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A Rough Path Perspective on Renormalization

Y. Bruned, I. Chevyrev, P. K. Friz, and R. Preiß

January 6, 2017

Abstract

We revisit (higher-order) translation operators on rough paths, in both the geometric and branched setting. As in Hairer's work on the renormalization of singular SPDEs we propose a purely algebraic view on the matter. Recent advances in the theory of regularity structures, especially the Hopf algebraic interplay of positive and negative renormalization of Bruned–Hairer–Zambotti (2016), are seen to have precise counterparts in the rough path context, even with a similar formalism (short of polynomial decorations and colourings). Renormalization is then seen to correspond precisely to (higher-order) rough path translation.

Contents

1 Introduction

1.1 Rough paths and regularity structures

The theory of rough paths deals with controlled differential equations of the form

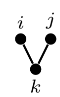
$$dY_t = \sum_{i=1}^{d} f_i\left(Y_t\right) dX_t^i \equiv f\left(Y_t\right) dX_t$$

where $X : [0,T] \to \mathbb{R}^d$ is a continuous path of low, say α-Hölder, regularity for $0 < \alpha \leq 1$. As may be seen by formal Picard iteration, solution can be expanded in terms of certain integrals. Assuming validity of the chain-rule, these are just iterated integrals of the form $\int dX^{i_1} \cdots dX^{i_n}$ with integration over a n-dimensional simplex. In geometric rough path theory one postulates the existence of such integrals, for sufficiently many words $w = (i_1, \ldots, i_n)$, namely $|w| = n \leq [1/\alpha]$, such as to regain analytic control: the collection of resulting objects

$$\langle \mathbf{X}, w \rangle = \int \ldots \int dX^{i_1} \ldots dX^{i_n} \text{ (integration over } s < t_1 < \cdots < t_n < t, \forall 0 \leq s < t \leq T)$$

subject to suitable analytic and algebraic constraints (in particular, *Chen's relation*, which describes the recentering $s \to \tilde{s}$) is then known as a (level-n) *geometric rough path* [Lyo98, LQ02, LCL07, FV10]. Without assuming a chain-rule (think: Itô), iterated integrals of the form $\int X^i X^j dX^k$ appear in the expansion, the resulting objects are then naturally indexed by trees, for example

$$\langle \mathbf{X}, \tau \rangle = \int X^i X^j dX^k \text{ with } \tau = [\bullet_i \bullet_j]_{\bullet_k} \equiv$$



The collection of all such objects, again for sufficiently many trees, $|\tau| = \#\text{nodes} \leq [1/\alpha]$ and subject to algebraic and analytic constraints, form what is known as a *branched rough path* [Gub10, HK15]; every geometric rough path can then be canonically viewed as a branched rough path. A basic result - known as the *extension theorem* [Lyo98, Gub10] - asserts that all "higher" iterated integrals, n-fold with $n > [1/\alpha]$, are automatically well-defined, with validity of all algebraic and analytic constraints in the extended setting.[1] Solving differential equations driven by such rough paths can

[1]This entire ensemble of iterated integrals is called the *signature* or the *fully lifted rough path.*

then be achieved, following [Gub04], see also [FH14], by formulating a fixed point problem in a space of controlled rough paths, essentially a (linear) space of good integrands for rough integration (mind that rough path spaces are, in contrast, fundamentally non-linear due to the afore-mentioned algebraic constraints). Given a rough differential equation (RDE) of the form

$$dY = f(Y)\, d\mathbf{X}$$

it is interesting to see the effect on Y induced by higher-order perturbations on $\mathbf{X}$. For instance, one can use Itô integration to lift a Brownian motion to a (level-2) random rough path, $\mathbf{X} = \mathbf{B}^{\text{Itô}}(\omega)$ of regularity $\alpha \in (1/3, 1/2)$, in which case the above RDE corresponds to a classical Itô SDE. However, we may perturb $\mathbf{B}^{\text{Itô}} = \left(B, \mathbb{B}^{\text{Itô}}\right)$ via $\mathbb{B}^{\text{Itô}}_{s,t} \mapsto \mathbb{B}^{\text{Itô}}_{s,t} + \frac{1}{2} I\,(t-s) =: \mathbb{B}^{\text{Strat}}_{s,t}$, without touching the underlying Brownian path B. The above RDE then becomes a Stratonovich SDE, the effect of this perturbation is then given in terms of an Itô-Stratonovich correction, here of the form $\frac{1}{2}\sum_{i=1}^{d} \nabla_{f_i} f_i$. Examples from physics (Brownian motion in a magnetic field) suggest second order perturbation of the form $\mathbb{B}^{\text{Strat}}_{s,t} \mapsto \mathbb{B}^{\text{Strat}}_{s,t} + a\,(t-s)$, for some $a \in \mathfrak{so}(d)$, the SDE then sees an additional drift vector-field of the form $\sum_{i,j} a^{ij}\,[f_i, f_j]$. All these examples are but the tip of an iceberg: higher order perturbations (which can be traced back to Sussmann's work on ODEs driven by highly oscillatory signals) were studied from a SDE/RDE perspective in [FO09], see also [FV10]. The situation is actually similar for (non-singular, though non-linear) stochastic/rough PDEs: as noted in [CFO11], tampering with higher-levels of the lifted noise affects the structure of the SPDE/RPDE in a way as one would expect from the methods of "rough" characteristics.

From rough paths to regularity structures. The theory of regularity structures [Hai14] extends rough path theory and then provides a remarkable framework to analyse (singular) semi-linear stochastic partial differential equations, e.g. of the form

$$(\partial_t - \Delta)\, u = f(u, Du) + g(u)\, \xi(t, x, \omega)$$

with $(d+1)$-dimensional space-time white noise. The demarche is similar as above: noise is lifted to a *model*, whose algebraic properties (especially with regard to recentering) are formulated with the aid of the *structure group*. Given an (abstract) model, a fixed point problem is solved and gives a solution flow in a (linear) space of *modelled distributions*. The abstract solution can then be mapped ("*reconstructed*") into an actual Schwartz distribution. In fact, one has the rather precise correspondences as follows (see [FH14] for explicit details in the level-2 setting):

rough path	$\longleftrightarrow$	model
Chen's relation	$\longleftrightarrow$	structure group
controlled rough path	$\longleftrightarrow$	modelled distribution
rough integration	$\longleftrightarrow$	reconstruction map

Table 1: Basic correspondences: rough paths $\longleftrightarrow$ regularity structures

Furthermore, one has similar results concerning continuity properties of the solution map as a function of the enhanced noise:

continuity of solution in (rough path $\longleftrightarrow$ model) topology

Unfortunately mollified lifted noise - in infinite dimensions - in general does not converge (as a model), hence *renormalization* plays a fundamental role in regularity structures. The algebraic formalism of how to conduct this renormalization then relies on heavy Hopf algebraic considerations [Hai14], pushed to a (seemingly) definite state of affairs in [BHZ16], see also [Hai16] for a summary. Our investigation was driven by two questions:

(1) *Are there meaningful (finite-dimensional) examples from stochastics which require renormalization?*

(2) *Do we have algebraic structures in rough path theory comparable with those seen in regularity structures?*

In contrast to common belief, perhaps, the answer to question (1) is yes: despite the fact that reasonable approximations to Brownian motion – including piecewise linear - mollifier - Karhunen-Loeve type - and random walk approximations – all converge to the (Stratonovich) Brownian rough path, there are perfectly meaningful (finite-dimensional, and even 1-dimensional!) situations which require renormalization. We sketch these in Section 1.2 below, together with precise references. The main part of this work then deals with, and towards, question (2): Following [BHZ16], the algebraic formalism in regularity structures relies crucially on two Hopf algebra, $\mathcal{T}_+$ and $\mathcal{T}_-$ (which are further in "cointeraction"). The first one helps to construct the structure group which, in turn, provides the recentering ("positive renormalization" in the language of [BHZ16]) and hence constitutes an abstract form of Chen's relation in rough path theory. In this sense, $\mathcal{T}_+$ was always present in rough path theory, the point being enforced in the case of branched rough paths [Gub10, HK15] where $\mathcal{T}_+$ is effectively given by the Connes-Kreimer Hopf algebra. (Making this link precise and explicit seems a worthwhile undertaking in its own right, we accomplish this *en passant* in the sequel.)

Question (2) is then reduced to the question if $\mathcal{T}_-$, built to carry out the actual renormalization of models, and subsequently SPDEs, ("negative renormalization" in the language of [BHZ16]), has any correspondence in rough path theory. Our answer is again affirmative and, very loosely speaking, a main insight of this paper is:

$$\boxed{\text{translation of rough paths} \longleftrightarrow \text{renormalization of models}}$$

Several remarks are in order.

- Much of this paper can be read from a rough path perspective only. That is, despite the looming goal to find a connection to regularity structures, we first develop the algebraic renormalization theory for rough paths in its own right; analytic considerations then take place in Section 5. The link to regularity structures and its renormalization theory is only made in Section 6.

- In applications to (singular) SPDEs, *renormalization* is typically a must-do (with some surprising exceptions such as the motion of a random loop [Hai16]), whereas in typical SDE situations, renormalization/translation is a can-do (and hence better called *translation*). That said, in the next section we present several examples, based on finite- (and even one-) dimensional Brownian motion, which do require genuine renormalization.

- Our approach actually differs from the work of Hairer and coauthors [Hai14, BHZ16], for we are taking a **primal view** which, in the case of branched rough paths, leads us to make crucial use of pre-Lie structures (a concept not present in [BHZ16]). Only after developing the corresponding **dual view**, and further restricting our general setup, do we start to see precise correspondences to [BHZ16]. (In Section 1.3 below we introduce primal resp. dual view in an elementary setting.)

- We have an explicit understanding of the renormalization group in terms of the Lie algebra associated to the Butcher group, equipped with vector space addition. (In particular, despite non-commutativity of the Butcher group, our renormalization group turns out to be abelian, corresponding to addition in the Lie algebra.)

- The existence of a finite-dimensional renormalization group is much related to the stationarity of the (lifted) noise, see [Hai14] and forthcoming work [CH16]. In the rough path context, this amounts to saying that a random (branched) rough path $\mathbf{X} = \mathbf{X}(\omega)$, with values in a (truncated) Butcher (hence finite-dimensional Lie) group $\mathcal{G}$, actually has independent increments with respect to the Grossmann-Larson product $\star$ (dual to the Connes-Kreimer coproduct $\Delta_\star$). In other words, $\mathbf{X}$ is a (continuous) $\mathcal{G}$-valued Lévy process. This invites a comparison with [FS14, Che15] and in Section 4.2 we shall see how Lévy triplets behave under renormalization.

1.2 Higher-order translation and renormalization in finite-dimensions

Physical Brownian motion in a (large) magnetic field. It was shown in [FGL15] that the motion of a charged Brownian particle, in the zero mass limit, in a magnetic field naturally leads to a perturbed second level, of the form $\bar{\mathbb{B}}_{s,t} = \mathbb{B}^{\text{Strat}}_{s,t} + a(t-s)$ for some $0 \neq a \in \mathfrak{so}(d)$. Here a is proportional to the strength of the magnetic field. One can set up the evolution of the physical (finite mass ε) system, with trajectories B^ε, in a way that the magnetic field scales as a power of $1/\varepsilon$, as a method to model magnetic fields which are very large in comparison to the (very small) mass. Doing so leads to approximations B^ε of Brownian motion, whose canonical rough path lifts $\mathbf{B}^\varepsilon$ do not converge in rough path space (due to divergence of the Lévy's area). This can be rectified by replacing $\mathbf{B}^\varepsilon$ with a translated rough path $M_{v^\varepsilon}\mathbf{B}^\varepsilon$, with a suitable second-level perturbation $v^\varepsilon \in \mathfrak{so}(d)$, so that $M_{v^\varepsilon}\mathbf{B}^\varepsilon$ converges in rough path space. Moreover, v^ε can be chosen so that the renormalized limit is indeed $\mathbf{B}^{\text{Strat}}$. Details of this second-level renormalization can be found in [BCF17]. We also point out that higher order renormalization can be expected in the presence of highly oscillatory fields, which also points to some natural connections with homogenization theory.

Higher-order Feynman-Kac theory. Building on works of V. Yu. Krylov, the forthcoming work by Sina Nejad [Nej] constructs higher order analogues to the classical Feynman-Kac formula, which utilises a notion of Lip^γ functions on paths inspired by the work of Lyons-Yang [LY16]. Switching between non-divergence form (Itô) and sum-of-squares (Stratonovich) generators then has an analogue for higher order operators, for which a formulation in terms of our rough path translation operators naturally appear.

Rough stochastic volatility and robust Itô integration. Applications from quantitative finance recently led to the pathwise study of the (1-dimensional) Itô-integral,

$$\int_0^T f(\hat{B}_t)dB_t \text{ with } \hat{B}_t = \int_0^t |t-s|^{H-1/2}\, dB_s$$

where $f : \mathbb{R} \to \mathbb{R}$ is of the form $x \mapsto \exp(\eta x)$. When $H \in (0, 1/2)$, the case relevant in applications, this stochastic integration is singular in the sense that the mollifier approximations actually diverge (infinite Itô-Stratonovich correction, due to infinite quadratic variation of $\hat{B}$ when $H < 1/2$). The integrand $f(B_t^H)$, which plays the role of a stochastic volatility process ($\eta > 0$ is a volatility-of-volatility parameter) is *not* a controlled rough path, nor has the pair $(\hat{B}, B)$ a satisfactory rough path lift (the Itô integral $\int \hat{B} dB$ is well-defined, but $\int B d\hat{B}$ is not). The correct "Itô rough path" in this context is then an $\mathbb{R}^{n+1}$-valued "partial" branched rough path of the form

$$\left(B, \hat{B}, \int \hat{B} d\hat{B}, ... \int \hat{B}^n dB \right)$$

where $n \sim 1/H$. Again, mollifier approximations will diverge but it is possible to see that one can carry out a renormalization which restores convergence to the Itô limit. (Note the similarity with SPDE situations like KPZ.) See the forthcoming work [BFG$^+$] for details.

Fractional delay / Hoff process Viewed as two-dimensional rough paths, Brownian motion and its ε-delay, $t \mapsto (B_t, B_{t-\varepsilon})$, does not converge to (B, B), with - as one may expect - zero area. Instead, the quadratic variation of Brownian motion leads to a rough path limit of the form $(B, B; A)$ with area of order one. It is then possible to check that, replacing B by a fBm with Hurst parameter $H < 1/2$, the same construction will yield exploding Lévy area as $\varepsilon \downarrow 0$. The same phenomena is seen in lead-lag situations, popular in time series analysis. As in the case of physical Brownian motion in a (large) magnetic field, these divergences can be cured by applying suitable (second-level) translation / renormalization operators. See e.g. [FV10, Ch.13], [FHL16] for the Brownian case, [BCF17] for the (singular) fractional case $H < 1/2$.

1.3 Translation of paths: primal vs. dual description

Consider a d-dimensional path X_t, written with respect to the standard basis $e_1, \dots, e_d$ of $\mathbb{R}^d$,

$$X_t = \sum_{i=1}^{d} X_t^i e_i.$$

We are interested in constant speed perturbations, of the form

$$T_v X_t := X_t + tv, \quad \text{with } v = \sum_{i=1}^{d} v^i e_i \in \mathbb{R}^d.$$

In coordinates, $(T_v X_t)^i = X_t^i + tv^i$ for $i = 1, \dots, d$, which is just the *dual point of view*,

$$\langle T_v X, e_i^* \rangle = \langle X_t, e_i^* \rangle + \langle tv, e_i^* \rangle .$$

For a *primal point of view*, take $e_0, e_1, \dots, e_d$ to be the standard basis of $\mathbb{R}^{1+d}$, and consider the $\mathbb{R}^{1+d}$-valued "time-space" path

$$\bar{X}_t = X_t + X_t^0 e_0 = \sum_{i=0}^{d} X_t^i e_i$$

with scalar-valued $X_t^0 \equiv t$. We can now write

$$T_v \bar{X}_t = \bar{X}_t + tv = X_t + X_t^0 \left(e_0 + v \right)$$

which identifies T_v as linear map on $\mathbb{R}^{1+d}$, which maps $e_0 \mapsto e_0 + v$, and $e_i \mapsto e_i$ for $i = 1, \ldots, d$. This is our primal view. We then can (and will) also look at general endomorphisms of the vector space $\mathbb{R}^{1+d}$, which we still write in the form

$$\begin{aligned} e_j &\mapsto e_j + v_j,\ j = 0, \ldots, d \\ v_j &= \sum_{i=0}^{d} v_j^i e_i \in \mathbb{R}^{1+d}. \end{aligned}$$

(The initially discussed case corresponds to $(v_0, v_1, \ldots, v_d) = (v_0, 0, \ldots, 0)$, with $v_0 \perp e_0^*$, and much of the sequel, especially with regard to the dual view, will take advantage of this additional structure.)

We shall be interested to understand how such perturbations propagate to higher level iterated integrals, whenever X has sufficient structure to make this meaningful.

For instance, if $X = B(\omega)$, a d-dimensional Brownian motion, an object of interest would be, with repeated (Stratonovich) integration over $\{(r, s, t) : 0 \leq r \leq s \leq t \leq T\}$,

$$(T_v B)^{ijk} := \int \circ (dB^i + v^i\, dr) \circ (dB^j + v^j\, ds) \circ (dB^k + v^k\, dt) = B_t^{ijk} + \ldots$$

where the omitted terms (dots) can be spelled out in terms of contractions of v (resp. tensor-powers of v) and iterated integrals of $(1 + d)$-dimensional time-space Brownian motion "(t, B)". (Observe that we just gave a dual description of this perturbation, as seen on the third level, yet the initial perturbation took place at the first level: v is vector here.)

There is interest in higher-level perturbations. In particular, given a 2-tensor $v = \sum_{i,j=1}^{d} v^{ij} e_i \otimes e_j$, we can consider the level-2-perturbation with no effect on the first level, i.e., $(T_v B_t)^i \equiv B_t^i$, while for all $i, j = 1, ..., d$,

$$(T_v B_t)^{ij} = B_t^{ij} + v^{ij}\, t$$

For instance, writing $B^{I;w}$ for iterated Itô integrals, in contrast to B^w defined by iterated Stratonovich integration, we have with $v := \frac{1}{2} I^d$ where $(I^d)^{ij} = \delta^{ij}$, i.e., the identity matrix,

$$(T_v B_t)^{I;ij} = B_t^{ij}.$$

This is nothing but a restatement of the familiar formula $\int_0^t B^i \circ dB^j = \int_0^t B^i dB^j + \frac{1}{2}\delta^{ij} t$. It is a non-trivial exercise to understand the Itô-Stratonovich correction at the level of higher iterated integrals, cf. [BA89] and a "branched" version thereof briefly discussed in Section 4.1 below. Further examples were already given in the previous section, notably the case $\bar{B}^{ij} = (T_a B_t)^{ij}$ with an anti-symmetric 2-tensor $a = (a^{ij})$ which arises in the study of Brownian particles in a magnetic field.

1.4 Organization of the paper

This note is organized as follows. In Section 2, we first discuss renormalization/translation in the by now well established setting of geometric rough paths. The algebraic background is found for instance in [Reu93]. We then, in Section 3, move to branched rough paths [Gub10], in the notation and formalism from Hairer-Kelly [HK15]. In Section 4 we illustrate the use of the (branched) translation operator (additional examples were already mentioned in Section 1.2.), while in Section

5 we describe the analytic and algebraic effects of such translations on rough paths and associated RDEs. Lastly, Section 6 is devoted to the systematic comparison of the translation operator and "negative renormalization" introduced in [BHZ16].

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2 Translation of geometric rough paths

We review the algebraic setup for geometric rough paths, as enhancements of $X = (X_0, X_1, ..., X_d)$, a signal with values in $V = \mathbb{R}^{1+d}$. Recall the natural state-space of such rough paths is $T((V))$, a space of tensor series (resp. a suitable truncation thereof related to the regularity of X). Typically $\dot{X} \equiv (\xi_0, \xi_1, ..., \xi_d)$ models noise. Eventually, we will be interested in $X_0(t) = t$, so that X is a time-space (rough) path, though this plays little role in this section.

2.1 Preliminaries for tensor series

We first establish the notation and conventions used throughout the paper. Most algebraic aspects used in this section may be found in [Reu93] and [FV10] Chapter 7.

Throughout the paper we let $\{e_0, e_1, \dots, e_d\}$ be a basis for $\mathbb{R}^{1+d}$. Let

$$(T((\mathbb{R}^{1+d})), \dot{\otimes}, \Delta_{⧢}, \alpha)$$

denote the Hopf algebra of tensor series equipped with tensor product $\dot{\otimes}$, the coproduct $\Delta_{⧢}$ which is dual to to the shuffle product $⧢$ on $T(\mathbb{R}^{1+d})$, and antipode α. Recall that $\Delta_{⧢}$ is explicitly given as the unique algebra morphism such that

$$\Delta_{⧢} : T((\mathbb{R}^{1+d})) \mapsto T((\mathbb{R}^{1+d})) \otimes T((\mathbb{R}^{1+d}))$$
$$\Delta_{⧢} : v \mapsto v \otimes 1 + 1 \otimes v, \quad \forall v \in \mathbb{R}^{1+d}.$$

We shall often refer to elements $e_{i_1} \dot{\otimes} \dots \dot{\otimes} e_{i_k}$ as words consisting of the letters $e_{i_1}, \dots, e_{i_k} \in \{e_0, \dots, e_d\}$, and shall write $e_{i_1,\dots,i_k} = e_{i_1} \dot{\otimes} \dots \dot{\otimes} e_{i_k}$.

