Valid Inequalities for Two-Period Relaxations of Big-Bucket Lot-Sizing Problems: Zero Setup Case

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Mahdi Doostmohammadi
School of Mathematics, University of Edinburgh, Edinburgh EH9 3FD, UK,
mahdi.doostmohammadi@ed.ac.uk

Kerem Akartunalı
Dept. of Management Science, University of Strathclyde, Glasgow, G4 0GE, UK,
kerem.akartunali@strath.ac.uk

In this paper, we investigate two-period subproblems for big-bucket lot-sizing problems, which have shown a great potential for obtaining strong bounds. In particular, we investigate the special case of zero setup times and identify two important mixed integer sets representing relaxations of these subproblems. We analyze the polyhedral structure of these sets, deriving several families of valid inequalities and presenting their facet-defining conditions. We then extend these inequalities in a novel fashion to the original space of two-period subproblems, and also propose a new family of valid inequalities in the original space. In order to investigate the true strength of the proposed inequalities, we propose and implement exact separation algorithms, which are computationally tested over a broad range of test problems. In addition, we develop a heuristic framework for separation, in order to extend computational tests to larger instances. These computational experiments indicate the proposed inequalities can be indeed very effective improving lower bounds substantially.

Key words: Production; Lot-Sizing; Integer Programming; Polyhedral Analysis; Valid Inequalities.

Mathematics Subject Classification (2000): 90B06; 90C10; 90C59.

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1. Introduction

The lot-sizing problem aims to determine an optimal production plan detailing how much to produce and stock in each time period of the planning horizon, given manufacturing system limitations such as machine capacities and customer orders/forecasted demand. Due to its strong impact on manufacturing companies' performance in terms of customer service quality and operating costs, lot-sizing has been a very active research area for many decades with significant attention from researchers as well as practitioners. Due to its practical importance

and limited knowledge in the literature, we focus in this paper on the *multi-item* lot-sizing problem with *big-bucket* capacities, where each resource is shared by multiple items and more than one type of item can be produced in any time period. We study this problem from a theoretical perspective, where we analyze a two-period relaxation of this problem and characterize its important properties. Our main contributions are (i) several families of new valid inequalities and their facet-defining properties for the relaxations of the two-period subproblem, (ii) novel extensions of these inequalities into the original space of the two-period relaxation, as well as new valid inequalities for the original space, and (iii) exact separation algorithms designed to test the practical strength of the proposed inequalities. We also develop a simple but effective heuristic approach in order to extend computational experiments. Our computational results show that the proposed inequalities have great potential to strengthen the lower bounds significantly.

1.1 Literature Review

Most lot-sizing problems are inherently difficult problems: from the theoretical complexity perspective, even the multi-item problem with a single joint capacity and without setup times is strongly $\mathcal{NP}$-hard (Chen and Thizy, 1990). From a computational perspective, problems with multiple items and capacities, in particular of industrial scale, remain notoriously difficult to solve to optimality, often resulting in high duality gaps (Buschkühl et al., 2010). Therefore, there is a wide spectrum of research on lot-sizing problems, ranging from practically efficient heuristics (e.g., Federgruen et al. (2007)) and meta-heuristics (e.g., Jans and Degraeve (2007)) to mathematical programming techniques, which we discuss next in more detail due to their relevance to our study.

Because of their complexity, most researchers in the mathematical programming community studied special cases of lot-sizing problems, which still provide valuable insights on some inherent structures of more general problems and hence support the solution methodologies proposed. The exact approaches most often employed either defining valid inequalities (e.g., Barany et al. (1984); Pochet and Wolsey (1994); Küçükyavuz and Pochet (2009)) or extended reformulations (e.g., Krarup and Bilde (1977); Eppen and Martin (1987); Pochet and Wolsey (2010)) for variants of single-item problem, some of which were also extended to multi-item problems (e.g., Belvaux and Wolsey (2000)). There are also few studies using other techniques such as Lagrangian relaxation (e.g., Billington et al. (1986)) and Dantzig-Wolfe decomposition (e.g., Degraeve and Jans (2007)). Pochet and Wolsey (2006) provide a

thorough review of different variants of lot-sizing problems, their complexities and solution methods used. Most recently, there have been insightful polyhedral results on multi-level problems, such as the valid inequalities of Zhang et al. (2012), and the compact formulations of Van Vyve et al. (2014) for *small-bucket* capacities, i.e., items do not share resources.

Despite this extensive literature, the research explicitly investigating complications arising from multiple items competing for the limited capacities inherent in big-bucket problems is rather limited, and only few exceptions exist to the best of our knowledge. The polyhedral analysis of a single-period relaxation by Miller et al. (2000, 2003) provided us some insightful properties of this polyhedron including new valid inequalities. The study of Jans and Degraeve (2004) presented various decompositions and indicated that period decompositions provide stronger bounds, which is recently investigated further by the branch-and-cut framework of de Araujo et al. (2015) resulting in promising computational results with regards to gaps. The work of Van Vyve and Wolsey (2005) obtained strong lower bounds, most often stronger than any previous results, by applying approximate extended reformulations only for a small number of periods. The extensive computational study of Akartunalı and Miller (2012) noted the bottleneck in big-bucket problems as the lack of a good understanding of the convex hull of single-machine, multi-period problems. This motivated the novel framework of Akartunalı et al. (2016), where the smallest such problem, a two-period relaxation, is used to separate all violated inequalities by generating the extreme points of its convex hull, without pre-defining families of inequalities. The computational results of this study have shown great promise to significantly close duality gaps for big-bucket problems in general, which motivated us to study such a two-period relaxation from a polyhedral perspective.

