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Run and tumble particle under resetting: a renewal approach

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Abstract. We consider a particle undergoing run and tumble dynamics, in which its velocity stochastically reverses, in one dimension. We study the addition of a Poissonian resetting process occurring with rate r. At a reset event the particle's position is returned to the resetting site X_r and the particle's velocity is reversed with probability η. The case $\eta = 1/2$ corresponds to position resetting and velocity randomization whereas $\eta = 0$ corresponds to position-only resetting. We show that, beginning from symmetric initial conditions, the stationary state does not depend on η i.e. it is independent of the velocity resetting protocol. However, in the presence of an absorbing boundary at the origin, the survival probability and mean time to absorption do depend on the velocity resetting protocol. Using a renewal equation approach, we show that the the mean time to absorption is always less for velocity randomization than for position-only resetting.

1. Introduction

Stochastic processes with correlated noise have a long history in physics beginning with the Ornstein Uhlenbeck process which generates a finite correlation time for Brownian motion [1, 2]. More recently, active matter has been described by stochastic processes with correlated noise such as the run and tumble dynamics used to model bacterial motion. Run and tumble dynamics is, in turn, the continuum limit of persistent random walkers [3, 4] commonly used to model animal movement [5] and search processes [6]. Also the bidirectional motion of cellular cargoes is, in general, a correlated random walk [7].

A particle under run and tumble dynamics, in one dimension, obeys a Langevin equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_0\sigma(t) \tag{1}$$

Here σ is a stochastic process which switches between two states $\sigma = \pm 1$ with rate γ. Thus a run and tumble process is sometimes referred to as 'telegraphic noise' to describe the evolution of σ [8]. Non-Poissonian switching has been considered in [9].

Run and tumble dynamics have revealed a number of interesting nonequilibrium properties such as clustering at boundaries [10, 11], novel stationary states [12, 13] and first-passage properties [14, 15]. In this paper we investigate the effect of resetting [16] on run and tumble dynamics.

Resetting is the procedure of restarting a stochastic process from a given initial condition [17]. The initial condition can be fixed [16] or chosen from some resetting distribution [18]. In this way the system is held away from any long time stationary state and a nonequilibrium stationary state is generated [19]. Interesting transient properties of the relaxation to this state have been revealed [20]. Also the resetting process can be considered as a realisation of an intermittent search process [21, 22] where a reset event is a long range move. The resetting process is usually considered to be a Poisson process with exponentially distributed waiting times between resetting events, however more general waiting time distributions including power law distributions have been considered [23, 24, 25]. Moreover, it has been shown that a deterministic resetting period may be optimal in the minimisation of mean search times or mean first passage times [24, 26, 27, 28]. Resetting with memory, where a walker resets only to previously visited sites with a certain distribution, have also been studied [29, 30, 31, 32]. While some interesting properties of the mean first-passage time and its fluctuations for Markov processes with resetting (i.e., without any memory of the pre-resetting history) have been derived [18, 33, 34, 27, 28], many fundamental questions concerning the full first-passage probability under resetting, with or without memory effects, still remain open [35, 31].

Recent variations on the resetting theme have been to consider: resetting of discrete-time Lévy flights [36] and continuous-time Lévy walks [37, 38], resetting for random walks in a bounded domain [39, 38], resetting of extended systems such as fluctuating interfaces [40] and a reaction diffusion process in one dimension [41], Michaelis-Menten reaction schemes [42, 34]; the thermodynamics of resetting [43, 44] and large deviations of the additive functionals of resetting processes [45, 46, 47], interaction-driven resetting [48], resetting with branching [49] and fractional Brownian motion with resetting [50]. Very recently, resetting dynamics in quantum systems have also been studied [51, 52].

Stochastic process of the form (1) open up new possibilities for resetting as the velocity variable σ as well as the position x may be reset. (Previously orientation resets have been considered in continuous time random walks with drifts [53].) We focus on the case where the position is reset to a fixed resetting point X_r and simultaneously the velocity undergoes a resetting protocol. Initially, we consider velocity randomization in which the velocity is reversed with probability 1/2. Later we generalise this to the

protocol where the velocity is reversed with probability η which allows interpolation to the case of position-only resetting where $\eta = 0$.

We employ a renewal equation approach first noted in [16] (see also [36] for the computation of first-passage probability using the renewal approach and [54] for recent work). For the velocity randomization case we may use a simple renewal equation for the survival probability (29), which is applicable to many systems resetting to their initial conditions. In the general velocity resetting case a system of renewal equations for joint probabilities of survival and velocity is required (55).

Our study reveals that resetting of position and velocity of run and tumble particle results in nonequilibrium stationary state that is a Laplace distribution (symmetric, exponential decay) which is of the same form as a diffusive particle with position resetting. However the survival probability of the path particle and the mean first passage time do depend on the velocity resetting parameter η.

The paper is organised as follows. In section 2 we review run and tumble dynamics as described by a Master equation system. We then consider position resetting and velocity randomization and compute the stationary state in section 3, survival probability in section 4 and mean time to absorption in section 5. In section 6 we consider general velocity resetting parametrised by η and present a general renewal scheme. We work out particular formulae for the mean time to absorption for position-only resetting and compare to the velocity randomization case. We conclude in section 7.

