and decreasing values for the regularizer weight $\alpha_{pro} \in \{10, 1, \frac{1}{10}, \frac{1}{100}\}$. Solid lines with star (respectively, circle) markers indicate the prospective (respectively, reference) schedules. The shaded region highlights where there is a positive value of instantaneous reserve u_{cap}. When α_{pro} is large, the prospective schedule u_{pro} tries to minimize the overall cost of consumption subject to the temperature constraints, similarly to the reference schedule u_{ref}. However, as α_{pro} decreases u_{pro} increasingly tries to stay below u_{ref}, leading to rapid bang-bang control behaviour. Parameter values are $x(0) = 27$ and $\hat{X} = 18$ for the initial and pre-cooling temperatures respectively. Temperature and control constraints are shown using dotted horizontal lines whilst $\bar{t}_2$, the approximate time from which the reference schedule is at maximum power, is shown using the dashed vertical line.

suitable for a decremental reserve service such as DTU since it is too variable and does not sustain a high level of reserve. Nevertheless, it may be possible to obtain a practicable and profitable prospective control schedule by using a less variable sub-optimal control schedule, or by aggregating the decremental reserve from multiple buildings to obtain a smoother control signal, a topic which is outside the scope of this work. We also note that the prospective control schedule in Figure 2(b) can still provide some total decremental reserve, albeit not with constant power, during the relatively long period where it is more regular.

Case 3: $\frac{R}{P} = \frac{5}{4}$

In Figure 2(c) the benefit-cost ratio satisfies $\frac{R}{P} > 1$. The control u_{pro} initially applies maximum power to steer the internal temperature to its minimum allowed value X_{min}. Let $\check{t}$ denote the first time that the temperature hits X_{min}. Analogous to (17), $\check{t}$ is

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given approximately by,

$$\check{t} = \tau \log\left(\frac{X_{max} - X_{on}}{X_{min} - X_{on}}\right).$$

The control u_{pro} keeps the temperature at X_{min} from $\check{t}$ until the time $\bar{t}_2$ by applying power that is on average approximately equal to (cf. (14)),
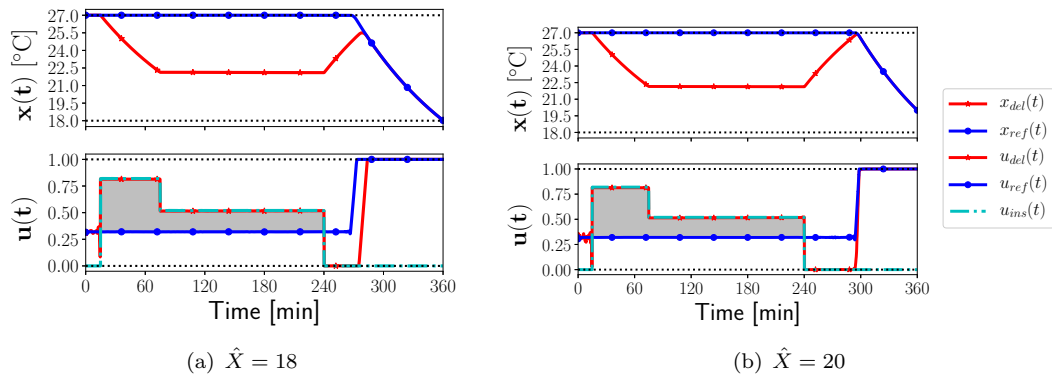
$$u_{pro}(t) = \frac{X_{off} - X_{min}}{X_{off} - X_{on}}, \quad t \in [\check{t}, \bar{t}_2].$$

This prospective schedule provides two contiguous periods of constant reserve power at two different levels,

$$u_{cap}(t) = \begin{cases} 1, & t \in [0, \check{t}], \\ \frac{X_{max} - X_{min}}{X_{off} - X_{on}}, & t \in [\check{t}, \bar{t}_2], \end{cases}$$

and is thus suitable for a decremental reserve service such as DTU.