We likewise denote by

$$(T(\mathbb{R}^{1+d}), ⧢, \Delta_{\dot{\otimes}}, \tilde{\alpha})$$

the shuffle Hopf algebra. Recall that by identifying $\mathbb{R}^{1+d}$ with its dual through the basis $\{e_0, \dots, e_d\}$, there is a natural duality between $T(\mathbb{R}^{1+d})$ and $T((\mathbb{R}^{1+d}))$ in which $⧢$ is dual to $\Delta_{⧢}$, and $\dot{\otimes}$ is dual to $\Delta_{\dot{\otimes}}$.

We let $G(\mathbb{R}^{1+d})$ and $\mathcal{L}((\mathbb{R}^{1+d}))$ denote the set of group-like and primitive elements of $T((\mathbb{R}^{1+d}))$ respectively. Recall that $\mathcal{L}((\mathbb{R}^{1+d}))$ is precisely the space of Lie series over $\mathbb{R}^{1+d}$, and that

$$G(\mathbb{R}^{1+d}) = \exp_{\dot{\otimes}}(\mathcal{L}((\mathbb{R}^{1+d}))).$$

For any integer $N \geq 0$, we denote by $T^N(\mathbb{R}^{1+d})$ the truncated algebra obtained as the quotient of $T((\mathbb{R}^{1+d}))$ by the ideal consisting of all series with no words of length less than N (we keep in mind that the tensor product is always in place on $T^N(\mathbb{R}^{1+d})$). Similarly we let $G^N(\mathbb{R}^{1+d}) \subset T^N(\mathbb{R}^d)$

and $\mathcal{L}^N(\mathbb{R}^{1+d}) \subset T^N(\mathbb{R}^{1+d})$ denote the step-N free nilpotent Lie group and Lie algebra over $\mathbb{R}^{1+d}$ respectively, constructed in analogous ways.

Finally, we identify $\mathbb{R}^d$ with the subspace of $\mathbb{R}^{1+d}$ with basis $\{e_1, \ldots, e_d\}$. From this identification, we canonically treat all objects discussed above built from $\mathbb{R}^d$ as subsets of their counterparts built from $\mathbb{R}^{1+d}$. For example, we treat the algebra $T((\mathbb{R}^d))$ and Lie algebra $\mathcal{L}^N(\mathbb{R}^d)$ as a subalgebra of $T((\mathbb{R}^{1+d}))$ and a Lie subalgebra of $\mathcal{L}^N(\mathbb{R}^{1+d})$ respectively.

2.2 Translation of tensor series

Definition 1. *For a collection of Lie series $v = (v_0, \ldots, v_d) \subset \mathcal{L}((\mathbb{R}^{1+d}))$, define the algebra morphism $T_v : T((\mathbb{R}^{1+d})) \mapsto T((\mathbb{R}^{1+d}))$ as the unique extension of the linear map*

$$T_v : \mathbb{R}^{1+d} \mapsto \mathcal{L}((\mathbb{R}^{1+d}))$$
$$T_v : e_i \mapsto e_i + v_i, \quad \forall i \in \{0, \ldots, d\}.$$

In the sequel we shall often be concerned with the case that $v_i = 0$ for $i = 1, \ldots, d$ and v_0 takes a special form. We shall make precise whenever such a condition is in place by writing, for example, $v = v_0 \in \mathcal{L}^N(\mathbb{R}^d)$.

We observe the following immediate properties of T_v:

- Since T_v is an algebra morphism which preserves the Lie algebra $\mathcal{L}((\mathbb{R}^{1+d}))$, it holds that T_v maps $G(\mathbb{R}^{1+d})$ into $G(\mathbb{R}^{1+d})$;
- $T_v \circ T_u = T_{v+T_v(u)}$, where we write $T_v(u) := (T_v(u_0), \ldots, T_v(u_d))$. In particular, $T_{v+u} = T_v \circ T_u$ for all $v = v_0, u = u_0 \in \mathcal{L}((\mathbb{R}^d))$;
- For every integer $N \geq 0$, T_v induces a well-defined algebra morphism $T_v^N : T^N(\mathbb{R}^{1+d}) \mapsto T^N(\mathbb{R}^{1+d})$, which furthermore maps $G^N(\mathbb{R}^{1+d})$ into itself.

The following lemma moreover shows that T_v respects the Hopf algebra structure of $T((\mathbb{R}^{1+d}))$.

Lemma 2. *The map $T_v : T((\mathbb{R}^{1+d})) \mapsto T((\mathbb{R}^{1+d}))$ is a Hopf algebra morphism.*

Proof. By construction, T_v respects the product $\dot{\otimes}$. To show that $(T_v \otimes T_v)\Delta_{\sqcup\!\sqcup} = \Delta_{\sqcup\!\sqcup} T_v$, note that both $(T_v \otimes T_v)\Delta_{\sqcup\!\sqcup}$ and $\Delta_{\sqcup\!\sqcup} T_v$ are algebra morphisms, and so they are equal provided they agree on $e_0, \ldots, e_d$. Indeed, we have

$$\Delta_{\sqcup\!\sqcup} T_v(e_i) = \Delta_{\sqcup\!\sqcup}(e_i + v_i) = 1 \otimes (e_i + v_i) + (e_i + v_i) \otimes 1$$

(here we used that each v_i is a Lie element, i.e., primitive in the sense $\Delta_{\sqcup\!\sqcup} v_i = 1 \otimes v_i + v_i \otimes 1$) and

$$(T_v \otimes T_v)\Delta_{\sqcup\!\sqcup}(e_i) = (T_v \otimes T_v)(1 \otimes e_i + e_i \otimes 1) = 1 \otimes (e_i + v_i) + (e_i + v_i) \otimes 1,$$

as required. So far, we have shown T_v is a bialgebra morphism. It remains to show that T_v commutes with the antipode α. Actually, this is implied by general principles ([Pre16] Theorem 2.14), but as it is short to spell out, we give a direct argument: consider the opposite algebra $(T((\mathbb{R}^{1+d})))^{\mathrm{op}}$ (same set and vector space structure as $T((\mathbb{R}^{1+d}))$ but with reverse multiplication). Then $\alpha : T((\mathbb{R}^{1+d})) \mapsto (T((\mathbb{R}^{1+d})))^{\mathrm{op}}$ is an algebra morphism, and again it suffices to check that αT_v and $T_v \alpha$ agree on $e_0, \ldots, e_d$. Indeed, since $v_i \in \mathcal{L}((\mathbb{R}^{1+d}))$, we have $\alpha(v_i) = -v_i$, and thus

$$\alpha T_v(e_i) = \alpha(e_i + v_i) = -e_i - v_i$$

and

$$T_v\alpha(e_i) = T_v(-e_i) = -e_i - v_i.$$

□

2.3 Dual action on the shuffle Hopf algebra $T(\mathbb{R}^{1+d})$

We now wish to describe the dual map $T_v^* : T(\mathbb{R}^{1+d}) \mapsto T(\mathbb{R}^{1+d})$ for which

$$\langle T_v x, y\rangle = \langle x, T_v^* y\rangle, \ \ \forall x \in T((\mathbb{R}^{1+d})), \ \ \forall y \in T(\mathbb{R}^{1+d}).$$

We note immediately that Lemma 2 implies T_v^* is a Hopf algebra morphism from $(T(\mathbb{R}^{1+d}), ⧢, \Delta_{\dot\otimes}, \tilde\alpha)$ to itself.

For simplicity, and as this is the case most relevant to us, we only consider in detail the case $v = v_0 \in \mathcal{L}((\mathbb{R}^{1+d}))$, i.e., $v_i = 0$ for $i = 1, \dots, d$ (but see Remark 4 for a description of the general case).

Let $\mathcal{S}$ denote the unital free commutative algebra generated by the non-empty words $e_{i_1,\dots,i_k} = e_{i_1}\dot\otimes \dots \dot\otimes e_{i_k}$ in $T(\mathbb{R}^{1+d})$. We let $\mathbf{1}$ and $\cdot$ denote the unit element and product of $\mathcal{S}$ respectively. For example,

$$\begin{aligned} e_{0,1}\cdot e_2 = e_2 \cdot e_{0,1} \in \mathcal{S}, \\ e_0 \cdot e_{1,2} \neq e_0 \cdot e_{2,1} \in \mathcal{S}. \end{aligned}$$

For a word $w \in T(\mathbb{R}^{1+d})$, we let $D(w)$ denote the set of all elements

$$w_1 \cdot \ldots \cdot w_k \otimes \tilde w \in \mathcal{S} \otimes T(\mathbb{R}^{1+d})$$

where $w_1, \dots, w_k$ is formed from disjoint subwords of w and $\tilde w$ is the word obtained by replacing every w_i in w with e_0 (note that $\mathbf{1} \otimes w$, corresponding to $k = 0$, is also in $D(w)$).

Consider the linear map $S : T(\mathbb{R}^{1+d}) \mapsto \mathcal{S} \otimes T(\mathbb{R}^{1+d})$ defined for all words $w \in T(\mathbb{R}^{1+d})$ by

$$S(w) = \sum_{w_1\cdot\ldots\cdot w_k\otimes\tilde w \in D(w)} w_1 \cdot \ldots \cdot w_k \otimes \tilde w.$$

For example

$$\begin{aligned} S(e_{0,1,2}) =& \mathbf{1} \otimes (e_{0,1,2,}) \\ &+ (e_0) \otimes (e_{0,1,2}) + (e_1) \otimes (e_{0,0,2}) + (e_2) \otimes (e_{0,1,0}) \\ &+ (e_0 \cdot e_1) \otimes (e_{0,0,2}) + (e_0 \cdot e_2) \otimes (e_{0,1,0}) + (e_1 \cdot e_2) \otimes (e_{0,0,0}) \\ &+ (e_0 \cdot e_1 \cdot e_2) \otimes (e_{0,0,0}) + (e_{0,1}) \otimes (e_{0,2}) + (e_{1,2}) \otimes (e_{0,0}) \\ &+ (e_{0,1} \cdot e_2) \otimes (e_{0,0}) + (e_0 \cdot e_{1,2}) \otimes (e_{0,0}) \\ &+ (e_{0,1,2}) \otimes (e_0). \end{aligned}$$

Proposition 3. *Let $v = v_0 \in \mathcal{L}((\mathbb{R}^{1+d}))$. The dual map $T_v^* : T(\mathbb{R}^{1+d}) \mapsto T(\mathbb{R}^{1+d})$ is given by*

$$T_v^* w = (v \otimes \mathrm{id})S(w),$$

where $v(w_1 \cdot \ldots \cdot w_k) := \langle w_1, v\rangle \ldots \langle w_k, v\rangle$ and $v(\mathbf{1}) := 1$.

In principle, Proposition 3 can be proved algebraically by showing that the adjoint of $\Phi := (v \otimes \mathrm{id})S$ is an algebra morphism from $T((\mathbb{R}^{1+d}))$ to itself, and check that $\Phi^*(e_i) = T_v(e_i)$ for every generator e_i. Indeed this is the method used in Section 3.3 to prove the analogous result for the translation map on branched rough paths. However in the current setting of geometric rough paths, we can provide a direct combinatorial proof.

Proof. Note that the claim is equivalent to showing that for every two words $u, w \in T(\mathbb{R}^{1+d})$ (treating $u \in T((\mathbb{R}^{1+d}))$)

$$\langle T_v u, w \rangle = \sum_{w_1 \cdot \ldots \cdot w_k \otimes \tilde{w} \in D(w)} \langle w_1, v \rangle \ldots \langle w_k, v \rangle \langle \tilde{w}, u \rangle. \tag{1}$$

Consider a word $u = e_{i_1} \dot{\otimes} \ldots \dot{\otimes} e_{i_k} \in T(\mathbb{R}^{1+d})$. Then

$$T_v(u) = e_{i_1} \dot{\otimes} \ldots \dot{\otimes} (e_0 + v) \dot{\otimes} \ldots \dot{\otimes} e_{i_k},$$

where every occurrence of e_0 in u is replaced by $e_0 + v$. We readily deduce that for every $w \in T(\mathbb{R}^{1+d})$

$$\langle T_v(u), w \rangle = \sum_{\substack{w_1 \cdot \ldots \cdot w_k \otimes \tilde{w} \in D(w) \\ u = \tilde{w}}} \langle w_1, v \rangle \ldots \langle w_k, v \rangle. \tag{2}$$

For example, with $v = [e_1, e_2] = e_{1,2} - e_{2,1}$ and $u = e_{0,1,2}$, we have

$$T_v(u) = e_{0,1,2} + e_{1,2,1,2} - e_{2,1,1,2},$$

and we see that indeed for

$$w \in A := \{e_{0,1,2}, e_{1,2,1,2}, e_{2,1,1,2}\},$$

the right hand side of (2) gives $\langle T_v(u), w \rangle$, whilst $\langle w_1, v \rangle \ldots \langle w_k, v \rangle = 0$ for all $w \in T(\mathbb{R}^{1+d}) \setminus A$ and $w_1 \cdot \ldots \cdot w_k \otimes \tilde{w} \in D(w)$ such that $u = \tilde{w}$. But now (2) immediately implies (1). □

Remark 4. A similar result to Proposition 3 holds for the general case $v = (v_0, \ldots, v_d)$. The definition of S changes in the obvious way that in the second tensor, instead of replacing every subword by the letter e_0, one instead replaces every combination of subwords by all combinations of e_i, $i \in \{0, \ldots, d\}$, while in the first tensor, one marks each extracted subword w_j with the corresponding label $i \in \{0, \ldots, d\}$ that replaced it, which gives $(w_j)_i$ (so the left tensor no longer belongs to $\mathcal{S}$ but instead to the free commutative algebra generated by $(w)_i$, for all words $w \in T(\mathbb{R}^{1+d})$ and labels $i \in \{0, \ldots, d\}$). Finally the term $\langle w_1, v \rangle \ldots \langle w_k, v \rangle$ would then be replaced by $\langle (w_1)_{i_1}, v_{i_1} \rangle \ldots \langle (w_k)_{i_k}, v_{i_k} \rangle$.

3 Translation of branched rough paths

In the previous section we studied the translation operator T, in the setting relevant for geometric rough path. Here we extend these results to the branched rough path setting, calling the translation operator M to avoid confusion. Our construction of M faces new difficulties, which we resolve with pre-Lie structures. The dual view then leads us to an extraction procedure of subtrees (a concept familiar from regularity structures, to be explored in Section 6).

3.1 Preliminaries for forest series

As in the preceding section, we first introduce the notation used throughout the section. We mostly follow the notation of Hairer-Kelly [HK15] and refer the reader to [GBVF01] Chapter 14 for relevant algebraic material.

We let $\mathcal{B}$ denote the real vector space spanned by the set of unordered rooted trees with vertices labelled from the set $\{0, \ldots, d\}$. We denote by $\mathcal{B}^*$ its algebraic dual, which we identify with the space of formal series of labelled trees. We canonically identify with $\mathbb{R}^{1+d}$ the subspace of $\mathcal{B}$ (and of $\mathcal{B}^*$) spanned by the trees with a single vertex $\{\bullet_0, \ldots, \bullet_d\}$.

We further denote by $\mathcal{H}$ the vector space spanned by (unordered) forests composed of trees (including the empty forest denoted by 1), and let $\mathcal{H}^*$ denote its algebraic dual which we identify with the space of formal series of forests. We canonically treat $\mathcal{B}^*$ as a subspace of $\mathcal{H}^*$. Following commonly used notation (e.g., [HK15]), for trees $\tau_1, \ldots, \tau_n \in \mathcal{B}$ we let $[\tau_1 \ldots \tau_n]_{\bullet_i} \in \mathcal{B}$ denote the forest $\tau_1 \ldots \tau_n \in \mathcal{H}$ grafted onto the vertex $\bullet_i$.

We equip $\mathcal{H}^*$ with the structure of the Grossman-Larson Hopf algebra

$$(\mathcal{H}^*, \star, \Delta_\odot, \alpha)$$

and $\mathcal{H}$ with the structure of the dual graded Hopf algebra (the Connes-Kreimer Hopf algebra)

$$(\mathcal{H}, \odot, \Delta_\star, \tilde{\alpha}).$$

In other words, $\mathcal{H}$ is the free commutative algebra over $\mathcal{B}$ equipped with a coproduct $\Delta_\star$, and graded by the number of vertices in a forest. We shall often drop the product $\odot$ and simply write $\tau \odot \sigma = \tau\sigma$.

The coproduct $\Delta_\star$ may be described in terms of admissible cuts, for which we use the convention to keep the "trunk" on the right: for every tree $\tau \in \mathcal{B}$

$$\Delta_\star \tau = \sum_c \tau_1^c \ldots \tau_k^c \otimes \tau_0^c,$$

where we sum over all admissible cuts c of τ, and denote by τ_0^c the trunk and by $\tau_1^c \ldots \tau_k^c$ the branches of the cut respectively.

In the sequel, we shall also find it convenient to treat the space $\mathcal{H}$ equipped with $\star$ as a subalgebra of $\mathcal{H}^*$, in which case we explicitly refer to it as the algebra $(\mathcal{H}, \star)$.

Recall that the space of series $\mathcal{B}^*$ is exactly the set of primitive elements of $\mathcal{H}^*$. We let $\mathcal{G}$ denote the group-like elements of $\mathcal{H}^*$, often called the Butcher group, for which it holds that

$$\mathcal{G} = \exp_\star(\mathcal{B}^*).$$

All the objects introduced above play an analogous role to those of the previous section. To summarise this correspondence, it is helpful to keep the following picture in mind [2]

"Primal space" ...	$\mathcal{H}^*$	$\longleftrightarrow$	$T((\mathbb{R}^{1+d}))$
"Dual space" ...	$\mathcal{H}$	$\longleftrightarrow$	$T(\mathbb{R}^{1+d})$
Lie elements ...	$\mathcal{B}^* \subset \mathcal{H}^*$	$\longleftrightarrow$	$\mathcal{L}((\mathbb{R}^{1+d}))$
Group-like elements ...	$\mathcal{G} \subset \mathcal{H}^*$	$\longleftrightarrow$	$G(\mathbb{R}^{1+d})$.

As in the previous section, for any integer $N \geq 0$ we let $\mathcal{H}^N$ denote "truncated" algebra obtained by the quotient of $\mathcal{H}^*$ by the ideal consisting of all series with no forests having less than N vertices (we keep in mind that the product $\star$ is always in place for the truncated objects). Similarly we let $\mathcal{G}^N \subset \mathcal{H}^N$ and $\mathcal{B}^N \subset \mathcal{H}^N$ denote the step-N Butcher Lie group over $\mathbb{R}^{1+d}$ its and Lie algebra respectively, constructed in analogous ways.

Finally, as before, we use $(\mathbb{R}^d)$ to denote the analogous objects built from $\mathbb{R}^d$, and which we treat as subsets of their "full" counterparts built from $\mathbb{R}^{1+d}$ (by identifying $\mathbb{R}^d$ with the subspace of $\mathbb{R}^{1+d}$ with basis $\{e_1, \ldots, e_d\}$). For example, we treat $\mathcal{H}^*(\mathbb{R}^d)$ and $\mathcal{B}^N(\mathbb{R}^d)$ as a subalgebra of $\mathcal{H}^*$ and a Lie subalgebra of $\mathcal{B}^N$ respectively.

3.2 Translation of forest series

3.2.1 Non-uniqueness of algebra extensions

In the previous section, we defined a map T_v which "translated" elements in $T((\mathbb{R}^{1+d}))$ in directions $(v_0, \ldots, v_d) \subset \mathcal{L}((\mathbb{R}^{1+d}))$, and which mapped the set of group-like elements $G(\mathbb{R}^{1+d})$ into itself. In the same spirit, we aim to define a map M_v which translates elements in $\mathcal{H}^*$ in directions $(v_0, \ldots, v_d) \subset \mathcal{B}^*$, and which likewise maps $\mathcal{G}$ into itself.

Note that our construction of T_v relied on the fact that any linear map $M : \mathbb{R}^{1+d} \mapsto T((\mathbb{R}^{1+d}))$ extended uniquely to an algebra morphism $M : T((\mathbb{R}^{1+d})) \mapsto T((\mathbb{R}^{1+d}))$ (for the product $\dot{\otimes}$). We note here that no such universal property holds for $\mathcal{H}^*$; indeed, there exists a canonical injective algebra morphism

$$
\begin{aligned}
\imath &: T((\mathbb{R}^{1+d})) \mapsto \mathcal{H}^* \\
\imath &: e_i \mapsto \bullet_i
\end{aligned}
\tag{3}
$$

which embeds $T((\mathbb{R}^{1+d}))$ into a *strict* subalgebra of $\mathcal{H}^*$.

