RIGIDITY OF LINEARLY-CONSTRAINED FRAMEWORKS

JAMES CRUICKSHANK, HAKAN GULER, BILL JACKSON, AND ANTHONY NIXON

ABSTRACT. We consider the problem of characterising the generic rigidity of bar-joint frameworks in $\mathbb{R}^d$ in which each vertex is constrained to lie in a given affine subspace. The special case when $d = 2$ was previously solved by I. Streinu and L. Theran in 2010. We will extend their characterisation to the case when $d \geq 3$ and each vertex is constrained to lie in an affine subspace of dimension t, when $t = 1, 2$ and also when $t \geq 3$ and $d \geq t(t-1)$. We then point out that results on body-bar frameworks obtained by N. Katoh and S. Tanigawa in 2013 can be used to characterise when a graph has a rigid realisation as a d-dimensional body-bar framework with a given set of linear constraints.

1. INTRODUCTION

A (bar-joint) framework (G, p) in $\mathbb{R}^d$ is a finite graph $G = (V, E)$ together with a realisation $p : V \to \mathbb{R}^d$. The framework (G, p) is *rigid* if every edge-length preserving continuous motion of the vertices arises as a congruence of $\mathbb{R}^d$.

It is NP-hard to determine whether a given framework is rigid [1], but this problem becomes more tractable for generic frameworks. It is known that the rigidity of a generic framework (G, p) in $\mathbb{R}^d$ depends only on the underlying graph G, see [2]. We say that G *is rigid in* $\mathbb{R}^d$ if some/every generic realisation of G in $\mathbb{R}^d$ is rigid. Combinatorial characterisations of generic rigidity in $\mathbb{R}^d$ have been obtained when $d \leq 2$, see [9], and these characterisations give rise to efficient combinatorial algorithms to decide if a given graph is rigid. In higher dimensions, however, no combinatorial characterisation or algorithm is yet known.

Motivated by numerous potential applications, notably in mechanical engineering, rigidity has also been considered for frameworks with various kinds of pinning constraints [4, 8, 13, 14, 16]. Most relevant to this paper is the work of Streinu and Theran [14] on slider-pinning, which we describe below.

Throughout this paper we will use the term *graph* to describe a graph which may contain multiple edges and loops and denote such a graph by $G = (V, E, L)$ where V, E, L are the sets of vertices, edges and loops, respectively. We will use the terms *simple graph* to describe a graph which contains neither multiple edges nor loops, and *looped simple graph* to describe a graph which contains no multiple edges but may contain loops.

A *linearly-constrained framework in* $\mathbb{R}^d$ is a triple (G, p, q) where $G = (V, E, L)$ is a graph, $p : V \to \mathbb{R}^d$ and $q : L \to \mathbb{R}^d$. For $v_i \in V$ and $e_j \in L$ we put $p(v_i) = p_i$ and $q(e_j) = q_j$. It is *generic* if (p, q) is algebraically independent over $\mathbb{Q}$.

Date: July 2, 2018.

2010 *Mathematics Subject Classification.* 52C25, 05C10 and 53A05.

Key words and phrases. rigidity, linearly-constrained framework, pinned framework, slider joint, framework on a surface.

An *infinitesimal motion* of (G,p,q) is a map $\dot{p} : V \to \mathbb{R}^d$ satisfying the system of linear equations:

$$(p_i - p_j) \cdot (\dot{p}_i - \dot{p}_j) = 0 \text{ for all } v_i v_j \in E \tag{1.1}$$

$$q_j \cdot \dot{p}_i = 0 \text{ for all incident pairs } v_i \in V \text{ and } e_j \in L. \tag{1.2}$$

The second constraint implies that the infinitesimal velocity of each $v_i \in V$ is constrained to lie on the hyperplane through p_i with normal q_j for each loop e_j incident to v_i.

The *rigidity matrix* $R(G,p,q)$ of the linearly-constrained framework (G,p,q) is the matrix of coefficients of this system of equations for the unknowns $\dot{p}$. Thus $R(G,p,q)$ is a $(|E|+|L|) \times d|V|$ matrix, in which: the row indexed by an edge $v_iv_j \in E$ has $p(u) - p(v)$ and $p(v) - p(u)$ in the d columns indexed by v_i and v_j, respectively and zeros elsewhere; the row indexed by a loop $e_j = v_iv_i \in L$ has q_j in the d columns indexed by v_i and zeros elsewhere. The $|E| \times d|V|$ sub-matrix consisting of the rows indexed by E is the *bar-joint rigidity matrix* $R(G-L,p)$ of the bar-joint framework $(G-L,p)$.

The framework (G,p,q) is *infinitesimally rigid* if its only infinitesimal motion is $\dot{p} = 0$, or equivalently if $\operatorname{rank} R(G,p,q) = d|V|$. We say that the graph G is *rigid* in $\mathbb{R}^d$ if $\operatorname{rank} R(G,p,q) = d|V|$ for some realisation (G,p,q) in $\mathbb{R}^d$, or equivalently if $\operatorname{rank} R(G,p,q) = d|V|$ for all *generic* realisations (G,p,q) i.e. all realisations for which (p,q) is algebraically independent over $\mathbb{Q}$.

Streinu and Theran [14] characterised the graphs G which are rigid in $\mathbb{R}^2$. We need to introduce some terminology to describe their result. Let $G = (V,E,L)$ be a graph. For $F \subseteq E \cup L$, let V_F denote the set of vertices incident to F.

Theorem 1.1. *A graph can be realised as an infinitesimally rigid linearly-constrained framework in $\mathbb{R}^2$ if and only if it has a spanning subgraph $G = (V,E,L)$ such that $|E|+|L| = 2|V|$, $|F| \le 2|V_F|$ for all $F \subseteq E \cup L$ and $|F| \le 2|V_F| - 3$ for all $\emptyset \ne F \subseteq E$.*

The results of this paper extend Theorem 1.1 to $\mathbb{R}^d$ under the assumption that the number of linear constraints at each vertex is sufficiently large compared to d. We need some further definitions to state our main results. A graph $G = (V,E,L)$ is *(k,ℓ)-sparse* if $|F| \le k|V_F| - \ell$ holds for all $\emptyset \ne F \subseteq E \cup L$. The graph G is *(k,ℓ)-tight* if it is (k,ℓ)-sparse and $|E| + |L| = k|V| - \ell$. We use $G^{[k]}$ to denote the graph obtained from G by adding k loops to each vertex of G.