2. Run and tumble particle dynamics

In this section we review the dynamics of a run and tumble particle (see for example [3, 15, 4]). The system of forward master equations (in the absence of resetting) read

$$\frac{\partial P_+(x,t)}{\partial t} = -v_0 \frac{\partial P_+(x,t)}{\partial x} - \gamma P_+(x,t) + \gamma P_-(x,t) \tag{2}$$

$$\frac{\partial P_-(x,t)}{\partial t} = +v_0 \frac{\partial P_-(x,t)}{\partial x} - \gamma P_-(x,t) + \gamma P_+(x,t) \tag{3}$$

where $P_\sigma(x,t)$ is the probability density for the particle to have velocity σ and be at position x at time t. The terms proportional to γ originate from the switching of velocity with rate γ; the correlation time of the velocity is thus $1/\gamma$. We note that the system is invariant under time reversal: $\sigma \to -\sigma$ and $v_0 \to -v_0$.

It will be convenient to have at our disposal the form of the Laplace transforms of $P_\pm$, which are defined as

$$\tilde{P}_\pm(x,s) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-st} P_\pm(x,t) \,. \tag{4}$$

Taking the Laplace transform of (2,3) we obtain the system

$$-P_+(x,0) + v_0 \frac{\mathrm{d}\tilde{P}_+}{\mathrm{d}x} + (s+\gamma)\tilde{P}_+ - \gamma \tilde{P}_- = 0 \tag{5}$$

$$-P_-(x,0) - v_0 \frac{\partial \tilde{P}_+}{\partial x} + (s+\gamma)\tilde{P}_- - \gamma \tilde{P}_+ = 0 \,. \tag{6}$$

We need to fix initial conditions which we choose to be at the origin and symmetric

$$P_+(x,0) = P_-(x,0) = \frac{1}{2}\delta(x) \,, \tag{7}$$

i.e. the particle begins at the origin with equal probability for the velocity $\sigma(0) = \pm 1$.

By taking a further spatial derivative and rearranging, we may turn the first-order system (5, 6) into *decoupled* second-order equations which read (away from $x = 0$)

$$v_0^2 \frac{d^2 \tilde{P}_\pm}{dx^2} - \left[(s+\gamma)^2 - \gamma^2\right] \tilde{P}_\pm = 0 \,. \tag{8}$$

The solutions which respect the boundary conditions that $P_\pm$ remain finite as $x \to \pm\infty$ are

$$\tilde{P}_\pm = A_\pm e^{-\lambda x} \quad \text{for} \quad x > 0 \tag{9}$$
$$\tilde{P}_\pm = B_\pm e^{+\lambda x} \quad \text{for} \quad x < 0 \tag{10}$$

where

$$\lambda = \left(\frac{s(s+2\gamma)}{v_0^2}\right)^{1/2} \,. \tag{11}$$

In order to fix the coefficients $A_\pm$, $B_\pm$ we go back to (5, 6) and obtain conditions

$$(s + \gamma - \lambda v_0)A_+ = \gamma A_- \tag{12}$$
$$(s + \gamma + \lambda v_0)B_+ = \gamma B_- \,. \tag{13}$$

Also, as the initial condition is symmetric around $x = 0$ and the dynamics is invariant under time reversal, the total probability $P(x,t) = P_+(x,t) + P_-(x,t)$ must be symmetric about $x = 0$. This implies

$$A_+ + A_- = B_+ + B_- \,. \tag{14}$$

Finally, normalisation of probability dictates

$$\int dx \left[\tilde{P}_+ + \tilde{P}_-\right] = \frac{1}{s} \tag{15}$$

which implies

$$A_+ + A_- = B_+ + B_- = \frac{\lambda}{2s} \tag{16}$$

so that

$$\tilde{P}(x,t) = \tilde{P}_+(x,t) + \tilde{P}_-(x,t) = \frac{\lambda}{2s}e^{-\lambda|x|} \,. \tag{17}$$

The Laplace transform (17) is sufficient for our purposes in the next section. However, it is possible to invert the Laplace transform [3] to obtain the time-dependent distribution

$$P(x,t) = \frac{e^{-\gamma t}}{2}\left\{\delta(x - v_0 t) + \delta(x + v_0 t) + \frac{\gamma}{2v_0}\left[I_0(\rho) + \frac{\gamma I_1(\rho)}{\rho}\right]\Theta(v_0 t - |x|)\right\} \tag{18}$$

where

$$\rho = \sqrt{v_0^2 t^2 - x^2}\frac{\gamma}{v_0} \tag{19}$$

and $I_0(\rho)$ and $I_1(\rho)$ are modified Bessel functions of the first kind.

Equation (17) is the main result of this section. For completeness we present the coefficients appearing in (9, 10) which can be found by solving (12, 13, 16)

$$A_+ = \frac{\lambda\gamma}{2s(s + 2\gamma - \lambda v_0)} \quad A_- = \frac{\lambda(s + \gamma - \lambda v_0)}{2s(s + 2\gamma - \lambda v_0)} \tag{20}$$

$$B_+ = \frac{\lambda\gamma}{2s(s + 2\gamma + \lambda v_0)} \quad B_- = \frac{\lambda(s + \gamma + \lambda v_0)}{2s(s + 2\gamma + \lambda v_0)} . \tag{21}$$

3. Run and tumble particle under position resetting and velocity randomization

We now add a resetting process to the dynamics, which comprises simultaneous resetting both the position and the velocity. In this section the resetting position is taken to be the origin. With rate r the particle resets its initial position and the velocity σ is chosen to be ± 1 with probability $1/2$, i.e. the velocity is randomized. We refer to this resetting protocol as position resetting and velocity randomization. We shall consider more general resetting protocols in section 6.