Specifically, we can see that $\imath$ is injective by considering the space $\mathcal{B}^*_\ell \subset \mathcal{B}^*$ of linear trees, i.e., trees of the form $[\ldots[\bullet_{i_1}]_{\bullet_{i_2}}]\ldots]_{\bullet_{i_k}}$. Then there is a natural projection $\pi_\ell : \mathcal{H}^* \mapsto \mathcal{B}^*_\ell$, and one can readily see that $\pi_\ell \circ \imath$ is a vector space isomorphism (this is the same isomorphism described in Remark 2.7 of [HK15]). To see further that the image of $T((\mathbb{R}^{1+d}))$ under $\imath$ is not all of $\mathcal{H}^*$, it suffices to observe that the linear tree $[\bullet_i]_{\bullet_j}$ is not in the algebra generated by $\{\bullet_i\}_{i=1}^{1+d}$.

[2]The reader may wish for a more unified notation of these spaces. However, it does not appear wise to deviate from established notation in the geometric rough path literature, nor do we want to change notation from [HK15], which is our main source for branched rough paths.

Also, our use of the adjectives "primal" and "dual" is motivated by viewing the rough path state space as primal, see Section 1.3, whereas words/trees are viewed as dual objects. This is somewhat the other way round in [HK15], wherer these spaces were introduced as $\mathcal{H}^*$ and $\mathcal{H}$, respectively. No confusion will arise of this.

Remark 5. The embedding $\imath$ arises naturally in the context of branched rough paths as this is essentially the embedding used in [HK15] to realise geometric rough paths as branched rough paths (though note $\imath$ in [HK15] denotes $\pi_\ell \circ \imath$ in our notation).

Remark 6. While the above argument shows that $(\mathcal{B}, [\cdot,\cdot])$ is clearly not isomorphic to $\mathcal{L}(\mathbb{R}^{1+d})$ as a Lie algebra, it is a curious and non-trivial fact that $(\mathcal{B}, [\cdot,\cdot])$ *is* isomorphic to a free Lie algebra generated by another subspace of $\mathcal{B}$. Correspondingly, $(\mathcal{H}, \star)$, being isomorphic to the universal enveloping algebra of $\mathcal{B}$, is isomorphic to a tensor algebra (see [Foi02] Section 8, or [Cha10]).

It follows form the above discussion that given a map $M : \mathbb{R}^{1+d} \mapsto \mathcal{H}^*$, even one whose range is in $\mathcal{B}^*$, there is in general no canonical choice of how to extend M to elements outside $\imath(T((\mathbb{R}^{1+d})))$ if we only demand that the extension $M : \mathcal{H}^* \mapsto \mathcal{H}^*$ is an algebra morphsim (moreover, without calling on Remark 6, it is *a priori* not even clear that such an extension always exists).

Example 7. *Consider the* $1+d=1$ *(i.e., a single label* 0*), and the map* $M : \{\bullet_0, [\bullet_0]_{\bullet_0}\} \mapsto \mathcal{B}^*$ *given by*

$$M : \bullet_0 \mapsto \bullet_0$$
$$M : [\bullet_0]_{\bullet_0} \mapsto \bullet_0 .$$

Since

$$\bullet_0 \star \bullet_0 = [\bullet_0]_{\bullet_0} + 2\,\bullet_0\,\bullet_0,$$

we may extend M *to an algebra morphism on the truncated space* $\mathcal{H}^2 \mapsto \mathcal{H}^2$ *by setting*

$$M(\bullet_0 \bullet_0) = \frac{1}{2}\left([\bullet_0]_{\bullet_0} + 2\,\bullet_0\,\bullet_0 - \bullet_0\right).$$

This example shows that, on the level of the truncated algebras, there is not a unique algebra morphism above the identity map $\mathrm{id} : \bullet_0 \mapsto \bullet_0$*.*

Of course it is not clear from the above that the identity map $\mathrm{id} : \bullet_0 \mapsto \bullet_0$ *can extend in a non-trivial way to an algebra morphism on all of* $\mathcal{H}^* \mapsto \mathcal{H}^*$*, but such extensions will always exist due to Remark 6.*

In what follows, we address this non-uniqueness issue by demanding a finer structure on the extension of M, namely that $M : \mathcal{B}^* \mapsto \mathcal{B}^*$ is a *pre-Lie algebra* morphism. The notion of a pre-Lie algebra will be recalled in the following subsection, and the significance of preserving the pre-Lie product on $\mathcal{B}^*$ will be made precise in Section 5.2. For now we simply state that this is a natural condition to demand given the role of pre-Lie algebras in control theory and Butcher series [CEFM11, Man11].

3.2.2 The free pre-Lie algebra over $\mathbb{R}^{1+d}$

Definition 8. *A (left)* pre-Lie algebra *is a vector space* V *with a bilinear map* $\triangleright : V \times V \mapsto V$*, called the* pre-Lie product*, such that*

$$(x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = (y \triangleright x) \triangleright z - y \triangleright (x \triangleright z), \quad \forall x, y, z \in V.$$

That is, the associator $(x, y, z) := (x \triangleright y) \triangleright z - x \triangleright (y \triangleright z)$ *is invariant under exchanging* x *and* y*.*

One can readily check that every pre-Lie algebra $(V, \triangleright)$ induced a Lie algebra $(V, [\cdot,\cdot])$ consisting of the same vector space V with bracket $[x, y] := x \triangleright y - y \triangleright x$.

Example 9. *A basic example of a pre-Lie algebra is the space of smooth vector fields on* $\mathbb{R}^e$ *with the product* $(f_i\partial_i) \triangleright (f_j\partial_j) := (f_i\partial_i f_j)\partial_j$. *The induced bracket is the usual Lie bracket of vector fields.*

The space of trees $\mathcal{B}$ can be equipped with a (non-associative) pre-Lie product $\curvearrowright: \mathcal{B} \times \mathcal{B} \mapsto \mathcal{B}$ defined by

$$\tau_1 \curvearrowright \tau_2 = \sum_\tau n(\tau_1, \tau_2, \tau)\tau,$$

where the sum is over all trees $\tau \in \mathcal{B}$ and $n(\tau_1, \tau_2, \tau)$ is the number of *single* admissible cuts of τ for which the branch is τ_1 and the trunk is τ_2. Equivalently, $\curvearrowright$ is given in terms of $\star$ by

$$\tau_1 \curvearrowright \tau_2 = \pi_{\mathcal{B}}(\tau_1 \star \tau_2),$$

where $\pi_{\mathcal{B}} : \mathcal{H} \mapsto \mathcal{B}$ is the projection onto $\mathcal{B}$.

It holds that $(\mathcal{B}, \curvearrowright)$ indeed defines a Lie algebra for which

$$[\tau_1, \tau_2] := \tau_1 \curvearrowright \tau_2 - \tau_2 \curvearrowright \tau_1 = \tau_1 \star \tau_2 - \tau_2 \star \tau_1,$$

i.e., the Lie algebra structures on $\mathcal{B}$ induced by $\star$ and $\curvearrowright$ coincide. Moreover since $\curvearrowright$ respects the grading of $\mathcal{B}$, we can naturally extend $\curvearrowright$ to a bilinear map on the space of series, so that $(\mathcal{B}^*, \curvearrowright)$ is also a pre-Lie algebra.

We now recall the following universal property of $(\mathcal{B}, \curvearrowright)$ first established by Chapoton and Livernet [CL01] Corollary 1.10 (see also [DL02] Theorem 6.3).

Theorem 10. *The space* $(\mathcal{B}, \curvearrowright)$ *is the free pre-Lie algebra over* $\mathbb{R}^{1+d}$.

An equivalent formulation of Theorem 10 is that for any pre-Lie algebra $(V, \triangleright)$ and linear map $M : \mathbb{R}^{1+d} \mapsto V$, there exists a unique extension of M to a pre-Lie algebra morphism $M : (\mathcal{B}, \curvearrowright) \mapsto (V, \triangleright)$.

3.2.3 Construction of the translation map

An immediate consequence of Theorem 10 is the following.

Theorem 11. *Every linear map* $M : \mathbb{R}^{1+d} \mapsto \mathcal{B}^*$ *extends to a unique algebra morphism* $M : \mathcal{H}^* \mapsto \mathcal{H}^*$ *whose restriction to* $\mathcal{B}^*$ *is a pre-Lie algebra morphism from* $\mathcal{B}^*$ *to itself.*

Proof. By Theorem 10, M extends uniquely to a pre-Lie algebra morphism $M : \mathcal{B} \mapsto \mathcal{B}^*$. Recall that for a Lie algebra $\mathfrak{g}$, the universal enveloping algebra $U(\mathfrak{g})$ is the (unique up to isomorphism) algebra such that $\mathfrak{g}$ embeds into $U(\mathfrak{g})$, and for any algebra A, every Lie algebra morphism $f : \mathfrak{g} \mapsto A$ extends uniquely to an algebra morphism $f : U(\mathfrak{g}) \mapsto A$. Recall also that, by the Milnor-Moore theorem, $(\mathcal{H}, \star)$ is isomorphic to the universal enveloping algebra of $(\mathcal{B}, [\cdot, \cdot])$.

It thus follows that M extends further to a unique algebra morphism $M : (\mathcal{H}, \star) \mapsto (\mathcal{H}^*, \star)$. Finally, since M necessarily does not decrease the degree of every element $x \in \mathcal{H}$, we obtain a unique extension $M : \mathcal{H}^* \mapsto \mathcal{H}^*$ for which the restriction $M : \mathcal{B}^* \mapsto \mathcal{B}^*$ is a pre-Lie algebra morphism as desired. □

We can finally define a natural translation map $M_v : \mathcal{H}^* \mapsto \mathcal{H}^*$ analogous to T_v.

Definition 12. *For $v = (v_0, \ldots, v_d) \subset \mathcal{B}^*$, define $M_v : \mathcal{H}^* \mapsto \mathcal{H}^*$ as the unique algebra morphism obtained in Theorem 11 from the linear map*

$$\begin{aligned} &M_v : \mathbb{R}^{1+d} \mapsto \mathcal{B}^* \\ &M_v : \bullet_i \mapsto \bullet_i + v_i, \ \ \forall i \in \{0, \ldots, d\}. \end{aligned}$$

Example 13. *Let us illustrate how the construction works in the case of two nodes with a single label 0. Since M_v is constructed as pre-Lie algebra morphism, we compute*

$$M_v\left([\bullet_0]_{\bullet_0}\right) = M_v\left(\bullet_0 \curvearrowright \bullet_0\right) = M_v\left(\bullet_0\right) \curvearrowright M_v\left(\bullet_0\right) = \left(\bullet_0 + v_0\right) \curvearrowright \left(\bullet_0 + v_0\right).$$

Since M_v is in addition an algebra morphism w.r.t. $\star$ we have

$$\left(\bullet_0 + v_0\right) \star \left(\bullet_0 + v_0\right) = \left(M_v \bullet_0\right) \star \left(M_v \bullet_0\right) = M_v\left(\bullet_0 \star \bullet_0\right) = M_v\left(2\,\bullet_0\,\bullet_0 + [\bullet_0]_{\bullet_0}\right)$$

from which we can uniquely determine $M_v\left(\bullet_0 \bullet_0\right)$.

As in the previous section, we shall often be concerned with the case that $v_i = 0$ for $i = 1, \ldots, d$ and v_0 takes a special form. We again make precise whenever such a condition is in place by writing, for example, $v = v_0 \in \mathcal{B}^N(\mathbb{R}^d)$.

We observe the following immediate properties of M_v, analogous to those of T_v:

- Since M_v is an algebra morphism which preserves the Lie algebra $\mathcal{B}^*$, it holds that M_v maps $\mathcal{G}$ into $\mathcal{G}$;
- $M_v \circ M_u = M_{v+M_v(u)}$, where we write $M_v(u) = (M_v(u_0), \ldots, M_v(u_d))$. In particular, $M_{v+u} = M_v \circ M_u$ for all $v = v_0, u = u_0 \in \mathcal{B}^*(\mathbb{R}^d)$;
- For every integer $N \geq 0$, M_v induces a well-defined algebra morphism $M_v^N : \mathcal{H}^N \mapsto \mathcal{H}^N$, which maps $\mathcal{G}^N$ into $\mathcal{G}^N$;
- Recall the embedding $\imath : T((\mathbb{R}^{1+d})) \mapsto \mathcal{H}^*$ from (3). Then for all $v = (v_0, \ldots, v_d) \subset \mathcal{L}^*$, it holds that $M_{\imath(v)} \circ \imath = \imath \circ T_v$ (as both are morphisms from $T((\mathbb{R}^{1+d}))$ to $\mathcal{H}^*$ which agree on $e_0, \ldots, e_d$).

Lemma 14. *The map $M_v : \mathcal{H}^* \mapsto \mathcal{H}^*$ is a Hopf algebra morphism.*

Remark 15. We note that in the following proof, we only use the fact that M_v is an algebra morphism from $\mathcal{H}^*$ to itself which preserves the space of primitive elements $\mathcal{B}^*$, and so do not directly use the fact that M_v preserves the pre-Lie product of $\mathcal{B}^*$.

Proof. By construction, M_v respects the product $\star$ of $\mathcal{H}^*$. To show that the maps $(M_v \otimes M_v)\Delta_\odot$ and $\Delta_\odot M_v$ agree, note that it suffices to show they agree in $\mathcal{H}$. In turn, their restrictions to $\mathcal{H}$ are algebra morphisms $(\mathcal{H}, \star) \mapsto (\mathcal{H}^*, \star)$, and, since $(\mathcal{H}, \star)$ is the universal enveloping algebra of its space of primitive elements $\mathcal{B}$ by the Milnor-Moore theorem, it suffices to show that

$$(M_v \otimes M_v)\Delta_\odot \tau = \Delta_\odot M_v \tau, \ \ \forall \tau \in \mathcal{B}.$$

But this is immediate since M_v maps $\mathcal{B}^*$ into itself and $M_v(1) = 1$.

We have now shown that M_v is a bialgebra morphism. It now follows from general principles ([Pre16] Theorem 2.14) that M_v also commutes with the antipode (though this also follows from a direct argument identical to the proof of Lemma 2). □

3.3 Dual action on the Connes–Kreimer Hopf algebra $\mathcal{H}$

As in Section 2.3, we now wish to describe the dual map $M_v^* : \mathcal{H} \mapsto \mathcal{H}$ for which

$$\langle M_v x, y \rangle = \langle x, M_v^* y \rangle, \quad \forall x \in \mathcal{H}^*, \quad \forall y \in \mathcal{H}.$$

For simplicity, we again consider in detail only the special case $v_i = 0$ for $i = 1, \dots, d$ (but see Remark 19 for a description of the general case).

Let $\mathcal{A}$ denote the unital free commutative algebra generated by the trees $\tau \in \mathcal{B}$. We let $\mathbf{1}$ and $\cdot$ denote the unit element and product of $\mathcal{A}$ respectively. The algebra $\mathcal{A}$ plays here the same role as the algebra $\mathcal{S}$ in Section 2.3.

Remark 16. Although the algebras $(\mathcal{A}, \cdot)$ and $(\mathcal{H}, \odot)$ are isomorphic, they should be thought of as separate spaces and thus we make a clear distinction between the two.

For a tree $\tau \in \mathcal{B}$, we let $D(\tau)$ denote the set of all elements

$$\tau_1 \cdot \dots \cdot \tau_k \otimes \tilde{\tau} \in \mathcal{A} \otimes \mathcal{B}$$

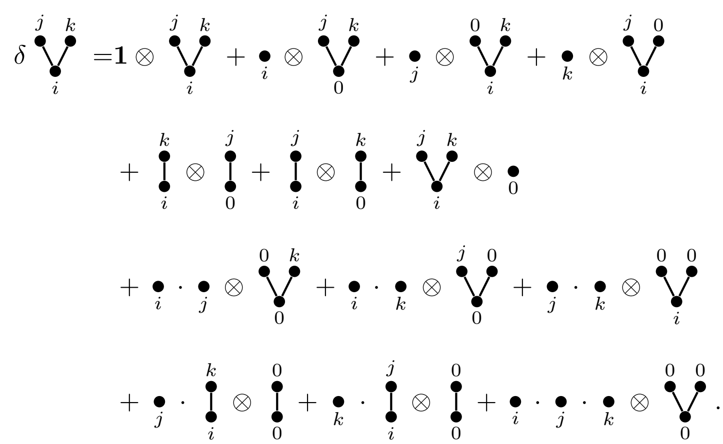
where $\tau_1, \dots, \tau_k$ is formed from all non-empty disjoint collections of subtrees of τ (including subtrees consisting of a single vertex), and $\tilde{\tau}$ is the tree obtained by contracting every subtree τ_i to a single node which is then labelled by 0 (note that $\mathbf{1} \otimes \tau$, corresponding to $k = 0$, is also in $D(\tau)$).

Consider the linear map $\delta : \mathcal{H} \mapsto \mathcal{A} \otimes \mathcal{H}$ defined for all trees $\tau \in \mathcal{B}$ by

$$\delta\tau = \sum_{\tau_1 \cdot \dots \cdot \tau_k \otimes \tilde{\tau} \in D(\tau)} \tau_1 \cdot \dots \cdot \tau_k \otimes \tilde{\tau},$$

and then extended multiplicatively to all of $\mathcal{H}$, where we canonically treat $\mathcal{A} \otimes \mathcal{H}$ as an algebra with multiplication $\mathcal{M}_{\mathcal{A} \otimes \mathcal{H}}(\tau_1 \otimes \hat{\tau}_1 \otimes \tau_2 \otimes \hat{\tau}_2) := (\tau_1 \cdot \tau_2) \otimes (\hat{\tau}_1 \odot \hat{\tau}_2)$ for $\tau_1, \tau_2 \in \mathcal{A}$, $\hat{\tau}_1, \hat{\tau}_2 \in \mathcal{H}$.

For example,



(4)

Proposition 17. *Let $v = v_0 \in \mathcal{B}^*$. The dual map $M_v^* : \mathcal{H} \mapsto \mathcal{H}$ is given by*

$$M_v^* \tau = (v \otimes \mathrm{id}) \circ \delta(\tau),$$

where $v(\tau_1 \cdot \dots \cdot \tau_k) := \langle \tau_1, v \rangle \dots \langle \tau_k, v \rangle$ and $v(\mathbf{1}) := 1$.

For the proof of Proposition 17, we require the following combinatorial lemma. We note that similar "cointeraction" results appear for closely related algebraic structures in [CEFM11, Thm 8] and [BHZ16, Thm 5.37]. We will particularly discuss in further detail the link with the work of [BHZ16] in Section 6.

Lemma 18. *Let* $\curvearrowright^*\colon \mathcal{B} \mapsto \mathcal{B}\otimes\mathcal{B}$ *denote the adjoint of* $\curvearrowright$*. It holds that*

$$(\mathrm{id}\otimes \curvearrowright^*)\delta = \mathcal{M}_{1,3}(\delta\otimes\delta)\curvearrowright^*, \tag{5}$$

where $\mathcal{M}_{1,3} : \mathcal{A}\otimes\mathcal{B}\otimes\mathcal{A}\otimes\mathcal{B} \mapsto \mathcal{A}\otimes\mathcal{B}\otimes\mathcal{B}$ *is the linear map defined by* $\mathcal{M}_{1,3}(\tau_1\otimes\tau_2\otimes\tau_3\otimes\tau_4) = \tau_1\tau_3\otimes\tau_2\otimes\tau_4$*.*

Proof. Note that

$$\curvearrowright^* \tau = \sum_c b_c\otimes\tau_c$$

where the sum runs of all *single* admissible cuts c of τ, and b_c is the branch, τ_c the trunk of c. Consider a single cut c of τ across an edge e. Let τ^c denote the sum of the terms of $(\mathrm{id}\otimes \curvearrowright^*)\delta\tau$ obtained by contracting all collections of subtrees of τ which do not contain e, followed by a cut (on the second tensor) along the edge e (which necessarily remains). One immediately sees that τ^c is equivalently given by first cutting along e, and then contracting along all collections of subtrees of b_c and τ_c, and then grouping the extracted subtrees together, i.e., $\tau^c = \mathcal{M}_{1,3}(\delta\otimes\delta)(b_c\otimes\tau_c)$. It finally remains to observe that summing over all single cuts c gives (5). □

Proof of Proposition 17. Denote by

$$\Phi = (v\otimes\mathrm{id})\circ\delta : \mathcal{B}\mapsto\mathcal{B}.$$

Since M_v is a Hopf algebra morphism by Lemma 14, it follows that so is M_v^*. In particular, it suffices to show that $\Phi\tau = M_v^*\tau$ for every tree $\tau\in\mathcal{B}$.