3.3. *Optimal reserve service delivery*

(a) $\hat{X} = 18$ (b) $\hat{X} = 20$

Figure 4. Optimized internal temperature x and delivery, reference, and minimum instructed control schedules, u_{del}, u_{ref} and u_{min}, corresponding to two different values for the pre-cooling temperature $\hat{X}$. Delivery (respectively, reference) schedules are displayed using solid lines with circle (respectively, star) markers. The shaded region highlights the additional normalized power that is delivered. Reserve service instructions call for an increase in normalized power by 0.5 between minutes 15 and 75, then 0.2 between minutes 75 and 240. The schedule u_{del} uses minimal power to satisfy the temperature constraints and reserve service instructions. Moreover, the building is pre-cooled after the service has been delivered. Parameters values are $x(0) = 27$ for the initial temperature and $\alpha_{ref} = \alpha_{del} = 0.01$ for the regularizer weights. Temperature and control constraints are shown using dotted horizontal lines.

In this section we summarize the numerical results for the optimal reserve service delivery problem (13). As Figure 4 illustrates, the optimized delivery schedule uses minimal power to satisfy the temperature constraints and reserve service instructions. After delivering the service, the internal temperature is lower than the reference level, and the cooling equipment can be turned off temporarily while the temperature rises within its permitted range. The cooling equipment remains off for a longer duration if less pre-cooling is required at the control horizon's end.

4. Summary and recommendations

As the energy transition transforms power grids across the globe, high levels of intermittent renewable generation complicate the job of continuously balancing power supply and demand, which is necessary for the grid's stability. New ancillary services have emerged in this regard, such as National Grid UK's Demand Turn Up (DTU) (National Grid UK 2018), which is a reserve service that incentivizes large energy consumers to increase their electricity demand, for example during overnight periods of high output from renewable generation and low overall demand. In this paper we explore the optimal participation of a commercial building, through the control of its temperature cooling equipment, in such an ancillary service initiative. We provide a computational framework for solving this problem that takes into account the economic incentives given. The framework has three main outputs:

(1) an optimal reference night-time control schedule for the cooling system when it does not provide DTU.
(2) an optimal schedule of DTU reserve power relative to the reference for a given remuneration.
(3) an optimal night-time control schedule to fulfil DTU instructions.

The framework also takes into account the building's relaxation dynamics, so that DTU requests are used as an opportunity to optimally pre-cool the building. In addition to the DTU payment, this pre-cooling reduces energy consumption during the subsequent morning peak period, which is a financial benefit to the customer and also reduces stress on the grid.

The optimal control schedule used as reference or to fulfil the DTU instructions is intuitive and satisfies the temperature and power constraints with minimal cost. Consistent with previous studies, we find that the level of participation in the ancillary service is affected by the dynamics and constraints for the internal temperature, and how well the building manager is remunerated. Moreover, participation in DTU can be profitable even in a case where the utilization payment is lower than the night-time electricity price. This is because simply shifting pre-planned HVAC operation to a time earlier in the night leads to increased demand at the earlier time, attracting compensation under DTU. However, the optimal control strategy becomes more complex as the night-time electricity price further exceeds the utilization payment, fluctuating more frequently between minimum and maximum power as it "hunts" for profit. Rapid power fluctuations may be problematic for system stability, and frequent or prolonged demand reductions are undesirable as they undermine the purpose of DTU. Therefore, in order to economically incentivize a building manager to provide DTU practicably, the level of remuneration must be sufficiently high.

Possible future extensions of our work would include controlling an ensemble of possibly heterogeneous thermostatic loads (Tindemans, Trovato, and Strbac 2015) and considering measured temperature relaxation dynamics for each member of the ensemble. Heterogeneity, which can come from different zones in a single building or from an aggregation of buildings, may be exploited to obtain a less variable and practicable aggregate response for the ancillary service. Our model may also be extended to consider the uncertainty in parameters affecting the internal temperature, such as the external weather conditions, or the uncertainty in being called to provide DTU.

Disclosure statement

There is no potential conflict of interest, financial interest or benefit that has arisen from the direct applications of this work.