Theorem 1.2. *Suppose G is a graph and d,t are positive integers with $d \ge \max\{2t, t(t-1)\}$. Then $G^{[d-t]}$ can be realised as an infinitesimally rigid linearly-constrained framework in $\mathbb{R}^d$ if and only if G has a $(t,0)$-tight looped simple spanning subgraph.*

The complete simple graph K_{2t+1} shows that the condition $d \ge 2t$ in Theorem 1.2 is, in some sense, best possible: K_{2t+1} is $(t,0)$-tight and $K_{2t+1}^{[d-t]}$ does not have an infinitesimally rigid realisation in $\mathbb{R}^d$ when $d = 2t-1$. This follows from the fact that the rows of the bar-joint rigidity matrix of any realisation of K_{2t+1} in $\mathbb{R}^{2t-1}$ are dependent. Hence the rank of the rigidity matrix of any realisation of $K_{2t+1}^{[t-1]}$ in $\mathbb{R}^{2t-1}$ will be at most $|E(K_{2t+1}^{[t-1]})| - 1 = (2t-1)|V(K_{2t+1}^{[t-1]})| - 1$. We do not know whether the conclusion of Theorem 1.2 still holds if the condition $d \ge t(t-1)$ is removed - it is conceivable that this condition is an artifact of our proof technique.

Linearly-constrained frameworks naturally arise when considering the infinitesimal rigidity of a bar-joint framework (G,p) in $\mathbb{R}^d$ under the additional constraint that the vertices of G lie on a smooth algebraic variety $\mathcal{V}$. We can model this situation as a linearly-constrained framework $(G^{[d-t]}, p, q)$ in which $t = \dim \mathcal{V}$ and q is chosen so that the image of the loops of

$G^{[d-t]}$ at each vertex v span the orthogonal complement of the tangent space of $\mathcal{V}$ at $p(v)$. In this context, the continuous isometries of $\mathcal{V}$ will always induce infinitesimal motions of (G,p). We say that (G,p) *is infinitesimally rigid on* $\mathcal{V}$ if these are the only infinitesimal motions of (G,p). Equivalently (G,p) is infinitesimally rigid on $\mathcal{V}$ if $\operatorname{rank} R(G^{[d-t]},p,q) = d|V| - \alpha$ where α is the *type* of $\mathcal{V}$ i.e. the dimension of the space of infinitesimal isometries of $\mathcal{V}$. The special case when $\mathcal{V}$ is an irreducible surface in $\mathbb{R}^3$ was previously studied by Nixon, Owen and Power [11, 12]. They characterised generic rigidity for all such surfaces of types 3, 2 and 1.

Theorem 1.3. *Let $G=(V,E)$ be a simple graph and let $\mathcal{M}$ be an irreducible surface in $\mathbb{R}^3$ of type $\alpha \in \{1,2,3\}$. Then a generic framework (G,p) on $\mathcal{M}$ is infinitesimally rigid on $\mathcal{M}$ if and only if G has a $(2,\alpha)$-tight spanning subgraph.*

Theorem 1.2 characterises the graphs G which can be realised as an infinitesimally rigid linearly constrained framework on some type 0, t-dimensional variety in $\mathbb{R}^d$, whenever $d \geq \max\{2t, t(t-1)\}$. The special case when $t = d-2$ and $d \geq 4$ characterises the graphs which have an infinitesimally rigid realisation as a linearly-constrained framework on some type 0 surface in $\mathbb{R}^d$. A characterisation for the case when $d=2$ is given by Theorem 1.1. Our next result gives a solution to the remaining case when $d=3$.

Theorem 1.4. *Suppose G is a graph. Then $G^{[1]}$ can be realised as an infinitesimally rigid linearly-constrained framework in $\mathbb{R}^3$ if and only if G has a $(2,0)$-tight looped simple spanning subgraph which contains no copy of K_5.*

Theorem 1.4 completes the characterisation of infinitesimal rigidity of generic *plane-constrained frameworks* i.e. frameworks in which each vertex is constrained to lie on a given plane in $\mathbb{R}^d$. In this context it is natural to consider the infinitesimal rigidity of *line-constrained frameworks* in $\mathbb{R}^d$. Given a graph G and an integer $d \geq 2$, Theorem 1.2 tells us that $G^{[d-1]}$ can be realised as an infinitesimally rigid linearly-constrained framework in $\mathbb{R}^d$ if and only if G has a $(1,0)$-tight looped simple spanning subgraph. We will prove a more general result. We consider the case when we are given a graph G and a map q from the set of loops of $G^{[d-1]}$ to $\mathbb{R}^d$ such that the image of the set of loops at each vertex of $G^{[d-1]}$ spans a subspace of dimension at least $d-1$. We then characterise when there exists a map $p : V \to \mathbb{R}^d$ such that (G,p,q) is infinitesimally rigid.

Our final results concern *body-bar frameworks in* $\mathbb{R}^d$ i.e. frameworks consisting of d-dimensional rigid bodies joined by rigid bars. Tay [15] characterised when a graph has an infinitesimally rigid realisation as a body-bar framework in $\mathbb{R}^d$. We point out that two results of Katoh and Tanigawa [8] immediately extend Tay's result to linearly-constrained body-bar frameworks in $\mathbb{R}^d$.

2. Linearly-constrained frameworks

We will prove Theorem 1.2. The necessity part of the theorem follows immediately from:

Lemma 2.1. *Suppose G is a graph. If $G^{[d-t]}$ can be realised as an infinitesimally rigid linearly-constrained framework in $\mathbb{R}^d$ then G has a $(t,0)$-tight looped simple spanning subgraph.*

Proof. We may suppose that $(G^{[d-t]},p,q)$ is an infinitesimally rigid generic realisation of $G^{[d-t]}$ in $\mathbb{R}^d$. Let S be a set of loops of $G^{[d-t]}$ consisting of exactly $d-t$ loops at each vertex. It is not difficult to see that the rows of $R(G^{[d-t]},p,q)$ labeled by S are linearly independent and hence we can choose a spanning subgraph $H=(V,E_H,L_H)$ of G such that the rows of $R(H^{[d-t]}, p, q|_{H^{[d-t]}})$ are linearly independent and rank $R(H^{[d-t]}, p, q|_{H^{[d-t]}}) = d|V|$. The linear independence of the rows of $R(H^{[d-t]}, p, q|_{H^{[d-t]}})$ immediately implies that H has no multiple edges. If H had a subgraph $F = (V_F, E_F, L_F)$ with $|E_F| + |L_F| > t|V_F|$ then we would have rank $R(F^{[d-t]}, p, q|_{F^{[d-t]}}) \leq d|V_F| < |E_{F^{[d-t]}}| + |L_{F^{[d-t]}}|$. This would contradict the fact that the rows of $R(H^{[d-t]}, p, q|_{H^{[d-t]}})$ are linearly independent. Hence H is $(t, 0)$-tight. $\square$

Our proof of sufficiency is inductive and is based on a reduction lemma which reduces a $(t, 0)$-tight graph to a smaller such graph and an extension lemma which extends a graph which has a generically infinitesimally rigid realisation in $\mathbb{R}^d$ to a larger such graph.