In this paper, we present our work investigating the special case of zero setup times. This does not only enable us to analytically study inherent structures and hence provide useful insights that can be potentially extended to more complicated problems, but also improve our understanding about this multi-item problem with zero setup times that has been actively studied for many years in the lot-sizing literature, e.g., the earlier work of Dixon and Silver (1981) proposed heuristic approaches for this problem, and essentially the motivation for the seminal work of Padberg et al. (1985) stemmed from this problem. From a practical perspective, it is worth noting that setup times are not necessarily zero, however, using zero setup times has been a very effective modelling approach in case of "negligible setup times", i.e., very low setup times compared to processing times, in order to reduce the problem complexity, see, e.g., Kuik et al. (1994) for a discussion. Negligible

setup times can be observed in various manufacturing settings, e.g., assembly operations in automobile industry and packing operations in food industry. Moreover, as noted by Olaitan and Geraghty (2013) technological advances such as agile tooling and material handling make it possible to produce different products on the same set of machines and therefore enable production line designers reduce setup costs significantly. We finally refer the interested reader to the review of Jans and Degraeve (2008) for a thorough discussion about this special case, achievements and open challenges.

In the next section, we present the problem formulation and the two-period relaxation X^{2PL}, originally proposed by Akartunalı et al. (2016), and study some of its polyhedral properties, including the special case with no setup times. In Section 3, we present two relaxations of X^{2PL}, propose a number of valid inequalities for these relaxations and discuss their facet-defining properties. Section 4 presents novel extensions of these inequalities into the original space of X^{2PL}, as well as a new family of valid inequalities. We present exact separation algorithms in Section 5, and computationally test the strength of the inequalities in Section 6, which show promising results for their effectiveness. We also develop a simple but effective heuristic separation approach in Section 7 enabling experimentation with larger instances, where further encouraging results are obtained. We conclude the paper with a discussion of possible extensions and generalizations. We note that all essential proofs are provided in the Online Supplement due to their lengthy and involved nature.

2. A Two-Period Relaxation for Big-Bucket Lot-Sizing Problem

Before we define and study the two-period relaxation of interest, we first provide the mathematical formulation of the multi-item lot-sizing problem with big-bucket capacities. We let T, I and R denote the sets of *time periods*, *items*, and *machine (resource) types*, respectively. We represent the production, setup, and inventory variables for item i in period t by x_t^i, y_t^i, and s_t^i, respectively.

$$\min \sum_{t\in T}\sum_{i\in I} f_t^i y_t^i + \sum_{t\in T}\sum_{i\in I} h_t^i s_t^i \tag{1}$$

$$\text{s.t. } x_t^i + s_{t-1}^i - s_t^i = d_t^i \qquad t \in T, i \in I \tag{2}$$

$$\sum_{i\in I}(a_r^i x_t^i + ST_r^i y_t^i) \le C_t^r \qquad t \in T, r \in R \tag{3}$$

$$x_t^i \le M_t^i y_t^i \qquad t \in T, i \in I \tag{4}$$

$$y \in \{0,1\}^{|T|\text{x}|I|}; x, s \ge 0 \tag{5}$$

The objective function (1) minimizes total cost, where f_t^i and h_t^i indicate the setup and inventory cost coefficients, respectively. The flow balance constraints (2) ensure that the demand for each item i in period t, denoted by d_t^i, is satisfied. We note that the model can be generalized to involve multiple levels (see, e.g., Akartunalı and Miller (2012)), however, we omit this for the sake of simplicity. The big-bucket capacity constraints (3) ensure that the capacity C_t^r of machine r is not exceeded in time period t, where a_r^i and ST_r^i represent the per unit production time and setup time for item i, respectively. The constraints (4) guarantee that the setup variable is equal to 1 if production occurs, where M_t^i represents the maximum number of item i that can be produced in period t, which is the minimum of either the remaining cumulative demand or the capacity available. Finally, the integrality and non-negativity constraints are given by (5).

2.1 A Two-Period Relaxation: X^{2PL}

Next, we present the feasible region of a two-period, single-machine relaxation, as originally proposed by Akartunalı et al. (2016), of the multi-item production planning problem with big-bucket capacities, denoted by X^{2PL}.

$$x_{t'}^i \le \widetilde{M}_{t'}^i y_{t'}^i \qquad i \in I, t' = 1,2$$

$$x_{t'}^i \le \widetilde{d}_{t'}^i y_{t'}^i + s^i \qquad i \in I, t' = 1,2$$

$$x_1^i + x_2^i \le \widetilde{d}_1^i y_1^i + \widetilde{d}_2^i y_2^i + s^i \qquad i \in I$$

$$x_1^i + x_2^i \le \widetilde{d}_1^i + s^i \qquad i \in I$$

$$\sum_{i\in I}(a^i x_{t'}^i + ST^i y_{t'}^i) \le \widetilde{C}_{t'} \qquad t' = 1,2$$

$$x, s \ge 0, y \in \{0,1\}^{2\times|I|}$$

As we consider a single machine $r \in R$ in this relaxation, we dropped the index r. We note that, for a given time period t, the choice for the "horizon" of this two-period subproblem will be $t+\alpha$ with $\alpha = 1, \ldots, NT - t$. An obvious choice for α would be 1, i.e., $t' = 1, 2$ relate to the periods of $t, t+1$. The parameters can be associated with the original problem parameters using the relations $\widetilde{M}^i_{t'} = M^i_{t+(t'-1)\alpha}$, $\widetilde{C}_{t'} = C^r_{t+(t'-1)\alpha}$, and $\widetilde{d}^i_{t'} = d^i_{t+(t'-1)\alpha,t+\alpha}$ for all i and $t' = 1, 2$. We refer the interested reader to Akartunalı et al. (2016) for a detailed discussion of structuring two-period subproblems. Next, we note some polyhedral properties of X^{2PL}.