Given that the initial condition of the particle is also at the origin with the velocity σ is chosen to be ± 1 with probability $1/2$, we may write down a renewal equation [16] for the total probability density in the presence of resetting which we denote $P_r(x, t)$:

$$P_r(x, t) = e^{-rt} P_0(x, t) + r \int_0^t d\tau e^{-r\tau} P_0(x, \tau) \int dx' P_r(x', t - \tau) \tag{22}$$

$$= e^{-rt} P_0(x, t) + r \int_0^t d\tau e^{-r\tau} P_0(x, \tau) . \tag{23}$$

Here, $P_0(x, t)$ is the probability density without resetting considered in section 2. The first term on the r.h.s of (22) is the contribution from trajectories in which there is no resetting, which occurs with probability e^{-rt}; the second term integrates the contributions from trajectories in which the last reset occurs at time $t - \tau$ and at position x' and there is no resetting from time $t - \tau$ to t, which occurs with probability $e^{-r\tau}$. The second equality (23) comes from the fact that when there are no absorbing boundaries probability is conserved which implies $\int dx' P_r(x', t - \tau) = 1$.

3.1. Stationary State and limits

The stationary distribution

$$P_r^{st}(x) = \lim_{t \to \infty} P_r(x, t) \tag{24}$$

is easily obtained from the limit $t \to \infty$ of (23) from which we learn, using the result (17) of section 2, that

$$P_r^{st}(x) = r\tilde{P}(x, r) = \frac{\lambda_r}{2} e^{-\lambda_r |x|} . \tag{25}$$

where λ_r is given by

$$\lambda_r = \left(\frac{r(r+2\gamma)}{v_0^2} \right)^{1/2} . \tag{26}$$

The distribution is a double exponential distribution, also known as Laplace distribution, with decay length

$$\ell \equiv \frac{1}{\lambda_r} = \left(\frac{v_0^2}{r(r+2\gamma)} \right)^{1/2} . \tag{27}$$

Thus the decay length increases with the speed v_0 but decreases with switching rate γ and resetting rate r.

It is of interest to consider the various limits of the process and the form of the decay length in these limits. First as $r \to 0$, the limit of no resetting, the decay length diverges as $r^{-1/2}$ indicating that there is no stationary state. The *ballistic limit* is when the switching rate $\gamma \to 0$ in which case $\ell \to \frac{v_0}{r}$ which is the mean distance travelled between resets. Finally the *diffusive limit* occurs when both v_0 and γ diverge but

$$\lim_{v_0,\gamma\to\infty} \frac{v_0^2}{\gamma} = 2D \tag{28}$$

where D is the diffusion coefficient. Then $\ell \to \sqrt{\frac{D}{r}}$ which recovers the expression for diffusive resetting [16].

4. Survival Probability

We now consider the survival probability of the persistent random walker in the presence of an absorbing boundary at the origin (and under position resetting to $X_r \geq 0$ and velocity randomization as in section 3). In the context of a search we refer to the origin as the target; clearly, the event of particle touching the boundary corresponds to the event of a searcher locating the target.

We again take advantage of a renewal equation. We first define $Q_r(x_0, t)$ as the survival probability in the *presence* of resetting and $Q_0(x_0, t)$ as the survival probability in the *absence* of resetting, for a particle having started from initial position $x = x_0 \geq 0$ with initial velocity chosen to be $\sigma = \pm 1$ with equal probability $1/2$. Note that $Q_r(x_0, t)$ implies an integration over the final position of the particle. Also note that the initial position x_0 is a variable which, at the end of the calculation, we may set equal to X_r.

Then we have a renewal equation analogous to (22)

$$Q_r(x_0, t) = e^{-rt} Q_0(x_0, t) + r \int_0^t d\tau e^{-r\tau} Q_0(X_r, \tau) Q_r(x_0, t-\tau) . \tag{29}$$

Again, the first term on the r.h.s is the contribution from survival trajectories in which there is no resetting; the second term integrates the contributions from survival trajectories in which the last reset occurs at time $t - \tau$.

Taking the Laplace Transform

$$\tilde{Q}_*(x_0, s) = \int_0^\infty dt\, e^{-st} Q_*(x_0, t) . \tag{30}$$

where $*$ indicates 0 or r, we readily obtain from (29)

$$\tilde{Q}_r(x_0, s) = \frac{\tilde{Q}_0(x_0, r + s)}{1 - r\tilde{Q}_0(X_r, r + s)} \tag{31}$$

and, in particular, setting the initial position $x_0 = X_r$,

$$\tilde{Q}_r(X_r, s) = -\frac{1}{r} + \frac{1}{1 - r\tilde{Q}_0(X_r, r + s)} . \tag{32}$$

Equation (32) is an equation of rather general applicability, which applies whenever resetting to the initial conditions is a Poisson process with rate r.