To this end, observe that Lemma 18 implies $\curvearrowright^* \Phi = (\Phi\otimes\Phi)\curvearrowright^*$, from which it follows that $\Phi^* : \mathcal{B}^*\mapsto\mathcal{B}^*$ is a pre-Lie algebra morphism. Furthermore, for every tree $\tau\in\mathcal{B}$

$$\forall i\in\{1,\dots,d\},\ \langle\Phi^*\bullet_i,\tau\rangle = \langle\bullet_i,\Phi\tau\rangle = \langle\bullet_i,\tau\rangle = \langle M_v\bullet_i,\tau\rangle;$$
$$\langle\Phi^*\bullet_0,\tau\rangle = \langle\bullet_0,\Phi\tau\rangle = \langle\bullet_0,\tau\rangle + \langle v,\tau\rangle = \langle M_v\bullet_0,\tau\rangle.$$

It follows that Φ^* is a pre-Lie algebra morphism on $(\mathcal{B}^*,\curvearrowright)$ which agrees with M_v on the set $\{\bullet_0,\dots,\bullet_d\}\subset\mathcal{B}^*$. Hence, by the universal property of $(\mathcal{B},\curvearrowright)$ (Theorem 10), Φ^* agrees with M_v on all of $\mathcal{B}^*$, which concludes the proof. □

Remark 19. A similar result to Proposition 17 holds for the general case $v = (v_0,\dots,v_d)$. The definition of δ changes in the obvious way that in the second tensor, instead of replacing every subtree by the node $\bullet_0$, one instead replaces every combination of subtrees by all combinations of $\bullet_i$, $i\in\{0,\dots,d\}$, while in the first tensor, one marks each extracted subtree τ_j with the corresponding label $i\in\{0,\dots,d\}$ that replaced it, which gives $(\tau_j)_i$ (so the left tensor no longer belongs to $\mathcal{A}$ but instead to the free commutative algebra generated by $(\tau)_i$, for all trees $\tau\in\mathcal{B}$ and labels $i\in\{0,\dots,d\}$). Finally the term $\langle\tau_1,v\rangle\dots\langle\tau_k,v\rangle$ would then be replaced by $\langle(\tau_1)_{i_1},v_{i_1}\rangle\dots\langle(\tau_k)_{i_k},v_{i_k}\rangle$.

4 Examples

4.1 Itô-Stratonovich conversion

As an application of Proposition 17, we illustrate how to re-express iterated Stratonovich integrals (and products thereof) over some interval $[s,t]$ as Itô integrals. Consider the $\mathbb{R}^{1+d}$-valued process $B_t = (B_t^0, B_t^1, \dots, B_t^d)$, where $(B_t^1, \dots, B_t^d)$ is a standard $\mathbb{R}^d$-valued Brownian motion, and $B_t^0 \equiv t$ denotes the time component. Let $\mathbf{B}^{\mathrm{Strat}}$ denote the enhancement of B_t to an α-Hölder branched rough path, $\alpha \in (0, 1/2)$, using Stratonovich iterated integrals. For example,

$$\begin{aligned} \left\langle \mathbf{B}^{\mathrm{Strat}}_{s,t}, \tau \right\rangle &= \int \cdots \int_{s<t_1<\cdots<t_m<t} \circ\, dB^{i_1}_{t_1} \circ \cdots \circ dB^{i_m}_{t_m} \qquad (6) \\ \text{for the linear tree } \tau &= [\dots[\bullet_{i_1}]_{\bullet_{i_2}} \dots]_{\bullet_{i_m}}, \ \ i_1, \dots, i_m \in \{0, \dots, d\}, \end{aligned}$$

and

$$\begin{aligned} \left\langle \mathbf{B}^{\mathrm{Strat}}_{s,t}, \tau \right\rangle &= \int_s^t B^j_u B^k_u \circ dB^i_u \\ \text{for } \tau &= [\bullet_j \bullet_k]_{\bullet_i}, \ \ i, j, k \in \{0, \dots, d\}. \end{aligned}$$

Similarly, we define $\mathbf{B}^{\mathrm{Itô}}$ in exactly the same way using Itô integrals.

Remark 20. In view of the Hölder regularity, only the first 2 levels of $\mathbf{B}^{\mathrm{Strat}}$, call this $\mathbf{B}^{(2);\mathrm{Strat}}$, need to be constructed by stochastic integration. The "full" $T((\mathbb{R}^d))$-valued (geometric) rough paths $\mathbf{B}^{\mathrm{Strat}}$ is then obtained from $\mathbf{B}^{(2);\mathrm{Strat}}$ via the extension theorem in rough path theory. Strictly speaking, however, one needs to verify that the so supplied rough path, on $[s,t]$ and evaluated against τ, for $|\tau| > 2$, agrees a.s. with the object defined by iterated Stratonovich integration in (6). This can done by showing $\left\langle \mathbf{B}^{\mathrm{Strat}}_{s,t}, \tau \right\rangle \lesssim |t-s|^{\alpha|\tau|}$ a.s., together with appealing to the uniqueness part of the extension theorem. [3]

For a tree $\tau \in \mathcal{B}$, recall the definition of $D(\tau) \subset \mathcal{A} \otimes \mathcal{B}$ from Section 3.3 (which was used to define δ). Consider the subset $\hat{D}(\tau) \subseteq D(\tau)$ containing $\mathbf{1} \otimes \tau$ and all $\tau_1 \cdot \ldots \cdot \tau_k \otimes \tilde{\tau} \in D(\tau)$ for which $\tau_j \in \{[\bullet_1]_{\bullet_1}, \dots, [\bullet_d]_{\bullet_d}\}$ for all $1 \le j \le k$.

Proposition 21. *For every tree $\tau \in \mathcal{B}$ it holds that*

$$\langle \mathbf{B}^{\mathrm{Strat}}_{s,t}, \tau \rangle = \sum_{\tau_1 \cdot \ldots \cdot \tau_k \otimes \tilde{\tau} \in \hat{D}(\tau)} \left(\frac{1}{2}\right)^k \langle \mathbf{B}^{\mathrm{Itô}}_{s,t}, \tilde{\tau} \rangle. \qquad (7)$$

Proof. Consider the sum of linear trees $v = v_0 = \frac{1}{2}\sum_{i=1}^d [\bullet_i]_{\bullet_i} \in \mathcal{B}^2(\mathbb{R}^d)$. One can readily verify that $\mathbf{B}^{\mathrm{Strat}} = M_v(\mathbf{B}^{\mathrm{Itô}})$, understood in the pointwise sense $\mathbf{B}^{\mathrm{Strat}}_{s,t} = M_v(\mathbf{B}^{\mathrm{Itô}}_{s,t})$. Indeed, both $\mathbf{B}^{\mathrm{Strat}}$ and $M_v(\mathbf{B}^{\mathrm{Itô}})$ are a.s. "full" α-Hölder rough path, where this fact - in the case of $M_v(\mathbf{B}^{\mathrm{Itô}})$ - either requires an (easy) check by hand, or an appeal to Theorem 28, (ii), below. Since, by construction, both agree on the first two levels, and $\alpha \in (1/2, 1/3)$, we see that $\mathbf{B}^{\mathrm{Strat}}$ and $M_v(\mathbf{B}^{\mathrm{Itô}})$ must be equal, a.s., thanks to the uniqueness part of the extension theorem.

[3] In the geometric rough path case only, one can give an argument based on the fact that the full rough path is the solution to a linear RDE, knowing that solutions to RDEs driven by Stratonovich Brownian rough paths are solutions to the corresponding Stratonovich SDEs. No such argument works in the branched case.

It then follows by Proposition 17 that

$$\langle \mathbf{B}^{\mathrm{Strat}}_{s,t}, \tau\rangle = \langle \mathbf{B}^{\mathrm{Itô}}_{s,t}, M_v^*\tau\rangle = \sum_{\tau_1\cdot\ldots\cdot\tau_k\otimes\tilde\tau\in D(\tau)} \langle \mathbf{B}^{\mathrm{Itô}}_{s,t}, \langle v,\tau_1\rangle\ldots\langle v,\tau_k\rangle\tilde\tau\rangle.$$

Since $\langle v, \mathbf{1}\rangle = 1$, while $\langle v,\tau_j\rangle = 1/2$ if $\tau_j \in \{[\bullet_1]_{\bullet_1},\ldots,[\bullet_d]_{\bullet_d}\}$ and zero otherwise, we obtain precisely (7). □

Example 22. *Consider the case $\tau = [\bullet_j\bullet_k]_{\bullet_i}$ so that*

$$\langle \mathbf{B}^{\mathrm{Strat}}_{s,t}, \tau\rangle = \int_s^t B^j_u B^k_u \circ dB^i_u.$$

Recalling the definition of $\hat D(\tau)$ and the explicit form of $\delta\tau$ in (4)*, we see that if i is distinct from both j,k, then only $\mathbf{1}\otimes\tau$ remains in $\hat D(\tau)$, and so (in trivial agreement with stochastic calculus)*

$$\langle \mathbf{B}^{\mathrm{Strat}}_{s,t}, \tau\rangle = \langle \mathbf{B}^{\mathrm{Itô}}_{s,t}, \tau\rangle.$$

On the other hand,if $i = j \neq k$, an additional term $[\bullet_i]_{\bullet_i}\otimes[\bullet_k]_{\bullet_0}$ appears in $\hat D(\tau)$, and so

$$\begin{aligned}\langle \mathbf{B}^{\mathrm{Strat}}_{s,t}, \tau\rangle &= \langle \mathbf{B}^{\mathrm{Itô}}_{s,t}, \tau\rangle + \frac{1}{2}\iint_{s<t_1<t_2<t} dB^k_{t_1} dB^0_{t_2}\\ &= \langle \mathbf{B}^{\mathrm{Itô}}_{s,t}, \tau\rangle + \frac{1}{2}\int_s^t B^k_u du.\end{aligned}$$

The case $i = k \neq j$ is identical. At last, in the case $i = j = k$, looking at $\delta\tau$ shows that

$$\begin{aligned}\langle \mathbf{B}^{\mathrm{Strat}}, \tau\rangle &= \langle \mathbf{B}^{\mathrm{Itô}}, \tau\rangle + \frac{1}{2}\iint_{s<t_1<t_2<t} dB^i_{t_1} dB^0_{t_2} + \frac{1}{2}\iint_{s<t_1<t_2<t} dB^i_{t_1} dB^0_{t_2}\\ &= \langle \mathbf{B}^{\mathrm{Itô}}, \tau\rangle + \int_s^t B^i_u du.\end{aligned}$$

Remark 23. When $\tau = [\ldots[\bullet_{i_1}]_{\bullet_{i_2}}\ldots]_{\bullet_{i_m}}$ is a linear tree, this is in agreement with [BA89] Proposition 1. In fact, by considering general semi-martingales $X^1_t,\ldots,X^d_t$ and adding extra labels $\bullet_{i,j}$, $1 \leq i \leq j \leq d$ (thus increasing the underlying dimension from d to $d + d(d+1)/2$) to encode the quadratic variants $[X_i, X_j]$, the above procedure (in the more general setting with elements $v_{ij} = [\bullet_i]_{\bullet_j} \in \mathcal{B}^2(\mathbb{R}^d)$, see Remark 19), immediately provides an Itô-Stratonovich conversion formula for general semi-martingales.

4.2 Lévy rough paths

Note that the example in the previous section can be viewed as follows: $\mathbf{B}^{\mathrm{Itô}}$ and $\mathbf{B}^{\mathrm{Strat}}$ are both $\mathcal{G}^2$-valued Lévy processes which are branched p-rough paths, $2 < p < 3$, and one can recover the signature of one from the other by a suitable (deterministic) translation map $M_v : \mathcal{G} \mapsto \mathcal{G}$. We now consider a generalisation of this setting to arbitrary $\mathcal{G}^N$-valued Lévy processes, which have already been studied in the context of rough paths in [FS14, Che15].

Let $\tau_1, \ldots, \tau_m$ be a basis for $\mathcal{B}^N$ consisting of trees, which we identify with left-invariant vector fields on $\mathcal{G}^N$, where we suppose for convenience that $\tau_1 = \bullet_0$. Recall that $\mathcal{G}^N$ is a homogenous group in the sense of [FS82] (cf. [HK15] Remark 2.15).

Recall that to every (left) Lévy process $\mathbf{X}$ in $\mathcal{G}^N$ without jumps and with identity starting point (i.e., $\mathbf{X}_0 = 1_{\mathcal{G}^N}$ a.s.) there is an associated Lévy triplet $(A, B, 0)$, where $B = \sum_{i=1}^m B^i \tau_i$ is an element of $\mathcal{B}^N$ and $(A^{i,j})_{i,j=1}^m$ is a correlation matrix. Then the generator of $\mathbf{X}$ is given for all $f \in C_0^2(\mathcal{G}^N)$ by (see, e.g., [Lia04])

$$\lim_{t \to 0} t^{-1} \mathbb{E}\left[f(x \star \mathbf{X}_t) - f(x)\right] = \sum_{i=1}^m B^i (\tau_i f)(x) + \frac{1}{2} \sum_{i,j=1}^m A^{i,j} (\tau_i \tau_j f)(x).$$

Remark 24. We have assumed here that $\mathbf{X}$ is without jumps only for simplicty. Indeed, one can treat any $\mathcal{G}^N$-valued càdlàg process $\mathbf{X}$ of finite p-variation (in th rough path sense) as a branched p-rough path (albeit in general non-canonically) using the notion of a path function [Che15].

Lemma 25. *Let $M : \mathcal{H}^N \mapsto \mathcal{H}^N$ be an algebra morphism which preserves $\mathcal{G}^N$ and $\mathbf{X}$ a Lévy process in $\mathcal{G}^N$ with Lévy triplet $(A, B, 0)$.*

Then $M(\mathbf{X})$ is the (unique in law) $\mathcal{G}^N$-valued (left) Lévy process with generator given for all $f \in C_0^2(\mathcal{G}^N)$ by

$$\lim_{t \to 0} t^{-1} \mathbb{E}\left[f(x \star M\mathbf{X}_t) - f(x)\right] = \sum_{i=1}^m B^i (M\tau_i f)(x) + \frac{1}{2} \sum_{i,j=1}^m A^{i,j} (M\tau_i M\tau_j f)(x). \tag{8}$$

Proof. The fact that $M\mathbf{X}$ is a Lévy process is immediate from the fact that $\mathbf{X}$ is a Lévy process and that $M : \mathcal{G}^N \mapsto \mathcal{G}^N$ is a (continuous) group morphism. It thus only remains to show (8), where we may suppose without loss of generality that $x = 1_{\mathcal{G}^N}$. To this end, define $h = f \circ M$ and observe that

$$\lim_{t \to 0} t^{-1} \mathbb{E}\left[f(M\mathbf{X}_t) - f(1_{\mathcal{G}^N})\right] = \sum_{i=1}^m B^i (\tau_i h)(1_{\mathcal{G}^N}) + \frac{1}{2} \sum_{i,j=1}^m A^{i,j} (\tau_i \tau_j h)(1_{\mathcal{G}^N})$$

(note that in general h might fail to decay at infinity and thus not be an element of $C_0^2(\mathcal{G}^N)$, however the above limit is readily justified by taking suitable approximations). Using the fact that $(\tau h)(x) = \frac{d}{dt} h(x \star e^{t\tau}) \mid_{t=0}$, one can easily verify that for all $\sigma, \tau \in \mathcal{B}^N$ and $x \in \mathcal{G}^N$

$$(\tau h)(x) = (M\tau f)(Mx),$$
$$(\sigma\tau h)(x) = ((M\sigma)(M\tau) f)(Mx),$$

from which (8) follows. □

We now specialise to the case that $(A^{i,j})_{i,j=1}^m$ is a correlation matrix for which $A^{i,i} = 0$ whenever τ_i has more than $\lfloor N/2 \rfloor$ nodes, which is a necessary and sufficient condition for the sample paths of $\mathbf{X}$ to a.s. have finite p-variation for all $N < p < N + 1$ [Che15]. Assume also that $A^{i,i} = 0$ whenever τ_i contains a node with label 0, and that $B = \tau_1 = \bullet_0$, so that for all $f \in C_0^2(\mathcal{G}^N)$

$$\lim_{t \to 0} t^{-1} \mathbb{E}\left[f(x \star \mathbf{X}_t) - f(x)\right] = (\tau_1 f)(x) + \frac{1}{2} \sum_{i,j=1}^m A^{i,j} (\tau_i \tau_j f)(x).$$

The drift term $(\tau_1 f)(x)$ should be interpreted as the time component of the branched rough path $\mathbf{X}$ (which also explains the zero-diffusion condition in the direction of trees with a label 0).

Any other $\mathcal{G}^N$-valued Lévy process $\tilde{\mathbf{X}}$ without jumps and the same correlation matrix $(A^{i,j})_{i,j=1}^m$ is also a branched p-rough, and its generator differs from that of $\mathbf{X}$ only by a drift term. As a consequence of Lemma 25, we see that every such $\tilde{\mathbf{X}}$ can be constructed by applying a (deterministic) translation map M_v to $\mathbf{X}$. In particular, the full signature of $\tilde{\mathbf{X}}$ can be recovered from that of $\mathbf{X}$, generalising the example from Section 4.1.

Corollary 26. *Let $v = v_0 \in \mathcal{B}^N$ and $M_v : \mathcal{H}^N \mapsto \mathcal{H}^N$ the truncation of the translation map from Section 3.2.3.*

Then $M_v(\mathbf{X})$ is the (unique in law) $\mathcal{G}^N$-valued (left) Lévy process with generator given for all $f \in C_0^2(\mathcal{G}^N)$ by

$$\lim_{t\to 0} t^{-1}\mathbb{E}\left[f(x \star M_v(\mathbf{X}_t)) - f(x)\right] = (\bullet_0 + v)f(x) + \frac{1}{2}\sum_{i,j=1}^{m} A^{i,j}(\tau_i\tau_j f)(x).$$

Remark 27. The statement of the corollary likewise holds for every algebra morphism $M : \mathcal{H}^N \mapsto \mathcal{H}^N$ satisfying $M\bullet_0 = \bullet_0 + v$ and $M\tau = \tau$ for all forests $\tau \in \mathcal{H}^N$ without a label 0, which is a manifestation of the final point of the upcoming Theorem 28 (ii).

5 Rough differential equations

5.1 Translated rough paths are rough paths

We now show that the maps T_v and M_v act on the spaces of weakly geometric and branched rough paths. Throughout, we regard these rough paths as fully lifted, as can always (and uniquely) be done thanks to the extension theorem. The action of our translation operator is then pointwise, i.e.

$$(M_v\mathbf{X})_{s,t} := M_v(\mathbf{X}_{s,t}),$$

and similarly for the geometric rough path translation operator T. In the following, we let $|w|$ denotes the length of a word $w \in T(\mathbb{R}^{1+d})$ (resp. number of nodes in a forest $w \in \mathcal{H}$), and equip the space of α-Hölder weakly geometric (resp. branched) rough paths with the inhomogeneous Hölder norm

$$||\mathbf{X}||_{\alpha\text{-Höl};[s,t]} = \max_{|w|\le\lfloor 1/\alpha\rfloor} \sup_{u\neq v\in[s,t]} \frac{|\langle \mathbf{X}_{u,v}, w\rangle|}{|v-u|^{|w|\alpha}},$$

where the max runs over all words $w \in T(\mathbb{R}^{1+d})$ (resp. forests $w \in \mathcal{H}$) with $|w| \le \lfloor 1/\alpha \rfloor$.

Theorem 28. *Let $\alpha \in (0,1]$ and $\mathbf{X}$ a α-Hölder weakly geometric (resp. branched) rough path over $\mathbb{R}^{1+d}$*

(i) *Let $v = (v_0, v_1 \dots, v_d)$ be a collection of elements in $\mathcal{L}^N(\mathbb{R}^{1+d})$ (resp. in $\mathcal{B}^N$).*

Then $T_v\mathbf{X}$ (resp. $M_v\mathbf{X}$) is a α/N-Hölder weakly geometric (resp. branched) rough path satisfying

$$||T_v\mathbf{X}||_{\alpha/N\text{-Höl};[s,t]} \ (\text{resp. } ||M_v\mathbf{X}||_{\alpha/N\text{-Höl};[s,t]}) \ \le C_v \,||\mathbf{X}||_{\alpha\text{-Höl};[s,t]} \tag{9}$$

for a constant C_v depending polynomially on v.