Proposition 2.1 (Akartunalı et al. (2016)) *W.l.o.g., we assume* $0 < \widetilde{M}^i_{t'}$ *and* $ST^i < \widetilde{C}_{t'}$ *hold* $\forall i \in I, t' = 1, 2$. *Then,* $conv(X^{2PL})$ *is full-dimensional.*

Proposition 2.2 *The trivial facet-defining inequalities for* $conv(X^{2PL})$ *and their facet-defining conditions (if any) are:*

1. $x^i_{t'} \geq 0, i \in I, t' = 1, 2$.
2. $y^i_{t'} \leq 1, i \in I, t' = 1, 2$.
3. $s^i \geq 0, i \in I$.
4. $x^i_{t'} \leq \widetilde{M}^i_{t'} y^i_{t'}, i \in I, t' = 1, 2$.
5. $x^i_{t'} \leq \widetilde{d}^i_{t'} y^i_{t'} + s^i, i \in I, t' = 1, 2$ *(if* $\widetilde{d}^i_{t'} < \widetilde{M}^i_{t'}$*)*.
6. $x^i_1 + x^i_2 \leq \widetilde{d}^i_1 y^i_1 + \widetilde{d}^i_2 y^i_2 + s^i, i \in I$ *(if* $\widetilde{d}^i_{t'} < \widetilde{M}^i_{t'}, \forall t' \in \{1, 2\}$*)*.
7. $\sum_{i \in I}(a^i x^i_{t'} + ST^i y^i_{t'}) \leq \widetilde{C}_{t'}, t' = 1, 2$ *(if for* $t' \in \{1, 2\}$, $\sum_{i \in I}(a^i \widetilde{M}^i_{t'} + ST^i) \geq \widetilde{C}_{t'} + (a^k \widetilde{M}^k_{t'} + ST^k), \forall k \in I$*)*.

We omit the proof for the sake of simplicity of the presentation. Next, we present some non-trivial facets of $conv(X^{2PL})$. W.l.o.g., we assume $a^i = 1$, $\forall i \in I$ in the remainder of the paper since variables can be scaled as needed.

Proposition 2.3 *For* $i \in I$,

1. *The following inequality is valid for* X^{2PL}*:*

$$x^i_1 + x^i_2 \leq \widetilde{d}^i_1 y^i_1 + d^i_2 + s^i$$

If $\widetilde{d}^i_{t'} + ST^i \leq \widetilde{C}_{t'}, \forall t' \in \{1, 2\}$*, it defines a facet of* $conv(X^{2PL})$.

2. *If* $ST^i \leq \widetilde{C}_2$, *then the following inequality is valid for* X^{2PL}*:*

$$x_1^i + x_2^i \leq \left(\widetilde{d}_1^i - (\widetilde{C}_2 - ST^i)\right) y_1^i + (\widetilde{C}_2 - ST^i) + s^i$$

If $\widetilde{d}_1^i + ST^i \leq \widetilde{C}_1$ *and* $\widetilde{d}_2^i + ST^i > \widetilde{C}_2$, *it is facet-defining for* $conv(X^{2PL})$.

3. *If* $d_1^i + ST^i \leq \widetilde{C}_1$, *then the following inequality is valid for* X^{2PL}*:*

$$x_1^i + x_2^i \leq d_1^i y_1^i + \left(\widetilde{d}_1^i - (\widetilde{C}_1 - ST^i)\right) y_2^i + \left((\widetilde{C}_1 - ST^i) - d_1^i\right) + s^i$$

If $\widetilde{d}_1^i + ST^i > \widetilde{C}_1$ *and* $\widetilde{d}_2^i + ST^i \leq \widetilde{C}_2$, *it defines a facet of* $conv(X^{2PL})$.

4. *If* $\widetilde{d}_1^i + 2ST^i \leq \widetilde{C}_1 + \widetilde{C}_2$, *then the following inequality is valid for* X^{2PL}*:*

$$x_1^i + x_2^i \leq \left(\widetilde{d}_1^i - (\widetilde{C}_2 - ST^i)\right) y_1^i + \left(\widetilde{d}_1^i - (\widetilde{C}_1 - ST^i)\right) y_2^i + \left((\widetilde{C}_1 - ST^i) + (\widetilde{C}_2 - ST^i) - \widetilde{d}_1^i\right) + s^i$$

If $\widetilde{d}_1^i + ST^i > \widetilde{C}_1$ *and* $\widetilde{d}_2^i + ST^i > \widetilde{C}_2$, *it defines a facet of* $conv(X^{2PL})$.