4.1. Survival probability in the absence of resetting

In view of (32) we just need to compute $Q_0(X_r, s)$, the Laplace transform of the survival probability in the absence of resetting. To do so it is convenient to introduce $Q_0^+(x_0, t)$ and $Q_0^-(x_0, t)$ as the survival probability (without resetting) for a particle having started from position $x = x_0 \geq 0$ with initial velocity ± 1 respectively.

We can write down a system of *backward* evolution equations for these survival probabilities

$$\frac{\partial Q_0^+(x_0, t)}{\partial t} = v_0 \frac{\partial Q_0^+(x_0, t)}{\partial x_0} - \gamma Q_0^+(x_0, t) + \gamma Q_0^-(x_0, t) \tag{33}$$

$$\frac{\partial Q_0^-(x_0, t)}{\partial t} = -v_0 \frac{\partial Q_0^-(x_0, t)}{\partial x_0} - \gamma Q_0^-(x_0, t) + \gamma Q_0^+(x_0, t) . \tag{34}$$

which needs to be solved in the positive half-space $x_0 \geq 0$. The initial conditions are $Q_0^+(x_0, 0) = Q_0^-(x_0, 0) = 1$ and the boundary condition, which imposes an absorbing boundary at $x = 0$, is just $Q_0^-(0, t) = 0$. This is because if the particle starts at the origin with a negative initial velocity it can not survive up to finite time t. In contrast, if it starts with a positive velocity, it can survive and $Q_0^+(0, t)$ is therefore unspecified and has to be determined a posteriori. In fact, as we will see below that just the single condition $Q_0^-(0, t) = 0$ is sufficient to provide a unique solution to this system of coupled equations.

Taking the Laplace transform of (33,34) yields

$$+v_0 \frac{\partial \tilde{Q}_0^+}{\partial x_0} - (s + \gamma)\tilde{Q}_0^+ + \gamma\tilde{Q}_0^- = -1 \tag{35}$$

$$-v_0 \frac{\partial \tilde{Q}_0^-}{\partial x_0} - (s + \gamma)\tilde{Q}_0^- + \gamma\tilde{Q}_0^+ = -1 \tag{36}$$

from which a further spatial derivative and rearrangement yields the decoupled equation

$$v_0^2 \frac{\partial^2 \tilde{Q}_0^-}{\partial x_0^2} - s(s + 2\gamma)\tilde{Q}_0^- = -\frac{(2\gamma + s)}{v_0} . \tag{37}$$

The solution which satisfies the boundary condition $Q_-(0, t) = 0$ is

$$\tilde{Q}_0^-(x_0, s) = \frac{1}{s}\left[1 - e^{-\lambda x_0}\right] \tag{38}$$

where λ is given by (11). Substituting back into (36) yields

$$\tilde{Q}_0^+(x_0, s) = \frac{1}{s} + \frac{1}{\gamma s}\left[v_0\lambda - (s+\gamma)\right]e^{-\lambda x_0} . \tag{39}$$

Given the symmetric velocity initial condition, we have

$$\tilde{Q}_0(x_0, s) \equiv \frac{1}{2}\left[\tilde{Q}_0^+(x_0, s) + \tilde{Q}_0^-(x_0, s)\right]$$

$$= \frac{1}{s} + \frac{1}{2\gamma s}\left[v_0\lambda - (s+2\gamma)\right]e^{-\lambda x_0} . \tag{40}$$

Inserting (40) into (32) yields the result

$$\tilde{Q}_r(X_r, s) = -\frac{1}{r} + \frac{1}{r}\left[\frac{2\gamma(s+r)e^{\lambda_{s+r}X_r}}{2\gamma s e^{\lambda_{s+r}X_r} - r\left[v_0\lambda_{r+s} - (r+s+2\gamma)\right]}\right] \tag{41}$$

where

$$\lambda_{r+s} = \left(\frac{(r+s)(r+s+2\gamma)}{v_0^2}\right)^{1/2} . \tag{42}$$

5. Mean first passage time

The mean first passage time to the origin (or equivalently the mean time to absorption at the origin), $T(X_r)$, is conveniently given by

$$T(X_r) = \tilde{Q}(X_r, s = 0) . \tag{43}$$

In the $s \to 0$ limit it can be checked that (41) reduces to

$$T(X_r) = -\frac{1}{r} + \frac{2\gamma}{r}\left[\frac{e^{\lambda_r X_r}}{r + 2\gamma - (r(r+2\gamma))^{1/2}}\right] \tag{44}$$

where λ_r is given by (26).

First let us check the diffusive limit (28) in which case $\lambda_r \to (r/D)^{1/2}$ and

$$T(X_r) \to -\frac{1}{r} + \frac{e^{(r/D)^{1/2}X_r}}{r} \tag{45}$$

recovering the result of [16].

We also note that $T(X_r)$ diverges as $r^{-1/2}$ as $r \to 0$ and also diverges exponentially in r as $r \to \infty$ implying a minimum value at intermediate r. In order to analyse where this minimum occurs it is useful to introduce reduced variables

$$R = \frac{r}{2\gamma} \tag{46}$$

$$\xi = \frac{2\gamma X_r}{v_0} . \tag{47}$$

R is half the ratio of resetting rate to velocity switching rate whereas ξ is twice the ratio of distance to the target to mean distance travelled between reversals of velocity (the factors of two are included for later convenience). In terms of these variables

$$\lambda_r X_r = (R(R+1))^{1/2}\xi \tag{48}$$

and one obtains

$$2\gamma T(R,\xi) = -\frac{1}{R} + \frac{e^{(R(1+R))^{1/2}\xi}}{R\left[1 + R - (R(1+R))^{1/2}\right]} .$$
(49)

We may minimise this expression with respect to R at fixed ξ. It has a unique minimum. A plot of $T(R, \xi = 1)$ vs. R is shown in Fig. (1).