(ii) *Let* $v = (v_0, 0, \dots 0)$ *for* $v_0 \in \mathcal{L}^N\left(\mathbb{R}^{1+d}\right)$ *(resp.* $v_0 \in \mathcal{B}^N$*). Suppose that* $\mathbf{X}$ *satisfies*

$$||\mathbf{X}||_{(1,\alpha)\text{-Höl};[s,t]} := \max_{|w|\le\lfloor 1/\alpha\rfloor} \sup_{u\neq v\in[s,t]} \frac{|\langle \mathbf{X}_{u,v}, w\rangle|}{|v-u|^{(1-\alpha)|w|_0+\alpha|w|}} < \infty, \tag{10}$$

where the max *runs over all words* $w \in T(\mathbb{R}^{1+d})$ *(resp. forests* $w \in \mathcal{H}$*) with* $|w| \le \lfloor 1/\alpha \rfloor$ *and* $|w|_0$ *denotes the number of times the letter* e_0 *(resp. label* 0*) appears in* w*.*

Then $T_v\mathbf{X}$ *(resp.* $M_v\mathbf{X}$*) is a* $\alpha \wedge (1/N)$*-Hölder weakly geometric (resp. branched) rough path over* $\mathbb{R}^{1+d}$ *satisfying*

$$||T_v\mathbf{X}||_{\alpha\wedge(1/N)\text{-Höl};[s,t]} \ \left(\text{resp. } ||M_v\mathbf{X}||_{\alpha\wedge(1/N)\text{-Höl};[s,t]}\right) \le C_v\, ||\mathbf{X}||_{(1,\alpha)\text{-Höl};[s,t]}$$

for a constant C_v *depending polynomially on* v*.*

Finally, in the setting of branched rough paths, let $M : \mathcal{H}^* \mapsto \mathcal{H}^*$ *be any algebra morphism which preserves* $\mathcal{G}$ *and such that* $M\tau = \tau$ *for every forest* $\tau \in \mathcal{H}$ *without a label* 0*, and* $M\bullet_0 = M_v\bullet_0 = \bullet_0 + v_0$*. Then* $M\mathbf{X} = M_v\mathbf{X}$*.*

Before the proof of the theorem, several remarks are in order.

Remark 29. In Theorem 28 we treat α-Hölder weakly geometric rough paths as already enhanced with their iterated integrals. Thus $\mathbf{X}_{s,t}$ is an element of $T((\mathbb{R}^{1+d}))$ and $(T_v\mathbf{X})_{s,t}$ is just the image of $\mathbf{X}_{s,t}$ under T_v. Therefore the statement of the proposition is that not only does $(T_v\mathbf{X})_{s,t}$ have the correct regularity on the first $n = \lfloor 1/\alpha \rfloor$ levels to qualify as a rough path but that all further iterated integrals are already given, in a purely algebraic way, by $(T_v\mathbf{X})$. That said, if one takes the level-n view, writing $\pi_n(T_v\mathbf{X})$ for the translation only defined as a level-n rough path, the extension theorem asserts that there is a unique full rough path lift, say $\mathbf{Z}$. But then, by the uniqueness part of the extension theorem, $\mathbf{Z} = T_v\mathbf{X}$ so that our construction is compatible with the rough path extension.

The same remark applies to branched rough paths, where we recall that, as a particular consequence of the sewing lemma, every α-Hölder branched rough paths admits a unique lift (extension) to all of $\mathcal{H}^*$ ([Gub10] Theorem 7.3, or [HK15] p.223). We would also like to point out that Boedihardjo [Boe15] recently extended a result on the factorial decay of lifts of geometric rough paths (first shown in [Lyo98]) to the branched setting, answering a conjecture in [Gub10].

Remark 30. In the case of geometric rough paths the previous remark points to an alternative (analytic) construction of the translation operator, first defined on a smooth path X identified with its full lift $X \equiv (1, X^1, X^2, ...)$, and subsequently extended to geometric rough paths by continuity. We stick to the case of one Lie polynomial $v_0 = v = (v^1, v^2, ...v^N)$ which we want to add at constant speed to X. At level 1, obviously $(T_vX)^1_{s,t} = X^1_{s,t} + (t-s)v^1$ and (T_vX) is a Lipschitz path (a 1-rough path). We then perturb the canonically obtained (extended) 2-rough path which in turn we can perturb on the second level by adding $(t-s)v^2$, thereby obtaining a (non-canonical) 2-rough path. Iterating this construction allows us to "feed in, level-by-level" the perturbation v until we arrive at a rough path $T_v\mathbf{X}$ with regularity α-Höl $\wedge\,(1/N)$. We leave it to the reader to check that this construction yields indeed $T_v\mathbf{X}$. The downside of this construction is its restricted to geometric rough paths, not to mention its repeated use of the (analytic) extension theorem, in a situation that is within reach of purely algebraic methods.

Remark 31. The condition (10) on $\mathbf{X}$ is very natural and arises by "colifting" a Lipschitz path X^0 with a d-dimensional α-Hölder weakly geometric rough path. Moreover, this is a special case of a weakly geometric (p,q)-rough path (see [FV10] Section 9.4), and the statement can readily be extended to this general setting. One can also make a statement about the continuity of the maps $(v, \mathbf{X}) \mapsto T_v\mathbf{X}$ and $(v, \mathbf{X}) \mapsto M_v\mathbf{X}$ in suitable rough path topologies. However these points will not be explored here further.

Remark 32. The proof of Theorem 28 part (i) will reveal that the only properties required of T_v (resp. M_v) is that it be an algebra morphism, preserves group-like (or equivalently primitive) elements, is upper-triangular (increases grading), and that it increases the grade of every word of length k (resp. forest with k nodes) to at most Nk. While already the first of these conditions uniquely determines T_v once $T_v(e_i) = e_i + v_i$ is chosen, we emphasise that without demanding that M_v is a pre-Lie algebra morphism, there is freedom to how M_v can be extended to satisfy these properties even after $M_v(\bullet_i) = \bullet_i + v_i$ is chosen.

In general, different choices of M_v will give rise to different branched rough paths $M_v(\mathbf{X})$. There is a notable exception to this, which is when $\mathbf{X}$ is the canonical lift of a Lipschitz (or more generally α-Hölder, $\alpha \in (1/2, 1]$) path in $\mathbb{R}^{1+d}$. Then for every algebra morphism $M : \mathcal{H}^* \mapsto \mathcal{H}^*$ such that $M\bullet_i = M_v\bullet_i = \bullet_i + v_i$, it holds that $M\mathbf{X} = M_v\mathbf{X}$. Indeed, in this case $\mathbf{X}$ is necessarily in the image of $G(\mathbb{R}^{1+d}) \subset T((\mathbb{R}^{1+d}))$ under the embedding (3), and since M and M_v agree on the generators $\bullet_i$, it follows that $M\mathbf{X} = M_v\mathbf{X}$ (this discussion relates of course to the final point of Theorem 28 part (ii), where upon demanding additional structure on $\mathbf{X}$, we see that all maps M satisfying the specified properties agree on $\mathbf{X}$).

Remark 33. In [BCF17], two examples are studied of families of random bounded variation paths $(X^\varepsilon)_{\varepsilon>0}$ whose canonical lifts to geometric rough paths $(\mathbf{X}^\varepsilon)_{\varepsilon>0}$ diverge as $\varepsilon \to 0$. In particular, ODEs driven by X^ε in general also fail to converge. However, for suitably chosen $v^\varepsilon = v_0^\varepsilon \in \mathcal{L}^N(\mathbb{R}^d)$, for which in general $\lim_{\varepsilon\to 0} |v^\varepsilon| = \infty$, one obtains convergence of the translated rough paths $T_{v^\varepsilon}\mathbf{X}^\varepsilon$. In particular, it follows from the upcoming Theorem 36 that solutions to modified ODEs driven by X^ε, with terms generally diverging as $\varepsilon \to 0$, converge to well-defined limits. In this specific context, the translation maps T_{v^ε} are precisely the renormalization maps occurring in regularity structures when applied to the setting of SDEs; we shall make this connection precise in Section 6.

Remark 34. Observe that the level-N lift of a weakly geometric rough path is precisely the solution to the linear RDE

$$dY_t = L(Y_t)d\mathbf{X}_t, \;\; Y_0 = 1 \in T^N(\mathbb{R}^{1+d}),$$

where $L = (L_0, \ldots, L_d)$ are the linear vector fields on $T^N(\mathbb{R}^{1+d})$ given by right-multiplication by $(e_0, \ldots, e_d)$ respectively. In much the same way, the level-N truncation of the translated path $Y_t := \pi_N(T_v\mathbf{X}_t)$ is the solution to the modified linear RDE

$$dY_t = L^v(Y_t)d\mathbf{X}_t, \;\; Y_0 = 1 \in T^N(\mathbb{R}^{1+d}),$$

where now $L^v = (L_{e_0+v_0}, \ldots, L_{e_d+v_d})$ are given by right-multiplication by $(e_0 + v_0, \ldots, e_d + v_d)$ (which is a special case of the upcoming Theorem 36).

We note however that the same conclusion does not hold for branched rough paths. Indeed, even the level-N lift of a branched rough path $\mathbf{X}$, $N \geq \lfloor 1/\alpha \rfloor$, is in general not the solution of a linear RDE driven by $\mathbf{X}$, which can easily be seen from the fact that linear RDEs are completely determined by the values $\langle \mathbf{X}_{s,t}, \tau\rangle$ where τ ranges over all linear trees $\tau = [\ldots[\bullet_{i_1}]_{\bullet_{i_2}}\ldots]_{\bullet_{i_m}}$ (see, e.g., [HK15] Example 3.11). A simple example is any branched rough path $\mathbf{X}$ for which $\langle \mathbf{X}, \tau\rangle \equiv 0$ for all linear trees τ (e.g., the $\frac{1}{3}$-Hölder branched rough path for which $\langle \mathbf{X}_{s,t}, \tau\rangle = t - s$ for some $\tau = [\bullet_i\bullet_j]_{\bullet_k}$ and zero for every other tree τ of size $|\tau| \leq 3$), so that every linear RDE driven by $\mathbf{X}$ is constant.

Proof of Theorem 28. (i) We are required to show that

1. $T_v\mathbf{X}$ takes values in $G(\mathbb{R}^{1+d})$,
2. Chen's relation $(T_v\mathbf{X})_{s,t}\dot{\otimes}(T_v\mathbf{X})_{t,u} = (T_v\mathbf{X})_{s,u}$ holds, and
3. the analytic condition (9).

The first two properties follow immediately from the analogous properties of $\mathbf{X}$ and the fact that $T_v\big|_{G(\mathbb{R}^{1+d})} : G(\mathbb{R}^{1+d}) \mapsto G(\mathbb{R}^{1+d})$ is group morphism. To verify the final property, fix a word $w \in T(\mathbb{R}^{1+d})$. It readily follows from Proposition 3 and Remark 4 that $T_v^* w = \sum_i \lambda_i w_i$ where $\lambda_i \in \mathbb{R}$ and w_i is a word which satisfies $N|w_i| \geq |w|$. However

$$|\langle \mathbf{X}_{s,t}, w_i \rangle| \leq ||\mathbf{X}||_{\alpha\text{-Höl};[s,t]} |t-s|^{\alpha|w_i|},$$

and thus

$$|\langle (T_v\mathbf{X})_{s,t}, w \rangle| = |\langle \mathbf{X}_{s,t}, T_v^* w \rangle| \leq C\, ||\mathbf{X}||_{\alpha\text{-Höl};[s,t]} |t-s|^{\alpha|w|/N}$$

with C depending only on w and (polynomially) on v. It follows that $T_v\mathbf{X}$ is indeed a α/N-Hölder rough path, and the desired estimate (9) follows by running over all w with $|w| \leq \lfloor N/\alpha \rfloor$. The proof for the case of branched rough paths is identical, using now Proposition 17.

The proof of the first statement of (ii) is virtually the same, except we now observe that Proposition 3 and the condition $v = v_0 \in \mathcal{L}^N(\mathbb{R}^{1+d})$ imply that $T_v^* w = \sum_i \lambda_i w_i$ where $\lambda_i \in \mathbb{R}$ and w_i is a word which satisfies

$$N|w_i|_0 + (|w_i| - |w_i|_0) \geq |w|.$$

The first statement of (ii) now follows from (10), and the proof for the case of branched rough paths is again identical.

To show the last point of (ii), consider the subspace $\mathcal{H}^k(\mathbb{R}^d) \oplus \langle \bullet_0 \rangle \subset \mathcal{H}^k$ spanned by $\bullet_0$ and all forests $\tau \in \mathcal{H}^k$ without a label 0. Observe that it suffices to show that for every $k \geq 0$, the level-k truncation $\pi_k\mathbf{X}$ takes values in the subalgebra of $\mathcal{H}^k$ generated by $\mathcal{H}^k(\mathbb{R}^d) \oplus \langle \bullet_0 \rangle$.

To this end, consider the space $\tilde{C}^\infty$ defined as the collection of all piecewise smooth paths $\mathbf{x} : [0,T] \mapsto \mathcal{G}^k$ for which $\dot{\mathbf{x}} \in \mathcal{H}^k(\mathbb{R}^d) \oplus \langle \bullet_0 \rangle$ (so that in fact $\dot{\mathbf{x}} \in \mathcal{B}^k(\mathbb{R}^d) \oplus \langle \bullet_0 \rangle$). For every partition $D = (t_0, \ldots, t_m) \subset [0,T]$, we can construct $\mathbf{x}^D \in \tilde{C}^\infty$ as the piecewise geodesic path (for the Riemannian structure of $\mathcal{G}^k$) whose increment over $[t_i, t_{i+1}]$ is $\exp(\pi_{\mathcal{B}^k(\mathbb{R}^d)\oplus\langle \bullet_0 \rangle} \log \mathbf{X}_{t_i,t_{i+1}})$. One can verify that condition (10) guarantees that $\mathbf{x}^D \to \pi_k\mathbf{X}$ uniformly as $|D| \to 0$. The conclusion now follows since, by construction, $\mathbf{x}^D$ takes values in the subalgebra generated by $\mathcal{B}^k(\mathbb{R}^d) \oplus \langle \bullet_0 \rangle$. $\square$

5.2 Effects of translations on RDEs

Throughout this section, we assume that $f = (f_0, \ldots, f_d)$ is a collection of vector fields on $\mathbb{R}^e$ which are as regular as required for all stated operations and RDEs to make sense.

Observe that f induces a canonical map from $\mathcal{L}^N(\mathbb{R}^{1+d})$ to the space of vector fields $\mathrm{Vect}(\mathbb{R}^e)$ which extends the map $e_i \mapsto f_i$. Write f_v for the image of $v \in \mathcal{L}^N\left(\mathbb{R}^d\right)$ under this map, e.g., for $v = [e_1, e_2]$, we have the vector field $f_{[e_1,e_2]} \equiv [f_1, f_2]$. Given a collection $v = (v_0, \ldots, v_d) \subset \mathcal{L}^N(\mathbb{R}^{1+d})$, we write

$$f^v = (f_0^v, \ldots, f_d^v) = (f_{e_0+v_0}, \ldots, f_{e_d+v_d}).$$

Similarly, f induces a canonical map from $\mathcal{B}^N$ to $\mathrm{Vect}(\mathbb{R}^e)$ which extends $\bullet_i \mapsto f_i$ using the pre-Lie product $\triangleright$ on $\mathrm{Vect}(\mathbb{R}^e)$ (recall from Example 9 that in coordinates $\left(f^i\partial_i\right) \triangleright \left(g^j\partial_j\right) \equiv \left(f^i\partial_i g^j\right)\partial_j$).

Once more write f_v for the image of $v \in \mathcal{B}^N$ under this map, e.g., for $v = [\bullet_1]_{\bullet_2} = \bullet_1 \curvearrowright \bullet_2$, we have the vector field

$$f_{\bullet_1 \curvearrowright \bullet_2} = f_{[\bullet_1]_{\bullet_2}} \equiv f_1 \triangleright f_2$$

(note that our notation f_v agrees with that of [HK15] Section 3). Again given a collection $v = (v_0, \ldots, v_d) \subset \mathcal{B}^N$, we write

$$f^v = (f_0^v, \ldots, f_d^v) = (f_{\bullet_0 + v_0}, \ldots, f_{\bullet_d + v_d}).$$

Remark 35. Treating $\mathcal{L}^N(\mathbb{R}^{1+d})$ (resp. $\mathcal{B}^N$) as a nilpotent Lie (resp. pre-Lie) algebra, the map considered above is not in general a Lie (resp. pre-Lie) algebra morphism into $\mathrm{Vect}(\mathbb{R}^e)$.

Theorem 36. (i) *Let notation be as in Theorem 28 part* (i)*. Then Y is an RDE solution flow to*

$$dY = f(Y)\, d(T_v\mathbf{X}) \quad \text{(resp. } dY = f(Y)\, d(M_v\mathbf{X})\text{)}$$

if and only if Y is an RDE solution flow to

$$dY = f^v(Y)\, d\mathbf{X}.$$

(ii) *Let notation be as in Theorem 28 part* (ii)*. Then Y is an RDE solution flow to*

$$dY = f(Y)\, d(T_v\mathbf{X}) \quad \text{(resp. } dY = f(Y)\, d(M_v\mathbf{X})\text{)}$$

if and only if Y is an RDE solution flow to

$$dY = f^v(Y) d\mathbf{X} \equiv f(Y)\, d\mathbf{X} + f_{v_0}(Y)\, dX^0.$$

Remark 37. Since the space of weakly geometric rough paths embeds into the space of branched rough paths using the map (3), the statements in Theorem 36 for weakly geometric rough paths are a special case of those for branched rough paths. We make a distinction between the two cases only for clarity.

Proof. For clarity, we first prove the statement for geometric rough paths and then generalise to branched rough paths (although by Remark 37, it suffices to prove the statement only in the branched case).

Observe that for weakly geometric rough paths, (i) will follow directly from the usual Euler RDE estimate ([FV10] Corollary 10.15) once we show that

$$\sum_{|u| \le \lfloor 1/\alpha \rfloor} \langle \mathbf{X}_{s,t}, u \rangle f_u^v(y) = \sum_{|u| \le \lfloor N/\alpha \rfloor} \langle T_v \mathbf{X}_{s,t}, u \rangle f_u(y) + r_{s,t}, \quad \forall y \in \mathbb{R}^e, \quad \forall s,t \in [0,T], \tag{11}$$

where $|r_{s,t}| = o(|t-s|)$ and where the sums run over any orthonormal basis of $\mathcal{L}^{\lfloor 1/\alpha \rfloor}(\mathbb{R}^{1+d})$ and $\mathcal{L}^{\lfloor N/\alpha \rfloor}(\mathbb{R}^{1+d})$ respectively.

Consider for the moment that $f = (f_0, \ldots, f_d)$ is a collection of smooth vector fields, so that $\Phi_f : u \mapsto f_u$ is a genuine Lie algebra morphism from $\mathcal{L}(\mathbb{R}^{1+d})$ into $\mathrm{Vect}^\infty(\mathbb{R}^e)$. Hence, whenever f are smooth, the maps $\Phi_f \circ T_v$ and Φ_{f^v} are both Lie algebra morphisms from $\mathcal{L}(\mathbb{R}^{1+d})$ into $\mathrm{Vect}^\infty(\mathbb{R}^e)$ which furthermore agree on the generators e_i. Thus $\Phi_f \circ T_v = \Phi_{f^v}$, and so

$$\sum_u \langle x, u \rangle f_u^v = \sum_u \langle T_v x, u \rangle f_u, \quad \forall x \in \mathcal{L}(\mathbb{R}^{1+d}), \tag{12}$$

where both sums run over any orthonormal basis of $\mathcal{L}(\mathbb{R}^{1+d})$. This proves (11) for smooth $f = (f_0, \ldots, f_d)$. For the general case where f are only sufficiently regular for the stated RDEs to make sense, we note that equality (12) is purely algebraic, so (11) can be readily deduced by truncation.

To extend this argument to the case of branched rough paths, (i) will follow directly from the Euler estimate derived in [HK15] Proposition 3.8 once we show that

$$\sum_{\tau \in \mathcal{B}^{\lfloor 1/\alpha \rfloor}} \langle \mathbf{X}_{s,t}, \tau \rangle f^v_\tau(y) = \sum_{\tau \in \mathcal{B}^{\lfloor N/\alpha \rfloor}} \langle M_v \mathbf{X}_{s,t}, \tau \rangle f_\tau(y) + r_{s,t}, \quad \forall y \in \mathbb{R}^e, \quad \forall s,t \in [0,T], \tag{13}$$

where $|r_{s,t}| = o(|t-s|)$ and where the sums run over all trees τ in $\mathcal{B}^{\lfloor 1/\alpha \rfloor}$ and $\mathcal{B}^{\lfloor N/\alpha \rfloor}$ respectively.

As before, suppose first that $f = (f_0, \ldots, f_d)$ is a collection of smooth vector fields, so that $\Phi_f : x \mapsto f_x \equiv \sum_{\tau \in \mathcal{B}} \langle x, \tau \rangle f_\tau$ is a pre-Lie algebra morphism from $\mathcal{B}$ into $\mathrm{Vect}^\infty(\mathbb{R}^e)$. Hence, whenever f are smooth, the maps $\Phi_f \circ M_v$ and Φ_{f^v} are both pre-Lie algebra morphisms from $\mathcal{B}$ into $\mathrm{Vect}^\infty(\mathbb{R}^e)$ which furthermore agree on the generators $\bullet_i$. Thus $\Phi_f \circ M_v = \Phi_{f^v}$, and so

$$\sum_{\tau \in \mathcal{B}} \langle x, \tau \rangle f^v_\tau = \sum_{\tau \in \mathcal{B}} \langle M_v x, \tau \rangle f_\tau, \quad \forall x \in \mathcal{B}.$$

As the above equality is purely algebraic, we again deduce (13) by truncation in the general case where f are only sufficiently regular for the stated RDEs to make sense.

