

CARTAN SUBALGEBRAS IN C^* -ALGEBRAS. EXISTENCE AND UNIQUENESS

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ABSTRACT. We initiate the study of Cartan subalgebras in C^* -algebras, with a particular focus on existence and uniqueness questions. For homogeneous C^* -algebras, these questions can be analysed systematically using the theory of fibre bundles. For group C^* -algebras, while we are able to find Cartan subalgebras in C^* -algebras of many connected Lie groups, there are classes of (discrete) groups, for instance non-abelian free groups, whose reduced group C^* -algebras do not have any Cartan subalgebras. Moreover, we show that uniqueness of Cartan subalgebras usually fails for classifiable C^* -algebras. However, distinguished Cartan subalgebras exist in some cases, for instance in nuclear uniform Roe algebras.

1. INTRODUCTION

The construction of groupoid C^* -algebras is an extremely powerful and general method to produce C^* -algebras. Many C^* -algebras admit descriptions as groupoid C^* -algebras, either directly by construction or indirectly by – often explicitly given – identifications. Having groupoid models at hand is very helpful to analyse the C^* -algebraic structure, be it for determining ideals, for nuclearity, computing K -theory, the UCT etc. With this in mind, it is natural to ask the following questions: Which C^* -algebras admit groupoid models? And to what extent are these groupoid models unique? As we shall explain below, these two questions are closely related to the existence and uniqueness questions for Cartan subalgebras in C^* -algebras.

Our goal in this paper is to initiate a systematic study of Cartan subalgebras in C^* -algebras, with a particular focus on existence and uniqueness. Classification of C^* -algebras has seen tremendous advances recently [69, 70, 37, 38, 58, 23, 15, 65]. However, finer structures given by Cartan subalgebras have not yet been analysed in detail. Apart from being of interest for C^* -algebra theory itself, the notion of Cartan subalgebras also builds a bridge to topological dynamical systems, as it is closely related to the notion of continuous orbit equivalence [32]. The latter, in turn, establishes a connection to geometric group theory via dynamic characterizations of quasi-isometry [34]. The loop is closed by the observation in Theorem 6.2, building on [60, 34], that quasi-isometry can also be characterized using Cartan subalgebras in uniform Roe algebras.

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The situation we are describing here is completely analogous to the measurable framework, where a fruitful interplay between Cartan subalgebras in von Neumann algebras (vN-algebras), orbit equivalence for measure-preserving dynamical systems and geometric group theory has been established [59, 21, 20]. Indeed, the comparison between C*-algebraic and vN-algebraic Cartan subalgebras is one of the leading principles behind our present work.

Coming back to our original questions on existence and uniqueness of groupoid models, we have to make our question more precise in order to get a satisfactory answer. What kind of groupoids should be allowed? For example, group C*-algebras, which are amongst the most challenging C*-algebras, are groupoid C*-algebras by definition; however in some cases, e.g. abelian or $(ax + b)$ -groups, they may admit more revealing groupoid realisations. Usually, one looks for a groupoid which has as little isotropy as possible. In the case of vN-algebras, it suffices to consider groupoids without isotropy (also known as principal groupoids or equivalence relations). In the theory of C*-algebras, we allow some isotropy and consider topologically principal groupoids (this means that the points without isotropy are dense) to cover C*-algebras arising from foliations or from pseudo-groups. In this paper, we work with groupoids which are topologically principal, locally compact, Hausdorff and étale. The reason why we choose such groupoids is that the reduced C*-algebras of such groupoids, equipped possibly with a twist, admit a convenient C*-algebraic characterisation, namely the existence of a Cartan subalgebra, a notion borrowed from the theory of vN-algebras: An abelian sub-C*-algebra B of a C*-algebra A is called a *Cartan subalgebra* if it contains an approximate unit of A , is maximal abelian, regular and there exists a faithful conditional expectation of A onto B . By [55], every Cartan pair (A, B) consisting of a Cartan subalgebra B in a C*-algebra A is of the form $(C_r^*(G, \Sigma), C_0(G^{(0)}))$, where the groupoid G is as above, $G^{(0)}$ is its unit space and Σ is a twist over G . The case when G is an equivalence relation had been worked out earlier by A. Kumjian: G has no isotropy if and only if the subalgebra B has in addition the unique extension property; then it is called a *diagonal*. The original definition of a Cartan subalgebra in the theory of vN-algebras is an abelian subalgebra \mathbf{A} of a vN-algebra \mathbf{M} which is a masa, regular and there exists a faithful normal conditional expectation of \mathbf{M} onto \mathbf{A} . The generic Cartan pair is $(\mathbf{M}, \mathbf{A}) = (\mathbf{M}(R, \sigma), L^\infty(X))$, where R is a countable standard measured equivalence relation on X and σ is a 2-cocycle (see [18]). Existence and uniqueness of Cartan subalgebras in vN-algebras is an active field of research [67, 43, 44, 45, 49, 50, 4]. The examples presented in this paper will show that the investigation of Cartan subalgebras in C*-algebras is no less interesting. It turns out that, unless a C*-algebra contains a Cartan subalgebra from its very construction, showing the existence of a Cartan subalgebra involves a variety of techniques. It will also turn out that, when Cartan subalgebras exist, there are usually infinitely many of them, at least in case the ambient C*-algebra is classifiable. Let us briefly describe the content of the paper section by section.

The second section studies Cartan subalgebras in n -homogeneous C*-algebras. When n is finite, it is shown that a Cartan pair is necessarily locally trivial. This puts the

problem into the general theory of fibre bundles: the n -homogeneous C^* -algebra A admits a Cartan subalgebra if and only if the principal $\text{Aut}(M_n)$ -bundle associated to A can be reduced to the subgroup $\text{Aut}(M_n, D_n)$. Using this theory, we give a complete enumeration of Cartan subalgebras of an n -homogeneous C^* -algebra over a k -sphere. This gives a conceptual explanation of the result of K.D. Gregson [24] and T. Natsume (see appendix of [30]) that for $k > 2$, only the trivial n -homogeneous C^* -algebra $C(S^k, M_n)$ admits a Cartan subalgebra; moreover, our study shows that this Cartan subalgebra is essentially unique. A similar study is made when n is infinite, under the assumption that the pair (A, B) is locally trivial.

The third section gives the existence of Cartan subalgebras in group C^* -algebras of certain connected Lie groups, building on [19, 40, 27, 66, 46]. Namely, it is shown that the full group C^* -algebras of $SL(2, \mathbb{C})$, $SL(2, \mathbb{R})$ as well as the reduced group C^* -algebras of every connected semi-simple Lie group with real rank one and finite center and of every connected complex semi-simple Lie group have a Cartan subalgebra. The proof uses the description of these C^* -algebras as sub- C^* -algebras of $C_0(M, \mathcal{K}(\mathcal{H}))$, where M is a locally compact Hausdorff space and \mathcal{H} is an infinite dimensional Hilbert space.

In the fourth section, we show that certain reduced group C^* -algebras do not have Cartan subalgebras. This is in particular the case for non-abelian free groups. Here the strategy is to use the known corresponding results for the group vN -algebras [67, 25, 43, 44, 45, 49, 50, 4]. Note that some extra work is needed to show the non-existence of Cartan subalgebras rather than diagonals.

The fifth section illustrates various techniques to construct infinitely many Cartan subalgebras in a given C^* -algebra. The first one uses the recent construction of Deeley, Putnam and Strung [10] of the Jiang-Su algebra \mathcal{Z} as a groupoid C^* -algebra. It applies to unital C^* -algebras which have a Cartan subalgebra whose spectrum has finite covering dimension and which are \mathcal{Z} -stable. The second one, described by A. Kumjian in [29], uses non-stable K -theory and applies to irrational rotation C^* -algebras. In both cases, the Cartan subalgebras are shown to be non-isomorphic. The last result is that unital (resp. stable) UCT Kirchberg algebras have infinitely many inequivalent Cartan subalgebras which are all isomorphic to the algebra of continuous functions on the Cantor space (or its locally compact non-compact analogue). It relies on a groupoid realisation of these algebras due to J. Spielberg [61].

The phenomena presented in Section 5 show that in general, we cannot expect Cartan subalgebras in C^* -algebras to be rigid, i.e., uniqueness fails. However, it turns out that in some cases, we can find distinguished Cartan subalgebras. This means that within a class of C^* -algebras, there are special Cartan subalgebras which are distinguished in the sense that the existence of an arbitrary C^* -algebra isomorphism implies the existence of a C^* -algebra isomorphism preserving our special Cartan subalgebras. In Section 6, we present such examples by combining rigidity results by Spakula and Willett [60] and by Whyte [68] with recent results of the first named author: Under the assumption that the finitely generated groups Γ and Λ are exact, the isomorphism of their stable

uniform Roe algebras implies the isomorphism of their stable groupoids; if moreover Γ and Λ are non-amenable, this – and hence in particular isomorphism of uniform Roe algebras – implies the isomorphism of their groupoids.

2. CARTAN SUBALGEBRAS IN HOMOGENEOUS C^* -ALGEBRAS

Let A be an n -homogeneous C^* -algebra over a locally compact Hausdorff space T . A is the C^* -algebra of continuous sections of a C^* -bundle \mathcal{A} over T with fibre $M_n := M_n(\mathbb{C})$ (see [11, 12, 13, 19]). We know that \mathcal{A} is locally trivial (see for instance [19, Theorem 3.2]). Furthermore, assume that B is a Cartan subalgebra of A . Then B is the C^* -algebra of continuous sections of a sub- C^* -bundle \mathcal{B} of \mathcal{A} over the base space T . The fibres of \mathcal{B} are given by the C^* -algebra D_n , which is the up-to-conjugacy unique Cartan subalgebra of M_n (see for instance [24, § 2.1]).

Lemma 2.1. *There exists a local trivialization for $(\mathcal{A}, \mathcal{B})$. This means that for every $t \in T$, there exists a neighbourhood W of t and an isomorphism of C^* -bundles $\mathcal{A}|_W \xrightarrow{\cong} W \times M_n$ which identifies $\mathcal{B}|_W$ with $W \times D_n$.*

Proof. Fix $t \in T$. By local triviality, there exists an open neighbourhood V of t such that $\mathcal{A}|_V$ is trivial, i.e., we can find an isomorphism $\mathcal{A}|_V \xrightarrow{\cong} V \times M_n$, $a \mapsto (s, a(s))$. Using functional calculus, we can find elements $b_1, \dots, b_n \in B$ and a neighbourhood W of t with $W \subseteq V$ such that for every $s \in W$, $\{b_1(s), \dots, b_n(s)\}$ are non-zero pairwise orthogonal projections. Let $\{\zeta_1, \dots, \zeta_n\}$ be an orthonormal basis of \mathbb{C}^n with $b_i(t)\zeta_i = \zeta_i$ for $1 \leq i \leq n$. Then we must have $b_i(t)\zeta_j = \delta_{ij}\zeta_i$ for all $1 \leq i, j \leq n$. By replacing W by a smaller neighbourhood if necessary, we may assume that $b_i(s)\zeta_i \neq 0$ for all $1 \leq i \leq n$ and $s \in W$. Define

$$\eta_i(s) := \frac{b_i(s)\zeta_i}{\|b_i(s)\zeta_i\|}$$

for all $s \in W$. Since $\{b_1(s), \dots, b_n(s)\}$ are pairwise orthogonal, $\{\eta_1(s), \dots, \eta_n(s)\}$ must be an orthonormal basis of \mathbb{C}^n for all $s \in W$. Moreover, $b_i(s)\xi = \langle \xi, \eta_i(s) \rangle \eta_i(s)$ for all $s \in W$ and $1 \leq i \leq n$. Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product of \mathbb{C}^n , linear in the first variable. Define the unitary operator $U(s) : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $(\lambda_i)_i \mapsto \sum_i \lambda_i \eta_i(s)$. We obtain a C^* -bundle isomorphism

$$\mathcal{A}|_W \xrightarrow{\cong} W \times M_n \xrightarrow{\cong} W \times M_n, a \mapsto (s, a(s)) \mapsto (s, U(s)^* a(s) U(s)).$$

Under this isomorphism, b_i is identified with $U(s)^* b_i(s) U(s)$. If $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of \mathbb{C}^n , then

$$U(s)^* b_i(s) U(s) e_j = U(s)^* b_i(s) \eta_j(s) = \delta_{ij} U(s)^* \eta_i(s) = \delta_{ij} e_i.$$

Thus, $U(s)^* b_i(s) U(s)$ is the rank one projection corresponding to e_i . This shows that $\mathcal{B}|_W$ is identified with $W \times D_n$ under the isomorphism $\mathcal{A}|_W \xrightarrow{\cong} W \times M_n$ above. \square

In the following, by a local trivialization of $(\mathcal{A}, \mathcal{B})$, we mean a covering of T by open subspaces W together with isomorphisms $\mathcal{A}|_W \xrightarrow{\cong} W \times M_n$ which identify $\mathcal{B}|_W$ with $W \times D_n$.