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Notation

Constant quantities

Notation	Description	Units
T	length of control horizon, the period of time during which the building's temperature is controlled for the ancillary service	min
P	night-time electricity price	p/kWh
R	utilization payment	p/kWh
X_{min}	night-time lower temperature limit	°C
X_{max}	night-time upper temperature limit	°C
$\hat{X}$	temperature limit used for pre-cooling at time T	°C
X_{off}	asymptotic temperature for the building "off" state	°C
X_{on}	asymptotic temperature for the building "on" state	°C
τ	thermal time constant	min
C_{max}	maximum power limit for the building	kW
α	importance weighting for a regularization term in the control problem, distinguished by a subscript (α_{ref}, for example)	1
$\mathcal{U}$	set of normalized cooling power consumption variables u	–

Variable quantities

Notation	Description	Units
$\mathbf{1}_A(t)$	indicator function of a set A: $\mathbf{1}_A(t) = 1$ if $t \in A$ and $\mathbf{1}_A(t) = 0$ if $t \notin A$	–
$x(t)$	building internal temperature at time t	°C
$C(t)$	power consumption of the cooling equipment	kW
$C_{ref}(t)$	reference cooling power consumption	kW
$C_{pro}(t)$	prospective power consumption used to calculate the reserve capability relative to C_{ref}	kW
$C_{cap}(t)$	declared level of reserve capability	kW
$C_{ask}(t)$	reserve service instructions, a schedule of additional power consumption which must be delivered	kW
$C_{del}(t)$	power consumption when delivering the reserve service according to the instructions C_{ask}	kW
$u(t)$	normalized power cooling consumption	1
$u_{ref}(t)$	normalized reference cooling power consumption	1

$u_{pro}(t)$	normalized prospective cooling power consumption	1
$u_{del}(t)$	normalized delivery cooling power consumption	1

References

Adkins, William A, and Mark G Davidson. 2012. *Ordinary Differential Equations (Undergraduate Texts in Mathematics)*. New York, NY, USA: Springer.

ASHRAE. 2016. "Standard 90.4-2016 – Energy Standard for Data Centers." .

Blum, David H., Tea Zakula, and Leslie K. Norford. 2017. "Opportunity Cost Quantification for Ancillary Services Provided by Heating, Ventilating, and Air-Conditioning Systems." *IEEE Transactions on Smart Grid* 8 (3): 1264–1273.

Cesari, Lamberto. 1983. *OptimizationTheory and Applications*. New York, NY, USA: Springer New York.

Clarke, Francis. 2013. *Functional Analysis, Calculus of Variations and Optimal Control*. Vol. 264 of *Graduate Texts in Mathematics*. London, England: Springer London.

Dayarathna, Miyuru, Yonggang Wen, and Rui Fan. 2016. "Data Center Energy Consumption Modeling: A Survey." *IEEE Communications Surveys & Tutorials* 18 (1): 732–794.

De Coninck, Roel, and Lieve Helsen. 2016. "Quantification of flexibility in buildings by cost curves Methodology and application." *Applied Energy* 162: 653–665.

Deng, Ruilong, Zaiyue Yang, Mo-Yuen Chow, and Jiming Chen. 2015. "A Survey on Demand Response in Smart Grids: Mathematical Models and Approaches." *IEEE Transactions on Industrial Informatics* 11 (3): 570–582.

Hirst, Eric, and Brendan Kirby. 1998. *Unbundling Generation and Transmission Services for Competitive Electricity Markets: Examining Ancillary Services*. Technical Report. Columbus, OH, USA: The National Regulatory Research Institute, The Ohio State University.

Ihara, Satoru, and Fred Schweppe. 1981. "Physically Based Modeling of Cold Load Pickup." *IEEE Transactions on Power Apparatus and Systems* PAS-100 (9): 4142–4150.

International Energy Agency (IEA). 2018. "Energy in Buildings and Communities Programme (EBC) Annex 67." Accessed 04/09/2018. http://www.annex67.org/.

Jensen, Søren Østergaard, Anna Marszal-Pomianowska, Roberto Lollini, Wilmer Pasut, Armin Knotzer, Peter Engelmann, Anne Stafford, and Glenn Reynders. 2017. "IEA EBC Annex 67 Energy Flexible Buildings." *Energy and Buildings* 155: 25–34.

Junker, Rune Grønborg, Armin Ghasem Azar, Rui Amaral Lopes, Karen Byskov Lindberg, Glenn Reynders, Rishi Relan, and Henrik Madsen. 2018. "Characterizing the energy flexibility of buildings and districts." *Applied Energy* 225 (May): 175–182.