The desired result in (ii) for both geometric and branched rough paths follows in the same way. □

6 Link with renormalization in regularity structures

We now recall several notions from the theory of regularity structures and draw a link between the map δ from Section 3.3 and the coproduct Δ^- associated to negative renormalization [BHZ16, Hai16]. In particular, we demonstrate how negative renormalization maps on the regularity structure associated to branched rough paths carry a natural interpretation as rough path translations (see Theorem 47 below).

6.1 Regularity structures

Regularity structures usually deal with (e.g. SPDE solutions) $u = u(z)$ where $z \in \mathbb{R}^n$ (e.g. space-time), u takes values in $\mathbb{R}$ (or $\mathbb{R}^e$). Equations further involve a β-regularizing kernel, and there are d sources of noise, say $\xi_1, ..., \xi_d$, of arbitrary (negative) order $\alpha_{\min}$, as long as the equation is subcritical.

6.1.1 Generalities

We review the general (algebraic) setup in the case $n = 1$, $\beta = 1$ and $\alpha_{\min} \in (-1, 0)$.

In the spirit of Hairer's formalism, consider the equation

$$u(t) = u(0) + \left(K * \sum_{i=1}^{d} f_i(u(\cdot))\xi_i(\cdot) \right)(t), \quad t \in \mathbb{R}, \tag{14}$$

where $u(t)$ is a real-valued function for which we solve, $\xi_i(t)$ are driving noises, f_i are smooth functions on $\mathbb{R}$ (one could readily extend to the case that u takes values in $\mathbb{R}^e$ and f_i are vector fields on $\mathbb{R}^e$), and K is a kernel which improves regularity by order $\beta = 1$.

Remark 38. The example to have in mind here is $K(s) = \exp(-\lambda s)1_{s>0}$, which allows to incorporate an additional linear drift term ("$-\lambda u dt$"), or of course the case $\lambda = 0$, i.e. the Heaviside step function, which leads to the usual setting of controlled differential equations. We shall indeed specialize to the Heaviside case in subsequent sections, as this simplifies some algebraic constructions and so provides a clean link to rough path structures. For the time being, however, we find it instructive to work with a general 1-regularizing K, as this illustrates the need for polynomials decorations as well as symbols $\mathcal{J}_k$, representing k-th derivatives of the kernel.

Our driving noises $\xi_i(t)$ should be treated as distributions on $\mathbb{R}$ of regularity $C^{\alpha-1}$ for some $\alpha \in (0, 1)$ (which will later correspond to the case of α-Hölder branched rough paths). In the case that $\alpha \le 1/2$, due to the product $f_i(u)\xi_i$, (14) is singular and thus cannot in general be solved analytically. However the equation is evidently sub-critical in the sense of [Hai14], and so one can build an associated regularity structure.

Introducing the symbols

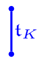
We first collect all the symbols of the regularity structure required to solve (14) and which is stable under the renormalization maps in the sense of [BHZ16]. Define the linear space

$$\mathcal{T} = \langle \mathcal{W} \rangle,$$

where $\mathcal{W}$ is the set of all rooted trees where every node carries a decoration $k \in \mathbb{N} \cup \{0\}$ and where every edge which ends on a leaf may be (but is not necessarily) assigned a type $\mathfrak{t}_{\Xi_i}$, $i \in \{1, \ldots, d\}$.

An edge with type i corresponds to the driving noise ξ_i. Every other edge has a type $\mathfrak{t}_K$ which means that it is associated to the kernel K. (For now, we only assume K is 1-regularizing, later we will take it to be the Heaviside step function.) Also, each node has at most one incoming edge with type belonging to $\{1, \ldots, d\}$.[4]

To avoid confusion between the different meaning of trees in $\mathcal{W}$ and those introduced in Section 3, we will color every tree in $\mathcal{W}$ blue. Every such tree has a corresponding symbol representation, e.g.,



$\leftrightarrow$  $\leftrightarrow \mathcal{I}$, 

$\leftrightarrow$

$\leftrightarrow \Xi_i$,

$\leftrightarrow X^k$,

$\leftrightarrow \mathcal{I}(\mathcal{I}(\Xi_1)\mathcal{I}(\Xi_2 X^5)\mathcal{I}(X^7))X^6$,

where we implicitly drop the 0 decoration ($\leftrightarrow X^0$) from the nodes. It is instructive to check that $\mathcal{W}$ provides an example of a structure built from a subcritical complete rule (in the sense of [BHZ16] Section 5) arising from the equation (14). Indeed, we can give the set of rules used for the construction of

$$R(\Xi_i) = \{()\}, \quad R(\mathcal{I}) = \{([\mathcal{I}]_\ell), ([\mathcal{I}]_\ell, \Xi_i), \ell \in \mathbb{N} \cup \{0\}, i \in \{1, ..., d\}\}.$$

The notation $[\mathcal{I}]_\ell$ is a shorthand notation for $\mathcal{I}, ..., \mathcal{I}$ where $\mathcal{I}$ is repeated ℓ times.

We define a degree $|\cdot|$ associated to an edge type and a decorated tree. For edge types and polynomials, we have

$$|\Xi_i| = \alpha - 1, \quad |\mathcal{I}| = 1, \quad |X^k| = k.$$

Then by recursion,

$$|\mathcal{I}(\tau)| = |\tau| + |\mathcal{I}|, \qquad \left|\prod_i \tau_i\right| = \sum_i |\tau_i|.$$

For a non-recursive definition see [BHZ16] where the degree is described through a summation over all the edge types and the decorations in the tree.

Remark 39. Remark that $\mathcal{W} \equiv \mathcal{W}_{\text{BHZr}}$ (the "r" in BHZr refers to *reduced*, in the terminology of [BHZ16] these are trees without any extended decorations) will contain certain symbols which do *not* arise if one follows the original procedure of [Hai14] (which, in some sense, is the most economical way to build the structure):

$$\mathcal{W}_{\text{Hai14}} \subset \mathcal{W}_{\text{BHZr}} \subset \mathcal{W}_{\text{BHZ}}.$$

Indeed in [Hai14], the set of rules is not necessarily complete so one has to add terms by hand coming from the renormalization procedure and in the end one works with a space $\bar{\mathcal{W}}_{\text{Hai14}}$ lying between $\mathcal{W}_{\text{Hai14}}$ and $\mathcal{W}_{\text{BHZr}}$. For example, $\mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j)$, $\mathcal{I}(\mathcal{I}(\Xi_k)$, and $\mathcal{I}() \equiv \mathcal{I}(X^0)$ do not appear in $\mathcal{W}_{\text{Hai14}}$, but all of these appear in $\mathcal{W}_{\text{BHZr}}$. These in turn are embedded in $\mathcal{W}_{\text{BHZ}}$, a set of trees with extended decorations on the nodes and also colourings of the nodes which give more algebraic properties. In the setting of [BHZ16], we would work with an additional symbol $\mathbf{1}_\alpha$ for $\alpha \in \mathbb{R}$, representing an extended decoration, which provides information on some "singular" (negative degree) tree which has been removed, and all of these symbols are would be placed using a complete set of rules.

[4]This rules out symbols corresponding to products of noise, such as $\Xi_i \Xi_j$ with $i, j \in \{1, \ldots, d\}$.

Introducing $\mathcal{T}_-$

We define the space $\mathcal{T}_-$ as

$$\mathcal{T}_- = \{\tau_1 \bullet \cdots \bullet \tau_n,\ \tau_i \in \mathcal{W},\ |\tau_i| < 0\}. \tag{15}$$

where $\bullet$ is the forest product and the unit is given by the empty forest. (In other words, $\mathcal{T}_-$ is the free unital commutative algebra generated by elements in $\mathcal{W}$ of negative degree.) We now recall that $\mathcal{T}_-$ can be equipped with a Hopf algebra structure $\mathcal{T}_-$ for which there exists a coaction $\Delta^- : \mathcal{T} \mapsto \mathcal{T}_- \otimes \mathcal{T}$ such that $(\mathcal{T}, \Delta^-)$ is a (left) comodule over $\mathcal{T}_-$. Then the action of a character $\ell \in \mathcal{T}_-^*$ on $x \in \mathcal{T}$, termed "negative renormalization", is given by $M_\ell x = (\ell \otimes \mathrm{id})\Delta^- x$.

Following [Hai16] Section 2 we can describe the coaction Δ^- as follows. Fix a tree $\tau \in \mathcal{W}$, consider a subforest $A \subset \tau$, i.e., an arbitrary subgraph of τ which contains no isolated vertices. We then write $R_A\tau$ for the tree obtained by contracting the connected components of A in τ. With this notation at hand, we then define a linear map, the coaction,

$$\Delta^- : \mathcal{T} \to \mathcal{T}_- \otimes \mathcal{T}$$

by setting, for $\tau \in \mathcal{W}$,

$$\Delta^- \tau = \sum_{A \subset \mathcal{T}_-} A \otimes R_A \tau. \tag{16}$$

Unfortunately, this is not quite the correct coaction as it does not handle correctly the powers of X. However, upon restriction to $\tilde{\mathcal{T}} \subset \mathcal{T}$, as done in detail in the next section, this is precisely the form of the coaction (now on $\tilde{\mathcal{T}}$). When moving to a coproduct this fortunately plays no role (since $\mathcal{T}_-$ does not contain any non-zero powers of X or a factor of the form $\mathcal{I}()$). By abuse of notation, Δ^- also acts as a coproduct, that is

$$\Delta^- : \mathcal{T}_- \to \mathcal{T}_- \otimes \mathcal{T}_-. \tag{17}$$

To be explicit, given $f = \tau_1 \cdots \tau_n \in \mathcal{T}$, we have $\Delta^-(f) = \Delta^-(\tau_1)...\Delta^-(\tau_n)$ with each $\Delta^-(\tau_i)$ as defined above, but with an additional projection to the negative trees on the right-hand side of the tensor-product.

Remark 40. The spaces $\mathcal{T}_- \equiv \mathcal{T}^-_{\mathrm{BHZr}}$, $\mathcal{T}^-_{\mathrm{BHZ}}$ and $\mathcal{T}^-_{\mathrm{Hai14}}$ are the same *in this framework* (cf. assumptions from the beginning of this subsection). Indeed, all negative trees of $\mathcal{W}$ have a degree of the form $N\alpha - 1$. Then if we remove one negative subtree, of degree $M\alpha - 1$ say, from a negative tree, we obtain a degree $(N-M)\alpha$ which is positive and hence the "cured" tree does not belong to $\mathcal{T}_-$.

Introducing $\mathcal{T}_+$

In order to describe the space $\mathcal{T}_+$ as in [BHZ16], we need to associate to each edge a decoration $k \in \mathbb{N} \cup \{0\}$ viewed as a derivation of the kernels or the driving noises. Such a decoration does not appear in $\mathcal{T}$. Thus we will replace the letter $\mathcal{I}$ by $\mathcal{J}$ in this context. We do not give any graphical notation for $\mathcal{J}_k$, the edge with type $\mathfrak{t}_K$ and decoration k representing $K^{(k)}$, because these symbols ultimately will not appear in our context.

We define $\mathcal{T}_+$ as the linear span of

$$\{X^k \prod_{i=1}^n \mathcal{J}_{k_i}(\tau_i) \mid k, n \in \mathbb{N} \cup \{0\}, k_i \in \mathbb{N} \cup \{0\}, \tau_i \in \mathcal{W},\ |\tau_i| + 1 - k_i > 0\}.$$

(In other words, $\mathcal{T}_+$ is the free unital commutative algebra generated by $\{X\} \cup \{\mathcal{J}_k\tau \mid \tau \in \mathcal{W}, |\tau| + 1 - k > 0\}$). We use a different letter $\mathcal{J}$ to stress that $\mathcal{W}$ is different from $\mathcal{W}^+$. Moreover, the use of this letter is viewed in [BHZ16] as a colouration of the root and plays a role in the sequel. We also define the degree of a term

$$\tau = X^k \prod_{i=1}^{n} \mathcal{J}_{k_i}(\tau_i) \in \mathcal{T}_+, \quad |\tau| = k + \sum_{i=1}^{n} 1 - k_i + |\tau_i|.$$

The space $\mathcal{T}_+$ is used in the description of the structure group associated to $\mathcal{T}$. More precisely, recall that $\mathcal{T}_+$ can be equipped with a Hopf algebra structure for which there exists a coaction $\Delta^+ : \mathcal{T} \mapsto \mathcal{T} \otimes \mathcal{T}_+$ such that $(\mathcal{T}, \Delta^+)$ is a (right) comodule over $\mathcal{T}_+$. Following Hairer's survey [Hai16], the coaction

$$\Delta^+ \colon \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}_+ \tag{18}$$

is given by

$$\Delta^+ X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i \ , \qquad \Delta^+ \Xi_i = \Xi_i \otimes \mathbf{1} \ , \tag{19}$$

and then recursively by

$$\Delta^+ \mathcal{I}(\tau) = (\mathcal{I} \otimes \mathrm{id})\Delta^+ \tau + \sum_{\ell \in \mathbb{N} \cup \{0\}} \frac{X^\ell}{\ell!} \otimes \mathcal{J}_\ell(\tau) \tag{20}$$

and

$$\Delta^+(\tau\bar{\tau}) = \Delta^+\tau \, \Delta^+\bar{\tau}. \tag{21}$$

The coproduct $\Delta^+ : \mathcal{T}_+ \to \mathcal{T}_+ \otimes \mathcal{T}_+$ is then defined in the same way by replacing (20) with

$$\Delta^+ \mathcal{J}_k(\tau) = (\mathcal{J}_k \otimes \mathrm{id})\Delta^+ \tau + \sum_{\ell \in \mathbb{N} \cup \{0\}} \frac{X^\ell}{\ell!} \otimes \mathcal{J}_{k+\ell}(\tau),$$

in which $\Delta^+\tau$ is understood as the coaction $\Delta^+ : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}_+$.

Then the action of a character $g \in \mathcal{T}_+^*$ on $x \in \mathcal{T}$, termed "positive renormalization", is given by $\Gamma_g x = (\mathrm{id} \otimes g)\Delta^+ x$.

Unfortunately, there is a problem here in that, with definition (20), a desirable cointeraction between Δ^+ and Δ^- fails (see Remark 42). The "official" remedy, following [BHZ16], is to use the extended decorations through another degree $|\cdot|_+$ which takes into account these decorations and behaves the same as $|\cdot|$ for the rest. For example, one has $|\mathcal{I}(\mathbf{1}_\beta\tau)|_+ = |\tau|_+ + 1 + \beta$. The "correct" coaction Δ^+ (see [BHZ16, (4.14)]) then also involves these extended decorations. In the present setting, however, we can get away by replacing (20) with the same formula, but only keeping $\ell = 0$ in the sum; that is, with $\mathcal{J} \equiv \mathcal{J}_0$,

$$\Delta^+ \mathcal{I}(\tau) = (\mathcal{I} \otimes \mathrm{id})\Delta^+ \tau + 1 \otimes \mathcal{J}(\tau). \tag{22}$$

Remark 41. The space $\mathcal{T}_+ \equiv \mathcal{T}^+_{\mathrm{BHZr}}$ depends strongly on the space $\mathcal{W}$. We have

$$\mathcal{T}^+_{\mathrm{Hai14}} \subset \mathcal{T}^+_{\mathrm{BHZr}} \subset \mathcal{T}^+_{\mathrm{BHZ}}.$$

These two inclusions are Hopf subalgebra inclusions. Indeed, as proved in [BHZ16], the second one, with $\mathcal{T}_+$ equipped with coproduct Δ^+ is a Hopf subalgebra inclusion (with Δ^+_{BHZ} found in [BHZ16, (4.14)]). The same is also true for $\mathcal{T}^+_{\text{Hai14}}$. The key point for the Hopf algebra structure is that, in the terminology of [BHZ16], the symbols defined in [Hai14] and [BHZ16] are obtained by a "normal rule" which guarantees the invariance under Δ^+. In the case of $\mathcal{T}^+_{\text{BHZ}}$, we use the degree $|\cdot|_+$ which is exactly $|\cdot|$ when we restrict ourselves to $\mathcal{T}^+_{\text{BHZr}}$.

Remark 42. The extended decorations are crucial in [BHZ16] for obtaining a cointeraction between the two Hopf algebras $(\mathcal{T}_+, \Delta^+)$ and $(\mathcal{T}_-, \Delta^-)$:

$$\mathcal{M}^{(13)(2)(4)} \left(\Delta^- \otimes \Delta^-\right) \Delta^+ = \left(\text{id} \otimes \Delta^+\right) \Delta^-$$

where $\mathcal{M}^{(13)(2)(4)}$ is given as $\mathcal{M}^{(13)(2)(4)} \left(\tau_1 \otimes \tau_2 \otimes \tau_3 \otimes \tau_4\right) = \left(\tau_1 \bullet \tau_3\right) \otimes \tau_2 \otimes \tau_4$. This identity is both true on $\mathcal{T}$ through the comodule structures and on $\mathcal{T}_+$ when the coproduct Δ^- is viewed as an action on $\mathcal{T}_+$. We have already crossed something similar in Lemma 18 but in that case the maps involved were not really coproducts. In our simple framework, this property is not satisfied if we just consider the reduced structure. One can circumvent this by changing the coproduct to (22) as mention above. This approach is possible in our context because we know *a priori* that each edge type $\mathcal{I}$ in the elements of $\mathcal{W}$ with negative degree has the same "Taylor expansion" of length 1 in (20) ($\ell = 0$). In general, we would use the extended decorations to maintain this property, however, in our setting, we can just fix the length in the coproduct and not use the extended decorations. We follow this approach in the sequel when we restrict ourselves to the rough path setting by choosing the Connes-Kreimer coproduct for Δ^+. We can also get rid of the colour when we have no derivatives on the edges at the root: if we want to extract from $\mathcal{I}(\tau_1\Xi_i)\mathcal{I}(\tau_2\Xi_j)$ all the negative subtrees, we observe that it is not possible to extract one at the root, and thus are only left with negative subtrees in $\tau_1\Xi_i$ and $\tau_2\Xi_j$, which ensures that

$$M_\ell\mathcal{I}(\tau_1\Xi_i)\mathcal{I}(\tau_2\Xi_j) = \mathcal{I}\left(M_\ell\left(\tau_1\Xi_i\right)\right)\mathcal{I}\left(M_\ell\left(\tau_2\Xi_j\right)\right).$$

In the setting of [BHZ16], this multiplicativity property is encoded by a colour at the root which avoids the extraction of a tree containing the root.

6.1.2 The case of rough differential equations

As in the last subsection: $n = 1$, $\beta = 1$ and noise degree $\alpha_{\min} \in (-1, 0) > -1$. We further specialize the algebraic set in that no symbols $\mathcal{J}_k$ and polynomials X^k with $k > 0$ are required in describing $\mathcal{T}_+$.

Assuming K to be the Heaviside step function, all derivatives (away from the origin) are zero, hence there is no need (with regard to $\mathcal{W}$) to have any polynomial symbols (X^k with $k > 0$). Removing these from $\mathcal{W}$ leaves us with $\tilde{\mathcal{W}} \subset \mathcal{W}$ which we may list as

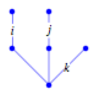
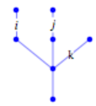
$$\begin{aligned}\tilde{\mathcal{W}} = \{&\Xi_i, ..., \mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j)\Xi_k, ..., 1, \mathcal{I}(\Xi_i), \mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j), ...\\ &..., \mathcal{I}(\mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j)\Xi_k), \mathcal{I}(\mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j)), ..., \mathcal{I}()\mathcal{I}(), \mathcal{I}(\mathcal{I}()), ...\},\end{aligned} \tag{23}$$

(all indices are allowed to vary from $1, ..., d$), with associated degrees $|\tau|$ as follows:[5]

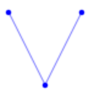
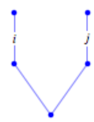
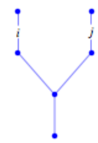
$$\alpha - 1, ..., 3\alpha - 1, ..., 0, \alpha, 2\alpha, ... \; ..., 3\alpha, 2\alpha + 1, ..., 2, 2, ...$$

[5] tacitly assuming $\alpha < 1/3$

As in the case of $\mathcal{W}$, elements of $\tilde{\mathcal{W}}$ can be viewed as rooted trees, but *without* node decorations. For instance,

 $\leftrightarrow \mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j)\Xi_k$,  $\leftrightarrow \mathcal{I}(\mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j)\Xi_k)$,

are trees ($\leftrightarrow$ symbols) contained in $\mathcal{W}$, and also in $\mathcal{W}_{\text{Hai14}}$, the symbols arising in the construction of [Hai14], whereas

 $\leftrightarrow \mathcal{I}()\mathcal{I}()$, $\leftrightarrow \mathcal{I}(\mathcal{I}())$,  $\leftrightarrow \mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j)$,  $\leftrightarrow \mathcal{I}(\mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j))$,

are contained in $\mathcal{W}$, following the above construction taken from [BHZ16], in order to obtain stability under the negative renormalization maps (but not included in $\mathcal{W}_{\text{Hai14}}$.)