2.1. Reformulation using the language of principal bundles. Lemma 2.1 allows us to reformulate the existence and uniqueness question for Cartan subalgebras using the language of principal bundles. In the following, let $\text{Aut}(M_n)$ be the automorphism group of the C^* -algebra M_n , and define

$$\text{Aut}(M_n, D_n) := \{\varphi \in \text{Aut}(M_n) : \varphi(D_n) = D_n\}.$$

As explained in [63, Part I, § 8.2], isomorphism classes of locally trivial M_n -bundles over T correspond to isomorphism classes of principal $\text{Aut}(M_n)$ -bundles over T .

Given a group G and two principal G -bundles β_1, β_2 over a base space T , we say that β_1 and β_2 are isomorphic over T if there is a bundle isomorphism between β_1 and β_2 which fixes (every point of) the base space. We say that β_1 and β_2 are isomorphic, written $\beta_1 \cong \beta_2$, if β_1 and β_2 are isomorphic via a bundle isomorphism which is allowed to induce a non-trivial homeomorphism on T . It is the latter notion which corresponds to isomorphism of C^* -algebras of continuous sections (see [11, 10.5.5]).

Proposition 2.2. *An n -homogeneous C^* -algebra A has a Cartan subalgebra if and only if the principal $\text{Aut}(M_n)$ -bundle of the M_n -bundle corresponding to A reduces to a principal $\text{Aut}(M_n, D_n)$ -bundle.*

Proof. Lemma 2.1 shows the implication “ \Rightarrow ”: If α is the principal $\text{Aut}(M_n)$ -bundle of the M_n -bundle \mathcal{A} corresponding to A , then Lemma 2.1 shows that there is a principal $\text{Aut}(M_n, D_n)$ -bundle β over T such that α is isomorphic to the principal $\text{Aut}(M_n)$ -bundles over T constructed from β and the canonical action $\text{Aut}(M_n, D_n) \curvearrowright \text{Aut}(M_n)$ by left multiplication (see for instance [26, Chapter 4, § 5] for the construction). This is equivalent to saying that α reduces to a principal $\text{Aut}(M_n, D_n)$ -bundle.

Conversely, assume that we can find a principal $\text{Aut}(M_n, D_n)$ -bundle β over T such that α is isomorphic to the principal $\text{Aut}(M_n)$ -bundles over T constructed from β and the canonical action $\text{Aut}(M_n, D_n) \curvearrowright \text{Aut}(M_n)$ by left multiplication. Now use β to construct an M_n -bundle \mathcal{A}' over T . As α is isomorphic to the principal $\text{Aut}(M_n)$ -bundles over T constructed from β and the canonical action $\text{Aut}(M_n, D_n) \curvearrowright \text{Aut}(M_n)$ by left multiplication, \mathcal{A}' can be identified with the M_n -bundle corresponding to A . Moreover, use β to construct a D_n -bundle \mathcal{B}' over T . \mathcal{B}' is a sub- C^* -bundle of \mathcal{A}' . Therefore, by going over to C^* -algebras of continuous sections, we obtain a sub- C^* -algebra B' of the C^* -algebra A' corresponding to \mathcal{A}' . B' is a Cartan subalgebra of A' because B' and A' correspond to locally trivial C^* -bundles by construction, and it is obvious that for an open subspace W of T , $C_0(W, D_n)$ is a Cartan subalgebra of $C_0(W, M_n)$. \square

Given two C^* -algebras A_1 and A_2 with Cartan subalgebras $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$, we write $(A_1, B_1) \cong (A_2, B_2)$ if there exists an isomorphism $\varphi : A_1 \xrightarrow{\cong} A_2$ satisfying $\varphi(B_1) = B_2$.

Proposition 2.3. *Let B_1 and B_2 be Cartan subalgebras of an n -homogeneous C^* -algebra A over T . As B_1 and B_2 are C^* -algebras of continuous sections of locally trivial D_n -bundles over T , they correspond to principal $\text{Aut}(M_n, D_n)$ -bundles β_1 and β_2 . We have $(A, B_1) \cong (A, B_2)$ if and only if $\beta_1 \cong \beta_2$ (as principal $\text{Aut}(M_n, D_n)$ -bundles).*

Proof. Let \mathcal{A} be the M_n -bundle corresponding to A , and $\mathcal{B}_1, \mathcal{B}_2$ the D_n -bundles corresponding to B_1, B_2 . If $(A, B_1) \cong (A, B_2)$, then there exists a bundle isomorphism $\mathcal{A} \xrightarrow{\cong} \mathcal{A}$ (possibly inducing a non-trivial homeomorphism on T) which sends \mathcal{B}_1 to \mathcal{B}_2 . Looking at local trivializations for $(\mathcal{A}, \mathcal{B}_1)$ and $(\mathcal{A}, \mathcal{B}_2)$, we see that $\beta_1 \cong \beta_2$ as principal $\text{Aut}(M_n, D_n)$ -bundles.

Conversely, if $\beta_1 \cong \beta_2$, then for the M_n -bundles \mathcal{A}_i and D_n -bundles \mathcal{B}_i attached to β_i ($i = 1, 2$), we obtain an isomorphism $\mathcal{A}_1 \xrightarrow{\cong} \mathcal{A}_2$ sending \mathcal{B}_1 to \mathcal{B}_2 . Passing over to the C^* -algebras of continuous sections, we get $(A, B_1) \cong (A, B_2)$ because the C^* -algebras of continuous sections for \mathcal{A}_1 and \mathcal{A}_2 are both isomorphic to A . \square

We summarize our discussion as follows: Fix a base space T (a locally compact Hausdorff space, as above). Consider the canonical map \mathcal{C} from isomorphism classes of principal $\text{Aut}(M_n, D_n)$ -bundles over T to isomorphism classes of principal $\text{Aut}(M_n)$ -bundles over T , sending the isomorphism class of a principal $\text{Aut}(M_n, D_n)$ -bundle α over T to the isomorphism class of the principal $\text{Aut}(M_n)$ -bundle constructed from α and the canonical action $\text{Aut}(M_n, D_n) \curvearrowright \text{Aut}(M_n)$ by left multiplication. Here our isomorphisms are allowed to induce non-trivial homeomorphisms on T .

By Proposition 2.2, the n -homogeneous C^* -algebras over T which have a Cartan subalgebra correspond exactly to the principal $\text{Aut}(M_n)$ -bundles which are in the image of \mathcal{C} . Given $[\alpha] \in \text{im}(\mathcal{C})$ (here we denote isomorphism classes by $[\cdot]$), let A be the corresponding n -homogeneous C^* -algebra. Two Cartan subalgebras B_1 and B_2 in A are called equivalent if $(A, B_1) \cong (A, B_2)$. By Proposition 2.3, the equivalence classes of Cartan subalgebras in A are in one-to-one correspondence with the pre-image $\mathcal{C}^{-1}[\alpha]$ of $[\alpha]$.

We can also rephrase this observation using the language of classifying spaces, as long as our base spaces are CW-complexes. Let $G = \text{Aut}(M_n)$ and $H = \text{Aut}(M_n, D_n)$. Clearly, $G \cong U(n)/\mathbb{T}$, while $H \cong (\mathbb{T}^n \rtimes S_n)/\mathbb{T} \cong (\mathbb{T}^n/\mathbb{T}) \rtimes S_n$. Here $U(n)$ is the (topological) group of unitary $n \times n$ -matrices over \mathbb{C} , and \mathbb{T} identifies with the center of $U(n)$. For H , S_n is the group of permutations of $\{1, \dots, n\}$, acting on \mathbb{T}^n in the canonical way. This gives rise to the semidirect product $\mathbb{T}^n \rtimes S_n$. Again, we embed \mathbb{T} into \mathbb{T}^n , and hence into $\mathbb{T}^n \rtimes S_n$ in a canonical way as diagonal matrices. Let $[\text{Homeo}(T)]$ be the image of the group of homeomorphisms of T in the set of homotopy classes $[T, T]$ of continuous maps from T to T . We view $[\text{Homeo}(T)]$ as a group under composition. Using [26, Chapter 4, § 13], the map \mathcal{C} from above becomes the following map induced by the inclusion $H \hookrightarrow G$:

$$(1) \quad [T, BH]/[\text{Homeo}(T)] \longrightarrow [T, BG]/[\text{Homeo}(T)]$$

where BH and BG are the classifying spaces of H and G , and $[\text{Homeo}(T)]$ acts on $[T, BH]$ and $[T, BG]$ by pre-composition. As we have seen above, the discussion of existence and uniqueness of Cartan subalgebras in n -homogeneous C^* -algebras over T boils down to studying the image and pre-images of the map (1), in exactly the same way as we explained above.

2.2. Cartan subalgebras in homogeneous C*-algebras over spheres. Let us now restrict to the case where our base space T is given by the k -sphere

$$S^k = \{(x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1} : x_1^2 + \dots + x_{k+1}^2 = 1\}.$$

As above, let $H = \text{Aut}(M_n, D_n)$ and $G = \text{Aut}(M_n)$. By [26, Chapter 8, Theorem 8.2], isomorphism classes of principal H -bundles over S^k are in one-to-one correspondence to $C(S^{k-1}, H)/\sim$. Here $C(S^{k-1}, H)$ is the set of continuous maps $S^{k-1} \rightarrow H$, and we define an equivalence relation by saying $c_1 \sim c_2$ if there is $h \in H$ such that c_1 and $hc_2^{\pm 1}h^{-1}$ are homotopic, for $c_1, c_2 \in C(S^{k-1}, H)$. Similarly, isomorphism classes of principal G -bundles over S^k are in one-to-one correspondence to $C(S^{k-1}, G)/\sim$. Note that all the maps in $C(S^{k-1}, H)$ and $C(S^{k-1}, G)$ are supposed to send a fixed based point of S^{k-1} to the identity (in H or G , respectively).

As explained in [26, Chapter 8, Theorem 8.2], these one-to-one correspondences are established by decomposing S^k into upper and lower hemispheres, finding a trivialization of our principal bundle over these hemispheres, and taking the \sim -class of the resulting transition function on S^{k-1} . Here we embed S^{k-1} into S^k via $(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0)$. Thus the map \mathcal{C} introduced above corresponds to the canonical map $C(S^{k-1}, H)/\sim \rightarrow C(S^{k-1}, G)/\sim$ induced by the canonical inclusion $H \hookrightarrow G$. Note that [26] only discusses isomorphism classes of principal bundles under isomorphisms which induce the identity on the base space. Now observe that every homeomorphism $S^k \xrightarrow{\cong} S^k$ is homotopic to the identity or to $S^k \rightarrow S^k$, $(x_1, \dots, x_{k+1}) \mapsto (x_1, \dots, -x_{k+1})$, because the homotopy class is determined by the degree. Only the second homeomorphism acts non-trivially by sending (the homotopy class of) a function c on S^{k-1} with values in G or H to (the homotopy class of) c^{-1} , the pointwise inverse of c .

In the following discussion, let us use the more precise notation $H_n := \text{Aut}(M_n, D_n)$ and $G_n := \text{Aut}(M_n)$. Our goal is to determine the (number of) isomorphism classes of principal H_n -bundles over spheres. We then apply our findings in § 2.1 to study existence and uniqueness of Cartan subalgebras in n -homogeneous C*-algebras over spheres.

Let us start with $k = 1$. It is clear that $|C(S^0, H_n)/\sim|$ is given by $p(n)$, the number of conjugacy classes of S_n , which is the same as the number of partitions of $\{1, \dots, n\}$. Therefore, we see that the n -homogeneous C*-algebra $C(S^1, M_n)$ has exactly $p(n)$ pairwise inequivalent Cartan subalgebras. An explicit construction of these Cartan subalgebras is given as follows: Decompose S^1 into $D_+^1 = \{(x_1, x_2) \in S^1 : x_2 > -\varepsilon\}$ and $D_-^1 = \{(x_1, x_2) \in S^1 : x_2 < \varepsilon\}$, for some fixed $\varepsilon \in (0, \frac{1}{4})$. Choose representatives $\sigma_1, \dots, \sigma_{p(n)}$ for the conjugacy classes of S_n . For every $1 \leq i \leq p(n)$, we obtain a D_n -bundle \mathcal{B}_i over S^1 by gluing together the trivial D_n -bundles on D_+^1 and D_-^1 using the identity on $\{(x_1, x_2) \in D_+^1 \cap D_-^1 : x_1 > 0\}$ and σ_i on $\{(x_1, x_2) \in D_+^1 \cap D_-^1 : x_1 < 0\}$. These \mathcal{B}_i give rise to a complete list of pairwise inequivalent Cartan subalgebras $B_1, \dots, B_{p(n)}$ of $C(S^1, M_n)$ up to equivalence. For $n = 2$, we obtain the trivial bundle and the Möbius bundle.