Reduction operation. We shall need the following well known result on tight subgraphs of a sparse graph $G = (V, E, L)$. It follows easily from the fact that the function $F \mapsto |V_F|$ for $F \subseteq E \cup L$ is submodular.

Lemma 2.2. *Suppose that H, K are $(t, 0)$-tight subgraphs of a $(t, 0)$-sparse graph. Then $H \cup K$ and $H \cap K$ are also $(t, 0)$-tight.* $\square$

Given a vertex v in a graph G, we use $N_G(v)$ to denote the *neighbour set* of v (i.e. the set of vertices of $G - v$ which are adjacent to v), $L_G(v)$ to denote the set of loops incident to v, and $\delta_G(v)$ to denote the number of edges and loops of G which are incident to v, counting each loop once. We will suppress the subscript 'G' when it is obvious which graph we are referring to.

Lemma 2.3. *Suppose t is a positive integer and v is a vertex of a $(t, 0)$-tight graph G. Let $k = \delta(v) - t$. Then there are distinct vertices $v_1, \dots, v_k \in N(v)$ such that $(G - v) \cup \{l_1, \dots, l_k\}$ is $(t, 0)$-tight, where l_i is a new loop added at v_i for all $1 \leq i \leq k$.*

Proof. We show that there are distinct vertices $v_1, \dots, v_i \in N(v)$ such that $H_i = (G - v) \cup \{l_1, \dots, l_i\}$ is $(t, 0)$-sparse for all $0 \leq i \leq k$ by induction on i. If $i = 0$ then $G - v$ is $(t, 0)$-sparse since it is a subgraph of G.

Suppose inductively that, for some $0 \leq r \leq k - 1$, we have vertices $v_1, \dots, v_r \in N(v)$ such that H_r is $(t, 0)$-sparse. For a contradiction, suppose that there is no neighbour $v_{r+1} \in N(v)$ such that H_{r+1} is also $(t, 0)$-sparse. Then for every $u \in N(v) - \{v_1, \dots, v_r\}$ there is some $(t, 0)$-tight subgraph of H_r that contains u. We can now use Lemma 2.2 to obtain a $(t, 0)$-tight subgraph H of H_r that contains $N(v) - \{v_1, \dots, v_r\}$. It is possible that H contains some of $l_1, \dots, l_r$. Without loss of generality, suppose that $l_1, \dots, l_b \in H$ and $l_{b+1}, \dots, l_r \notin H$. The subgraph of G induced by $V(H) \cup \{v\}$ is obtained from H by deleting the loops $l_1, \dots, l_b$, and adding the vertex v and $\delta(v) - (r - b)$ edges incident to v. Since H is $(t, 0)$-tight and

$$\delta(v) - (r - b) - b = k + t - r \geq k + t - (k - 1) > t,$$

this contradicts the fact that G is $(t, 0)$-sparse.

Thus there are vertices $v_1, \dots, v_k \in N(v)$ such that H_k is $(t, 0)$-sparse. Finally observe that H_k has $\delta(v) - k = t$ fewer edges than G and one less vertex and so is, in fact, $(t, 0)$-tight. $\square$

Extension operation. Let $H = (V, E, L)$ be a graph and $d \geq 1, 0 \leq k \leq d$ be integers. The *d-dimensional k-loop extension operation* forms a new graph G from H by deleting k loops incident to distinct vertices of H and adding a new vertex v and $d + k$ new edges and loops incident to v, with the proviso that at least k loops are added at v and exactly one new edge is added from v to each of the end-vertices of the k deleted loops. Note that, since $k \leq d$, these conditions imply that v is incident to at most d loops and at most d edges, see Figure 1.

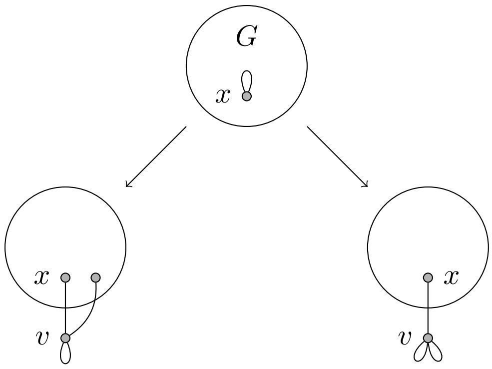

FIGURE 1. Possible 2-dimensional 1-loop extensions of a graph G.

Lemma 2.4. *Suppose that G is obtained from H by a d-dimensional k-loop extension operation which deletes a loop l_j at k distinct vertices v_j, $1 \le j \le k$, of H and adds a new vertex v. Suppose further that H has an infinitesimally rigid realisation as a linearly-constrained framework in $\mathbb{R}^d$, and that v_j is incident with at least $\left\lceil \frac{(k-1)d}{k} \right\rceil$ loops in H for all $1 \le j \le k$ when $k \ge 2$. Then G has an infinitesimally rigid realisation as a linearly-constrained framework in $\mathbb{R}^d$.*

Proof. Suppose that (H, p_H, q_H) is generic and rigid where $p_H : V_H \to \mathbb{R}^d$ and $q_H : L_H \to \mathbb{R}^d$. For each vertex $u \in V_H$, define $p_G(u) = p_H(u)$ and for any loop $l \in L_H$ at a vertex other than $v_1, \ldots, v_k$, define $q_G(l) = q_H(l)$.

For $1 \le j \le k$, let W_j be the linear span of $q_H(L_H(v_j))$, $A_j = p_H(v_j) + W_j$, and A be the affine span of $p_H(N_G(v))$. Since (p_H, q_H) is generic, the affine spaces A_j, $1 \le j \le k$, and A are in general position[1], $\dim(A_j) \ge \left\lceil \frac{(k-1)d}{k} \right\rceil$ and $\dim A = |N_G(v)| - 1 \le d - 1$. An elementary dimension counting argument now implies that $(\bigcap_{j=1}^k A_j) \setminus A \ne \emptyset$. Choose $z \in (\bigcap_{j=1}^k A_j) \setminus A$ and put $p_G(v) = z$.

We next define q_G on $L_G(v), L_G(v_1), \ldots, L_G(v_k)$. We first choose k loops $m_j \in L_G(v)$, $1 \le j \le k$, and put $q_G(m_j) = z - p_G(v_j) = z - p_H(v_j)$. By the construction of z and since p_H is generic, we see that $I = \{z - p_H(u) : u \in N_G(v)\}$ is a linearly independent set, and hence we may extend the definition of q_G to the whole of $L_G(v)$ in such a way that $I \cup q_G(L_G(v) \setminus \{m_1, m_2, \ldots, m_k\})$ is a basis for $\mathbb{R}^d$.