The proof is omitted for the sake of brevity. In this paper, we investigate the special case of zero setups, i.e., $ST^i = 0$, $\forall i \in I$. We note that a companion paper (Akartunalı et al., 2017) studies the polyhedral properties of the general case of non-zero setups as well as structuring an effective computational framework. In the next section, we establish two relaxations of X^{2PL} and study their polyhedral structures. We first present the known facet-defining inequalities, and then derive several classes of valid inequalities and establish their facet-defining conditions.

3. Polyhedral Analysis of the Relaxations of X^{2PL}

First, we make necessary definitions for the remainder of the paper.

Definition 3.1 *For a given* t*:*

- *A* ***cover*** *of* I *for period* t *is a set* S_t *such that* $\lambda_t = \sum_{i \in S_t} \widetilde{d}_t^i - \widetilde{C}_t > 0$.
- *For given non-empty sets* $S_t \subseteq I$ *and* $T_t' \subseteq I \backslash S_t$, *we define the* ***partition slack*** *as* $\xi_t = \sum_{i \in T_t'} \widetilde{M}_t^i + \sum_{i \in S_t} \widetilde{d}_t^i - \widetilde{C}_t$.

- *We define the set* S_t^+ *of* ***strictly positive cover/partition elements*** *as follows:*

$$S_t^+ = \begin{cases} \{i \in S_t | \widetilde{d}_t^i > \lambda_t\} & \text{if } S_t \text{ is a cover.} \\ \{i \in S_t | \widetilde{d}_t^i > \xi_t\} & \text{if } S_t \text{ is part of a partition.} \end{cases}$$

- *The positive maximum function as* $(b)^+ = \max\{b, 0\}$.

First, for a given t, we define the following relaxation, denoted by PIR_0, for X^{2PL}, since it is studied in the literature by various researchers.

$$\begin{aligned} & x^i \leq M^i y^i, \quad \forall i \in I \\ & \sum_{i \in I} x^i \leq C \\ & x \geq 0, y \in \{0,1\}^{|I|} \end{aligned}$$

We dropped here all the t indices as well as $\widetilde{\ }$ for the sake of simplicity. We note that Definition 3.1 remains valid in the same fashion that we use same definitions for this relaxation with all the t indices as well as $\widetilde{\ }$ dropped.

Next, we present known facet-defining inequalities for PIR_0.

Proposition 3.1 (Flow cover inequalities Padberg et al. (1985)) *Let* S *be a cover, and* $\sum_S M^i = C + \lambda$. *Assume that* $\overline{M} = \max_{i \in S} M^i > \lambda$. *Then,*

$$\sum_{i \in S} x^i - \sum_{i \in S} (M^i - \lambda)^+ y^i \leq C - \sum_{i \in S} (M^i - \lambda)^+ \tag{6}$$

is valid and defines a facet of $conv(PIR_0)$. *Moreover, for* $L \subseteq I \backslash S$ *and* $\overline{M}^i = \max(M^i, \overline{M})$, *the inequality*

$$\sum_{i \in S \cup L} x^i - \sum_{i \in S} (M^i - \lambda)^+ y^i - \sum_{i \in L} (\overline{M}^i - \lambda) y^i \leq C - \sum_{i \in S} (M^i - \lambda)^+ \tag{7}$$

is valid and defines a facet of $conv(PIR_0)$ *if* $0 < \overline{M} - \lambda < M^i \leq \overline{M}$, $\forall i \in L$.

In addition to this known relaxation and its facet-defining inequalities, we present a second relaxation of X^{2PL} for a given t. We call this as PIR_1 and study important properties of it in the remainder of this section:

$$\begin{aligned} & x^i \leq M^i y^i, \quad \forall i \in I \\ & x^i \leq d^i y^i + s^i, \quad \forall i \in I \\ & \sum_{i \in I} x^i \leq C \\ & x, s \geq 0, y \in \{0,1\}^{|I|} \end{aligned}$$

Similar to the previous relaxation, we dropped here all the t indices as well as $\tilde{}$, and Definition 3.1 remains valid in the same fashion. Since the structure of the set X^{2PL} is quite complex from a polyhedral perspective, this relaxation of X^{2PL} with a simpler polyhedral structure enables us to potentially derive valuable insights on the inherent structure of X^{2PL}. Next, we note some obvious properties of this polyhedron, including the full dimensionality and trivial facets of $conv(PIR_1)$. These propositions can be easily proven and therefore, we omit detailed proofs here for the sake of brevity.

Proposition 3.2 *$dim(conv(PIR_1)) = 3|I|$.*

Proposition 3.3 *The following inequalities are the trivial facets of PIR_1:*

1. $x^i \geq 0$, $\forall i \in I$.
2. $y^i \leq 1$, $\forall i \in I$.
3. $s^i \geq 0$, $\forall i \in I$.
4. $x^i \leq M^i y^i$, $\forall i \in I$.
5. $x^i \leq d^i y^i + s^i$ *if* $d^i < M^i, \forall i \in I$
6. $\sum_{i \in I} x^i \leq C$ *if* $\sum_{i \in I} M^i \geq C + M^k, \forall k \in I$.

Next, we discuss families of valid inequalities and establish their facet-defining conditions for PIR_1.