Let us consider $k = 2$. As we only consider continuous maps in $C(S^1, H_n)$ sending a fixed based point of S^1 to the identity of H_n , we have $C(S^1, H_n)/\sim = C(S^1, \mathbb{T}^n/\mathbb{T})/\sim$. Here, given $c_1, c_2 \in C(S^1, \mathbb{T}^n/\mathbb{T})$, we set $c_1 \sim c_2$ if there exists $\sigma \in S_n$ such that c_1 and $\sigma c_2^{\pm 1} \sigma^{-1}$ are homotopic. It follows that $C(S^1, \mathbb{T}^n/\mathbb{T})/\sim \cong (\{\pm \text{id}\} \times S_n) \backslash (\mathbb{Z}^n/\mathbb{Z})$. Moreover, combining this observation with the discussion in [28], it is clear that the map $C(S^1, H_n)/\sim \rightarrow C(S^1, G_n)/\sim$ induced by the canonical inclusion $H_n \hookrightarrow G_n$ can be identified with the map $(\{\pm \text{id}\} \times S_n) \backslash (\mathbb{Z}^n/\mathbb{Z}) \rightarrow \{\pm \text{id}\} \backslash (\mathbb{Z}/n\mathbb{Z})$ given by summation. So over S^2 , every n -homogeneous C^* -algebra has (countably) infinitely many pairwise inequivalent Cartan subalgebras.

Finally, we turn to the remaining case $k > 2$. As in the case $k = 2$, we have $C(S^{k-1}, H_n)/\sim = C(S^{k-1}, \mathbb{T}^n/\mathbb{T})/\sim$. Since $\pi_{k-1}(\mathbb{T})$ is trivial for $k > 2$, we conclude that $|C(S^{k-1}, \mathbb{T}^n/\mathbb{T})/\sim| = 1$. This shows that over S^k , for $k > 2$, only the trivial n -homogeneous C^* -algebra $C(S^k, M_n)$ has a Cartan subalgebra. Moreover, up to equivalence, $C(S^k, M_n)$ has a unique Cartan subalgebra given by $C(S^k, D_n)$. This gives a conceptual explanation for Gregson's and Natsume's examples of homogeneous C^* -algebras over spheres which do not admit Cartan subalgebras (see [24, § 2.2.5] and the appendix of [30]).

Exactly the same methods allow us to study Cartan subalgebras of \aleph_0 -homogeneous C^* -algebras over spheres, at least those where the Cartan pair admits a local trivialization. First of all, fix a countably infinite dimensional Hilbert space \mathcal{H} and let $\mathcal{K} := \mathcal{K}(\mathcal{H})$ be the C^* -algebra of compact operators on \mathcal{H} . \aleph_0 -homogeneous C^* -algebras over spheres correspond to locally trivial \mathcal{K} -bundles, and two such \aleph_0 -homogeneous C^* -algebras are isomorphic if their Dixmier-Douady invariants coincide (see for instance [3, Theorem IV.1.7.15]). We know that $H^3(S^k, \mathbb{Z}) \cong \{0\}$ if $k \neq 3$ and $H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$. Hence over S^k , for $k \neq 3$, every \aleph_0 -homogeneous C^* -algebra is trivial, i.e., isomorphic to $C(S^k, \mathcal{K})$, while there are countably infinitely many pairwise non-isomorphic \aleph_0 -homogeneous C^* -algebras over S^3 . Moreover, by [53] (see also [55, § 6.4]), every \aleph_0 -homogeneous C^* -algebra over S^k has a Cartan subalgebra. Since \aleph_0 -homogeneous C^* -algebras A over a finite dimensional second countable Hausdorff locally compact space T are classified up to isomorphism by their Dixmier-Douady invariant $\delta(A) \in H^3(T, \mathbb{Z})$, it suffices to construct a groupoid model for such an algebra. We realize the class $\delta(A)$ in $H^3(T, \mathbb{Z})$ as a Čech cocycle (σ_{ijk}) on an open cover $(U_i)_{i \in I}$ of T ; thus σ_{ijk} is a continuous \mathbb{T} -valued function defined on $U_i \cap U_j \cap U_k$. In order to get infinite fibers, we replace the index set I by $\tilde{I} := I \times \mathbb{N}$ and write $U_{(i,n)} := U_i$. We let $X := \{(\tilde{i}, t) \in \tilde{I} \times T : t \in U_{\tilde{i}}\}$ be the disjoint union of the $U_{\tilde{i}}$'s and $\pi : X \rightarrow T$ be the second projection. It is a local homeomorphism with infinite fibers. The fibered product $X \times_T X$ over T , which is a closed subgroupoid of the pair groupoid $X \times X$, is an étale equivalence relation. It carries the twist given by the 2-cocycle $\sigma \in Z^2(R, \mathbb{T})$ defined by $\sigma((\tilde{i}, t), (\tilde{j}, t), (\tilde{k}, t)) = \sigma_{ijk}(t)$. The twisted groupoid C^* -algebra $C^*(R, \sigma)$ has $C(X)$ as a Cartan subalgebra. Since $\pi : X \rightarrow T$ is usually not a covering map, the corresponding Cartan pair usually does not admit local trivializations. Also, another choice of open cover may give another

Cartan pair. However, as explained in [31, Section 5], the twisted groupoids are all Morita equivalent.

Note that the C^* -algebras attached to groupoids twisted by a 2-cocycle are a special case of C^* -algebras of twisted groupoids. [52, Chapter I, Proposition 1.14] explains how to construct the twist $\Sigma = E_\sigma$ from a 2-cocycle σ . In that case, Σ is a trivial principal \mathbb{T} -bundle. In fact, a twist Σ is of the form E_σ if and only if it is a trivial principal \mathbb{T} -bundle. As an example of a twist which is not given by a 2-cocycle, just consider a non-trivial principal \mathbb{T} -bundle over a space $G = X$.

Let us study existence and uniqueness of Cartan subalgebras of \aleph_0 -homogeneous C^* -algebras where the corresponding Cartan pairs admit local trivialisations. Let \mathcal{D} be a fixed Cartan subalgebra of \mathcal{K} . It does not matter which one we choose as every two such Cartan subalgebras are equivalent. Given an \aleph_0 -homogeneous C^* -algebra A over T and a Cartan subalgebra B , we let \mathcal{A} and \mathcal{B} be the corresponding \mathcal{K} - and \mathcal{D} -bundles over T . Recall that a local trivialisation of $(\mathcal{A}, \mathcal{B})$ is a covering of T by open subsets W together with isomorphisms $\mathcal{A}|_W \xrightarrow{\cong} W \times \mathcal{K}$ sending $\mathcal{B}|_W$ to $W \times \mathcal{D}$. Let $G = \text{Aut}(\mathcal{K}) \cong U(\mathcal{H})/\mathbb{T}$ and $H = \text{Aut}(\mathcal{K}, \mathcal{D}) = (\mathbb{T}^\infty/\mathbb{T}) \rtimes S_\infty$. Here S_∞ is the group of permutations of $\{1, 2, \dots\}$. The same argument as in the finite homogeneous case shows that over $T = S^k$, \aleph_0 -homogeneous C^* -algebras are in one-to-one correspondence with $C(S^{k-1}, G)/\sim$. Moreover, an \aleph_0 -homogeneous C^* -algebra has a Cartan subalgebra such that the corresponding Cartan pair has a local trivialisation if and only if it corresponds to an element in $C(S^{k-1}, G)/\sim$ which lies in the image of the canonical map $C(S^{k-1}, H)/\sim \rightarrow C(S^{k-1}, G)/\sim$. Given such an element in $C(S^{k-1}, G)/\sim$, equivalence classes of Cartan subalgebras in the corresponding \aleph_0 -homogeneous C^* -algebra are in one-to-one correspondence with the pre-image of that element under the canonical map $C(S^{k-1}, H)/\sim \rightarrow C(S^{k-1}, G)/\sim$.

For $k = 1$, it is easy to see that $C(S^{k-1}, H)/\sim$ is in one-to-one correspondence with conjugacy classes in S_∞ , and there are uncountably many of these. For $k \geq 2$, we have $|C(S^{k-1}, H)/\sim| = 1$.

This shows that for $k = 1$, the up-to-isomorphism unique \aleph_0 -homogeneous C^* -algebra $C(S^1, \mathcal{K})$ over S^1 has uncountably many Cartan subalgebras whose Cartan pairs admit local trivialisations. For $k = 2$ or $k \geq 4$, we see that the up-to-isomorphism unique \aleph_0 -homogeneous C^* -algebra $C(S^k, \mathcal{K})$ over S^k has $C(S^k, \mathcal{D})$ as its up-to-equivalence unique Cartan subalgebra whose Cartan pair admit a local trivialisation. For $k = 3$, the only \aleph_0 -homogeneous C^* -algebra over S^3 which admits a Cartan subalgebra whose Cartan pair has a local trivialisation is the trivial one, i.e., isomorphic to $C(S^3, \mathcal{K})$. And up to equivalence, the Cartan subalgebra of $C(S^3, \mathcal{K})$ whose Cartan pair admit a local trivialisation is given by $C(S^3, \mathcal{D})$. All the other non-trivial \aleph_0 -homogeneous C^* -algebra over S^3 also have Cartan subalgebras, as we mentioned above. Therefore, these Cartan subalgebras do not have the property that the corresponding Cartan pairs admit local trivialisations.

The case $k = 3$ shows that not every Cartan subalgebra of an \aleph_0 -homogeneous C^* -algebra is such that the corresponding Cartan pair admits a local trivialisation. It is also

interesting to point out that for $k = 4$, by [28], there exist non-trivial n -homogeneous C^* -algebras over S^4 , for every $n \geq 2$. By our previous discussion, these non-trivial n -homogeneous C^* -algebras cannot have Cartan subalgebras. However, after stabilizing, i.e., tensoring with \mathcal{K} , we obtain the trivial \aleph_0 -homogeneous C^* -algebra over S^4 , because its Dixmier-Douady invariant has to vanish. And this trivial \aleph_0 -homogeneous C^* -algebra obviously has a Cartan subalgebra. Thus we have just obtained an example of a C^* -algebra which has no Cartan subalgebra, but whose stabilization admits a Cartan subalgebra.

We summarize our findings as follows.

Theorem 2.4. *Let $n \in \mathbb{Z}$ with $n \geq 2$.*

Over S^1 , the up-to-isomorphism unique n -homogeneous C^ -algebra $C(S^1, M_n)$ has exactly $p(n)$ pairwise inequivalent Cartan subalgebras.*

Over S^2 , every n -homogeneous C^ -algebra has countably infinitely many pairwise inequivalent Cartan subalgebras.*

Over S^k with $k > 2$, only the trivial n -homogeneous C^ -algebra $C(S^k, M_n)$ has a Cartan subalgebra, which is unique up to equivalence and given by $C(S^k, D_n)$.*

Moreover, over S^1 , the up-to-isomorphism unique \aleph_0 -homogeneous C^ -algebra $C(S^1, \mathcal{K})$ has uncountably many inequivalent Cartan subalgebras whose Cartan pairs admit local trivialisations.*

Over S^2 or S^k with $k \geq 4$, the up-to-isomorphism unique \aleph_0 -homogeneous C^ -algebra $C(S^k, \mathcal{K})$ over S^k has $C(S^k, \mathcal{D})$ as its up-to-equivalence unique Cartan subalgebra whose Cartan pair admits a local trivialization.*

Over S^3 , only the trivial \aleph_0 -homogeneous C^ -algebra $C(S^3, \mathcal{K})$ has a Cartan subalgebra whose Cartan pair has a local trivialization, and this Cartan subalgebra is unique up to equivalence and given by $C(S^3, \mathcal{D})$.*

Remark 2.5. In all the cases in the above theorem, it is possible to follow the recipe as in the case $k = 1$, $n < \infty$ to construct models for the Cartan subalgebras.