Now, observe that for $1 \le j \le k$, $q_G(m_j)$ is a nonzero element of W_j. Thus we may define $q_G : L_G(v_j) \to \mathbb{R}^d$ so that $q_G(L_G(v_j)) \cup q_G(m_j)$ is a spanning set of W_j. (Note that $L_G(v_j) = L_H(v_j) \setminus \{l_j\}$).

We will show that (G, p_G, q_G) is a rigid realisation of G in $\mathbb{R}^d$. Suppose that $\dot{p} : V_G \to \mathbb{R}^d$ is an infinitesimal motion of (G, p_G, q_G). Then, $(\dot{p}(v) - \dot{p}(v_j)) \cdot (p_G(v) - p_G(v_j)) = 0 = \dot{p}(v) \cdot q_G(m_j)$ for all $1 \le j \le k$. Since $q_G(m_j) = z - p_H(v_j) = p_G(v) - p_G(v_j)$, this gives $\dot{p}(v_j) \cdot q_G(m_j) = 0$ for all $1 \le j \le k$. Since we also have $\dot{p}(v_j) \cdot q_G(l) = 0$ for all $l \in L_G(v_j)$ and $q_G(L_G(v_j)) \cup q_G(m_j)$ is a spanning set of W_j, $\dot{p}(v_j) \in W_j^\perp$ for all $1 \le j \le k$. This

[1] A set $\{B_1, B_2, \ldots, B_s\}$ of affine subspaces of $\mathbb{R}^d$ is in *general position* in $\mathbb{R}^d$ if for any two disjoint subsets S, T of $\{1, 2, \ldots, s\}$ such that $\bigcap_{i \in S} B_i \ne \emptyset$ and $\bigcap_{i \in T} B_i \ne \emptyset$, we have $\dim \bigcap_{i \in S \cup T} B_i = \dim \bigcap_{i \in S} B_i + \dim \bigcap_{i \in T} B_i - d$ when $\dim \bigcap_{i \in S} B_i + \dim \bigcap_{i \in T} B_i \ge d$, and $\bigcap_{i \in S \cup T} B_i = \emptyset$ when $\dim \bigcap_{i \in S} B_i + \dim \bigcap_{i \in T} B_i \le d - 1$.

implies that $\dot{p}$ restricts to an infinitesimal motion of (H, p_H, q_H). Thus $\dot{p}(u) = 0$ for all $u \in V_H$.

Finally, observe that since $I \cup q_G(L_G(v) \setminus \{m_1, m_2, \ldots, m_k\})$ is a basis for $\mathbb{R}^d$ and $\dot{p}(v)$ is orthogonal to all vectors in this basis, it must also be zero. □

Proof of Theorem 1.2. Necessity follows immediately from Lemma 2.1, so it only remains to prove sufficiency. We may assume without loss of generality that $G = (V, E, L)$ is a $(t, 0)$-tight looped simple graph. We will show that $G^{[d-t]}$ has an infinitesimally rigid realisation in $\mathbb{R}^d$ by induction on $|V|$. If $|V| = 1$ then G has t loops. Hence $G^{[d-t]}$ has d loops and is generically rigid in $\mathbb{R}^d$.

Now suppose that $|V| \geq 2$ and that the theorem is true for all graphs with at most $|V| - 1$ vertices. Let $\delta^*(u)$ denote the number of loops incident with each vertex u of G. Since $|E| + |L| = t|V|$ and $\sum_{u \in V}(\delta(u) + \delta^*(u)) = 2(|E| + |L|)$ there is a $v \in V$ such that $\delta(v) + \delta^*(v) \leq 2t$. Also, since G is $(t, 0)$-tight, $\delta(v) \geq t$. Let $k = \delta(v) - t$.

By Lemma 2.3 there are distinct vertices $v_1, \ldots, v_k \in N(v)$ such that the graph H obtained from $G - v$ by adding a new loop l_i at v_i for all $1 \leq i \leq k$ is $(t, 0)$-tight. By induction $H^{[d-t]}$ is generically rigid in $\mathbb{R}^d$.

We will show that $G^{[d-t]}$ is a d-dimensional k-loop extension of $H^{[d-t]}$. This will follow from the hypothesis that $d \geq 2t$ and the facts that $k = \delta(v) - t$ and $t \leq \delta(v) \leq 2t$. These imply that $0 \leq k \leq t$ and hence $k \leq d$ and $\delta^*_{G^{[d-t]}}(v) \geq d - t \geq t \geq k$. In addition, for all $1 \leq j \leq k$, we have

$$\delta^*_{H^{[d-t]}}(v_j) \geq d - t + 1 = \frac{(t-1)d}{t} + \frac{d}{t} - t + 1 \geq \left\lceil \frac{(t-1)d}{t} \right\rceil \geq \left\lceil \frac{(k-1)d}{k} \right\rceil,$$

since $d \geq t(t-1)$ and $t \geq k$. We can now use Lemma 2.4 to deduce that $G^{[d-t]}$ has an infinitesimally rigid realisation in $\mathbb{R}^d$. □

3. Plane-constrained frameworks

We will prove Theorem 1.4. We define the *degree*, $\deg_G(x)$, of a vertex v in a graph G to be the number of edges and loops incident to v, counting each loop *twice*. We say that a graph G is *k-regular* if each vertex of G has degree k. Our proof technique of Theorem 1.4 uses induction on the order of G when G is not simple and 4-regular, combined with an ad hoc argument for this exceptional case. We will need some further results to deal with this case.

Lemma 3.1. *Let $G = (V, E)$ be a 4-regular simple graph. Then G is $(2, 0)$-tight. Moreover if G is connected then $|F| \leq 2|V_F| - 1$ for all $F \subsetneq E$.*

Proof. Since G is 4-regular we have $|E| = 2|V|$. Suppose G is not $(2, 0)$-sparse. Then there is some $F \subset E$ with $|F| > 2|V_F|$. This implies that $G[F]$ has average degree strictly greater than 4, contradicting the fact that G is 4-regular.

Now assume G is connected. Suppose $|F| = 2|V_F|$ for some $F \subsetneq E$. This implies $G[F]$ has average degree exactly 4. Since G is connected and $F \subsetneq E$ there exists a vertex $x \in V_F$ with $\deg_G(x) > 4$, contradicting the fact that G is 4-regular. □

The next result gives a sufficient condition for the rigidity matrix of a generic bar-joint framework in $\mathbb{R}^3$ to have independent rows.