Proposition 3.4 *Let S be a cover of I. Then the following inequality (called **cover inequality**) is valid for PIR_1:*

$$\sum_{i \in S} x^i - \sum_{i \in S} (d^i - \lambda)^+ y^i \leq \sum_{i \in S} s^i + C - \sum_{i \in S} (d^i - \lambda)^+ \tag{8}$$

*Moreover, if $K \subseteq I \backslash S$ such that $M^i \leq \bar{d}^i$ holds $\forall i \in K$, where $\bar{d} = max_{i \in S} d^i \geq \lambda$ and $\bar{d}^i = \max(d^i, \bar{d}), i \in K$, then the following inequality (called **item-extended cover inequality**) is valid for PIR_1:*

$$\sum_{i \in S \cup K} x^i - \sum_{i \in S} (d^i - \lambda)^+ y^i - \sum_{i \in K} (\bar{d}^i - \lambda) y^i \leq \sum_{i \in S} s^i + C - \sum_{i \in S} (d^i - \lambda)^+ \tag{9}$$

The validity of (8) and (9) can be shown by considering the mixed-integer set $\{\sum_{i\in I}(x^i - s^i) \leq C, (x^i - s^i) \leq d^i y^i, y \in \{0,1\}^{|I|}\}$, which is a single-node flow set without the non-negativity constraints as well as a special relaxation of PIR_1, and then deriving flow covers and extended flow covers from it. Intuitively speaking, since the items in the cover have a total demand that is strictly exceeding the capacity by λ, the cover inequality considers only items from this set that have an individual demand strictly higher than this excess of λ, as it is not possible to produce at least one such item to its full demand.

Proposition 3.5 *If $d^i < M^i$ $\forall i \in S$ and $|S^+| \geq 2$ hold, (8) defines a facet of conv(PIR_1). If, in addition, $0 < \bar{d} - \lambda < d^i \leq \bar{d}$ holds $\forall i \in K$, then (9) defines a facet of conv(PIR_1).*

We provide a detailed proof in the Online Supplement.

Proposition 3.6 *Let $S \neq \emptyset$ be a subset of I, $\overline{T} = I \backslash S$, and (T', T'') be a partition of $\overline{T}$ such that $T' \neq \emptyset$ and $\xi \geq 0$. Then the following inequality (called **partition inequality**) is valid for PIR_1:*

$$\sum_{i\in S\cup T'} x^i + \sum_{i\in S}(d^i - \xi)^+(1 - y^i) + \sum_{i\in T'}(M^i - \xi)^+(1 - y^i) \leq \sum_{i\in S} s^i + C \tag{10}$$

We note that if either $S = \emptyset$ or $T' = \emptyset$, then this reduces to either flow cover inequalities of Proposition 3.1 or cover inequalities of Proposition 3.4, respectively. The validity can be shown by considering the mixed-integer set $\{\sum_{i\in S}(x^i - s^i) + \sum_{i\in I\backslash S} x^i \leq C, (x^i - s^i) \leq d^i y^i, x^i \leq M^i y^i, y \in \{0,1\}^{|I|}\}$, which is a single-node flow set without the non-negativity constraints as well as a special relaxation of PIR_1, and then deriving flow covers from it.

Proposition 3.7 *Let $\xi > 0$, and assume that for $T'^+ = \{i \in T' | M^i > \xi\}$, $|T'^+| \geq 1$ holds. Moreover, assume that $d^i < M^i$ holds $\forall i \in S$. Then, (10) defines a facet of conv(PIR_1).*

For the proof, we refer the reader to the Online Supplement.

Proposition 3.8 *Let S be a non-empty subset of I, $\overline{T} = I \backslash S$, (T', T'') be a partition of $\overline{T}$, $K \subseteq T''$ and $\xi \geq 0$. We define*

$$p_i = \begin{cases} d^i & : i \in S \\ M^i & : i \in T' \end{cases}$$

and $\bar{p} = \max_{i \in S \cup T'} p^i \geq \xi$. *We also define* $\bar{p}^i = \max(M^i, \bar{p}), i \in K$. *Then the following inequality (called* ***item-extended partition inequality****) is valid for* PIR_1*:*

$$\sum_{i \in S \cup T' \cup K} x^i - \sum_{i \in S} (d^i - \xi)^+ y^i - \sum_{i \in T'} (M^i - \xi)^+ y^i - \sum_{i \in K} (\bar{p}^i - \xi) y^i \leq \sum_{i \in S} s^i + C - \sum_{i \in S} (d^i - \xi)^+ - \sum_{i \in T'} (M^i - \xi)^+ \quad (11)$$

The validity follows by deriving generalized flow covers for the mixed-integer set $\{\sum_{i \in S}(x^i - s^i) + \sum_{i \in I \setminus S} x^i \leq C, (x^i - s^i) \leq d^i y^i, x^i \leq M^i y^i, y \in \{0,1\}^{|I|}\}$, which is a single-node flow set without the non-negativity constraints as well as a special relaxation of PIR_1.

Proposition 3.9 *Assume that the conditions presented in Proposition 3.7 hold. Moreover, let* $0 < \bar{p} - \xi < M^i \leq \bar{p}$ *hold* $\forall i \in K$. *Then, (11) defines a facet of conv*(PIR_1).

For the proof, we refer the reader to the Online Supplement.