3. CARTAN SUBALGEBRAS IN GROUP C^* -ALGEBRAS OF CERTAIN CONNECTED LIE GROUPS

The structure of reduced and full group C^* -algebras of Lie groups have been studied by many authors (see for instance [19, 40, 27, 66, 46]). Our aim is to show that many of these group C^* -algebras have a Cartan subalgebra.

We start with the full group C^* -algebra $C^*(SL(2, \mathbb{C}))$ of $SL(2, \mathbb{C})$. Its structure has been determined in [19]. Recall that by [19, § 5], the C^* -algebra $C^*(SL(2, \mathbb{C}))$ can be described as follows: Let $Z_1 := \{(m, \rho) : m \in \mathbb{Z}_{\geq 0}, \rho \in \mathbb{R}, \rho \geq 0 \text{ if } m = 0\}$, and endow Z_1 with the subspace topology of $\mathbb{Z} \times \mathbb{R}$. Let $Z_2 = [0, 1]$ with the usual topology. Define Z to be the topological space obtained from the disjoint union of Z_1 and Z_2

by identifying $(0, 0) \in Z_1$ with $0 \in Z_2$. Moreover, let $H = \ell^2(\mathbb{Z}_{\geq 0})$ with canonical orthonormal basis e_0, e_1, e_2, \dots , and let K be the closed subspace of H generated by $\{e_n : n \geq 1\}$. Let $M : K \rightarrow H$ be the unitary given by $M(e_n) = e_{n-1}$ for all $n \geq 1$. We write $\mathcal{K}(H)$ for the C*-algebra of compact operators on H . Let A be the C*-algebra of all $a \in C_0(Z, \mathcal{K}(H))$ satisfying the condition that

$$a(1) \in \mathcal{K}(\mathbb{C}e_0) \oplus M^{-1}a(2, 0)M \subseteq \mathcal{K}(\mathbb{C}e_0) \oplus \mathcal{K}(K) \subseteq \mathcal{K}(H).$$

Then [19, Theorem 5.4] tells us that $C^*(SL(2, \mathbb{C})) \cong A$.

Let \mathcal{D} be the C*-algebra of diagonal compact operators with respect to $\{e_n : n \in \mathbb{Z}_{\geq 0}\}$, i.e., $\mathcal{D} = \{T \in \mathcal{K}(H) : Te_n \in \mathbb{C}e_n \text{ for all } n \in \mathbb{Z}_{\geq 0}\}$.

Proposition 3.1. *$B := A \cap C_0(Z, \mathcal{D})$ is a Cartan subalgebra in A . In particular, $C^*(SL(2, \mathbb{C}))$ admits a Cartan subalgebra.*

Proof. Since $C_0(Z, \mathcal{D})$ is obviously a Cartan subalgebra of $C_0(Z, \mathcal{K}(H))$, it is clear that B contains an approximate unit for A , that B is maximal abelian in A , and that there exists a faithful conditional expectation $A \rightarrow B$. It remains to show that the normalizer

$$N_A(B) = \{n \in A : nBn^* \subseteq B \text{ and } n^*Bn \subseteq B\}$$

generates A as a C*-algebra.

For $g \in C_0(\mathbb{R})$, $f_0 \in C_0(\mathbb{R})$ with $f_0(-1) = 1$ and $f_2 \in C_0(\mathbb{R})$ with $f_2(2) = 1$, and fixed indices i and j , consider the function $n : Z \rightarrow \mathcal{K}(H)$ given by

$$\begin{aligned} n(m, \rho) &= \delta_{m,2} f_2(\rho) e_{i,j} \text{ for } (m, \rho) \in Z_1, m > 0 \\ n(0, \rho) &= (g(\rho), f_0(\rho) e_{i+1, j+1}) \in \mathcal{K}(\mathbb{C}e_0) \oplus \mathcal{K}(K) \text{ for } (0, \rho) \in Z_1 \\ n(t) &= (g(-t), f_0(-t) e_{i+1, j+1}) \in \mathcal{K}(\mathbb{C}e_0) \oplus \mathcal{K}(K) \text{ for } t \in Z_2. \end{aligned}$$

Here $\{e_{i,j}\}_{i,j}$ are the canonical matrix units with respect to the orthonormal basis $\{e_n\}_n$.

It is clear that $n \in A$ and $n \in N_A(B)$. Moreover, given $a \in A$ arbitrary, we can find elements $a_i \in A$ in the linear span of elements n as above such that $\lim_{i \rightarrow \infty} (a - a_i)(2, 0) = 0$ and $\lim_{i \rightarrow \infty} (a - a_i)(1) = 0$. Therefore, it suffices to show that every $a \in A$ with $a(2, 0) = a(1) = 0$ lies in the C*-algebra generated by $N_A(B)$. The collection of these elements obviously forms the ideal $I := C_0(Z \setminus \{(2, 0), 1\}, \mathcal{K}(H))$. We have $I \cap B = I \cap C_0(Z, \mathcal{D}) = C_0(Z \setminus \{(2, 0), 1\}, \mathcal{D})$. It is clear that $I \cap B$ is a Cartan subalgebra of I . In particular, I is generated as a C*-algebra by $N_I(I \cap B)$. Hence all we have to show is $N_I(I \cap B) \subseteq N_A(B)$. Choose $n \in N_I(I \cap B)$ and $b \in B$. Moreover, let $h_\lambda \in I \cap B$ be an approximate unit for I . Then $nb n^* = \lim_\lambda n h_\lambda b h_\lambda n^* \in I \cap B$ as $h_\lambda b h_\lambda \in I \cap B$. Thus, indeed, $N_I(I \cap B) \subseteq N_A(B)$. \square

To cover more group C*-algebras of Lie groups, we prove the following general result: Let M be a locally compact Hausdorff space. Let $H = \ell^2 \mathbb{Z}$ and $\{e_i : i \in \mathbb{Z}\}$ be the canonical orthonormal basis of H . Let $\mathcal{K}(H)$ be the C*-algebra of compact operators on H and \mathcal{D} the C*-algebra of compact operators on H which are diagonal with respect to $\{e_i : i \in \mathbb{Z}\}$. Assume that for every $p \in M$, we are given a decomposition $H =$

$H_{1,p} \oplus \dots \oplus H_{k(p),p}$. Moreover, suppose that there exists an open covering \mathcal{V} of M such that for every $V \in \mathcal{V}$, there is a continuous map $V \rightarrow U(H)$, $p \mapsto U_p^V$ and

$$(2) \quad \text{there is a partition } \mathbb{Z} = P_{1,p} \sqcup \dots \sqcup P_{k(p),p} \text{ for every } p \in M \text{ with} \\ H_{i,p} = \langle \{U_p^V(e_j) : j \in P_{i,p}\} \rangle$$

for all $1 \leq i \leq k(p)$, and

$$(3) \quad \{U_p^V(e_i) : i \in \mathbb{Z}\} = \{U_p^W(e_i) : i \in \mathbb{Z}\}$$

for all $V, W \in \mathcal{V}$ and $p \in V \cap W$. The latter condition allows us to define, for every $p \in M$, $\mathcal{D}_p := U_p^V \mathcal{D}(U_p^V)^*$, where $V \in \mathcal{V}$ satisfies $p \in V$ (by (3), \mathcal{D}_p does not depend on the choice of V). Define

$$A := \{a \in C_0(M, \mathcal{K}(H)) : a(p) \in \mathcal{K}(H_{1,p}) \oplus \dots \oplus \mathcal{K}(H_{k(p),p}) \text{ for all } p \in M\},$$

$$B := \{b \in A : b(p) \in \mathcal{D}_p \text{ for all } p \in M\}.$$

Proposition 3.2. *B is a Cartan subalgebra of A.*

Proof. Our conditions allow us to work locally. And working locally, all we have to show is that given an open subset V of M , and a sub-C*-algebra A_V of $C_0(V, \mathcal{K}(H))$ containing $B_V := C_0(V, \mathcal{D})$, B_V is a Cartan subalgebra of A_V .

It is clear that B_V has an approximate unit for A_V , that B_V is maximal abelian in A_V , and that there is a faithful conditional expectation $A_V \rightarrow B_V$. All we have to show is that the normalizer $N_{A_V}(B_V) = \{n \in A_V : nB_Vn^* \subseteq B_V \text{ and } n^*B_Vn \subseteq B_V\}$ generates A_V as a C*-algebra. Given an arbitrary element $a \in A_V$, let $e_{i,j}$ be the canonical matrix units attached to the orthonormal basis $\{e_i : i \in \mathbb{Z}\}$, and let $d_i \in C_b(V, \mathcal{D})$ be given by $d_i(p) = e_{i,i}$ for all $p \in V$. Obviously, $a = \lim_{n \rightarrow \infty} \sum_{i,j=1}^n d_i a d_j$, and $d_i a d_j$ clearly lies in $N_{A_V}(B_V)$ for all $i, j \in \mathbb{Z}$. \square

Using this, let us describe a Cartan subalgebra in $C^*(SL(2, \mathbb{R}))$. The full C*-algebra of $SL(2, \mathbb{R})$ has been described in [40]. We recall the description: Define

$$K := \{(x, 0) \in \mathbb{C}^2 : x \in [0, \infty) \cdot i\} \\ \cup \{(y, 1) \in \mathbb{C}^2 : y \in [0, \infty) \cdot i\} \\ \cup \{(z, 0) \in \mathbb{C}^2 : z \in [0, 1]\}$$

equipped with the subspace topology of \mathbb{C}^2 , and

$$M := K \sqcup \mathbb{Z}_{\geq 2} \sqcup \mathbb{Z}_{\geq 2},$$

where $\mathbb{Z}_{\geq 2}$ is endowed with the discrete topology. As above, let $H = \ell^2 \mathbb{Z}$. We use the same notation as above. For all $p \in M$, $p \neq (0, 1)$, $p \neq (1, 0)$, let $k(p) = 1$ and $H_{1,p} = H$. For $p = (0, 1)$, let $k(p) = 2$ and $H_{1,p} = \ell^2(\mathbb{Z}_{<0}) \subseteq H$ and $H_{2,p} = \ell^2(\mathbb{Z}_{\geq 0}) \subseteq H$. For $p = (1, 0)$, set $k(p) := 3$ and $H_{1,p} = \ell^2(\mathbb{Z}_{<0})$, $H_{2,p} = \mathbb{C}e_0$ and $H_{3,p} = \ell^2(\mathbb{Z}_{>0})$. Then $A := \{a \in C_0(M, \mathcal{K}(H)) : a(p) \in \mathcal{K}(H_{1,p}) \oplus \dots \oplus \mathcal{K}(H_{k(p),p}) \text{ for all } p \in M\}$ is isomorphic to $C^*(SL(2, \mathbb{R}))$ by [40].

To show that we can apply Proposition 3.2, let $\mathcal{V} = \{M\}$ and $U_p^M := \text{id}_H$. Obviously, the conditions for Proposition 3.2 are satisfied. Let \mathcal{D} be the C^* -algebra of compact operators on H which are diagonal with respect to $\{e_i : i \in \mathbb{Z}\}$. We obtain

Corollary 3.3. *$C_0(M, \mathcal{D})$ is a Cartan subalgebra of A . In particular, $C^*(SL(2, \mathbb{R}))$ has a Cartan subalgebra.*

Let us describe a Cartan subalgebra of the full group C^* -algebra $C^*(G)$ of the universal covering group G of $SL(2, \mathbb{R})$. Recall the description of $C^*(G)$ in [27]: Let $M = \{z \in \mathbb{C} : 0 < |z| \leq 1\} \cup [1, 2]$ equipped with the subspace topology of \mathbb{C} . Let $H = \ell^2\mathbb{Z}$ and U the unilateral shift. Choose a self-adjoint operator $T \in \mathcal{L}(H)$ such that $U = e^{2\pi iT}$. Define $U : [0, 1] \rightarrow U(H)$, $t \mapsto e^{2\pi itT}$. Clearly, $U(0) = \text{id}_H$ and $U(1) = U$. For every $t \in [0, 1]$, set $H_t := U(t)^*(\ell^2(\mathbb{Z}_{>0}))$. Set $k(p) = 1$ if $p \in M$, $0 < |p| < 1$, and $H_{1,p} = H$. Set $k(p) = 2$ if $p \in M$, $|p| = 1$, $p \neq 1$, and $H_{1,p} = H_t^\perp$, $H_{2,p} = H_t$, where $p = e^{2\pi it}$ for a uniquely determined $t \in (0, 1)$. Set $k(p) = 3$ if $p \in [1, 2] \subseteq M$, and $H_{1,p} := \ell^2(\mathbb{Z}_{<0})$, $H_{2,p} := \mathbb{C}e_0$, $H_{3,p} := \ell^2(\mathbb{Z}_{>0})$. Let

$$A := \{a \in C_0(M, \mathcal{K}(H)) : a(p) \in \mathcal{K}(H_{1,p}) \oplus \dots \oplus cK(H_{k(p),p})\},$$

$$A_1 := \{a \in A : a(p)e_0 = a(1)e_0 \text{ for all } p \in [1, 2] \text{ and } a(2) \in \mathbb{C}e_{0,0}\}.$$

Then $A_1 \cong C^*(G)$ by [27].