Theorem 3.2. *(*[5, Theorem 3.5]*) Let $G = (V, E)$ be a connected simple graph on at least three vertices with minimum degree at most* 4 *and maximum degree at most* 5*, and (G, p) be a generic realisation of G as a bar-joint framework in $\mathbb{R}^3$. Then the rows of the bar-joint* *rigidity matrix $R(G,p)$ are linearly independent if and only if $|F| \leq 3|V_F| - 6$ for all $F \subseteq V$ with $|F| \geq 2$.*

Our next result concerns frameworks on surfaces. Suppose $\mathcal{M}$ is an irreducible surface in $\mathbb{R}^3$ defined by a polynomial $f(x,y,z) = r$ and $q = (x_1, y_1, z_1, \dots, x_n, y_n, z_n) \in \mathbb{R}^{3n}$. Then the *family of 'concentric' surfaces induced by* q, $\mathcal{M}^q$, is the family defined by the polynomials $f(x,y,z) = r_i$ where $r_i = f(x_i, y_i, z_i)$ for $1 \leq i \leq n$.

Lemma 3.3. *([7, Lemma 9]) Suppose (G,p) is an infinitesimally rigid framework on some surface $\mathcal{M}$ in $\mathbb{R}^3$. Then (G,q) is infinitesimally rigid on $\mathcal{M}^q$ for all generic $q \in \mathbb{R}^{3|V|}$.*

We need two additional concepts for our next lemma. First, recall that the edge sets of the simple $(2,1)$-sparse subgraphs of a graph $G = (V,E)$ are the independent sets of a matroid on E. We call this the *simple* $(2,1)$*-sparse matroid* for G. Secondly, we define an *equilibrium stress* for a linearly-constrained framework (G,p,q) in $\mathbb{R}^3$ to be a pair (ω, λ), where $\omega : E \to \mathbb{R}$, $\lambda : L \to \mathbb{R}$ and (ω, λ) belongs to the cokernel of $R(G,p,q)$.

Lemma 3.4. *Let $G = (V,E)$ a 4-regular connected simple graph which is distinct from K_5. Then $G^{[1]}$ can be realised as an infinitesimally rigid plane-constrained framework in $\mathbb{R}^3$.*

Proof. Let $\mathcal{E}$ be the surface in $\mathbb{R}^3$ defined by the equation $x^2 + 2y^2 = 1$. Then $\mathcal{E}$ is an elliptical cylinder centred on the z-axis and has type 1. Let $p : V \to \mathbb{R}^3$ be generic, and (G,p) be the corresponding framework on the family of concentric elliptical cylinders $\mathcal{E}^p$ induced by p. Lemma 3.1 implies that E is a circuit in the simple $(2,1)$-sparse matroid for G. Theorem 1.3 and Lemma 3.3 now imply that $(G-e,p)$ is infinitesimally rigid on $\mathcal{E}^p$ for all $e \in E$. Hence the only infinitesimal motions of $(G-e,p)$ on $\mathcal{E}^p$ are translations in the direction of the z-axis.

Let $(G^{[1]}, p, q)$ be the plane-constrained framework corresponding to (G,p) on $\mathcal{E}^p$. Then $(G^{[1]}, p, q)$ has the same (1-dimensional) space of infinitesimal motions as (G,p) on $\mathcal{E}^p$ and hence $\operatorname{rank} R(G^{[1]}, p, q) = \operatorname{rank} R(G^{[1]} - e, p, q) = 3|V| - 1$ for all $e \in E$. This implies that $(G^{[1]}, p, q)$ has a unique non-zero equilibrium stress (ω, λ) up to scalar multiplication. Since G is simple, 4-regular and distinct from K_5, we have $|F| \leq 3|V_F| - 6$ for all $F \subset E$ with $|F| \geq 2$. Theorem 3.2 now implies that the rows of $R(G^{[1]}, p, q)$ indexed by E are linearly independent and hence we must have $\lambda_f \neq 0$ for some $f \in L$. It follows that the matrix R_f obtained from $R(G^{[1]}, p, q)$ by deleting the row indexed by f has $\ker R_f = \ker R(G^{[1]}, p, q)$ and hence each $\dot{p} \in \ker R_f$ corresponds to a translation along the z-axis. Let $(G^{[1]}, p, \tilde{q})$ be the plane-constrained framework with $\tilde{q}(e) = q(e)$ for all $f \in L - f$ and $\tilde{q}(f) = (0,0,1)$. Then $\ker R(G, p, \tilde{q}) \subseteq \ker R_f$. The choice of $\tilde{q}(f)$ implies that no nontrivial translation along the z-axis can belong to $\ker R(G^{[1]}, p, \tilde{q})$. Hence $\ker R(G^{[1]}, p, \tilde{q}) = \{0\}$ and $(G^{[1]}, p, \tilde{q})$ is an infinitesimally rigid plane-constrained framework in $\mathbb{R}^3$. □

We can now prove Theorem 1.4.

Proof of Theorem 1.4. We first prove necessity. Suppose that $G^{[1]}$ can be realised as an infinitesimally rigid plane-constrained framework in $\mathbb{R}^3$. We can show as in the proof of Lemma 2.1 that G has a spanning looped simple $(2,0)$-tight subgraph H, with a realisation (H,p,q) such that the rows of $R(H^{[1]}, p, q)$ are linearly independent and $\operatorname{rank} R(H^{[1]}, p, q) = d|V|$. If H had a subgraph K which is isomorphic to K_5, then the fact that K_5 is generically dependent as a bar-joint framework in $\mathbb{R}^3$ would imply that the rows of $R(H^{[1]}, p, q)$ labelled by $E(K)$ are linearly dependent. Hence H contains no copy of K_5.

We next prove sufficiency. Suppose G has a spanning looped simple subgraph which is $(2,0)$-tight, and in addition, contains no copy of K_5. We will prove that $G^{[1]}$ can be realised as an infinitesimally rigid plane-constrained framework in $\mathbb{R}^3$ by induction on $|V|+|E|+|L|$. We may assume that G is connected and so $|E|+|L|=2|V|$. If G is the graph with one vertex and two loops, then it is easy to see that $G^{[1]}$ has an infinitesimally rigid realisation in $\mathbb{R}^3$, so we may assume that $|V|\geq 2$. Let v be a vertex of G chosen so that $\delta(v)$ is as small as possible. Since G is $(2,0)$-tight, $2\leq\delta(v)\leq 4$.

Suppose $\delta(v)\in\{2,3\}$. By Lemma 2.3, G can be reduced to smaller $(2,0)$-tight graph H by deleting v and adding $\delta(v)-2$ loops to $N(v)$. By induction $H^{[1]}$ has an infinitesimally rigid realisation in $\mathbb{R}^3$. We can now apply Lemma 2.4 to deduce that $G^{[1]}$ has an infinitesimally rigid realisation.