Example. Let $I = \{1, 2, 3\}$, and PIR_1 defined by:

$$x^1 \leq 14y^1, \quad x^2 \leq 10y^2, \quad x^3 \leq 11y^3$$
$$x^1 \leq 10y^1 + s^1, \quad x^2 \leq 6y^2 + s^2, \quad x^3 \leq 8y^3 + s^3$$
$$x^1 + x^2 + x^3 \leq 14$$

Consider $S = \{1\}$ and $T' = \{3\}$. Hence $\xi = 10 + 11 - 14 = 7$. Then, we can generate a facet-defining partition inequality as follows:

$$x^1 + x^3 - (10 - 7)y^1 - (11 - 7)y^3 \leq s^1 + 14 - 3 - 4$$
$$\implies x^1 + x^3 - 3y^1 - 4y^3 \leq s^1 + 7$$

With $S = \{1\}$ and $T' = \{3\}$, we note $p^1 = 10, p^3 = 11$. Hence, $\bar{p} = \max_{i \in S \cup T'} p^i = 11 > \xi = 7$. Let $K = \{2\}$ (and hence $\bar{p}^2 = \max(11, 10) = 11$). Then, we can derive the facet-defining item-extended partition inequality:

$$x^1 + \mathbf{x^2} + x^3 - 3y^1 - (\mathbf{11 - 7})\mathbf{y^2} - 4y^3 \leq s^1 + 7$$

where bold elements indicate all terms that are additional compared to the previous inequality. Using PORTA (Christof and Lobel, 2009), we can identify 6 facet-defining partition inequalities and 3 facet-defining item-extended partition inequalities for this set. □

In Section 5, we describe separation algorithms for these inequalities. Next, we discuss how we can extend the results of this section to the space of the two-period relaxation of X^{2PL}.

4. Valid Inequalities in the Original Space of X^{2PL}

We recall the "original" space defined earlier as X^{2PL}. We first map the inequalities developed for PIR_1 in the previous section to the original space of X^{2PL}, and also define new valid inequalities for X^{2PL}. Note that all the t indices as well as $\tilde{}$ are introduced here again, which were dropped in the previous section, since they will be part of the discussion here.

Corollary 4.1 *Let $t \in \{1,2\}$, and S_t be a cover of I in period t. Then the inequality (mapped from the cover inequality in PIR_1) is valid for X^{2PL}:*

$$\sum_{i \in S_t} x_t^i + \sum_{i \in S_t} (\widetilde{d}_t^i - \lambda_t)^+ (1 - y_t^i) \leq \sum_{i \in S_t} s^i + \widetilde{C}_t \tag{12}$$

We can also extend this inequality as follows.

Proposition 4.1 *Let $t, t' \in \{1,2\}$, $t \neq t'$, and S_t be a cover of I for period t. In addition, assume $L_{t'} \subseteq S_t$. Then the following inequality (called **period-extended cover inequality**) is valid for X^{2PL}:*

$$\sum_{i \in S_t} x_t^i + \sum_{i \in L_{t'}} x_{t'}^i + \sum_{i \in S_t} (\widetilde{d}_t^i - \lambda_t)^+ (1 - y_t^i) - \sum_{i \in L_{t'}} \widetilde{d}_{t'}^i y_{t'}^i \leq \sum_{i \in S_t} s^i + \widetilde{C}_t \tag{13}$$

We omit the proof here, as Proposition 4.2 covers a more general case and this is the special case when $K_t = \emptyset$. Next, we discuss item-extended cover inequalities and how they can be represented in the original space.

Corollary 4.2 *Let $t \in \{1,2\}$ and S_t be a cover of I for period t. Let $K_t \subseteq I \backslash S_t$ such that $\widetilde{M}_t^i \leq \bar{d}_t^i$ holds $\forall i \in K_t$, where $\bar{d}_t = \max_{i \in S_t} \widetilde{d}_t^i \geq \lambda_t$ and $\bar{d}_t^i = \max(\bar{d}_t, \widetilde{d}_t^i)$. Then the following inequality (mapped from the item-extended cover inequality in PIR_1) is valid for X^{2PL}:*

$$\sum_{i \in S_t \cup K_t} x_t^i + \sum_{i \in S_t} (\widetilde{d}_t^i - \lambda_t)^+ (1 - y_t^i) - \sum_{i \in K_t} (\bar{d}_t^i - \lambda_t) y_t^i \leq \sum_{i \in S_t} s^i + \widetilde{C}_t \tag{14}$$

The proof is straightforward as it follows the same logic as Proposition 3.4. We can also extend this inequality as follows.

Proposition 4.2 *In addition to the assumptions and definitions of Corollary 4.2, let* $t, t' \in \{1,2\}$, $t \neq t'$, *and* $L_{t'} \subseteq S_t$. *Then the following inequality (called* ***item-and-period-extended cover inequality****) is valid for* X^{2PL}*:*

$$\sum_{i \in S_t \cup K_t} x_t^i + \sum_{i \in L_{t'}} x_{t'}^i + \sum_{i \in S_t} (\widetilde{d}_t^i - \lambda_t)^+ (1 - y_t^i) - \sum_{i \in K_t} (\bar{d}_t^i - \lambda_t) y_t^i - \sum_{i \in L_{t'}} \widetilde{d}_{t'}^i y_{t'}^i \leq \sum_{i \in S_t} s^i + \widetilde{C}_t \quad (15)$$

For the proof, we refer the reader to the Online Supplement. Next, we discuss the extension of partition inequalities to original space.