Let

$$V := \left\{ re^{2\pi it} : r \in (0, 1], t \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \right\} \cup [1, 2]$$

and

$$W := \{re^{2\pi it} : r \in (0, 1], t \in (0, 1)\}.$$

It is clear that $M = V \cup W$. Define $U_p^V := e^{-2\pi itT}$ if $p = re^{2\pi it}$ ($t \in (-\frac{\pi}{4}, \frac{\pi}{4})$), $U_p^V := \text{id}_H$ if $p \in [1, 2]$ and $U_p^W := U(t)^*$ if $p = re^{2\pi it}$ ($t \in (0, 1)$). In this way, we obviously obtain continuous maps $V \rightarrow U(H)$, $p \mapsto U_p^V$ and $W \rightarrow U(H)$, $p \mapsto U_p^W$.

Let us check condition (2): For $p \in M$ with $0 < |p| < 1$, condition (2) obviously holds. For $p \in M$ with $p = e^{2\pi it}$, condition (2) is clearly satisfied for U^W . For U^V and $p = e^{2\pi it}$ with $t \in [0, \frac{\pi}{4})$, it is also obvious that (2) holds. For U^V and $p = e^{2\pi it}$ with $t \in (-\frac{\pi}{4}, 0)$, we have

$$\begin{aligned} H_{2,p} &= U(1+t)^*\ell^2(\mathbb{Z}_{\geq 0}) = e^{-2\pi i(1+t)T}\ell^2(\mathbb{Z}_{\geq 0}) = e^{-2\pi iT}e^{-2\pi itT}\ell^2(\mathbb{Z}_{\geq 0}) \\ &= U_p^V U^*\ell^2(\mathbb{Z}_{\geq 0}) = U_p^V \ell^2(\mathbb{Z}_{\geq -1}) \end{aligned}$$

and $H_{1,p} = H_{2,p}^\perp = U_p^V \ell^2(\mathbb{Z}_{\leq -2})$. Hence (2) holds. For $p \in [1, 2] \subseteq M$, it is again obvious that condition (2) is true.

Let us check condition (3). We have

$$V \cap W = \left\{ re^{2\pi it} \in \mathbb{C} : r \in (0, 1], t \in \left(-\frac{\pi}{4}, 0\right) \cup \left(0, \frac{\pi}{4}\right) \right\}.$$

Clearly, (3) holds for $p \in V \cap W$ of the form $p = re^{2\pi it}$ with $r \in (0, 1]$ and $t \in (0, \frac{\pi}{4})$. For $p = re^{2\pi it}$ with $r \in (0, 1]$ and $t \in (-\frac{\pi}{4}, 0)$, we have

$$\begin{aligned} & \{U_p^V(e_i : i \in \mathbb{Z})\} = \{e^{-2\pi itT}(e_i) : i \in \mathbb{Z}\} = \{e^{-2\pi itT}(e_{i-1}) : i \in \mathbb{Z}\} \\ & = \{e^{-2\pi itT}e^{-2\pi iT}(e_i) : i \in \mathbb{Z}\} = \{e^{-2\pi i(t+1)T}(e_i) : i \in \mathbb{Z}\} = \{U_p^W(e_i) : i \in \mathbb{Z}\}. \end{aligned}$$

Hence condition (3) holds as well.

Therefore, Proposition 3.2 applies and tells us that if we set $\mathcal{D}_p := U_p^V \mathcal{D}(U_p^V)^*$ for $p \in V$ and $\mathcal{D}_p := U_p^W \mathcal{D}(U_p^W)^*$ for $p \in W$, then $B := \{b \in A : b(p) \in \mathcal{D}_p \text{ for all } p \in M\}$ is a Cartan subalgebra of A . Using this, it is now straightforward to check, using similar methods as in the proof of Proposition 3.1, that $B \cap A_1$ is a Cartan subalgebra of A_1 . Thus, we obtain

Corollary 3.4. *The full C^* -algebra of the universal covering group of $SL(2, \mathbb{R})$ has a Cartan subalgebra.*

Using the description of full group C^* -algebras for covering groups with finite center of $SO_0(2, 1)$ in [27], the same explanation as for Corollary 3.3 gives

Corollary 3.5. *The full group C^* -algebra of a covering group with finite center of $SO_0(2, 1)$ has a Cartan subalgebra.*

Remark 3.6. Looking at the conjecture at the end of [27], it seems that our method can be applied to show that for many connected semisimple Lie groups, the full group C^* -algebras have Cartan subalgebras.

We now turn to reduced group C^* -algebras of certain Lie groups. We use the descriptions of these C^* -algebras in [5, 66, 46].

Corollary 3.7. *For every connected semi-simple Lie group with real rank one and finite center, the reduced group C^* -algebra has a Cartan subalgebra.*

Proof. By the description of these reduced group C^* -algebras in [5, 66], we see that they are of the form as in Proposition 3.2, but with $k(p)$ equal to 1 or 2 for all $p \in M$, and if $k(p) = 2$, we can always choose $H_{1,p} = \ell^2(\mathbb{Z}_{\leq -1})$ and $H_{2,p} = \ell^2(\mathbb{Z}_{\geq 0})$. Thus, Proposition 3.2 applies because we can simply choose the trivial cover $\mathcal{V} = \{M\}$ and the constant map $U_M \equiv \text{id}_H$. \square

Corollary 3.8 (Corollary to Proposition 4.1 in [46]). *For every connected, complex semi-simple Lie group, the reduced group C^* -algebra has a Cartan subalgebra.*

Proof. Because of [46, Proposition 4.1], these reduced group C^* -algebras are trivial \aleph_0 -homogeneous C^* -algebras. Hence they obviously admit Cartan subalgebras. \square

Question 3.9. Is there a conceptual explanation why the group C^* -algebras of (many) connected Lie groups have Cartan subalgebras?

4. NON-EXISTENCE OF C*-DIAGONALS AND CARTAN SUBALGEBRAS IN CERTAIN
REDUCED GROUP C*-ALGEBRAS

Using non-existence results for Cartan subalgebras in certain group vN-algebras [67, 25, 43, 44, 45, 49, 50, 4], we show that certain reduced group C*-algebras have no diagonals, or even stronger, no Cartan subalgebras.

Since C*-algebras which have a Cartan subalgebra can be described as reduced twisted groupoid C*-algebras, we first complement some results from [52] on traces on these C*-algebras.

4.1. Traces on twisted groupoid C*-algebras. Let G be an étale second countable locally compact Hausdorff groupoid, and suppose that we are given a twist $\mathbb{T} \times G^{(0)} \xrightarrow{i} \Sigma \xrightarrow{j} G$. For the notion of twist and the construction of $C_r^*(G, \Sigma)$, we refer to [55] and the references therein. We assume for simplicity that $X := G^{(0)}$ is compact. Let $S \subseteq G$ be an open bisection, i.e., the range and source maps restrict to homeomorphisms $r|_S : S \rightarrow r(S)$ and $s|_S : S \rightarrow s(S)$. Let α_S be the homeomorphism $s(S) \rightarrow r(S)$, $x \mapsto r((s|_S)^{-1}(x))$. It is easily checked that a Borel measure μ on X is invariant in the sense of [52, Definition I.3.12] if and only if for every open bisection S in G and every $f \in C_0(r(S))$, we have $\int_{r(S)} f \, d\mu = \int_{s(S)} f \circ \alpha_S \, d\mu$. In other words, we have $\mu|_{r(S)} = \alpha_S(\mu|_{s(S)})$. Now let τ be a trace (we mean here a tracial state) on $C_r^*(G, \Sigma)$. $\tau|_{C(X)}$ corresponds to a probability measure μ on X . The following lemma is certainly well-known, but we include this result for completeness and because we were not able to locate a reference.

Lemma 4.1. *μ is invariant.*

Proof. Let $f \in C_c(s(S))$ be positive. Define $g \in C_c(j^{-1}(S))$ by $g(\zeta) = f^{\frac{1}{2}}(r(j(\zeta)))$. Then $g * g^*, g^* * g \in C(X)$. For $x \in s(S)$, let $\gamma \in S$ be determined by $s(\gamma) = x$. We have $g * g^*(x) = \left(f^{\frac{1}{2}}(r(\gamma^{-1}))\right)^2 = f(s(\gamma)) = f(x)$. At the same time, since $\alpha_S(x) = r(\gamma)$, we get $g^* * g(x) = \left(f^{\frac{1}{2}}(r(\gamma))\right)^2 = f(r(\gamma)) = f(\alpha_S(x))$. Thus, as τ is a trace, $\int_{r(S)} f \, d\mu = \tau(f) = \tau(g * g^*) = \tau(g^* * g) = \tau(f \circ \alpha_S) = \int_{s(S)} f \circ \alpha_S \, d\mu$. \square

Now let $E : C_r^*(G, \Sigma) \rightarrow C(X)$ be the canonical conditional expectation, and μ an invariant probability measure on X , viewed as a state on $C(X)$.

Lemma 4.2. *(see also [52, Proposition II.5.4]) $\mu \circ E$ is a trace on $C_r^*(G, \Sigma)$.*

Proof. As in [55], we denote by $C_c(G, \Sigma)$ the set of compactly supported, continuous functions $f : \Sigma \rightarrow \mathbb{C}$ such that $f(z\zeta) = f(\zeta)\bar{z}$. Given $f \in C_c(G, \Sigma)$, we set $\text{supp}(f) := \{\gamma \in G : f(\zeta) \neq 0 \text{ for all } \zeta \in \Sigma \text{ with } j(\zeta) = \gamma\}$. Now, to prove the lemma, it suffices to show that $(\mu \circ E)(f * g) = (\mu \circ E)(g * f)$ for all $f, g \in C_c(G, \Sigma)$ such that $\text{supp}(f) \subseteq S$ and $\text{supp}(g) \subseteq T$ for some open bisections S and T . For $x \in X$, we have $(f * g)(x) = f(\zeta^{-1})g(\zeta)$ if $x \in s(S^{-1} \cap T)$ and $\zeta \in \Sigma$ satisfies $s(j(\zeta)) = x$, $j(\zeta) \in S^{-1} \cap T$, and $(f * g)(x) = 0$ if $x \notin s(S^{-1} \cap T)$. Similarly, for $y \in X$, we have $(g * f)(y) = g(\eta^{-1})f(\eta)$

if $y \in s(S \cap T^{-1})$ and $\eta \in \Sigma$ satisfies $s(j(\eta)) = y$, $j(\zeta) \in S \cap T^{-1}$, and $(g * f)(y) = 0$ if $y \notin s(S \cap T^{-1})$. Thus $(f * g)|_X \in C_0(s(S^{-1} \cap T))$ and $(g * f)|_X \in C_0(s(S \cap T^{-1})) = C_0(r(S^{-1} \cap T))$. Moreover, for $x \in s(S^{-1} \cap T)$, choose $\zeta \in \Sigma$ with $s(j(\zeta)) = x$, $j(\zeta) \in S^{-1} \cap T$. Then $\alpha_{S^{-1} \cap T}(x) = r(j(\zeta)) = s(j(\zeta^{-1}))$ and $j(\zeta^{-1}) \in S \cap T^{-1}$. Hence $(f * g)(x) = g(\zeta)f(\zeta^{-1}) = (g * f)(s(j(\zeta^{-1}))) = (g * f)(\alpha_{S^{-1} \cap T}(x))$. We conclude that $(\mu \circ E)(f * g) = \mu((f * g)|_X) = \mu((g * f)|_X \circ \alpha_{S^{-1} \cap T}) = \mu((g * f)|_X) = (\mu \circ E)(g * f)$. \square

Given a trace τ on $C_r^*(G, \Sigma)$, when do we have $\tau = \tau \circ E$?