It remains to consider the case when $\delta(v)=4$. Since G is $(2,0)$-tight, it must be 4-regular and simple. Since G is connected and not equal to K_5, Lemma 3.4 now implies that G has an infinitesimally rigid realisation in $\mathbb{R}^3$. □

4. Line-constrained frameworks

Given a graph G and an integer $d\geq 2$, Theorem 1.2 implies that $G^{[d-1]}$ can be realised as an infinitesimally rigid line-constrained framework in $\mathbb{R}^d$ if and only if G has a spanning $(1,0)$-tight subgraph. We will prove a more general result which allows nongeneric line constraints. We will need a lemma concerning non-generic d-dimensional 0-loop extensions, which we will refer to simply as *d-dimensional 0-extensions*. See Figure 2 for an illustration when $d=2$.

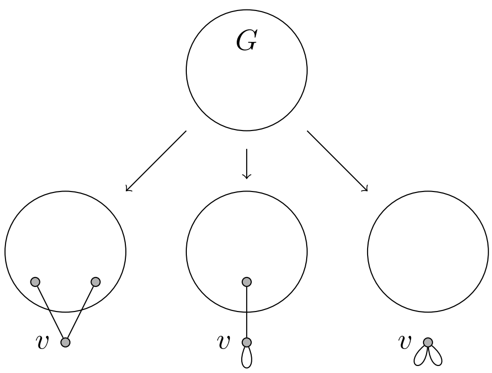

FIGURE 2. Possible 2-dimensional 0-extensions of a graph G.

Lemma 4.1. *Let G be a graph and H be constructed from G by a d-dimensional 0-extension operation which adds a new vertex v_0, new edges $v_0v_1, v_0v_2,\dots,v_0v_t$, and new loops $e_{t+1},e_{t+2},\dots,e_d$. Suppose (G,p,q) is a realisation of G in $\mathbb{R}^d$ and $(H,\hat{p},\hat{q})$ is a realisation of H with $\hat{p}|_G=p$ and $\hat{q}|_G=q$. Then $(H,\hat{p},\hat{q})$ is infinitesimally rigid if and only if (G,p,q) is infinitesimally rigid and $\{\hat{p}_0-\hat{p}_1,\hat{p}_0-\hat{p}_2,\dots,\hat{p}_0-\hat{p}_t,\hat{q}(e_{t+1}),\dots,\hat{q}(e_d)\}$ is linearly independent.*

Proof. The rigidity matrix for $(H,\hat{p},\hat{q})$ has the form

$$R(H,\hat{p},\hat{q})=\begin{pmatrix} A & * \\ 0 & R(G,p,q)\end{pmatrix}$$

where

$$A = \begin{pmatrix} \hat{p}_0 - \hat{p}_1 \\ \vdots \\ \hat{p}_0 - \hat{p}_t \\ \hat{q}(e_{t+1}) \\ \vdots \\ \hat{q}(e_d) \end{pmatrix}.$$

We first consider the case when $S = \{\hat{p}_0 - \hat{p}_1, \hat{p}_0 - \hat{p}_2, \ldots, \hat{p}_0 - \hat{p}_t, \hat{q}(e_{t+1}), \ldots, \hat{q}(e_d)\}$ is linearly independent. Then the rows of A are linearly independent and $\operatorname{rank} R(H, \hat{p}, \hat{q}) = \operatorname{rank} R(G, p, q) + \operatorname{rank} A = \operatorname{rank} R(G, p, q) + d$. Hence $(H, \hat{p}, \hat{q})$ is infinitesimally rigid if and only if (G, p, q) is infinitesimally rigid.

It remains to consider the case when S is not linearly independent. In this case $\dim\langle S\rangle \leq d - 1$ and we may define an infinitesimal motion $\dot{p}$ of $(H, \hat{p}, \hat{q})$ by choosing $\dot{p}(v_0)$ to be any nonzero vector in $\langle S\rangle^\perp$ and putting $\dot{p}(v) = 0$ for all $v \neq v_0$. Hence $(H, \hat{p}, \hat{q})$ is not infinitesimally rigid. □

Given a graph $G = (V, E, L)$ and $q : L \to \mathbb{R}^d$, let $W_v = \langle q(e) : e \in L(v)\rangle$ for all $v \in V$. We say that q is *line-admissible on G* if $\dim W_v \geq d - 1$ for all $v \in V$, and $\dim\langle q(L)\rangle = d$. A *cycle of length k* is a connected graph with k edges in which each vertex has degree two.

Lemma 4.2. *Let $d \geq 2$ be an integer, C be a cycle which is either a loop or has length at least three, $C^{[d-1]} = (V, E, L)$ and $q : L \to \mathbb{R}^d$. Then $(C^{[d-1]}, p, q)$ is infinitesimally rigid for some $p : V \to \mathbb{R}^d$ if and only if q is line-admissible on $C^{[d-1]}$.*

Proof. We first prove necessity. Suppose that q is not line-admissible on $C^{[d-1]}$. If $V = \{v\}$ then $\dim W_v \leq d - 1$ and any nonzero vector $\dot{p}_v \in W_v^\perp$ will be an infinitesimal motion of $(C^{[d-1]}, p, q)$ for all p. Hence we may assume that $|V| \geq 3$. If $\dim W_v \leq d - 2$ for some $v \in V$ then the rows of $R(C^{[d-1]}, p, q)$ indexed by $L(v)$ will be dependent and hence the rank of $R(C^{[d-1]}, p, q)$ will be less than $|E| + |L| = d|V|$ for all p. Hence we may assume that $\dim W_v = d - 1$ for all $v \in V$ and that $\dim\langle q(L)\rangle = d - 1$. Then $W_u = W_v$ for all $u, v \in V$. Let $W_v = W$, t be a nonzero vector in $W^\perp$, and $\dot{p} : V \to \mathbb{R}^d$ be defined by $\dot{p}(v) = t$ for all $v \in V$. Then $\dot{p}$ will be a nontrivial infinitesimal motion of $(C^{[d-1]}, p, q)$ for all p.