Corollary 4.3 *Let* $t \in \{1,2\}$ *and* S_t *be a non-empty subset of* I *in period* t. *Let* $\overline{T}_t = I \backslash S_t$, *and* (T_t', T_t'') *be a partition of* $\overline{T}_t$ *such that* $\xi_t \geq 0$. *Then the following inequality (mapped from the partition inequality in* PIR_1*) is valid for* X^{2PL}*:*

$$\sum_{i \in S_t \cup T_t'} x_t^i + \sum_{i \in S_t} (\widetilde{d}_t^i - \xi_t)^+ (1 - y_t^i) + \sum_{i \in T_t'} (\widetilde{M}_t^i - \xi_t)^+ (1 - y_t^i) \leq \sum_{i \in S_t} s^i + \widetilde{C}_t \quad (16)$$

The proof is straightforward as it follows a similar logic to the proof of Proposition 3.6. We can also extend this inequality as follows.

Proposition 4.3 *In addition to the assumptions and definitions of Corollary 4.3, let* $L_{t'} \subseteq S_t$, *where* $t, t' \in \{1,2\}$ *and* $t \neq t'$. *Then the following inequality (called* ***period-extended partition inequality****) is valid for* X^{2PL}*:*

$$\sum_{i \in S_t \cup T_t'} x_t^i + \sum_{i \in L_{t'}} x_{t'}^i + \sum_{i \in S_t} (\widetilde{d}_t^i - \xi_t)^+ (1 - y_t^i) + \sum_{i \in T_t'} (\widetilde{M}_t^i - \xi_t)^+ (1 - y_t^i) - \sum_{i \in L_{t'}} \widetilde{d}_{t'}^i y_{t'}^i \leq \sum_{i \in S_t} s^i + \widetilde{C}_t \quad (17)$$

We omit the proof as it follows a similar logic to the proof of Proposition 3.6 and Proposition 4.2. Next, we discuss item-extended partition inequalities.

Corollary 4.4 *Let $t \in \{1,2\}$, $S_t \neq \emptyset$ be a subset of I for period t, $\overline{T}_t = I \backslash S_t$ and (T'_t, T''_t) be a partition of $\overline{T}_t$ such that $\xi_t \geq 0$ and $K_t \subseteq T''_t$. Let*

$$p_t^i = \begin{cases} \widetilde{d}_t^i & : i \in S_t \\ \widetilde{M}_t^i & : i \in T'_t \end{cases}$$

$\bar{p}_t = \max_{i \in S_t \cup T'_t} p_t^i \geq \xi_t$, and $\bar{p}_t^i = \max(\widetilde{M}_t^i, \bar{p}_t), i \in K_t$. Then the inequality (mapped from the item-extended partition inequality in PIR_1) is valid for X^{2PL}:

$$\sum_{i \in S_t \cup T'_t \cup K_t} x_t^i + \sum_{i \in S_t} (\widetilde{d}_t^i - \xi_t)^+ (1 - y_t^i) + \sum_{i \in T'_t} (\widetilde{M}_t^i - \xi_t)^+ (1 - y_t^i) - \sum_{i \in K_t} (\bar{p}_t^i - \xi_t) y_t^i \leq \sum_{i \in S_t} s^i + \widetilde{C}_t \quad (18)$$

The proof is omitted as it follows a similar logic to the proof of Proposition 3.8. We can also extend this inequality further as follows.

Proposition 4.4 *In addition to the assumptions and definitions of Corollary 4.4, let $L_{t'} \subseteq S_t$, $t, t' \in \{1,2\}$ and $t \neq t'$. Then, the following inequality (called **item-and-period-extended partition inequality**) is valid for X^{2PL}:*

$$\sum_{i \in S_t \cup T'_t \cup K_t} x_t^i + \sum_{i \in L_{t'}} x_{t'}^i + \sum_{i \in S_t} (\widetilde{d}_t^i - \xi_t)^+ (1 - y_t^i) + \sum_{i \in T'_t} (\widetilde{M}_t^i - \xi_t)^+ (1 - y_t^i) - \sum_{i \in K_t} (\bar{p}_t^i - \xi_t) y_t^i - \sum_{i \in L_{t'}} \widetilde{d}_{t'}^i y_{t'}^i \leq \sum_{i \in S_t} s^i + \widetilde{C}_t \quad (19)$$

We omit the proof here since it is very similar to the proof of Proposition 4.3. Finally, we introduce in the next proposition a new family of valid inequalities that is not derived from any of the inequalities based on the single-period relaxation.

Proposition 4.5 *For $t, t' \in \{1,2\}$ and $t \neq t'$, let*

- *$S_t \subset I$ and $T'_t \subseteq I \backslash S_t$ such that $\theta_t = \sum_{i \in S_t} \widetilde{d}_t^i + \sum_{i \in T'_t} \widetilde{M}_t^i - \widetilde{C}_t \geq 0$, and*
- *$S_{t'} \subset I$ and $T'_{t'} \subseteq I \backslash S_{t'}$ such that $\theta_{t'} = \sum_{i \in S_{t'}} \widetilde{d}_{t'}^i + \sum_{i \in T'_{t'}} \widetilde{M}_{t'}^i - \widetilde{C}_{t'} \geq 0$.*