Lemma 4.3. (see also [52, Proposition II.5.4]) *Assume that G is principal. Then every trace τ on $C_r^*(G, \Sigma)$ satisfies $\tau = \tau \circ E$.*

Proof. It suffices to show that for $f \in C_c(G, \Sigma)$ with $f|_X \equiv 0$, we have $\tau(f) = 0$. Since G is principal, $G \setminus X$ can be covered by open bisections S with $s(S) \cap r(S) = \emptyset$. So we may assume that $\text{supp}(f) \subseteq S$ for such an open bisection S . Choose $h \in C(X)$ such that $\text{supp}(h) \subseteq s(S)$ and $h|_{s(\text{supp}(f))} \equiv 1$. Then $(f * h)(\zeta) = f(\zeta)$ for all $\zeta \in \Sigma$, i.e., $f * h = f$. Moreover, $(h * f)(\zeta) = h(\zeta \eta^{-1})f(\eta)$ for some $\eta \in j^{-1}(S)$ with $s(j(\eta)) = s(j(\zeta))$. As $s(j(\zeta \eta^{-1})) = s(j(\eta^{-1})) \in s(S^{-1}) = r(S)$ and $h|_{r(S)} \equiv 0$, we must have $h(\zeta \eta^{-1}) = 0$ and thus $h * f = 0$. Therefore, $\tau(f) = \tau(f * h) = \tau(h * f) = 0$. \square

Corollary 4.4. *Let A be a separable unital C^* -algebra. Let $B \subseteq A$ be a Cartan subalgebra with faithful conditional expectation $E : A \twoheadrightarrow B$. Assume that B is a C^* -diagonal or that A has unique trace. Then we have $\tau = \tau \circ E$ for every trace τ on A .*

4.2. A relationship between C^* -diagonals and Cartan subalgebras in vN -algebras. Let G be an étale second countable locally compact principal Hausdorff groupoid, and suppose that we are given a twist $\mathbb{T} \times G^{(0)} \xrightarrow{i} \Sigma \xrightarrow{j} G$. Again we assume for simplicity that the unit space $X = G^{(0)}$ is compact. We can identify G to the étale equivalence relation $R = \{(r(g), s(g)) \in X \times X : g \in G\}$. The twist can be represented by a normalized Borel 2-cocycle σ in the sense of [17, § 7]: according to the proof of [41, Lemma 3.2] (here second countability of G suffices), there exists a Borel map $c : G \rightarrow \Sigma$ such that $j \circ c = \text{id}_G$ (i.e., c is a section for j), $c|_X = \text{id}_X$ and $c(g^{-1}) = c(g)^{-1}$ for all $g \in G$. We then have for every $\zeta \in \Sigma$: $j(\zeta c(j(\zeta))^{-1}) = r(j(\zeta))$, so that there is a Borel map $t : \Sigma \rightarrow \mathbb{T}$ such that $\zeta = t(\zeta)c(j(\zeta))$ for all $\zeta \in \Sigma$. The desired 2-cocycle is $\sigma : G^{(2)} \rightarrow \mathbb{T}$, $(g, h) \mapsto t(c(gh)^{-1}c(g)c(h))$. Moreover, let μ be an invariant Borel probability measure on X . Then (X, R, μ) is a countable measured equivalence relation in the sense of Feldman-Moore. Set $B := C(X)$, $A := C_r^*(G, \Sigma)$ and let $E : A \twoheadrightarrow B$ be the unique faithful conditional expectation. Then $\tau := \mu \circ E$ is a trace on A by Lemma 4.2. Let $\pi := \pi_\tau$ be the GNS representation of A attached to τ , and denote by $H := H_\pi$ the underlying Hilbert space.

Lemma 4.5. *The von Neumann algebra $\pi(A)''$ generated by $\pi(A)$ in $\mathcal{L}(H)$ is isomorphic to the von Neumann algebra $\mathbf{M}(R, \sigma)$ constructed in [18].*

Proof. Let ν be the right counting measure of μ as in [17, Theorem 2]. First, we briefly recall the construction of $\mathbf{M}(R, \sigma)$ (see [18] for details). Given a bounded complex-valued Borel function a on R with

$$\sup_{x, y \in X} \# \{z \in X : a(x, z) \neq 0\} + \# \{z \in X : a(z, y) \neq 0\} < \infty,$$

the usual convolution formula defines a bounded linear operator L_a acting on $L^2(R, \nu)$. Now $\mathbf{M}(R, \sigma)$ is by definition the von Neumann algebra generated by all these operators L_a .

First of all, since we have a Borel isomorphism $G \cong R$, $g \mapsto (r(g), s(g))$, we may view ν as a measure on G or R , and we have $L^2(R, \nu) \cong L^2(G, \nu)$.

Secondly, the map $C_c(G, \Sigma) \rightarrow L^2(R, \nu)$, $f \mapsto \dot{f}$, where $\dot{f}(g) = f(c(g))$, extends to a unitary $U : H \cong L^2(R, \nu)$. Note that underlying this observation is the isomorphism of Borel groupoids $\mathbb{T} \times_\sigma G \rightarrow \Sigma$, $(z, g) \mapsto zc(g)$, with inverse given by $(t(\zeta), j(\zeta)) \leftarrow \zeta$. Here $\mathbb{T} \times_\sigma G$ is the groupoid with multiplication given by

$$(z_1, \gamma_1)(z_2, \gamma_2) = (z_1 z_2 \sigma(\gamma_1, \gamma_2), \gamma_1 \gamma_2).$$

Thirdly, it is straightforward to check that $U\pi(A)''U^* = \mathbf{M}(R, \sigma)$. □

Using [55, Proposition 5.11] and Lemma 4.3, we obtain

Corollary 4.6. *Let A be a separable unital C^* -algebra, $B \subseteq A$ a C^* -diagonal. Let τ be a trace on A . Let $\pi := \pi_\tau$ be the GNS representation of A attached to τ . Then $\pi(A)''$ has a Cartan subalgebra (in the sense of vN-algebras).*

In other words, if $\pi(A)''$ has no Cartan subalgebra (in the sense of vN-algebras), then A cannot admit C^* -diagonals. In particular, this applies to reduced group C^* -algebras if the corresponding group vN-algebras have no Cartan subalgebras. Classes of examples of groups with this property are given in [9, 43, 44, 45, 49, 50], to mention just a few. For instance, by combining Corollary 4.6 with the results in [44, 9], we obtain:

Corollary 4.7. *Let Γ be a countable discrete group which is non-amenable, weakly amenable and bi-exact. Then $C_r^*(\Gamma)$ has no C^* -diagonal.*

In particular, this applies to non-abelian free groups, and more generally, to icc hyperbolic groups. However, we will see later on that for non-abelian free groups (and actually for more classes of groups, for instance all groups satisfying the property strong (HH)⁺, in the sense of [45, Definitions 1]), the reduced group C^* -algebras have no Cartan subalgebras.

4.3. Non-existence of Cartan subalgebras in certain reduced group

C^* -algebras. Our goal now is to show non-existence of Cartan subalgebras, not only diagonals, in certain reduced group C^* -algebras. This builds on recent results about Cartan subalgebras in von Neumann algebras in [4], which go back to [67, 25, 43, 44, 45, 49, 50].

We start with the following observation: Let G be an étale second countable locally compact Hausdorff groupoid and $\mathbb{T} \times G^{(0)} \xrightarrow{i} \Sigma \xrightarrow{j} G$ a twist. Moreover, let $c : G \rightarrow \Sigma$ be a Borel section for j with $c|_X = \text{id}_X$ and $c(g^{-1}) = c(g)^{-1}$ for all $g \in G$ as before.

Assume that G is topologically principal, and let $E : C_r^*(G, \Sigma) \rightarrow C(X)$ be the faithful conditional expectation. Let τ be a trace on $C_r^*(G, \Sigma)$ and μ the corresponding measure on X . Let (π, H_π) be the GNS representation of τ . Let ν be the measure on G given by $\int_G f d\nu = \int_X \sum_{\gamma \in G^x} f(\gamma) d\mu(x)$. Just as in the proof of Lemma 4.5, we obtain

Lemma 4.8. *The map $C_c(G, \Sigma) \rightarrow C_c(G)$, $f \mapsto f \circ c$ extends to a unitary $H \xrightarrow{\cong} L^2(G, \nu)$.*

Lemma 4.9. *We have $\pi(C(X))'' \cong L^\infty(X, \mu)$.*

Proof. Let U be the unitary $H \xrightarrow{\cong} L^2(G, \nu)$ from Lemma 4.8. It is clear that for $f \in C(X)$, $(U\pi(f)U^*\xi)(\gamma) = f(r(\gamma))\xi(\gamma)$. Since we can cover G by countably many open bisections, we can find a cover of G by countably many pairwise disjoint Borel subsets $S_i \subseteq G$ which are bisections, i.e., $G = \bigsqcup_i S_i$. It is clear that $\nu|_{S_i}$ is the pushforward of $\mu|_{r(S_i)}$ under $(r|_{S_i})^{-1} : r(S_i) \xrightarrow{\cong} S_i$. For $j \in \mathbb{N} \cup \{\infty\} = \{1, 2, \dots\} \cup \{\infty\}$, define $X_j := \{x \in X : |G^x| = j\}$. Clearly, X_j are Borel subsets of G because all the S_i 's are. Define $I_j := \{1, \dots, j\}$ for $j \in \mathbb{N}$ and $I_j := \mathbb{N}$ if $j = \infty$. Then we have a canonical unitary

$$V : L^2(G, \nu) = \bigoplus_i L^2(S_i, \nu) \cong \bigoplus_i L^2(r(S_i), \mu|_{r(S_i)}) \cong \bigoplus_{j \in \mathbb{N} \cup \{\infty\}} L^2(X_j, \mu|_{X_j}) \otimes \ell^2(I_j).$$

Let M_j be the canonical representation of $L^\infty(X_j, \mu|_{X_j})$ on $L^2(X_j, \mu|_{X_j})$. Then for every $f \in C(X)$, $V\pi(f)V^* = (M_j(f) \otimes \text{id}_{\ell^2(I_j)})_{j \in \mathbb{N} \cup \{\infty\}}$. Hence,

$$\pi(C(X))'' = \prod_{j \in \mathbb{N} \cup \{\infty\}} L^\infty(X_j, \mu|_{X_j}) \cong L^\infty(X, \mu).$$

□

Lemma 4.10. *Assume that τ is a faithful trace on $C_r^*(G, \Sigma)$, and that $\pi(C_r^*(G, \Sigma))''$ is a II_1 -factor. Then μ is non-atomic. In particular, $\pi(C(X))''$ is diffuse.*

Proof. Assume that μ is atomic, so that $\mu\{x\} > 0$ for some $x \in X$. As μ is a probability measure on X which is $\mathcal{S}(G)$ -invariant by Lemma 4.1, we deduce that the orbit $O(x) := \{r(\gamma) : \gamma \in G_x\}$ is finite. But that would imply that the characteristic function $1_{O(x)}$ of the orbit of x lies in the center of $\pi(C_r^*(G, \Sigma))''$, which is given by $\mathbb{C}1$ by assumption. Hence $1_{O(x)} = 1$. This means that $X = O(x)$ is finite because τ is faithful. Hence G must be finite, because otherwise G could not be topologically principal. The conclusion is that $\pi(C_r^*(G, \Sigma))$ is finite dimensional and hence $\pi(C_r^*(G, \Sigma))''$ cannot be a II_1 -factor. □

Corollary 4.11. *Let Γ be a countable discrete group. Assume that Γ is torsion-free. Furthermore, suppose that Γ is non-amenable, and that Γ has the CMAP and admits a proper 1-cocycle into an orthogonal representation that is weakly contained in the regular representation. Then $C_\lambda^*(\Gamma)$ has no Cartan subalgebra.*

In particular, for $r \geq 2$, $C_\lambda^*(\mathbb{F}_r)$ has no Cartan subalgebra. More generally, for groups with the property strong $(\text{HH})^+$ in the sense of [45, Definitions 1], the reduced group

C*-algebras have no Cartan subalgebras. We refer the reader to [45, § 2] for concrete examples.

Proof. First of all, our assumptions imply that $C_\lambda^*(\Gamma)$ has unique trace: As Γ is non-amenable, and because Γ has the CMAP and admits a proper 1-cocycle into an orthogonal representation that is weakly contained in the regular representation, we know that Γ has the property strong $(\text{HH})^+$, in the sense of [45, Definitions 1]. Hence, by [45, Corollary 2.2], Γ has no infinite normal amenable subgroups. As Γ is in addition torsion-free, Γ has no non-trivial normal amenable subgroups. In other words, the amenable radical of Γ is trivial. By [6, Theorem 1.3], we conclude that $C_\lambda^*(\Gamma)$ has unique trace.