We next prove sufficiency. Suppose that q is line-admissible on $C^{[d-1]}$. If $V = \{v\}$ then $\dim W_v = d = \operatorname{rank} R(C^{[d-1]}, p, q)$ for all p. Hence we may assume that $|V| \geq 3$. Since $\dim\langle q(L)\rangle = d$, we may choose $u, v, w \in V$ such that $uv, uw \in E$ and $W_u \neq W_v$. Let $G = (V - u, E', L')$ be the graph obtained from $C^{[d-1]} - u$ by adding a loop e_0 at v. Define $q' : L' \to \mathbb{R}^d$ by putting $q'(e) = q(e)$ for all $e \in L \cap L'$ and $q'(e_0) = q(e)$ for some $e \in L(u)$ with $q(e) \notin W_v$. Then $\dim W'_v = d$ so the subgraph H of G induced by v has an infinitesimally rigid realisation $(H, p'', q'|_H)$ for any $p''(v) \in \mathbb{R}^d$. We can now use Lemma 4.1 recursively to construct $p' : V - u \to \mathbb{R}^d$ such that (G, p', q') is infinitesimally rigid. If necessary we may perturb this realisation slightly so that $p'(v) + q'(e_0)) - p'(w) \notin W_u$. Finally, we construct an infinitesimally rigid realisation $(C^{[d-1]}, p, q)$ from (G, p', q') by using the proof technique of Lemma 2.4. More precisely we construct $C^{[d-1]}$ from G by performing a d-dimensional 1-loop extension operation at the loop e_0 and choose $p : V \to \mathbb{R}^d$ such that $p|_{V-u} = p'$ and $p(u) = p'(v) + q'(e_0)$. □

Theorem 4.3. *Let $d \geq 2$ be an integer, G be a graph, $G^{[d-1]} = (V, E, L)$ and $q : L \to \mathbb{R}^d$ such that $\dim W_v \geq d - 1$ for all $v \in V$. Then $(G^{[d-1]}, p, q)$ is infinitesimally rigid for some*

$p : V \to \mathbb{R}^d$ if and only if every connected component of G has a cycle C of length not equal to two such that q is line-admissible on $C^{[d-1]}$.

Proof. We first prove necessity. Suppose that $G^{[d-1]}$ can be realised as an infinitesimally rigid line-constrained framework $(G^{[d-1]}, p, q)$ in $\mathbb{R}^d$. Then rank $R(G^{[d-1]}, p, q) = d|V|$. Since $\dim W_v \geq d-1$ for all $v \in V$, we can choose a subgraph $H = (V, E', L')$ of G such that the rows of $R(H^{[d-1]}, p, q|_{H^{[d-1]}})$ are linearly independent and rank $R(H^{[d-1]}, p, q|_{H^{[d-1]}}) = d|V|$. Then H is a looped simple graph of minimum degree at least one and $|E'| + |L'| = |V|$. If H has a vertex v of degree one, then we can apply Lemma 4.1 to $(H^{[d-1]}, p, q|_{H^{[d-1]}})$ to deduce that $((H-v)^{[d-1]}, p|_{V-v}, q|_{(H-v)^{[d-1]}})$ is infinitesimally rigid. We may then apply induction to $H - v$ to deduce that each component of H has a cycle C of length not equal to two such that q is line-admissible on $C^{[d-1]}$. It remains to consider the case when every vertex of H has degree two. Then each component of H is a cycle (which necessarily has length not equal to two since H is looped simple). In this case we may use Lemma 4.2 to deduce that q is line-admissible on $C^{[d-1]}$ for each component C of H. Since every connected component of H is contained in a component of G, each component of G has a cycle C of length not equal to two such that q is line-admissible on $C^{[d-1]}$.

We next prove sufficiency. Suppose that each connected component of G contains a cycle C of length not equal to two such that q is line-admissible on $C^{[d-1]}$. We may assume inductively that G is connected and that C is the unique cycle in G. We may now use Lemma 4.1 to reduce to the case when $G = C$. Lemma 4.2 now implies that $(G^{[d-1]}, p, q)$ is infinitesimally rigid for some p. □

Note that when $d = 1$, a graph $G = (V, E, L)$ with a given map $q : L \to \mathbb{R}$ has an infinitesimally rigid realisation (G, p, q) in $\mathbb{R}$ if and only if every connected component of G has a loop e with $q(e) \neq 0$. Thus Lemma 4.2 and Theorem 4.3 remain true when $d = 1$ if we adopt the convention that $\dim \emptyset = 0$.

5. Body-bar frameworks

Suppose we are given a graph $G = (V, E, L)$. We can realise G as a *linearly-point-constrained body-bar framework in* $\mathbb{R}^d$ by representing each vertex v by a d-dimensional rigid body B_v in $\mathbb{R}^d$, each edge uv as a distance constraint between two points in B_u and B_v, respectively, and each loop $e = vv$ as a constraint which restricts the motion of some point of B_v to be orthogonal to a given vector $q(e) \in \mathbb{R}^d$. A realisation of G as a *linearly-body-constrained body-bar framework* is defined similarly but in this case each loop $e = vv$ represents a constraint which restricts the motion of the whole body B_v to be orthogonal to a given vector $q(e) \in \mathbb{R}^d$.

Katoh and Tanigawa [8] consider closely related realisations of G as a body-bar framework with either bar or pin boundary in $\mathbb{R}^d$. In the first case each loop $e = vv$ constrains the infinitesimal motion of a point of the body B_v to be a rotation about a fixed point P_e. In the second case the loop constrains the infinitesimal motion of the whole body B_v to be a rotation about P_e. They characterise when G can be realised as an infinitesimally rigid body-bar framework with either a bar or pin boundary for any given set of boundary bar-directions or pins, [8, Corollary 6.2, Theorem 6.3]. Their approach uses projective geometry and considers an embedding of $\mathbb{R}^d$ into projective space $\mathbb{P}^d$. By choosing the projective points corresponding to each point P_e to lie on the hyperplane at infinity in $\mathbb{P}^d$ in the case of a pin boundary, their results immediately imply the following characterisations of graphs which can be realised as infinitesimally rigid linearly-constrained body-bar frameworks.

Theorem 5.1. *Let $G = (V, E, L)$ be a graph and $q : L \to \mathbb{R}^d \setminus \{0\}$. Then (G, q) can be realised as an infinitesimally rigid linearly-point-constrained body-bar framework in $\mathbb{R}^d$ if and only if*

$$\delta_G(\mathcal{P}) \geq \binom{d+1}{2}|\mathcal{P}| - \sum_{X \in \mathcal{P}} \dim\langle q(e)^* \, : \, e \in L(X)\rangle$$

for all partitions $\mathcal{P}$ of V, where $\delta_G(\mathcal{P})$ is the number of edges in E joining different sets in $\mathcal{P}$, $L(X)$ is the set of all loops in L incident to vertices of X, and $q(e)^* \in \mathbb{R}^{\binom{d+1}{2}}$ is the vector of Plücker coordinates of a line segment in $\mathbb{R}^d$ with direction $q(e)$.

Theorem 5.2. *Let $G = (V, E, L)$ be a graph and $q : L \to \mathbb{R}^d \setminus \{0\}$. Then (G, q) can be realised as an infinitesimally rigid linearly-body-constrained body-bar framework in $\mathbb{R}^d$ if and only if*

$$\delta_G(\mathcal{P}) \geq \binom{d+1}{2}|\mathcal{P}| - \sum_{X \in \mathcal{P}, L(X) \neq \emptyset} \sum_{i=1}^{d_X+1} (d - i + 1)$$

for all partitions $\mathcal{P}$ of V, where d_X is the dimension of the affine span of $\{q(e) \, : \, e \in L(X)\}$.