A linear subspace of $\mathcal{T} = \langle \mathcal{W} \rangle$ is then given by

$$\tilde{\mathcal{T}} := \langle \tilde{\mathcal{W}} \rangle. \tag{24}$$

Symbols for negative renormalization

Recall that, thanks to $\beta = 1$, noise degree $\alpha - 1 \in (-1, 0)$, no terms X, X^2 or $\mathcal{I}(), \ldots$ arise as symbol in $\mathcal{W}_- := \{\tau \in \mathcal{W} \mid |\tau| < 0\}$. (As a consequence, replacing $\mathcal{W}$ by $\mathcal{W}_{\text{Hai14}}, \tilde{\mathcal{W}}$ or $\mathcal{W}_{\text{BHZ}}$ in the definition of the negative symbols makes no difference.) In particular,

$$\mathcal{W}_- = \{\Xi_i, \mathcal{I}(\Xi_i)\Xi_j, \ldots, \mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j)\Xi_k, \ldots\}.$$

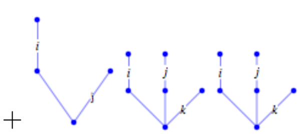
(where $\mathcal{W}_-$ "ends" right before the element 1 in (23) above) contains no powers of X, (hence no need to introduce "$\tilde{\mathcal{W}}_-$"). As previously defined (see (15)), we have

$$\mathcal{T}_- = \text{free unital commutative algebra generated by } \mathcal{W}_-.$$

For instance, writing $\bullet$ for the (free, commutative) product in $\mathcal{T}_-$,

$$2\Xi_i - \frac{1}{3}\Xi_i \bullet \Xi_j + \mathcal{I}(\Xi_i)\Xi_j \bullet (\mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j)\Xi_k)^{\bullet 2} \quad \in \mathcal{T}_-.$$

Interpreting $\bullet$ as the *forest product*, elements in $\mathcal{T}_-$ can then be represented as forests, such as

2 $-\frac{1}{3}$ $+$ 

One can readily verify that $\Delta^- : \mathcal{T} \to \mathcal{T}_- \otimes \mathcal{T}$ restricted to $\tilde{\mathcal{T}}$ maps $\tilde{\mathcal{T}} \to \mathcal{T}_- \otimes \tilde{\mathcal{T}}$, also denoted by Δ^- so that $(\tilde{\mathcal{T}}, \Delta^-)$ is a subcomodule of $(\mathcal{T}, \Delta^-)$.

Symbols for positive renormalization and $\mathcal{T}_+$.

Recall that $\mathcal{T}_+$ was generated, as a free commutative algebra, by

$$\mathcal{W}_+ := \{X\} \cup \{\mathcal{J}_k\tau \mid \tau \in \mathcal{W}, |\tau| + 1 - k > 0\}.$$

Writing $\mathcal{J} \equiv \mathcal{J}_0$ as usual, we define a subset $\tilde{\mathcal{W}}_+ \subset \mathcal{W}_+$ as follows

$$\begin{aligned}\tilde{\mathcal{W}}_+ &:= \{\mathcal{J}\tau \mid \tau \in \tilde{\mathcal{W}}\} \\ &= \{1, \mathcal{J}(\Xi_i), \mathcal{J}(\mathcal{I}(\Xi_i)\Xi_j), \mathcal{J}(\mathcal{I}(\mathcal{I}(\Xi_i)\Xi_j)\Xi_k), \mathcal{J}(\mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j)\Xi_k), ..., \mathcal{J}(\mathcal{I}(\Xi_i)\mathcal{I}(\Xi_j)), ...\}\end{aligned} \tag{25}$$

with degrees $0, \alpha, 2\alpha, 3\alpha, 3\alpha, ..., 2\alpha + 1, ...$ here.

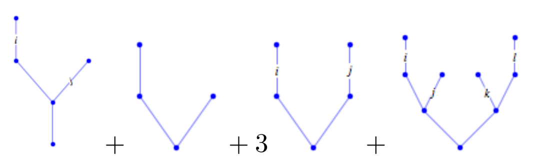
Recall that elements in $\mathcal{W}_+$ can be represented by *elementary* trees, in the sense that - disregarding the trivial (empty) tree 1 - only one edge departs from the root. The same is true for elements in $\tilde{\mathcal{W}}_+$. Set

$$\tilde{\mathcal{T}}_+ := \text{free unital commutative algebra generated by } \tilde{\mathcal{W}}_+.$$

For example, writing $\tau_1\tau_2$ for the (free, commutative) product of $\tau_1, \tau_2 \in \tilde{\mathcal{T}}_+$, an example of an element in this space would be

$$\mathcal{J}(\mathcal{I}(\Xi_i)\Xi_j) + \mathcal{J}(\mathcal{I}())\mathcal{J}(1) + 3\ \mathcal{J}(\Xi_i)\mathcal{J}(\Xi_j) + \mathcal{J}(\mathcal{I}(\Xi_i)\Xi_j)\mathcal{J}(\mathcal{I}(\Xi_k)\Xi_l) \quad \in \tilde{\mathcal{T}}_+.$$

Fortunately, every such element can still be represented as a tree; it suffices to interpret the free product in $\mathcal{T}_+$ as the "root-joining" product (which is possible since all constituting trees are elementary). The (abstract) unit element $1 \in \mathcal{T}_+$ is then indeed given by the (trivial) tree $\bullet \leftrightarrow X^0$, where we recall our convention to drop the node decoration "0". For instance, the above element becomes[6]


$\in \mathcal{T}_+$.

Remark 43. Though we used the same formalism to draw trees as in the case of $\tilde{\mathcal{W}}$ above, the interpretation here is slighly different in that all root-touching edges refer to $\mathcal{J}$ rather than $\mathcal{I}$. As mentioned before, in [BHZ16], this is indicated by a blue colouring of the root.

As before, we define a coaction of $\tilde{\mathcal{T}}_+$ on $\tilde{\mathcal{T}}$ (which we again denote $\Delta^+ : \tilde{\mathcal{T}} \to \tilde{\mathcal{T}} \otimes \tilde{\mathcal{T}}_+$) by (19), (21), and (22) as well as a coproduct $\Delta^+ : \tilde{\mathcal{T}}_+ \to \tilde{\mathcal{T}}_+ \otimes \tilde{\mathcal{T}}_+$ defined in the same way, but with $\mathcal{I}$ changed to $\mathcal{J}$ in (22). We note already that $(\tilde{\mathcal{T}}_+, \Delta^+)$ is isomorphic to the Connes-Kreimer Hopf algebra $\mathcal{H}$ arising from the identifications laid out in the following subsection (and which will be used crucially in the proof of the upcoming Proposition 45).

[6] Remark that $\mathcal{J}(1)$, which corresponds to the right branch of the second term, could also have been written as $\mathcal{J}()$, reflecting our convention to drop the decoration 0 from nodes (here: $1 \equiv X^0$). By the same logic, we could also write $\mathcal{I}()$, one of the symbols arising in $\mathcal{W}$, as $\mathcal{I}(1)$.

6.2 Link with translation of rough paths

6.2.1 Identification of spaces

We now give a precise description the map Δ^- *in our context as well as its connection to the map* δ *from Section 3.3. To do so, we first need to introduce several identification of vector spaces and algebras, as well as appropriately identify branched rough paths as models on a regularity structure.*

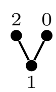
Recall the space $\mathcal{H} = \mathcal{H}(\bullet_0, ..., \bullet_d)$ from Section 3 spanned by labelled forests with label set $\{0, 1, \dots, d\}$. Consider now the enlarged vector space

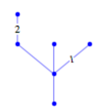
$$\tilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H}\Xi_1 \oplus .... \oplus \mathcal{H}\Xi_d. \tag{26}$$

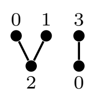
With $\tilde{\mathcal{T}}$ as defined in (24), we then have a vector space isomorphism

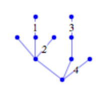
$$\tilde{\mathcal{H}} \leftrightarrow \tilde{\mathcal{T}}$$

obtained by adding an extra edge to indicate a noise Ξ_i, $i \neq 0$, and by "forgetting" the label 0 (which is equivalent to setting the noise Ξ_0 to the constant 1). For example,



$\leftrightarrow \mathcal{I}\left[\mathcal{I}(\Xi_1)\mathcal{I}(1)\Xi_2\right] =$




$\Xi_4 \leftrightarrow \mathcal{I}\left[\mathcal{I}(1)\mathcal{I}(\Xi_1)\Xi_2\right]\mathcal{I}\left[\mathcal{I}(\Xi_3)\right]\Xi_4 =$

.

Recall that $\mathcal{B} = \mathcal{B}(\bullet_0, ..., \bullet_d)$ denotes the subspace of $\mathcal{H}$ spanned by trees, and define

$$\mathcal{B}_- = \mathcal{B}_-(\bullet_1, ..., \bullet_d) \subset \mathcal{B} \subset \mathcal{H}$$

as the subspace of $\mathcal{B}$ spanned by trees with no label 0 and with at most $\lfloor 1/\alpha \rfloor$ nodes. Observe that there is a vector space isomorphism given by

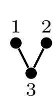
$$\tau \mapsto \phi(\tau) \equiv \dot{\tau},$$

where

$$\phi : \mathcal{B}_- \mapsto \langle \mathcal{W}_- \rangle \quad \subset \quad \mathcal{H}\Xi_1 \oplus .... \oplus \mathcal{H}\Xi_d \quad \subset \quad \tilde{\mathcal{H}}, \tag{27}$$

and where we have used the identification $\tilde{\mathcal{H}} \leftrightarrow \tilde{\mathcal{T}} \supset \langle \mathcal{W}_- \rangle$ for the first inclusion (and both inclusions being strict: for the first, just consider the element 

$\Xi_1 \notin \langle \mathcal{W}_- \rangle$). For example,

$\phi :$ 

$\mapsto \mathcal{I}(\Xi_1)\mathcal{I}(\Xi_2)\Xi_3,$

where we assume $\alpha \in (0, 1/3)$ so the tree appearing on the left is indeed an element in $\mathcal{B}_-$. Correspondingly, the symbol on the right has negative degree as an element of $\mathcal{W}$, hence is an element of $\mathcal{W}_-$.

Write $\mathcal{B}_-^*$ for the dual of the (finite-dimensional) vector space $\mathcal{B}_-$. Of course, $\mathcal{B}_-^* \cong \mathcal{B}_-$ which allows us to identify $\mathcal{B}_-^*$ with $\langle \mathcal{W}_- \rangle$. Recall that $(\mathcal{T}_-, \bullet)$ was defined as the free unital commutative algebra generated by $\mathcal{W}_-$, and that $\mathcal{G}_- \subset \mathcal{T}_-^*$ denotes the group of characters on $\mathcal{T}_-$. By definition of $\mathcal{T}_-$, we then have a bijection

$$\mathcal{B}_-^* \leftrightarrow \mathcal{G}_-. \tag{28}$$

To be fully explicit about this, recall that

$$\mathcal{T}_- = \langle \dot\tau_1 \bullet .... \bullet \dot\tau_n : \dot\tau_i \in \mathcal{W}_-,\ n = 1, 2, ... \rangle,$$

so writing $\tau_i = \phi^{-1}(\dot\tau_i) \in \mathcal{B}_-$, we have that associated to $v \in \mathcal{B}_-^*$ the character $\ell \in \mathcal{G}_-$ given explicitly by the formula

$$\ell(\dot\tau_1 \cdot .... \cdot \dot\tau_n) = \ell(\dot\tau_1)...\ell(\dot\tau_n) = \langle v, \tau_1 \rangle ... \langle v, \tau_n \rangle.$$

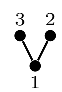
Define now

$$(\mathcal{H}_-, \cdot)$$

as the free commutative algebra generated by the subspace $\mathcal{B}_-$ of $\tilde{\mathcal{H}}$ (remark that the product in $\mathcal{H}_-$ has nothing to do with the product in $\mathcal{H}$ itself), so that there is an algebra isomorphism

$$\mathcal{H}_- \leftrightarrow \mathcal{T}_-.$$

A typical element of $\mathcal{H}_-$ looks like:

$\underset{2}{\bullet}\ \Xi_1 + \Xi_2 + \Xi_2 \cdot$  $\Xi_3,$

whereas one has $\underset{2}{\bullet} \notin \mathcal{H}_-$.

Note that we can also make the identification of algebras

$$\mathcal{H} \leftrightarrow \tilde{\mathcal{T}}_+.$$

For instance, using the bracket notation,

$$[\bullet_0]_{\bullet_0} \bullet_0 + [\bullet_i]_{\bullet_j} [\bullet_k]_{\bullet_l} \leftrightarrow \mathcal{J}(\mathcal{I}())\mathcal{J}(1) + \mathcal{J}(\mathcal{I}(\Xi_i)\Xi_j)\mathcal{J}(\mathcal{I}(\Xi_k)\Xi_l) \quad \in \tilde{\mathcal{T}}_+.$$

We denote by $\tilde{\mathcal{G}}_+ \subset \tilde{\mathcal{T}}_+^*$ the characters on $\tilde{\mathcal{T}}_+$ and note that there is also a bijection $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}_+$, where we recall that $\mathcal{G} \subset \mathcal{H}^*$ is the Butcher group over $\mathbb{R}^{1+d}$, i.e., the set of characters on $\mathcal{H}$.

To summarise, we have the following identifications in place

$$\begin{aligned}
\tilde{\mathcal{H}} &\leftrightarrow \tilde{\mathcal{T}}, \\
\mathcal{H}_- &\leftrightarrow \mathcal{T}_-, \\
\mathcal{H} &\leftrightarrow \tilde{\mathcal{T}}_+, \\
\mathcal{B}_-^* &\leftrightarrow \langle \mathcal{W}_- \rangle \leftrightarrow \mathcal{G}_- \subset \mathcal{T}_-^* \\
\mathcal{G} &\leftrightarrow \tilde{\mathcal{G}}_+ \subset \tilde{\mathcal{T}}_+^*.
\end{aligned}$$

6.2.2 Renormalization as rough path translations

It now only remains to identify (a family of) branched rough paths with a class of models on a suitable regularity structure. Define the index set $A := \{0\} \cup \alpha\mathbb{N} \cup (\alpha\mathbb{N} - 1)$. Recall that the action of $g \in \tilde{\mathcal{G}}_+$ on $\tilde{\mathcal{T}}$ is given exactly as before by

$$\Gamma_g \tau = (\mathrm{id} \otimes g)\Delta^+ \tau, \quad \forall \tau \in \tilde{\mathcal{T}}.$$

Note that Γ_g indeed maps $\tilde{\mathcal{T}}$ to itself due to the definition of $\tilde{\mathcal{G}}_+$. Note further that $\Gamma_g \Gamma_h$ (as a composition of linear maps) is exactly $\Gamma_{g \circ h}$ (with $\circ$ the product in $\tilde{\mathcal{G}}_+$ given as the dual of Δ^+), and so

$$G := \{\Gamma_g : g \in (\tilde{\mathcal{G}}_+, \circ)\}.$$

is indeed a group of endomorphisms of $\tilde{\mathcal{T}}$.

Recall now the definition of a regularity structure from [Hai14] Definition 2.1.

Lemma 44. *The triplet $(A, \tilde{\mathcal{T}}, G)$ is a regularity structure.*

Proof. The only non-trivial property to check is that for all $\tau \in \tilde{\mathcal{T}}$ of degree $\alpha \in A$ and $\Gamma \in G$, $\Gamma\tau - \tau$ is a linear combination of terms of degree strictly less than α, which in turn is a direct consequence of the definition of $\Delta^+ : \tilde{\mathcal{T}} \to \tilde{\mathcal{T}} \otimes \tilde{\mathcal{T}}_+$ from (22) (see end of Section 6.1.2). □

Recall also the definition of a model on a regularity structure (see [Hai14] Definition 2.17). Let $\mathscr{M}_{[0,T]}$ denote the set of all models (Π, Γ) for $(A, \tilde{\mathcal{T}}, G)$ on $\mathbb{R}$ such that

(i) $\Pi_t 1$ is the constant function 1 for all $t \in \mathbb{R}$,

(ii) $\Gamma_{st} = \mathrm{id}$ for $s, t \in (-\infty, 0]$ and for $s, t \in [T, \infty)$,

(iii) $(\Pi_t \mathcal{I} y)' = \Pi_t y$ for all $t \in \mathbb{R}$ and $y \in \tilde{\mathcal{T}}$. (Here $(..)'$ denotes the Schwartz derivative.).

On the other hand, let $\mathscr{R}^\alpha_{[0,T]}$ be the set of all $(1+d)$-dimensional α-Hölder branched rough paths $\mathbf{X} : [0,T]^2 \to \mathcal{G}$ whose zeroth component is time, i.e., $\langle \mathbf{X}_{s,t}, \bullet_0 \rangle = t - s$ and

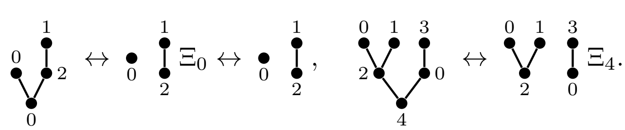
$$\langle \mathbf{X}_{s,t}, [\tau]_{\bullet_0} \rangle = \int_s^t \langle \mathbf{X}_{s,u}, \tau \rangle du, \quad \forall \tau \in \mathcal{H}, \; s, t \in [0,T]. \tag{29}$$

Observe that this condition necessarily implies that $\mathbf{X}$ satisfies condition (10) from Theorem 28 (cf. Remark 31). Note that $\mathbf{X}_{s,t}$ can be identified with an element of $\tilde{\mathcal{G}}_+$ due to the identification $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}_{+,}$.

Finally, observe that ϕ defined in (27) may be extended to an vector space isomorphism

$$\phi : \mathcal{B} \leftrightarrow \mathcal{H}\Xi_0 \oplus \mathcal{H}\Xi_1 \oplus .... \oplus \mathcal{H}\Xi_d \cong \mathcal{H} \oplus \mathcal{H}\Xi_1 \oplus .... \oplus \mathcal{H}\Xi_d \equiv \tilde{\mathcal{H}}$$

which maps a tree $\tau \in \mathcal{B}$ into a forest $\phi(\tau) \equiv \dot{\tau}$, as illustrated in



Conversely, ϕ^{-1} adds an extra node (which becomes the root) and should be thought of as taking the integral of a symbol in $\tilde{\mathcal{H}}$. The following result makes this precise by giving a bijection between $\mathscr{M}_{[0,T]}$ and $\mathscr{R}^\alpha_{[0,T]}$.

Proposition 45. *There is a bijective map* $I : \mathscr{R}^{\alpha}_{[0,T]} \to \mathscr{M}_{[0,T]}$ *which maps a branched rough path* $\mathbf{X}$ *to the unique model* $(\Pi, \Gamma) \in \mathscr{M}_{[0,T]}$ *with the property that*

$$(\Pi_s \mathcal{I}\dot{\tau})(t) = \langle \mathbf{X}_{s,t}, \tau \rangle \quad \forall \tau \in \mathcal{B} \quad \forall s, t \in [0,T],$$

where we have made the identifications $\phi(\tau) \equiv \dot{\tau} \in \tilde{\mathcal{H}} \leftrightarrow \tilde{\mathcal{T}}$*. Furthermore, the model* (Π, Γ) *satisfies* $\Gamma_{ts} = \Gamma_{\mathbf{X}_{s,t}}$ *(where we have made the identification* $\mathbf{X}_{s,t} \in \mathcal{G} \cong \tilde{\mathcal{G}}_+$*) and the multiplicativity property*

$$\Pi_t((\mathcal{I}y_1) \dots (\mathcal{I}y_n)) = \Pi_t(\mathcal{I}y_1) \dots \Pi_t(\mathcal{I}y_n), \quad \forall n \in \mathbb{N} \quad \forall y_i \in \tilde{\mathcal{T}}. \tag{30}$$

Proof. Consider $\mathbf{X} \in \mathscr{R}^{\alpha}_{[0,T]}$. For all $s, t \in [0,T]$ define $\Gamma_{ts} = \Gamma_{\mathbf{X}_{s,t}}$ and $(\Pi_s \mathcal{I}\dot{\tau})(t) = \langle \mathbf{X}_{s,t}, \tau \rangle$ for all $\tau \in \mathcal{B}$. Observe that we may further impose on (Π, Γ) that properties (i) and (ii) hold. Furthermore, for every $\tau \notin \mathcal{I}\tilde{\mathcal{T}}$, we may define $\Pi_t \tau = (\Pi_t \mathcal{I}\tau)'$, which completely characterises Π. It remains to verify (30), that property (iii) holds for all $\tau \in \mathcal{I}\tilde{\mathcal{T}}$, and that (Π, Γ) is indeed a model.