Write $A = C_\lambda^*(\Gamma)$. Assume that A has a Cartan subalgebra B , and let $E : A \rightarrow B$ be the faithful conditional expectation. Then we can find a groupoid G with compact unit space $X := G^{(0)}$ and a twist Σ of G such that $(A, B) \cong (C_r^*(G, \Sigma), C(X))$. Let μ be the probability measure on X corresponding to the state $\tau|_{C(X)}$. Since $C_\lambda^*(\Gamma)$ has unique trace, we must have $\tau = \tau \circ E$ (see Corollary 4.4). Moreover, let π be the GNS representation for τ , so that $\pi(C_\lambda^*(\Gamma))'' = L\Gamma$. By Lemma 4.9, we have $\pi(B)'' \cong L^\infty(X, \mu)$. Since $C_\lambda^*(\Gamma)$ has unique trace, Γ is icc, so that $L\Gamma$ is a II_1 -factor. Lemma 4.10 implies that $\pi(B)''$ is diffuse. Hence [4, Theorem 3.8] implies that $(s\mathcal{N}_{L\Gamma}(\pi(B)''))''$ is amenable, where

$$s\mathcal{N}_{L\Gamma}(\pi(B)'') = \{x \in L\Gamma : x\pi(B)''x^* \subseteq \pi(B)'' \text{ and } x^*\pi(B)''x \subseteq \pi(B)''\}.$$

At the same time, it is clear that $s\mathcal{N}_{L\Gamma}(\pi(B)'')$ contains $\pi(N_A(B))$, where

$$N_A(B) = \{n \in A : nBn^* \subseteq B \text{ and } n^*Bn \subseteq B\}.$$

Since B is a Cartan subalgebra in A , we know that $N_A(B)$ generates A as a C*-algebra, so that $L\Gamma = \pi(N_A(B))'' = (s\mathcal{N}_{L\Gamma}(\pi(B)''))''$ is amenable. This cannot be true as Γ is not amenable. \square

Question 4.12. Is there a C*-algebraic proof for the observation that for $r \geq 2$, $C_\lambda^*(\mathbb{F}_r)$ has no Cartan subalgebra?

5. NON-UNIQUENESS OF CARTAN SUBALGEBRAS

Now let us present several classes of C*-algebras which have infinitely many pairwise inequivalent Cartan subalgebras.

The following is a consequence of [10]:

Proposition 5.1. *Let A be a unital separable C*-algebra. Assume that A has a Cartan subalgebra B with $\dim \text{Spec}(B) < \infty$. If $A \cong A \otimes \mathcal{Z}$, then A has infinitely many pairwise inequivalent Cartan subalgebras.*

Here \dim stands for covering dimension, and \mathcal{Z} is the Jiang-Su algebra.

Proof. By [10, Theorem 2.9], the Jiang-Su algebra \mathcal{Z} has infinitely many Cartan subalgebras B_1, B_2, \dots with $\dim \text{Spec}(B_i) < \infty$ for all $i = 1, 2, \dots$ and $\lim_{i \rightarrow \infty} \dim \text{Spec} B_i =$

∞ . Then, by [1, Lemma 5.1], $B \otimes B_1, B \otimes B_2, \dots$ are Cartan subalgebras of $A \otimes \mathcal{Z}$. We have

$\dim \operatorname{Spec}(B \otimes B_i) = \dim(\operatorname{Spec}(B) \times \operatorname{Spec}(B_i)) \leq \dim \operatorname{Spec}(B) + \dim \operatorname{Spec}(B_i) < \infty$
for all $i = 1, 2, \dots$. Moreover,

$$\dim \operatorname{Spec}(B \otimes B_i) = \dim(\operatorname{Spec}(B) \times \operatorname{Spec}(B_i)) \geq 1 + \dim \operatorname{Spec}(B_i)$$

by [16, Chapter 2, § 6.3, (18)]. As $\lim_{i \rightarrow \infty} 1 + \dim \operatorname{Spec}(B_i) = \infty$, there must be infinitely many $j \in \mathbb{N}$ such that $\dim \operatorname{Spec}(B \otimes B_j)$ are pairwise distinct. Hence $A \otimes \mathcal{Z}$ contains infinitely many non-isomorphic Cartan subalgebras, and thus, so does A as $A \cong A \otimes \mathcal{Z}$. \square

Examples 5.2. Here are explicit examples where Proposition 5.1 applies:

- Generalized Bunce-Deddens algebras in the sense of [42], i.e., crossed products $C(\bar{\Gamma}) \rtimes_r \Gamma$ attached to odometer actions $\Gamma \curvearrowright \bar{\Gamma}$ of residually finite amenable groups on their profinite completions. \mathcal{Z} -stability follows from [42, 47] and [65, Corollary 6.4].
- Crossed products $C(X) \rtimes_r \Gamma$ attached to free minimal actions of finitely generated nilpotent infinite groups Γ on finite dimensional compact spaces X . \mathcal{Z} -stability follows from [64, 70].
- Examples of C*-algebras attached to equivalence relations constructed recently in [51]. By construction, these C*-algebras have Cartan subalgebras isomorphic to algebras of continuous functions on the Cantor space. It is shown in [51] that these C*-algebras are tracially AF, hence AH algebras of slow dimension growth (by a result of H. Lin, see [57, Theorem 3.3.5]), hence \mathcal{Z} -stable by [70, Last Corollary].

Remark 5.3. It was recently shown in [35] that every unital separable simple C*-algebra with finite nuclear dimension and satisfying the UCT has a Cartan subalgebra with finite dimensional spectrum. Hence every unital classifiable C*-algebra has infinitely many pairwise inequivalent Cartan subalgebras by Proposition 5.1.

Remark 5.4. Since the irrational rotation C*-algebras are \mathcal{Z} -stable, the previous proposition shows that these C*-algebras have infinitely many pairwise inequivalent Cartan subalgebras. A. Kumjian had already shown in his 1980 PhD thesis, published as [29], that these algebras have diagonals whose spectrum is a disjoint union of an arbitrary finite number of circles. He gives an explicit construction of these diagonals. The second author thanks him for explaining it to him. We fix an irrational number α and consider the subgroup $\Gamma = \mathbb{Z} + \alpha\mathbb{Z}$ of the additive group of real numbers \mathbb{R} . The groupoid of the action of Γ on \mathbb{R} by translation can be written a

$$G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \Gamma\}.$$

As a set, it is isomorphic to $\mathbb{R} \times \Gamma$; we equip it with the product topology of $\mathbb{R} \times \Gamma$, where Γ has the discrete topology. Let s be a positive element in Γ . We define a right action of $s\mathbb{Z}$ on G by $(x, y).sk = (x, y + sk)$ and denote by $Z(s) = G/s\mathbb{Z}$ the quotient space. We define a left action of G on $Z(s)$ by $(x, y)(y, z + s\mathbb{Z}) = (x, z + s\mathbb{Z})$. Since the action

of the subgroup $s\mathbb{Z}$ on \mathbb{R} is free and proper, the space $Z(s)$ is a principal G -space. The quotient space $G \backslash Z(s)$ can be identified with $\mathbb{R}/s\mathbb{Z}$ via the map $(x, y) \mapsto y + s\mathbb{Z}$. The construction of Section 3 of [54] produces a C^* -module $E(s) = C^*(Z(s))$ over $C^*(G)$. Its algebra of compact operators $A(s)$ is the C^* -algebra of the groupoid $G \backslash (Z(s) * Z(s))$, which can be identified with the groupoid of the action of the quotient group $\Gamma/s\mathbb{Z}$ on $\mathbb{R}/s\mathbb{Z}$ via the map $((x, y_1 + s\mathbb{Z}), (x, y_2 + s\mathbb{Z})) \mapsto (y_1 + s\mathbb{Z}, y_2 + s\mathbb{Z})$ and where the latter groupoid is written as

$$G(s) = \{(y_1 + s\mathbb{Z}, y_2 + s\mathbb{Z}) : y_1 - y_2 \in \Gamma\} \subset \Gamma/s\mathbb{Z} \times \mathbb{R}/s\mathbb{Z}.$$

It is an irrational rotation C^* -algebra if $s = p + \alpha q$ with (p, q) relatively prime. Otherwise, it is a matrix algebra over an irrational rotation C^* -algebra. The crucial Lemma of Section 9 of [29] constructs an isometry from $E(s + t)$ onto $E(s) \oplus E(t)$ for positive $s, t \in \mathbb{Z} + \alpha\mathbb{Z}$. The existence of such an isometry can also be deduced from the cancellation theorem for projective modules over the irrational rotation C^* -algebras of [56]. Let $B(s)$ be the canonical diagonal of $A(s)$, viewed as the groupoid C^* -algebra $C^*(G \backslash (Z(s) * Z(s)))$: its spectrum is the the unit space of the groupoid, namely the circle $\mathbb{R}/s\mathbb{Z}$. Then $A(s + t)$ has also a diagonal isomorphic to $B(s) \oplus B(t)$. In particular, a suitable decomposition $1 = s + t$ gives that $A(1)$, which is the C^* -algebra of the rotation $2\pi\alpha$, has also a diagonal whose spectrum is the disjoint union of two circles. Repeating the process, one can obtain a diagonal whose spectrum is the disjoint union of an arbitrary finite number of circles.

Remark 5.5. There are many other explicit examples of C^* -algebras with inequivalent Cartan subalgebras. Let us quote some. In [2, Section 7], B. Blackadar constructs an uncountable family of non-conjugate diagonals in the CAR algebra. In [22] (see the second remark after [22, Theorem 2.4]) and in [48], the authors give examples of non-conjugate minimal homeomorphisms (more precisely, their orbital equivalence relations are not isomorphic) having isomorphic crossed products C^* -algebras.

We present another non-uniqueness result for Cartan subalgebras. The difference to the previous result is that this time, we will find infinitely many pairwise isomorphic, but inequivalent Cartan subalgebras.

First of all, we need realizations of UCT Kirchberg algebras as groupoid C^* -algebras. This has been achieved by various authors. In the following, let us formulate a version which is suitable for our purposes. Moreover, we explain how this version follows from [61, 62]. The only difference to [61, 62] is that we want groupoid models (with prescribed additional properties) in the unital case as well.

Lemma 5.6. *For every unital UCT Kirchberg algebra A , there exists an étale second countable locally compact Hausdorff groupoid G such that*

- (i) $C_r^*(G) \cong A$;
- (ii) $G^{(0)}$ is homeomorphic to the Cantor space;
- (iii) G is topologically principal;
- (iv) for every $x \in G^{(0)}$, the stabilizer group G_x^x is free abelian of rank at most 2.

Proof. This is a slight modification of the construction in [61]. Set

$$(G_0, g, G_1) := (K_0(A), [1], K_1(A)).$$

Using [62], find countable, irreducible graphs E_0, E_1, F_0 and F_1 with a (unique) vertex emitting infinitely many edges, together with vertices v in E_0 and w in F_0 such that

- $(K_0(\mathcal{O}(E_0)), [P_v], K_1(\mathcal{O}(E_0))) \cong (G_0, g, \{0\})$;
- $(K_0(\mathcal{O}(F_0)), [P_w], K_1(\mathcal{O}(F_0))) \cong (\mathbb{Z}, 1, \{0\})$;
- $(K_0(\mathcal{O}(E_1)), K_1(\mathcal{O}(E_1))) \cong (G_1, \{0\})$;
- $(K_0(\mathcal{O}(F_1)), K_1(\mathcal{O}(F_1))) \cong (\{0\}, \mathbb{Z})$.