Kim, Young-Jin, David H. Blum, Nora Xu, Leo Su, and Leslie K. Norford. 2016. "Technologies and Magnitude of Ancillary Services Provided by Commercial Buildings." *Proceedings of the IEEE* 104 (4): 758–779.

Kirschen, Daniel S., and Goran Strbac. 2004. *Fundamentals of Power System Economics*. Chicester, England: John Wiley & Sons.

Lawrence, Thomas M., Marie-Claude Boudreau, Lieve Helsen, Gregor Henze, Javad Mohammadpour, Doug Noonan, Dieter Patteeuw, Shanti Pless, and Richard T. Watson. 2016. "Ten questions concerning integrating smart buildings into the smart grid." *Building and Environment* 108: 273–283.

Lin, Qun, Ryan Loxton, and Kok Lay Teo. 2014. "The control parameterization method for nonlinear optimal control: A survey." *Journal of Industrial and Management Optimization* 10 (1): 275–309.

Maurer, H. 1977. "On Optimal Control Problems with Bounded State Variables and Control Appearing Linearly." *SIAM Journal on Control and Optimization* 15 (3): 345–362.

National Grid UK. 2018. "Demand Turn Up." Accessed 20/08/2018. https://www.nationalgrid.com/uk/electricity/balancing-services/reserve-services/demand-turn.

Olivieri, Simon J., Gregor P. Henze, Chad D. Corbin, and Michael J. Brandemuehl. 2014. "Evaluation of commercial building demand response potential using optimal short-term

curtailment of heating, ventilation, and air-conditioning loads." *Journal of Building Performance Simulation* 7 (2): 100–118.

Paterakis, Nikolaos G., Ozan Erdinç, and João P.S. Catalão. 2017. "An overview of Demand Response: Key-elements and international experience." *Renewable and Sustainable Energy Reviews* 69: 871–891.

Pavlak, Gregory S., Gregor P. Henze, and Vincent J. Cushing. 2014. "Optimizing commercial building participation in energy and ancillary service markets." *Energy and Buildings* 81: 115–126.

Reddy, TA, LK Norford, and W Kempton. 1991. "Shaving residential air-conditioner electricity peaks by intelligent use of the building thermal mass." *Energy* 16 (7): 1001–1010.

Roth, Kurt, John Dieckmann, and James Brodrick. 2009. "Emerging Technologies: Using Off-Peak Precooling." *ASHRAE Journal* 51 (3).

Rothleder, Mark, and Clyde Loutan. 2014. "Case Study - Renewable Integration: Flexibility Requirement, Potential Overgeneration, and Frequency Response Challenges." In *Renewable Energy Integration - Practical Management of Variability, Uncertainty, and Flexibility in Power Grids*, edited by Lawrence E. Jones, Chap. 6, 67–79. New York: Elsevier Inc.

Szabó, Dávid Zoltán, and Randall Martyr. 2017. "Real option valuation of a decremental regulation service provided by electricity storage." *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* 375 (2100).

Teo, K. L., and C. J. Goh. 1991. *A unified computational approach to optimal control problems.* Essex, England: Longman Scientic & Technical.

Tindemans, Simon H., Vincenzo Trovato, and Goran Strbac. 2015. "Decentralized Control of Thermostatic Loads for Flexible Demand Response." *IEEE Transactions on Control Systems Technology* 23 (5): 1685–1700.

Vardakas, John S., Nizar Zorba, and Christos V. Verikoukis. 2015. "A Survey on Demand Response Programs in Smart Grids: Pricing Methods and Optimization Algorithms." *IEEE Communications Surveys & Tutorials* 17 (1): 152–178.

Wang, Shengwei, and Zhenjun Ma. 2008. "Supervisory and Optimal Control of Building HVAC Systems: A Review." *HVAC&R Research* 14 (1): 3–32.