6. General velocity resetting

We now consider a more general resetting process which comprises simultaneous resetting of both the position and the velocity. With rate r the particle resets its initial position at X_r and the velocity σ is reversed to $-\sigma$ with probability η or remains σ with probability $1 - \eta$. The case $\eta = 1/2$ corresponds to the velocity randomization considered in earlier sections and the case $\eta = 0$ corresponds to position-only resetting.

The first thing to note is that given a *symmetric* initial condition the stationary state is independent of η. The reason is that the velocity distribution will remain symmetric and is independent of η. To demonstrate this explicitly we let $P_*^{\sigma_f \sigma_i}(x, t)$ be the probability density of being at x at time t and having velocity σ_f given that the particle began at $t = 0$ at X_r with velocity σ_i; $*$ indicates 0 or r and corresponds to no resetting or with resetting respectively. We may then write down the following renewal equation system

$$P_r^{\sigma_f \sigma_i}(x, t) = e^{-rt} P_0^{\sigma_f \sigma_i}(x, t)$$
(50)
$$+ r \int_0^t d\tau e^{-r\tau} \int dx' \left\{ P_r^{\sigma_f \sigma_i}(x', t - \tau) \left[(1 - \eta) P_0^{\sigma_f \sigma_f}(x, \tau) + \eta P_0^{\sigma_f -\sigma_f}(x, \tau)\right] \right.$$
$$\left. + P_r^{-\sigma_f \sigma_i}(x', t - \tau) \left[(1 - \eta) P_0^{\sigma_f -\sigma_f}(x, \tau) + \eta P_0^{\sigma_f \sigma_f}(x, \tau)\right] \right\} .$$

Now let us fix the initial conditions at $t = 0$ as $\sigma = \pm 1$ with probability $1/2$ and define

$$P_*^{\sigma_f}(x, t) = \frac{1}{2} P_*^{\sigma_f +}(x, t) + \frac{1}{2} P_*^{\sigma_f -}(x, t) .$$
(51)

The system (50) becomes

$$P_r^{\sigma_f}(x, t) = e^{-rt} P_0^{\sigma_f}(x, t)$$
(52)
$$+ r \int_0^t d\tau e^{-r\tau} \int dx' \left\{ P_r^{\sigma_f}(x', t - \tau) \left[(1 - \eta) P_0^{\sigma_f \sigma_f}(x, \tau) + \eta P_0^{\sigma_f -\sigma_f}(x, \tau)\right] \right.$$
$$\left. + P_r^{-\sigma_f}(x', t - \tau) \left[(1 - \eta) P_0^{\sigma_f -\sigma_f}(x, \tau) + \eta P_0^{\sigma_f \sigma_f}(x, \tau)\right] \right\} .$$

Now due to the symmetric initial condition we have $\int dx' P_r^{\sigma_f}(x, t) = 1/2$ and we find that the terms with coefficient η in (52) cancel, leaving

$$P_r^{\sigma_f}(x, t) = e^{-rt} P_0^{\sigma_f}(x, t)$$
(53)
$$+ \frac{r}{2} \int_0^t d\tau e^{-r\tau} \left\{ P_0^{\sigma_f \sigma_f}(x, \tau) + P_0^{\sigma_f -\sigma_f}(x, \tau) \right\}$$
(54)

which recovers (23). Thus, the stationary state of the resetting run and tumble particle does not depend on the velocity resetting protocol.

However, as we shall now show the survival probability in the presence of an absorbing boundary does depend on η.

6.1. Survival probability

In order to solve for the survival probability in the general case we define $Q_r^{\sigma_f \sigma_i}(t)$ and $Q_0^{\sigma_f \sigma_i}(t)$ as the joint probability of survival and having velocity σ_f at time t, given that the particle began at X_r with velocity σ_i, with and without resetting respectively. To ease the notation we shall drop the dependence on initial position (which is always X_r) from the $Q_r^{\sigma_f \sigma_i}(t)$.

Then we may write down a renewal equation system as follows

$$Q_r^{\sigma_f \, \sigma_i}(t) = \mathrm{e}^{-rt} Q_0^{\sigma_f \, \sigma_i}(t) \tag{55}$$
$$+ r \int_0^t \mathrm{d}\tau \mathrm{e}^{-r\tau} \left\{ Q_r^{\sigma_f \, \sigma_i}(t - \tau) \left[(1 - \eta) Q_0^{\sigma_f \, \sigma_f}(\tau) + \eta Q_0^{\sigma_f \, -\sigma_f}(\tau) \right] \right.$$
$$\left. + Q_r^{-\sigma_f \, \sigma_i}(t - \tau) \left[(1 - \eta) Q_0^{\sigma_f \, -\sigma_f}(\tau) + \eta Q_0^{\sigma_f \, \sigma_f}(\tau) \right] \right\} .$$

Again this equation is easily understood: the first term represents surviving trajectories within which no resetting occurred; the second term integrates up the surviving trajectories which have the last reset at time $t - \tau$ and the coefficients $(1 - \eta)$ and η give the probability of a velocity switch occurring at that reset.