Here \mathcal{O} stands for graph algebra. Now apply the construction in [61]. We obtain a “graph-like” (or rather “2-graph-like”) object Ω out of the 2-graphs $E_0 \times F_0$ and $E_1 \times F_1$. For Ω we have a notion of finite paths, denoted by Ω^* , and a length function $l : \Omega^* \rightarrow \mathbb{N}^2$ (where $\mathbb{N} = \{0, 1, 2, \dots\}$). We also have infinite paths in Ω , and we can equip the set X of infinite paths with a natural topology such that X becomes a totally disconnected, second countable, locally compact Hausdorff space with no isolated points. Moreover, there is an obvious way to concatenate finite paths with infinite paths, denoted by μz for a finite path μ and an infinite path z . We can then define the groupoid \tilde{G} consisting of all triples $(x, n, y) \in X \times \mathbb{Z}^2 \times X$ for which there exist finite paths μ, ν and an infinite path z such that $x = \mu z, y = \nu z$ and $l(\mu) - l(\nu) = n$. Inversion and multiplication are given by $(x, n, y)^{-1} = (y, -n, x)$ and $(x, m, y)(y, n, z) = (x, m + n, z)$. In particular, the unit space of \tilde{G} is given by all triples $(x, 0, x), x \in X$, and is canonically homeomorphic to X . Spielberg shows in [61] that \tilde{G} is topologically principal and that $C_r^*(\tilde{G})$ is a stable UCT Kirchberg algebra and that there is canonical inclusion

$$i : (\mathcal{O}(E_0) \otimes \mathcal{O}(F_0)) \oplus (\mathcal{O}(E_1) \otimes \mathcal{O}(F_1)) \hookrightarrow C_r^*(\tilde{G}).$$

We do not need the precise form of i , but only the following properties:

- (i₁) i induces an isomorphism in K-theory;
- (i₂) for every finite path $\mu \in E_0^*$ in E_0 , $i(S_\mu S_\mu^* \otimes P_w)$ is of the form 1_Y for some compact open subspace $Y \subseteq X$.

In (i₂), S_μ is the partial isometry corresponding to μ in the graph C*-algebra $\mathcal{O}(E_0)$, and 1_Y is the characteristic function of Y , viewed as an element in $C_0(X) \cong C_0(\tilde{G}^{(0)}) \subseteq C_r^*(\tilde{G})$.

Let Z be the compact open subset of X such that $1_Z = i(P_v \otimes P_w)$. Then i induces an isomorphism

$$(K_0(C_r^*(\tilde{G})), [1_Z], K_1(C_r^*(\tilde{G}))) \cong (G_0, g, G_1).$$

Hence the canonical inclusion $C_r^*(\tilde{G}_Z^Z) \cong 1_Z C_r^*(\tilde{G}) 1_Z \hookrightarrow C_r^*(G)$ induces an isomorphism

$$(K_0(C_r^*(\tilde{G}_Z^Z)), [1], K_1(C_r^*(\tilde{G}_Z^Z))) \cong (G_0, g, G_1).$$

Therefore, we may set $G := \tilde{G}_Z^Z$, and the classification theorem due to Kirchberg and Phillips [57, Theorem 8.4.1] yields $C_r^*(G) \cong A$. The unit space of G is obviously given by Z and hence homeomorphic to the Cantor space as Z is totally disconnected, second countable, compact Hausdorff and has no isolated points. Moreover, G has properties (iii) and (iv) because it is equivalent to \tilde{G} which has the same properties. \square

With the help of this lemma, we obtain the desired non-uniqueness result for Cartan subalgebras in unital UCT Kirchberg algebras.

Proposition 5.7. *Every unital UCT Kirchberg algebra has infinitely many pairwise inequivalent Cartan subalgebras, whose spectra are all homeomorphic to the Cantor space.*

Proof. Let A be a unital UCT Kirchberg algebra. By Lemma 5.6, we can find a groupoid G with $G^{(0)}$ homeomorphic to the Cantor space such that $C_r^*(G) \cong A$ and for every $x \in G^{(0)}$, the stabilizer group $G_x^x = \{\gamma \in G : r(\gamma) = s(\gamma) = x\}$ is a free abelian group with $\sup_{x \in G^{(0)}} \text{rk } G_x^x \leq 2$. We set $\text{rk}_{\text{Stab}}(G) := \sup_{x \in G^{(0)}} \text{rk } G_x^x$. In particular, for \mathcal{O}_∞ , we obtain a Cartan subalgebra B_∞ and a groupoid G_∞ such that $(\mathcal{O}_\infty, B_\infty) \cong (C_r^*(G_\infty), C(G_\infty^{(0)}))$. We can choose G_∞ such that $G_\infty^{(0)}$ is homeomorphic to the Cantor space and $\text{rk}_{\text{Stab}}(G_\infty) = 1$ (see for instance [52, Chapter III, § 2]).

For every $i = 1, 2, \dots$, we know that $A \cong A \otimes \mathcal{O}_\infty^{\otimes i}$ because A is purely infinite. By [1, Lemma 5.1], we have $A \otimes \mathcal{O}_\infty^{\otimes i} \cong C_r^*(G \times G_\infty^i)$, where G_∞^i means here the i -fold Cartesian product. In this way, we obtain Cartan subalgebras B_i of A corresponding to $C(G^{(0)} \times (G_\infty^{(0)})^i)$ under the isomorphism $A \cong A \otimes \mathcal{O}_\infty^{\otimes i} \cong C_r^*(G \times G_\infty^i)$. Obviously, $\text{Spec}(B_i)$ is homeomorphic to the Cantor space for all $i = 1, 2, \dots$. As

$$\text{rk}_{\text{Stab}}(G \times G_\infty^i) = \text{rk}_{\text{Stab}}(G) + i \cdot \text{rk}_{\text{Stab}}(G_\infty) = \text{rk}_{\text{Stab}}(G) + i,$$

we know that the groupoids $\{G \times G_\infty^i : i = 1, 2, \dots\}$ must be pairwise non-isomorphic. Hence (A, B_i) , $i = 1, 2, \dots$, must be pairwise inequivalent. \square

An analogous non-uniqueness result for Cartan subalgebras holds for stable UCT Kirchberg algebras. The proof is the same as for Proposition 5.7. It is even simpler as we can directly use the groupoid models from [61, 62].

Proposition 5.8. *Every stable UCT Kirchberg algebra has infinitely many pairwise inequivalent Cartan subalgebras, whose spectra are all homeomorphic to the (up to homeomorphism) unique totally disconnected, second countable, locally compact non-compact Hausdorff space without isolated points.*

Remark 5.9. In [7], many (but not all) UCT Kirchberg algebras have been realized as C^* -algebras of principal – not only topologically principal – groupoids.

6. CARTAN SUBALGEBRAS IN UNIFORM ROE ALGEBRAS

Let us now observe that nuclear uniform Roe algebras have distinguished Cartan subalgebras (see Theorem 6.2). This follows from [60] and [34, § 2.5] (see also [39]). We refer to [60, 34] and the references therein for background material concerning notions from geometric group theory such as quasi-isometry or bilipschitz equivalence.

Let us start with the remark that although in [55], the groupoids are assumed to be second countable, the proof of [55, Proposition 4.13] works without any change for étale Hausdorff locally compact topologically principal groupoids which are σ -compact. Hence, combining this observation with [34, § 2.5], we obtain the following result. Let \mathcal{K} be the C^* -algebra of compact operators on $\ell^2\mathbb{Z}$ and $\mathcal{D} := C_0(\mathbb{Z}) \cong \mathcal{K} \cap \ell^\infty(\mathbb{Z})$.

Lemma 6.1. *Let Γ, Λ be finitely generated groups. The following are equivalent:*

- (QI1) Γ and Λ are quasi-isometric;
- (QI2) $\Gamma \curvearrowright \beta\Gamma$ and $\Lambda \curvearrowright \beta\Lambda$ are Kakutani equivalent, in the sense of [36, 34];
- (QI3) $\Gamma \curvearrowright \beta\Gamma$ and $\Lambda \curvearrowright \beta\Lambda$ are stably continuously orbit equivalent, in the sense of [8, 34];
- (QI4) *There exist projections $p \in C(\beta\Gamma) \subseteq C(\beta\Gamma) \rtimes_r \Gamma$ and $q \in C(\beta\Lambda) \subseteq C(\beta\Lambda) \rtimes_r \Lambda$ which are full in $C(\beta\Gamma) \rtimes_r \Gamma$ and $C(\beta\Lambda) \rtimes_r \Lambda$, respectively, such that*

$$(p(C(\beta\Gamma) \rtimes_r \Gamma)p, pC(\beta\Gamma)p) \cong (q(C(\beta\Lambda) \rtimes_r \Lambda)q, qC(\beta\Lambda)q);$$

- (QI5) $((C(\beta\Gamma) \rtimes_r \Gamma) \otimes \mathcal{K}, C(\beta\Gamma) \otimes \mathcal{D}) \cong ((C(\beta\Lambda) \rtimes_r \Lambda) \otimes \mathcal{K}, C(\beta\Lambda) \otimes \mathcal{D})$.

Moreover, the following are equivalent:

- (LIP1) Γ and Λ are bilipschitz equivalent;
- (LIP2) $\Gamma \curvearrowright \beta\Gamma$ and $\Lambda \curvearrowright \beta\Lambda$ are continuously orbit equivalent, in the sense of [32];
- (LIP3) $(C(\beta\Gamma) \rtimes_r \Gamma, C(\beta\Gamma)) \cong (C(\beta\Lambda) \rtimes_r \Lambda, C(\beta\Lambda))$.

Here, $\Gamma \curvearrowright \beta\Gamma$ and $\Lambda \curvearrowright \beta\Lambda$ are the canonical actions of Γ and Λ on their Stone-Ćech compactifications.

Let us now combine Lemma 6.1 with [60]. First of all, recall that for a finitely generated group Γ , if X_Γ denotes the coarse metric space attached to Γ , then the uniform Roe algebra $C_u^*(X_\Gamma)$ is canonically isomorphic to $C(\beta\Gamma) \rtimes_r \Gamma$, and that the stable uniform Roe algebra $C_s^*(X_\Gamma)$ is canonically isomorphic to $(C(\beta\Gamma) \rtimes_r \Gamma) \otimes \mathcal{K}$. We let $D_u(X_\Gamma)$ be the canonical Cartan subalgebra of $C_u^*(X_\Gamma)$ corresponding to the Cartan subalgebra $C(\beta\Gamma)$ of $C(\beta\Gamma) \rtimes_r \Gamma$. Similarly, $D_s(X_\Gamma)$ denotes the canonical Cartan subalgebra of $C_s^*(X_\Gamma)$ corresponding to the Cartan subalgebra $C(\beta\Gamma) \otimes \mathcal{D}$ of $(C(\beta\Gamma) \rtimes_r \Gamma) \otimes \mathcal{K}$.

Theorem 6.2. *Let Γ, Λ be finitely generated groups. Consider the following statements:*

- (i) Γ and Λ are bilipschitz equivalent;
- (ii) $(C_u^*(X_\Gamma), D_u(X_\Gamma)) \cong (C_u^*(X_\Lambda), D_u(X_\Lambda))$;
- (iii) $C_u^*(X_\Gamma) \cong C_u^*(X_\Lambda)$;
- (iv) Γ and Λ are quasi-isometric;
- (v) *There exist projections $p \in D_u(X_\Gamma) \subseteq C_u^*(X_\Gamma)$ and $q \in D_u(X_\Lambda) \subseteq C_u^*(X_\Lambda)$ which are full in $C_u^*(X_\Gamma)$ and $C_u^*(X_\Lambda)$, respectively, such that*

$$(pC_u^*(X_\Gamma)p, pD_u(X_\Gamma)p) \cong (qC_u^*(X_\Lambda)q, qD_u(X_\Lambda)q);$$

- (vi) $(C_s^*(X_\Gamma), D_s(X_\Gamma)) \cong (C_s^*(X_\Lambda), D_s(X_\Lambda))$;
- (vii) $C_s^*(X_\Gamma) \cong C_s^*(X_\Lambda)$.

We always have (i) \Leftrightarrow (ii) and (iv) \Leftrightarrow (v) \Leftrightarrow (vi). Moreover, (ii) \Rightarrow (iii), (vi) \Rightarrow (vii), (ii) \Rightarrow (v) and (iii) \Rightarrow (vii) are obvious. If Γ and Λ are exact, we have (vii) \Rightarrow (iv), so that (iv) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii). For non-amenable groups Γ and Λ , we have (iv) \Rightarrow (i). Hence for non-amenable, exact groups Γ and Λ , all these items (i) – (vii) are equivalent.

Proof. (i) \Leftrightarrow (ii) and (iv) \Leftrightarrow (v) \Leftrightarrow (vi) follow from Lemma 6.1. For exact groups, (vii) \Rightarrow (iv) is [60, Corollary 6.3]. Moreover, [68] tells us that (iv) \Rightarrow (i) holds for non-amenable groups. \square

Remark 6.3. We know by [14] that for amenable groups, (iv) does not imply (i) in general. It would be interesting to find examples of groups where (v) holds but not (iii), or where (iii) holds but not (ii). We know that for the groups in [14], at least one of the implications (v) \Rightarrow (iii) or (iii) \Rightarrow (ii) must fail. But which one(s)?

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