Appendix A. Description of the control parametrization method

In this section we describe the control parametrization method for solving optimal control problems of the form,

$$\text{minimize } J(u) = \int_0^T \Psi_0(t, x(t), u(t))dt + \Phi_0(x(T)) \text{ subject to:}$$

$$i) \quad \dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0;$$

$$ii) \quad u \in \mathcal{U}; \tag{A1}$$

$$iii) \quad C_\eta(u) \leq \varepsilon_\eta \text{ for } \eta = 1, \dots, N.$$

where $f(t, x(t), u(t))$ governs the dynamics of x according to (1), each $C_\eta(u)$ is the *canonical form* of a loss function associated with a constraint,

$$C_\eta(u) = \int_0^T \Psi_\eta(t, x(t), u(t)) \, dt + \Phi_\eta(x(T)), \tag{A2}$$

and each $\varepsilon_\eta \geq 0$ is a sufficiently small tolerance parameter. For more details on this numerical method please see the textbook Teo and Goh (1991) and the survey Lin, Loxton, and Teo (2014).

Let S^p, where $p \geq 1$ is an integer, denote a finite subset of the control horizon $[0, T]$ consisting of $n_p + 1$ *partitioning points* $t_0^p, \ldots, t_{n_p}^p$,

$$t_0^p = 0, \ t_{n_p}^p = T, \ \text{and} \ t_{k-1}^p < t_k^p \ \text{for} \ k = 1, \ldots, n_p.$$

An increasing sequence of sets $\{S^p\}_{p=1}^{\infty}$ is formed by taking successive refinements of partitioning points, and these sets should become dense in $[0, T]$ as p tends to infinity,

$$\lim_{p \to \infty} \max_{k=1,\ldots,n_p} |t_k^p - t_{k-1}^p| = 0.$$

For instance, we can use equidistant partitioning points, $t_k^p = \frac{k}{n_p} T$ for $k = 0, \ldots, n_p$, with the ratio $\frac{n_{p+1}}{n_p}$, $p \geq 1$, being a constant integer that is greater than 1 (a common choice is $\frac{n_{p+1}}{n_p} = 2$). We define $\mathcal{U}^p$ as the subset of control variables $u^p \in \mathcal{U}$ that are step functions and consistent with S^p in the following sense,

$$u^p(t) = \sum_{k=1}^{n_p} u_k^p \mathbf{1}_{[t_{k-1}^p, t_k^p)}(t), \quad u_k^p \in [0, 1].$$

Each control u^p is parametrized by an element $\boldsymbol{U}^p$ of the n_p-dimensional space $\mathbb{U}^p = \prod_{k=1}^{n_p} [0, 1]$. This induces equivalent state dynamics $\tilde{f}$, costs $\tilde{J}$ and constraints $\tilde{C}_\eta$ that are dependent on the parameter $\boldsymbol{U}^p$,

$$\dot{x}(t) = \tilde{f}(t, x(t), \boldsymbol{U}^p) = f(t, x(t), u^p(t)),$$
$$\tilde{J}(\boldsymbol{U}^p) = J(u^p),$$
$$\tilde{C}_\eta(\boldsymbol{U}^p) = C_\eta(u^p).$$

An approximate solution to the *infinite dimensional* optimal control problem (A1) is obtained by solving the following non-linear *finite dimensional* optimization problem.

$$\text{minimize } \tilde{J}(\boldsymbol{U}^p) \text{ subject to:}$$

$i)$ $\dot{x}(t) = \tilde{f}(t, x(t), \boldsymbol{U}^p), \quad x(0) = x_0;$

$ii)$ $\boldsymbol{U}^p \in \mathbb{U}^p;$ (A3)

$iii)$ $\tilde{C}_\eta(\boldsymbol{U}^p) \leq \varepsilon_\eta$ for $\eta = 1, \ldots, N.$

An optimization algorithm such as sequential quadratic programming can be used to solve this approximate problem. Such optimization algorithms are typically iterative, and the main computations carried out during each iteration are outlined below (see Section 6.6 of Teo and Goh (1991) for further details):

(1) Obtain a trajectory for the state variable x by numerically integrating its dynamics forward in time on the partitioning points S^p.
(2) Evaluate the cost $\tilde{J}(\boldsymbol{U}^p)$ and constraints $\tilde{C}_\eta(\boldsymbol{U}^p)$ using numerical integration.
(3) Compute the gradients of the cost $\tilde{J}(\boldsymbol{U}^p)$ and constraints $\tilde{C}_\eta(\boldsymbol{U}^p)$ according to the formulas given in Section 6.6 of Teo and Goh (1991).