We now take the Laplace transform

$$\tilde{Q}_*^{\sigma_f \, \sigma_i}(s) = \int_0^\infty \mathrm{d}t \, \mathrm{e}^{-st} Q_*^{\sigma_f \, \sigma_i}(t) \tag{56}$$

with $* = 0, r$, to obtain

$$\tilde{Q}_r^{\sigma_f \, \sigma_i}(s) = \tilde{Q}_0^{\sigma_f \, \sigma_i}(r + s) \tag{57}$$
$$+ r \left\{ Q_r^{\sigma_f \, \sigma_i}(s) \left[(1 - \eta) \tilde{Q}_0^{\sigma_f \, \sigma_f}(r + s) + \eta \tilde{Q}_0^{\sigma_f \, -\sigma_f}(r + s) \right] \right.$$
$$\left. + \tilde{Q}_r^{-\sigma_f \, \sigma_i}(s) \left[(1 - \eta) \tilde{Q}_0^{\sigma_f \, -\sigma_f}(r + s) + \eta \tilde{Q}_0^{\sigma_f \, \sigma_f}(r + s) \right] \right\}$$

As usual the initial conditions at $t = 0$ are $\sigma = \pm 1$ with probability $1/2$ and we define

$$Q_*^{\sigma_f}(t) = \frac{1}{2} Q_*^{\sigma_f \, +}(t) + \frac{1}{2} Q_*^{\sigma_f \, -}(t) \tag{58}$$

with similar definitions for the Laplace transforms. Then system (57) becomes (where we now write out explicitly the two equations)

$$\tilde{Q}_r^+(s) = \tilde{Q}_0^+(r + s) \tag{59}$$
$$+ r \left\{ Q_r^+(s) \left[(1 - \eta) \tilde{Q}_0^{+ \, +}(r + s) + \eta \tilde{Q}_0^{+ \, -}(r + s) \right] \right.$$
$$\left. + \tilde{Q}_r^-(s) \left[(1 - \eta) \tilde{Q}_0^{+ \, -}(r + s) + \eta \tilde{Q}_0^{+ \, +}(r + s) \right] \right\}$$
$$\tilde{Q}_r^-(s) = \tilde{Q}_0^-(r + s) \tag{60}$$
$$+ r \left\{ Q_r^-(s) \left[(1 - \eta) \tilde{Q}_0^{- \, -}(r + s) + \eta \tilde{Q}_0^{- \, +}(r + s) \right] \right.$$
$$\left. + \tilde{Q}_r^+(s) \left[(1 - \eta) \tilde{Q}_0^{- \, +}(r + s) + \eta \tilde{Q}_0^{- \, -}(r + s) \right] \right\}$$

This system is easily solved to give

$$\tilde{Q}_r^+(s) = \frac{1}{ad - bc} \left[d\tilde{Q}_0^+(r+s) - b\tilde{Q}_0^-(r+s) \right] \tag{61}$$

$$\tilde{Q}_r^-(s) = \frac{1}{ad - bc} \left[-c\tilde{Q}_0^+(r+s) + a\tilde{Q}_0^-(r+s) \right] \tag{62}$$

where

$$a = 1 - r \left[(1-\eta)\tilde{Q}_0^{++}(r+s) + \eta\tilde{Q}_0^{+-}(r+s) \right] \tag{63}$$

$$b = -r \left[\eta\tilde{Q}_0^{++}(r+s) + (1-\eta)\tilde{Q}_0^{+-}(r+s) \right] \tag{64}$$

$$c = -r \left[\eta\tilde{Q}_0^{--}(r+s) + (1-\eta)\tilde{Q}_0^{-+}(r+s) \right] \tag{65}$$

$$d = 1 - r \left[(1-\eta)\tilde{Q}_0^{--}(r+s) + \eta\tilde{Q}_0^{-+}(r+s) \right] . \tag{66}$$

Thus we obtain the general expression for the Laplace transform of the total survival probability

$$\tilde{Q}_r(s) \equiv \tilde{Q}_r^+(s) + \tilde{Q}_r^-(s) \tag{67}$$

$$= \frac{1}{ad - bc} \left[(d-c)\tilde{Q}_0^+(r+s) + (a-b)\tilde{Q}_0^-(r+s) \right] . \tag{68}$$

The solution (68) simplifies greatly when $\eta = 1/2$ in which case $d - c = 1$, $a - b = 1$ and $ad - bc = 1 - r[Q_0^+(r+s) + Q_0^-(r+s)] = 1 - rQ_0(r+s)$ and the result (32) is recovered.

In the case of general η we require the knowledge of the Laplace transforms $\tilde{Q}_0^{\sigma_f \sigma_i}$ which we now show how to compute.