6. Concluding remarks

1. Theorem 1.2 gives rise to an efficient algorithm for testing whether a graph can be realised as an infinitesimally rigid framework on a type 0, t-dimensional variety in $\mathbb{R}^d$ when $d \geq \max\{2t, t(t-1)\}$. Details may be found in [3, 10]. As discussed in the introduction, the bound $d \geq 2t$ is tight but the bound $d \geq t(t-1)$ may be far from best possible.

2. The proof of Theorem 1.1 in [14] is a direct proof using a related 'frame matroid'. We briefly describe how our inductive techniques can be adapted to give an alternative proof of their result. Suppose $G = (V, E, L)$ is a graph which satisfies the hypotheses of Theorem 1.1 i.e. $|E| + |L| = 2|V|$, $|F| \leq 2|V_F|$ for all $F \subseteq E \cup L$, and, when $\emptyset \neq F \subseteq E$, $|F| \leq 2|V_F| - 3$. Then G contains a vertex v which is incident to either 2 or 3 edges and loops. In the first case, it is easy to see that $G - v = (V', E', L')$ also satisfies the hypotheses of Theorem 1.1. Hence we may use induction and Lemma 4.1 to deduce the rigidity of G. In the latter case, either v is incident with at least one loop and we can use Lemmas 2.3 and 2.4 to show that G is rigid, or v has three incident edges. To reduce such a vertex we may use the fact that the function $f : 2^{E \cup L} \to \mathbb{Z}$ given by $f(F) = 2|V_F| - 3$ if $F \subseteq E$ and $f(F) = 2|V_F|$ if $F \not\subseteq E$, is nonnegative on nonempty sets, nondecreasing and crossing submodular, and hence induces a matroid on $E \cup L$, see for example [6, 14]. Let $r(G)$ denote the rank of this matroid and suppose that $N(v) = \{x, y, z\}$. Let G' be formed from G by deleting v (and its incident edges) and adding the edges xy, xz, yz. Suppose that $r(G') = r(G) - 3$. Let G'' be formed from G by adding xy, xz, yz. Then $r(G'') \leq r(G') + 2 = r(G) - 1$ since the edge set of K_4 is dependent in the matroid. This is a contradiction since G is a subgraph of G''. Hence $G - v + e$ satisfies the hypotheses of Theorem 1.1 for some $e \in \{xy, xz, yz\}$. We can now complete the proof by choosing a generic infinitesimally rigid realisation of $G - v + e$ and placing v on the line joining the end-points of e.

3. Theorem 1.1 was extended by Katoh and Tanigawa [8, Theorem 7.6] to allow specified directions for the linear constraints. More precisely they determine when a given graph $G = (V, E, L)$ and map $q : L \to \mathbb{R}^2$ can be realised as an infinitesimally rigid linearly-constrained framework (G, p, q) in $\mathbb{R}^2$. It is an open problem to decide if this result can be extended to plane-constrained frameworks in $\mathbb{R}^d$ for $d \geq 3$. In particular we offer

Conjecture 6.1. *Let G be a graph and S be a fixed type 0 smooth surface in $\mathbb{R}^d$. Then G has an infinitesimally rigid realisation on S if and only G has a $(2,0)$-tight looped simple spanning subgraph, which contains no copy of K_5 when $d = 3$.*

References

[1] T. Abbott, Generalizations of Kempe's universality theorem, *Master's thesis, Massachusetts Institute of Technology* (2008).

[2] L. Asimow and B. Roth, The rigidity of graphs, *Trans. Amer. Math. Soc.* 245 (1978), 279–289.

[3] A. R. Berg and T. Jordán, Algorithms for graph rigidity and scene analysis, in Algorithms - ESA 2003, volume 2832 of Lecture Notes in Comput. Sci., pages 7889. Springer, Berlin, 2003.

[4] Y. Eftekhari, B. Jackson, A. Nixon, B. Schulze, S. Tanigawa and W. Whiteley, Point-hyperplane frameworks, slider joints, and rigidity preserving transformations, *arXiv:1703.06844*.

[5] B. Jackson and T. Jordán, The d-dimensional rigidity matroid of sparse graphs, *J Combinatorial Theory(B)* 95, (2005) 118–133.

[6] B. Jackson and V. H. Nguyen, Graded sparse graphs and body-length-direction frameworks, *European J. Comb.* 46, (2015) 51–67.

[7] B. Jackson and A. Nixon, Stress matrices and global rigidity of frameworks on surfaces, *Discrete and Computational Geometry* 54:3 (2015), 586–609.

[8] N. Katoh and S. Tanigawa, Rooted-tree decompositions with matroid constraints and the infinitesimal rigidity of frameworks with boundaries, *SIAM J. Discrete Math.* 27:1 (2013), 155–185.

[9] G. Laman, On graphs and rigidity of plane skeletal structures, *J. Engineering Math.* 4 (1970), 331–340.

[10] A. Lee and I. Streinu, Pebble game algorithms and sparse graphs, *Discrete Math.*, 308 (2008), no. 8, 1425–1437.

[11] A. Nixon, J. Owen and S. Power, Rigidity of frameworks supported on surfaces, *SIAM Journal on Discrete Mathematics* 26 (2012), no. 4, 1733–1757.

[12] A. Nixon, J. Owen and S. Power, A characterization of generically rigid frameworks on surfaces of revolution, *SIAM Journal on Discrete Mathematics* 28 (2014), no. 4, 2008–2028.

[13] B. Servatius, O. Shai, and W. Whiteley, Combinatorial characterization of the Assur graphs from engineering, *European Journal of Combinatorics* 31 (2010), 1091–1104.

[14] I. Streinu and L. Theran, Slider-pinning rigidity: a Maxwell-Laman-type theorem, *Discrete and Computational Geometry* 44:4 (2010) 812–837.

[15] T.S. Tay, Rigidity of multi-graphs I Linking Bodies in n-space, *J. Combinatorial Th. B* 26 (1984) 95-112.

[16] L. Theran, A. Nixon, E. Ross, M. Sadjadi, B. Servatius and M. Thorpe, Anchored boundary conditions for locally isostatic networks, *Physical Review E* 92:5 (2015) 053306.

School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland

E-mail address: james.cruickshank@nuigalway.ie

School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, U.K.

E-mail address: hakanguler19@gmail.com

School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, U.K.

E-mail address: b.jackson@qmul.ac.uk

Department of Mathematics and Statistics, Lancaster University, LA1 4YF, U.K.

E-mail address: a.nixon@lancaster.ac.uk