The gradient of the cost $\tilde{J}(\boldsymbol{U}^p)$, for example, involves computation of the gradient of a *Hamiltonian function* $\tilde{\mathcal{H}}_0$ with respect to the parameter $\boldsymbol{U}^p$,

$$\frac{\partial \tilde{J}(\boldsymbol{U}^p)}{\partial \boldsymbol{U}^p} = \int_0^T \frac{\partial \tilde{\mathcal{H}}_0(t, x(t), \boldsymbol{U}^p, z(t))}{\partial \boldsymbol{U}^p} dt,$$

where z is the *costate variable* associated to the cost. The Hamiltonian is defined by,

$$\tilde{\mathcal{H}}_0(t, x(t), \boldsymbol{U}^p, z(t)) = \Psi_0(t, x(t), u^p(t)) + z(t) f(t, x(t), u^p(t)).$$

Dynamics for this costate variable are given by,

$$\begin{cases} \dot{z}(t) = -\dfrac{\partial \tilde{\mathcal{H}}_0(t, x(t), \boldsymbol{U}^p, z(t))}{\partial x} \\ z(T) = \frac{d}{dx} \Phi_0(x(T)), \end{cases}$$

and this differential equation is solved numerically *backwards in time* given a trajectory for x. A costate variable for each constraint function (A2) is defined similarly.

Loss functions for state and control constraints

Here we describe our specification of the constraint loss functions (A2), only presenting those for the reserve service delivery problem (13) since those for the other optimal control problems can be formulated analogously. First define $(t, x, u) \mapsto \psi_1(t, x, u)$ and $x \mapsto \phi_1(x)$ by,

$$\begin{cases} \psi_1(t, x, u) = (X_{max} - x)(x - X_{min}) \\ \phi_1(x) = (\hat{X} - x)(x - X_{min}) \end{cases} \tag{A4}$$

By definition, we say that the integral constraint ψ_1 is satisfied at (t, x, u) if and only if $\psi_1(t, x, u) \geq 0$. Similarly, the terminal constraint ϕ_1 is satisfied at x if and only if $\phi_1(x) \geq 0$. Equation (A4) corresponds to the time-dependent pure state constraints on the internal temperature over $[0, T]$ (cf. (4)) and at time T (cf. (5)). In a similar way we define integral and terminal constraints for DTU,

$$\begin{cases} \psi_2(t, x, u) = u - \sum_{i=1}^N [u_{ref}(t) + u_{i,ask}] \mathbf{1}_{[s_i, e_i)}(t) \\ \phi_2(x) = 0 \end{cases}$$

Using these constraints we define *loss rate functions* $(t, x, u) \mapsto \Psi_\eta(t, x, u)$ and *terminal loss functions* $x \mapsto \Phi_\eta(x)$, $\eta \in \{1, 2\}$, by,

$$\begin{cases} \Psi_\eta(t, x, u) = (\min(0, \psi_\eta(t, x, u)))^2 \\ \Phi_\eta(x) = \lambda_\eta (\min(0, \phi_\eta(x)))^2 \end{cases}$$

where $\lambda_\eta > 0$ is a weighting parameter. The loss functions Ψ_η and Φ_η are combined to create the *total loss* $C_\eta(u)$ for the constraint (cf. (A2)). The total loss C_η is non-negative by construction and is equal to zero if, equivalently, the relevant constraints

are satisfied on $[0, T]$. We relax this condition by requiring,

$$C_\eta(u) \le \varepsilon_\eta, \ \eta \in \{1, 2\}.$$

The SLSQP routine that we use to solve the optimization problem (A3) requires derivative information as input. However, some of the constraints and costs are expressed in terms of indicator and ramp (maximum) functions that are not smooth. Therefore, where necessary we approximate these functions smoothly as follows:

$$\mathbf{1}_{[0,\infty)}(y) \approx \frac{e^{\theta y}}{1 + e^{\theta y}}, \quad \mathbf{1}_{[a,b]}(y) \approx \left(\frac{e^{\theta(y-a)}}{1 + e^{\theta(y-a)}} \right) \left(\frac{e^{\theta(b-y)}}{1 + e^{\theta(b-y)}} \right) \ \text{ for } \ a < b,$$
$$\max(0, y) \approx \frac{1}{\theta} \log(1 + e^{\theta y}),$$

where $\theta > 0$ is a sufficiently large parameter.