6.2. Survival probabilities in absence of reset

We generalise the system (33, 34) of section 4.1 we write down a system of four backward equations as

$$\frac{\partial Q_0^{\sigma_f \sigma_i}(x_0, t)}{\partial t} = \sigma_i v_0 \frac{\partial Q_0^{\sigma_f \sigma_i}(x_0, t)}{\partial x_0} - \gamma Q_0^{\sigma_f \sigma_i}(x_0, t) + \gamma Q_0^{\sigma_f -\sigma_i}(x_0, t) \tag{69}$$

Note that we have kept here the explicit x_0 dependence in $Q_0^{\sigma_f \sigma_i}(x_0, t)$ since we use x_0 as a variable in the backward Fokker-Planck approach. Eq. (69) has to be solved in the domain $x_0 \geq 0$. The initial conditions are now

$$Q_0^{++}(x_0, 0) = Q_0^{--}(x_0, 0) = 1 \tag{70}$$

$$Q_0^{+-}(x_0, 0) = Q_0^{-+}(x_0, 0) = 0 \tag{71}$$

and the boundary condition corresponding to the absorbing boundary at $x_0 = 0$ is

$$Q_0^{+-}(0, t) = Q_0^{--}(0, t) = 0 . \tag{72}$$

As usual, the solution to the system (69) is obtained by Laplace transform which we write out explicitly to show that it breaks into two subsystems

$$-1 = v_0 \frac{\partial \tilde{Q}_0^{++}(x_0, s)}{\partial x_0} - (s + \gamma)\tilde{Q}_0^{++}(x_0, s) + \gamma\tilde{Q}_0^{+-}(x_0, s) \tag{73}$$

$$0 = -v_0 \frac{\partial \tilde{Q}_0^{+-}(x_0, s)}{\partial x_0} - (s + \gamma)\tilde{Q}_0^{+-}(x_0, s) + \gamma \tilde{Q}_0^{++}(x_0, s) \tag{74}$$

$$-1 = -v_0 \frac{\partial \tilde{Q}_0^{--}(x_0, s)}{\partial x_0} - (s + \gamma)\tilde{Q}_0^{--}(x_0, s) + \gamma \tilde{Q}_0^{-+}(x_0, s) \tag{75}$$

$$0 = v_0 \frac{\partial \tilde{Q}^{-+}(x_0, s)}{\partial x_0} - (s + \gamma)\tilde{Q}_0^{-+}(x_0, s) + \gamma \tilde{Q}_0^{--}(x_0, s) \tag{76}$$

Then equations (73) and (74) and equations (75) and (76) can be turned into decoupled second order equations

$$\frac{\partial^2 \tilde{Q}_0^{+-}(x_0, s)}{\partial x_0^2} = \lambda^2 \tilde{Q}_0^{+-}(x_0, s) - \frac{\gamma}{v_0^2} \tag{77}$$

$$\frac{\partial^2 \tilde{Q}_0^{--}(x_0, s)}{\partial x_0^2} = \lambda^2 \tilde{Q}_0^{--}(x_0, s) - \frac{s + \gamma}{v_0^2} \tag{78}$$

The solution satisfying the boundary condition (72) is

$$\tilde{Q}_0^{+-}(s) = \frac{\gamma}{s(s + 2\gamma)} \left[1 - e^{-\lambda x_0} \right] \tag{79}$$

$$\tilde{Q}_0^{++}(s) = \frac{s + \gamma}{s(s + 2\gamma)} \left[1 - e^{-\lambda x_0} \right] + \frac{v_0 \lambda}{s(s + 2\gamma)} e^{-\lambda x_0} \tag{80}$$

$$\tilde{Q}_0^{--}(s) = \frac{s + \gamma}{s(s + 2\gamma)} \left[1 - e^{-\lambda x_0} \right] \tag{81}$$

$$\tilde{Q}_0^{-+}(s) = -\frac{1}{\gamma} + \frac{(s + \gamma)^2}{\gamma s(s + 2\gamma)} \left[1 - e^{-\lambda x_0} \right] + \frac{v_0(s + \gamma)\lambda}{\gamma s(s + 2\gamma)} e^{-\lambda x_0} . \tag{82}$$

where we have dropped, as usual for brevity, the explicit x_0 dependence of $\tilde{Q}_0^{\sigma_f \sigma_i}(x_0, s) \equiv \tilde{Q}_0^{\sigma_f \sigma_i}(s)$.

6.3. Position-only resetting

As a specific example we present results for the case $\eta = 0$ i.e. position-only resetting.

After unilluminating algebra (which we do not present here) equation (68) may be reduced to:

$$\tilde{Q}_r(s) = -\frac{1}{r} + \frac{1}{r} \frac{1 - \frac{r}{2} \left[\tilde{Q}_0^{++}(r + s) + \tilde{Q}_0^{--}(r + s) - \tilde{Q}_0^{+-}(r + s) - \tilde{Q}_0^{-+}(r + s) \right]}{ad - bc} \tag{83}$$
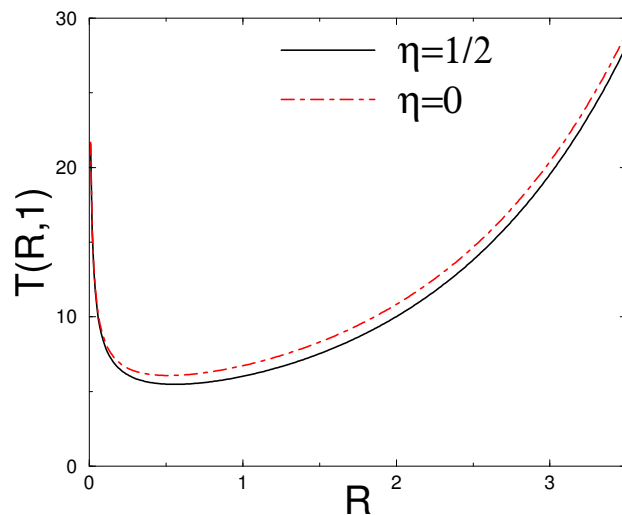
$$ad - bc = 1 - r \left[\tilde{Q}_0^{++}(r + s) + \tilde{Q}_0^{--}(r + s) \right]$$
$$+ r^2 \left[\tilde{Q}_0^{++}(r + s)\tilde{Q}_0^{--}(r + s) - \tilde{Q}_0^{+-}(r + s)\tilde{Q}_0^{-+}(r + s) \right] \tag{84}$$