For (30), note that from (29)

$$\begin{aligned}
\Pi_t(\mathcal{I}\dot{\tau}_1 \dots \mathcal{I}\dot{\tau}_n) &= (\Pi_t \mathcal{I}(\mathcal{I}\dot{\tau}_1 \dots \mathcal{I}\dot{\tau}_n))' \\
&= (\langle \mathbf{X}_{t,\cdot}, \phi^{-1}(\mathcal{I}\dot{\tau}_1 \dots \mathcal{I}\dot{\tau}_n) \rangle)' \\
&= (\langle \mathbf{X}_{t,\cdot}, [\tau_1 \dots \tau_n]_{\bullet_0} \rangle)' \\
&= \langle \mathbf{X}_{t,\cdot}, \tau_1 \dots \tau_n \rangle \\
&= \langle \mathbf{X}_{t,\cdot}, \tau_1 \rangle \dots \langle \mathbf{X}_{t,\cdot}, \tau_n \rangle = \Pi_t(\mathcal{I}\dot{\tau}_1) \dots \Pi_t(\mathcal{I}\dot{\tau}_n).
\end{aligned}$$

To show property (iii) for $\dot{\tau} = \mathcal{I}\dot{\bar{\tau}} \in \mathcal{I}\tilde{\mathcal{T}}$, where $\dot{\bar{\tau}} \in \tilde{\mathcal{T}}$, observe that $\phi([\bar{\tau}]_{\bullet_0}) = \dot{\tau}$, so that again by (29)

$$\begin{aligned}
\Pi_t \dot{\tau} &= \Pi_t \mathcal{I}\dot{\bar{\tau}} \\
&= \langle \mathbf{X}_{t,\cdot}, \bar{\tau} \rangle \\
&= (\langle \mathbf{X}_{t,\cdot}, [\bar{\tau}]_{\bullet_0} \rangle)' \\
&= (\Pi_t \mathcal{I}\phi([\bar{\tau}]_{\bullet_0}))' \\
&= (\Pi_t \mathcal{I}\dot{\tau})'.
\end{aligned}$$

It remains to show that (Π, Γ) is a model. We first verify that $\Pi_s \Gamma_{s,t} = \Pi_t$. Let $\tau \in \mathcal{B}$, so that $\mathcal{I}(\dot{\tau}) \in \tilde{\mathcal{T}}$. Recall that the Connes-Kreimer coproduct $\Delta_\star : \mathcal{H} \mapsto \mathcal{H} \otimes \mathcal{H}$ as was introduced in Section 3.1 can be defined recursively by

$$\Delta_\star[\tau_1 \dots \tau_n]_{\bullet_i} = [\tau_1 \dots \tau_n]_{\bullet_i} \otimes 1 + (\mathrm{id} \otimes [\cdot]_{\bullet_i})\Delta_\star(\tau_1 \dots \tau_n), \;\; \forall \tau_1, \dots, \tau_n \in \mathcal{B}, \; i \in \{0, \dots, d\}.$$

With this recursion, one can verify that

$$\Delta^+ : \mathcal{I}(\tilde{\mathcal{T}}) \mapsto \mathcal{I}(\tilde{\mathcal{T}}) \otimes \tilde{\mathcal{T}}_+$$

agrees with the "reversed" Connes-Kreimer coproduct

$$\sigma_{1,2}\Delta_\star : \mathcal{B} \mapsto \mathcal{B} \otimes \mathcal{H},$$

where $\sigma_{1,2} : \mathcal{H} \otimes \mathcal{B} \mapsto \mathcal{B} \otimes \mathcal{H}$, $\sigma_{1,2} : \tau \otimes \bar{\tau} \mapsto \bar{\tau} \otimes \tau$, and where we make the usual identification $\mathcal{H} \leftrightarrow \tilde{\mathcal{T}}_+$ as well as $\phi^{\mathcal{I}} : \mathcal{B} \mapsto \mathcal{I}(\tilde{\mathcal{T}})$ via $\phi^{\mathcal{I}} : \tau \mapsto \mathcal{I}(\dot{\tau})$ (which is of course just $\mathcal{I} \circ \phi$). Therefore, treating $\mathbf{X}_{s,t}$ as a character on $\mathcal{H} \leftrightarrow \tilde{\mathcal{T}}_+$, we have for all $\tau \in \mathcal{B}$

$$\begin{aligned}(\Pi_t \Gamma_{ts} \mathcal{I}\dot{\tau})(u) &= (\Pi_t(\mathrm{id} \otimes \mathbf{X}_{s,t})\Delta^+ \mathcal{I}\dot{\tau})(u) \\ &= \langle \mathbf{X}_{t,u}, (\phi^{\mathcal{I}})^{-1}(\mathrm{id} \otimes \mathbf{X}_{s,t})\Delta^+ \mathcal{I}\dot{\tau})\rangle \\ &= \langle \mathbf{X}_{t,u}, (\mathbf{X}_{s,t} \otimes \mathrm{id})\Delta_\star \tau)\rangle \\ &= \langle \mathbf{X}_{s,t} \otimes \mathbf{X}_{t,u}, \Delta_\star \tau\rangle \\ &= \langle \mathbf{X}_{s,t} \dot{\otimes} \mathbf{X}_{t,u}, \tau\rangle \\ &= \langle \mathbf{X}_{s,u}, \tau\rangle \\ &= \Pi_s(\mathcal{I}\dot{\tau})(u).\end{aligned} \tag{31}$$

Observe now that for $\tau \in \tilde{\mathcal{T}}$, we have

$$\Gamma_{ts}\mathcal{I}\tau = \mathcal{I}\Gamma_{ts}\tau + \langle \mathbf{X}_{s,t}, \mathcal{I}\tau\rangle 1,$$

where we emphasize the symbol $1 \in \tilde{\mathcal{T}}$. Therefore, by the (already established) properties (i) and (iii), it follows that for any $\tau \in \tilde{\mathcal{T}}$

$$\Pi_t\Gamma_{ts}\tau = (\Pi_t\mathcal{I}\Gamma_{ts}\tau)' = (\Pi_t(\Gamma_{ts}\mathcal{I}\tau - \langle \mathbf{X}_{s,t}, \mathcal{I}\tau\rangle 1))' = (\Pi_t\Gamma_{ts}\mathcal{I}\tau)' = (\Pi_s\mathcal{I}\tau)' = \Pi_s\tau,$$

which shows that $\Pi_t\Gamma_{ts} = \Pi_s$.

It remains to verify the analytic bounds on (Π, Γ). As in Theorem 28, denote by $|\tau|$ the number of nodes in τ and by $|\tau|_0$ the number of nodes with the label 0. It follows that the degree of $\mathcal{I}\dot{\tau}$ is given by $|\mathcal{I}\dot{\tau}| = |\tau|_0(1-\alpha) + |\tau|\alpha$. Since $\mathbf{X}$ satisfies (10), we have the analytic bound

$$|(\Pi_s\mathcal{I}\dot{\tau})(t)| = |\langle \mathbf{X}_{s,t}, \tau\rangle| \lesssim |t-s|^{|\mathcal{I}\dot{\tau}|}.$$

Since $\Pi_s\tau = (\Pi_s\mathcal{I}\tau)'$ by property (iii), we see that Π satisfies the correct analytic bounds. The exact same argument applies to Γ upon using the identification of Δ^+ with $\sigma_{1,2}\Delta_\star$ above. Therefore (Π, Γ) is a model in $\mathscr{M}_{[0,T]}$ as claimed.

Finally, it remains to observe that we may reverse the construction. Indeed, starting with a model (Π, Γ) in $\mathscr{M}_{[0,T]}$, we may define $\mathbf{X}$ by $\langle \mathbf{X}_{s,t}, \tau\rangle = (\Pi_s\mathcal{I}\dot{\tau})(t)$. The facts that $\mathbf{X}$ satisfies (29) follows from property (iii), while the required analytic bounds for $\mathbf{X}$ to be an α-Hölder branched rough path follow from the analytic bounds associated to Π. To conclude, it suffices to verify that $\mathbf{X}$ thus defined satisfies $\Gamma_{ts} = \Gamma_{\mathbf{X}_{s,t}}$ and $\mathbf{X}_{s,t}\dot{\otimes}\mathbf{X}_{t,u} = \mathbf{X}_{s,u}$. To this end, note that by definition of the structure group G, there exists $\gamma_{ts} \in \tilde{\mathcal{G}}_+ \cong \mathcal{G}$ such that $\Gamma_{ts} = (\mathrm{id} \otimes \gamma_{ts})\Delta^+$. Let $\tilde{\mathbf{X}}_{s,t} \in \mathcal{G}$ be the element associated to γ_{ts} in the identification $\tilde{\mathcal{G}}_+ \cong \mathcal{G}$, and we aim to show $\tilde{\mathbf{X}}_{s,t} = \mathbf{X}_{s,t}$. Indeed, from our identification $\mathcal{H} \leftrightarrow \tilde{\mathcal{T}}_+$, it follows that for all $\tau \in \mathcal{B}$

$$\langle \gamma_{ts}, \mathcal{J}\dot{\tau}\rangle = \langle \tilde{\mathbf{X}}_{s,t}, \tau\rangle.$$

On the other hand, we know that for all $\tau \in \mathcal{B}$

$$\langle \mathbf{X}_{s,t}, \tau\rangle = (\Pi_s\mathcal{I}\dot{\tau})(t) = (\Pi_t\Gamma_{ts}\mathcal{I}\dot{\tau})(t) = (\Pi_t(\mathrm{id} \otimes \gamma_{ts})\Delta^+\mathcal{I}\dot{\tau})(t) = \langle \gamma_{ts}, \mathcal{J}\dot{\tau}\rangle,$$

where for the last equality we have used property (i) and the fact that

$$\Delta^+\mathcal{I}\dot{\tau} = 1 \otimes \mathcal{J}\dot{\tau} + \sum \mathcal{I}(\dot{\tau}^{(1)}) \otimes \dot{\tau}^{(2)},$$

where every term $\mathcal{I}(\dot{\tau}^{(1)})$ is of positive degree, and so $(\Pi_t\mathcal{I}(\dot{\tau}^{(1)}))(t) = 0$. This concludes the proof that $\Gamma_{ts} = \Gamma_{\mathbf{X}_{s,t}}$. To verify that $\mathbf{X}_{s,t}\dot{\otimes}\mathbf{X}_{t,u} = \mathbf{X}_{s,u}$, we can now simply reorder the sequence of equalities (31). $\square$

Following [BHZ16] we introduce the *renormalization map* M_ℓ given by[7]

$$\mathrm{M}_\ell : \tilde{\mathcal{T}} \to \tilde{\mathcal{T}}, \ \ \tau \mapsto (\ell \otimes \mathrm{id})\, \Delta^- \tau,$$

for a given character $\ell \in \mathcal{G}_- \subset \mathcal{T}_-^*$. In our case, we have the fact that M_ℓ commutes with $\mathcal{I}$ (cf. end of Remark 42)

$$\mathrm{M}_\ell \mathcal{I} = \mathcal{I} \mathrm{M}_\ell, \tag{32}$$

which is readily verified by hand: $\mathcal{I}$ amounts to adding another edge to the root (thereby creating a new root), whereas M_ℓ amounts to extracting (negative) subtrees and maps them to $\mathbb{R}$ (via ℓ). Clearly, the afore-mentioned edge (of degree 1) can not possibly be part of any singular subtree, hence the desired commutation.

This map acts on a model $\mathbf{\Pi} = (\Pi, \Gamma)$ and yields the *renormalised model* (see [BHZ16] Theorem 6.15) given by

$$\Pi_s^{\mathrm{M}_\ell} := \Pi_s \mathrm{M}_\ell, \quad \Gamma_{t,s}^{\mathrm{M}_\ell} = \left(\mathrm{id} \otimes \gamma_{t,s}^{\mathrm{M}_\ell}\right) \Delta^+, \quad \gamma_{t,s}^{\mathrm{M}_\ell} = \gamma_{t,s} \mathrm{M}_\ell.$$

Recall from Section 3.3 the map $\delta : \mathcal{B} \mapsto \mathcal{A} \otimes \mathcal{B}$, where $\mathcal{A}$ is the free commutative algebra generated by $\mathcal{B}$ (thought of as an isomorphic but different space to $\mathcal{H}$). Let $\pi_- : \tilde{\mathcal{H}} \cong \mathcal{B} \mapsto \mathcal{B}_- \cong \langle \mathcal{W}_- \rangle$ denote the projection onto terms of negative degree, which we extend multiplicatively to an algebra morphism $\pi_- : \mathcal{A} \mapsto \mathcal{H}_-$. We now define the map

$$\delta^- = (\pi_- \otimes \mathrm{id})\delta : \tilde{\mathcal{H}} \mapsto \mathcal{H}_- \otimes \tilde{\mathcal{H}}.$$

For instance

$$\delta^- \bullet_0 = 1 \otimes \bullet_0,$$

whereas

$$\delta \bullet_0 = \bullet_0 \otimes \bullet_0 + 1 \otimes \bullet_0.$$

We are now ready to state the link between translation of branched rough paths and negative renormalization in the following two results.

Lemma 46. (i) *We have*

$$\Delta^- \dot{\tau} = \Delta^- \phi\,(\tau) = (\phi \otimes \phi)\, \delta^-\,(\tau)\,.$$

(ii) *For all $v \in \mathcal{B}_-^*$ it holds that*

$$\mathrm{M}_\ell \dot{\tau} = \mathrm{M}_\ell \phi\,(\tau) = \phi\,(M_v^* \tau)$$

Proof. (i) Let us consider $[\tau]_{\bullet_i} \in \mathcal{B}$. We then have the following identities:

$$\Delta^- \phi([\tau]_{\bullet_i}) = \Delta^- \tau \Xi_i = \sum_{C = A \cdot B \subset \tau} \left(C \otimes (\mathcal{R}_C \tau)\Xi_i + A \cdot B\Xi_i \otimes \mathcal{R}_C \tau\right). \tag{33}$$

[7] While we deliberately used the same letter, do not confuse $\mathrm{M}_\ell : \tilde{\mathcal{T}} \to \tilde{\mathcal{T}}$ with $M_v : \mathcal{H}^* \to \mathcal{H}^*$.

The sum is taken over all the couples (A, B) where A is a negative subforest of τ which does not include the root of τ and B is a subtree of τ at the root disjoint from A. In the sum in (33), the first term means that Ξ_i does not belong to the tree extracted at the root, while for the second term, Ξ_i belongs to the tree which comes from the product between Ξ_i and B giving a subtree of negative degree. One can derive the same identity for δ^-. We first rewrite δ^-:

$$\delta^- \tau = \sum_{A \subset \tau} A \otimes \tilde{\mathcal{R}}_A \tau,$$

where A is a subforest of τ and $\tilde{\mathcal{R}}_A \tau$ means that we contract the trees of A in τ and we leave a 0 decoration on their roots. Then the equivalent of (33) in that context is given by:

$$\begin{aligned}\delta^- [\tau]_{\bullet_i} &= \sum_{\tilde{C} = \tilde{A} \cdot \tilde{B} \subset \tau} \left(\tilde{C} \otimes [\tilde{\mathcal{R}}_{\tilde{C}} \tau]_{\bullet_i} + \tilde{A} \cdot [\tilde{B}]_{\bullet_i} \otimes \tilde{\mathcal{R}}_{\tilde{A} \cdot [\tilde{B}]_{\bullet_i}} [\tau]_{\bullet_i} \right) \\ (\phi \otimes \phi)\, \delta^- [\tau]_{\bullet_i} &= \sum_{\tilde{C} = \tilde{A} \cdot \tilde{B} \subset \tau} \left(\phi(\tilde{C}) \otimes (\tilde{\mathcal{R}}_{\tilde{C}} \tau) \Xi_i + \phi(\tilde{A}) \cdot \tilde{B} \Xi_i \otimes \phi\left(\tilde{\mathcal{R}}_{\tilde{A} \cdot [\tilde{B}]_{\bullet_i}} [\tau]_{\bullet_i} \right) \right).\end{aligned}$$

Now we have the following identifications:

$$\phi(\tilde{C}) \leftrightarrow C, \quad \tilde{B}\Xi_i \leftrightarrow B\Xi_i, \quad \phi\left(\tilde{\mathcal{R}}_{\tilde{A} \cdot [\tilde{B}]_{\bullet_i}} [\tau]_{\bullet_i} \right) = \tilde{\mathcal{R}}_{\tilde{C}} \tau \leftrightarrow \mathcal{R}_C \tau, \quad (\tilde{\mathcal{R}}_{\tilde{C}} \tau) \Xi_i \leftrightarrow (\mathcal{R}_C \tau) \Xi_i,$$

which gives the result.

(ii) Recall that $\delta^-(\tau)$ has an image of the form "forest $\otimes$ tree", and that $\ell \circ \phi = v$ (which is a "dual" tree and multiplicative over forests). Also note that $M_v^* \tau = (v \otimes \mathrm{id})\, \delta = (v \otimes \mathrm{id})\, \delta^-$ whenever $v \in \mathcal{B}_-^*$ (which not true for general $v \in \mathcal{B}^*$), so that

$$\begin{aligned} \mathrm{M}_\ell \dot{\tau} &= (\ell \otimes \mathrm{id})\, \Delta^- \dot{\tau} \\ &= (\ell \otimes \mathrm{id})\, \Delta^- \phi(\tau) \\ &= (v \otimes \phi)\, \delta^-(\tau) \\ &= \phi\left((v \otimes \mathrm{id})\, \delta^- \right) \\ &= \phi(M_v^* \tau). \end{aligned}$$

□

Theorem 47. (i) *It holds that the restriction* $\Delta^- : \tilde{\mathcal{T}} \mapsto \mathcal{T}_- \otimes \tilde{\mathcal{T}}$ *coincides with* $\delta^- : \tilde{\mathcal{H}} \mapsto \mathcal{H}_- \otimes \tilde{\mathcal{H}}$, *where we have made the identifications* $\tilde{\mathcal{H}} \leftrightarrow \tilde{\mathcal{T}}$ *and* $\mathcal{H}_- \leftrightarrow \mathcal{T}_-$ *as above.*

(ii) *Let* v *be an element of* $\mathcal{B}_-^*$ *and let* $\ell \in \mathcal{G}_-$ *by the associated element in* $\mathcal{G}_- \subset \mathcal{T}_-^*$, *as was detailed in* (28). *Then the following diagram commutes*

$$\begin{array}{ccc} \mathbf{X} & \longleftrightarrow & \mathbf{\Pi} \\ \downarrow & & \downarrow \\ M_v \mathbf{X} & \longleftrightarrow & \mathbf{\Pi}^{\mathrm{M}_\ell} \end{array}$$

(iii) *For* $v, v' \in \mathcal{B}_-$ *with associated characters* $\ell, \ell' \in \mathcal{G}_-$, *it holds that the character associated to* $v + v'$ *is* $\ell \circ \ell'$, *so that* $(\mathcal{B}_-, +) \cong (\mathcal{G}_-, \circ)$.

Remark 48. The final statement (iii) effectively says that the renormalization group associated to branched rough path is always abelian, despite the highly non-commutative nature of $\mathcal{H}^*$, the Grossman-Larson Hopf algebra.

Remark 49. The above theorem along with the discussion in Section 5.2 (see in particular Remark 33 and Theorem 36) provides a straightforward example of how S(P)DEs change under the negative renormalization maps in the sense of [BHZ16].

Proof of Theorem 47. Part (i) is a straightforward consequence of Lemma 46 (i).

To verify (ii), in view of Proposition 45, we only need to check

$$\Pi_s^{\mathrm{M}_l}\mathcal{I}\dot{\tau} = \langle M_v \mathbf{X}_{s,\cdot}, \tau\rangle = \langle \mathbf{X}_{s,\cdot}, M_v^*\tau\rangle \quad \forall \tau \in \mathcal{B}.$$

The LHS can be rewritten as, thanks to (32) and Lemma 46 (ii)

$$\begin{aligned} \Pi_s^{\mathrm{M}_\ell}\mathcal{I}\dot{\tau} &= \Pi_s \mathrm{M}_\ell \mathcal{I}\dot{\tau} \\ &= \Pi_s \mathcal{I}\mathrm{M}_\ell \dot{\tau} \\ &= \Pi_s \mathcal{I}\phi\left(M_v^*\tau\right). \end{aligned}$$

Applying Proposition 45 with $\dot{\tau} = \phi\left(M_v\tau\right)$ then shows that

$$\Pi_s \mathcal{I}\phi\left(M_v\tau\right) = \langle \mathbf{X}_{s,\cdot}, M_v^*\tau\rangle$$

which is what we wanted to show.

Finally, to show (iii), we note that

$$\langle \ell \circ \ell', \tau\rangle = \langle \ell \otimes \ell', \Delta^- \tau\rangle = \langle \ell, \tau\rangle + \langle \ell', \tau\rangle, \quad \forall \tau \in \mathcal{W}_-,$$

where the first equality follows by definition and the second from the fact that every element of $\mathcal{W}_-$ is primitive with respect to the coproduct Δ^-. Indeed from the Remark 40, we deduce that the coaction Δ^- maps every $\tau \in \mathcal{W}_-$ into $\tau \otimes \mathbf{1} + \sum_{(\tau)} \tau' \otimes \tau''$ such that τ'' is a tree of positive degree. However, Δ^- as coproduct on $\mathcal{T}_-$ (see (17)), will annihilate any term with τ'' of (strictly) positive degree. In particular then, $\Delta^-\tau = 1 \otimes \tau + \tau \otimes 1$ for all $\tau \in \mathcal{W}_-$, that is, any such τ is primitive. □

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