Using expressions (79–82) one eventually obtains

$$\tilde{Q}(s) = -\frac{1}{r} + \frac{1}{r^2 E} \left[\frac{s + 2\gamma}{r} + \frac{(1 - \beta)}{2} e^{-\lambda_{r+s} X_r} \right] \tag{85}$$

where

$$\beta = \frac{\gamma}{\lambda_{r+s} v_0 + \gamma + r + s} \tag{86}$$

Figure 1. Mean first-passage time $T(R, \xi = 1)$ vs. R given respectively in Eq. (49) for $\eta = 1/2$ (solid (black) line) and in Eq. (90) for $\eta = 0$ (dot-dashed (red) line). For convenience, we have set the parameter $\xi = 1$ for both $\eta = 1/2$ and $\eta = 0$. Clearly, in both cases $T(R, 1)$ has a unique minimum and also for all R, $T(R, 1)$ for $\eta = 1/2$ is always larger than $T(R, 1)$ for $\eta = 0$, indicating that the velocity randomization protocol i.e., $\eta = 1/2$ is more efficient than position-only resetting ($\eta = 0$).

and

$$E = \frac{(s + 2\gamma)s}{r(r + s)} + \frac{[(\beta + 1)\gamma + s]}{r(r + s)} \, \mathrm{e}^{-\lambda_{r+s}X_r} \tag{87}$$

where λ_{r+s} is given in Eq. (42). The mean first passage time to the origin (or equivalently the mean time to absorption at the origin), $T(X_r)$, is conveniently given by the $s \to 0$ limit of (85) which reduces to

$$T(X_r) = -\frac{1}{r} + \frac{1}{r + 2\gamma - (r(r + 2\gamma))^{1/2}} \left[\frac{2\gamma \mathrm{e}^{\lambda X_r}}{r} - \frac{r}{2\gamma} + \frac{(r(r + 2\gamma))^{1/2}}{2\gamma} \right] \tag{88}$$

where λ is given by (11).

Let us check the diffusive limit (28) in which case $\lambda \to (r/D)^{1/2}$ and

$$T(X_r) \to \frac{\mathrm{e}^{X_r(r/D)^{1/2}}}{r} - \frac{1}{r} \tag{89}$$

recovering the result of [16].

In terms of the reduced variables R (46) and ξ (47) we obtain

$$2\gamma T(R, \xi) = -\frac{1}{R} + \frac{\left[\mathrm{e}^{(R(1+R))^{1/2}\xi} - R^2 + R^{3/2}(1 + R)^{1/2} \right]}{R(1 + R - (R(1 + R))^{1/2})} \tag{90}$$

which is to be compared with the $\eta = 1/2$ case (49). As in the $\eta = 1/2$ case, $T(R, \xi)$ as a function of R for fixed ξ has a unique minimum, signalling an optimal resetting rate (see Fig. (1) for a plot). We see that for the same values of the reduced variables,

$T(R, \xi)$ is always greater for $\eta = 0$ than for $\eta = 1/2$. This shows that the velocity randomization protocol ($\eta = 1/2$) is more efficient in searching for a fixed target than the position-only protocol ($\eta = 0$), for the same parameter values such as X_r, γ and v_0.

7. Conclusion

In this paper we have studied the resetting of a run and tumble particle in one dimension. First we derived the stationary state for resetting to point X_r and simultaneous velocity resetting. It turns out that the stationary state does not depend on the resetting protocol. Indeed the stationary state distribution (25) has the same form as a diffusive process under resetting. The width of the stationary distribution decreases with γ and thus increases with increasing velocity correlation time.

However the velocity resetting protocol does affect the survival probability in the presence of an absorbing target at the origin. We have derived explicit expressions for the mean time to absorption in the case of position resetting and velocity randomization (49) and position-only resetting (90). For other parameters fixed, the position-only resetting gives a greater mean time to absorption. Writing the mean time to absorption in terms of the reduced variables R (46) and ξ (47) we see that there is an optimal value of R which minimises the mean time to absorption. It would be of interest to consider how these results generalise to the case of partial absorption of the particle by the boundary [55, 14].

Throughout we have used a renewal equation approach which facilitates the calculations. It would be interesting to see how this approach can be extended to study the resetting of a run and tumble particle in higher dimensions.

It would also be of interest to consider the resetting of other stochastic processes with correlated noise. For example, physical Brownian motion is described as an Ornstein-Uhlenbeck process [1]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v \qquad\qquad \frac{\mathrm{d}v}{\mathrm{d}t} = -\gamma v + \eta(t) \qquad\qquad (91)$$

where $\eta(t)$ is usual white noise. The renewal approach should again be applicable in this case.

Acknowledgments

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