Then, the following inequality (called ***two-period partition inequality****) is valid for* X^{2PL}*:*

$$\sum_{i\in S_t\cup T'_t} x^i_t + \sum_{i\in S_{t'}\cup T'_{t'}} x^i_{t'} + \sum_{i\in S_t}(\widetilde{d}^i_t - \theta_t)^+(1-y^i_t) + \sum_{i\in T'_t}(\widetilde{M}^i_t - \theta_t)^+(1-y^i_t)$$
$$+ \sum_{i\in S_{t'}}(\widetilde{d}^i_{t'} - \theta_{t'})^+(1-y^i_{t'}) + \sum_{i\in T'_{t'}}(\widetilde{M}^i_{t'} - \theta_{t'})^+(1-y^i_{t'}) \leq \widetilde{C}_t + \widetilde{C}_{t'} + \sum_{i\in S_t\cup S_{t'}} s^i \quad (20)$$

We provide a detailed proof in Online Supplement. We note that this inequality is non-dominated if $S_t \cap S_{t'} \neq \emptyset$.

5. Separation Algorithms for Relaxations and Original Space

The purpose of this section is to describe in detail the separation problems associated with all the families of inequalities defined in the previous sections. Since the main purpose of this paper is to investigate the true strength of the cuts generated, we focus on defining exact separation algorithms rather than their computational efficiency. Here, we follow the same structure and order of the previous two sections: we firstly define separation problems associated with families of inequalities defined for the relaxations of the problems, and then for those associated with the original space. In the remainder of this section, we let $(\bar{x}, \bar{y}, \bar{s})$ to represent a fractional solution vector in the associated space that is to be cut off. W.l.o.g., we assume all problem parameters to be integer valued.

5.1 Separation in the Relaxation Space

We start first with the family of cover inequalities as defined by (8). First, we note that we can rewrite these inequalities as follows:

$$\sum_{i\in S}\Big(x^i + (d^i - \lambda)^+(1-y^i) - s^i\Big) \leq C$$

Since S must be a cover, for a given value of $\lambda > 0$, we can find the most violated inequality (if any) by solving the following knapsack problem:

$$f_\lambda = \max\left\{\sum_{j\in I}\tau_j(\lambda)w_j | \sum_{i\in I} d^i w_i = C + \lambda, w \in \{0,1\}^{|I|}\right\}$$

where $\tau_j(\lambda) = \bar{x}^j + (d^j - \lambda)^+(1 - \bar{y}^j) - \bar{s}^j$. If $f_\lambda > C$, then a violated cover inequality is identified for the specified λ. We note that since $\lambda \in \mathbb{Z}_+$, one can solve this separation problem for any value of $\lambda \in [1, \sum_{i \in I} d^i - C]$.

Next, we discuss the separation procedure for the family of item-extended cover inequalities (9). We first rewrite these inequalities as follows:

$$\sum_{i \in S} \left(x^i + (d^i - \lambda)^+(1 - y^i) - s^i \right) + \sum_{i \in K} \left(x^i - (\bar{d}^i - \lambda) y^i \right) \leq C$$

For a given cover S, if $\bar{d} \geq \lambda$, then we can define the set K as follows:

$$K = \{ i \in I \backslash S | \bar{x}^i - (\bar{d}^i - \lambda)\bar{y}^i > 0, M^i \leq \bar{d}^i \}$$

Therefore, one can generate covers using the procedure defined for the cover inequalities, and then heuristically generate the set K. We note that this is a similar approach to the one proposed by Padberg et al. (1985) (p.854) for flow cover inequalities. Finally, we note that such a procedure is applied for the separation of the known inequalities (6) and (7).

Next, we discuss the separation procedure for the family of partition inequalities (10). First note that we can rewrite these inequalities as follows:

$$\sum_{i \in S} \left(x^i + (d^i - \xi)^+(1 - y^i) - s^i \right) + \sum_{i \in T'} \left(x^i + (M^i - \xi)^+(1 - y^i) \right) \leq C$$

For a given value of ξ, we define the following IP for the separation:

$$\begin{aligned} f_\xi = \max \sum_{i \in I} \left(\bar{x}^i + (d^i - \xi)^+(1 - \bar{y}^i) - \bar{s}^i \right) u_i + \sum_{i \in I} \left(\bar{x}^i + (M^i - \xi)^+(1 - \bar{y}^i) \right) v_i \\ \text{s.t.} \quad \sum_{i \in I} d^i u_i + \sum_{i \in I} M^i v_i = C + \xi \\ C > \sum_{i \in I} d^i u_i \\ u_i + v_i \leq 1, \ \forall i \in I \\ u_i, v_i \in \{0, 1\}, \ \forall i \in I \end{aligned}$$

Here, u_i and v_i variables indicate whether an item i belongs to set S or T' respectively. The first two constraints ensure that $\sum_{i \in I} v_i \geq 1$, i.e., $T' \neq \emptyset$. A violated inequality is found if $f_\xi > C$. Similar to the process for cover inequalities, since $\xi \in \mathbb{Z}_+$, one can solve this separation problem for any value of $\xi \in [1, \sum_{i \in I} \max\{d^i, M^i\} - C]$.

The separation procedure for item-extended partition inequalities (11) is similar to the procedure described for item-extended cover inequalities (9).

5.2 Separation in the Original Space

We start this section with the separation of the period-extended cover inequalities in the original space as defined by (13). First, we rewrite these inequalities as follows, where $S_t \subseteq I$ and $L_{t'} \subseteq S_t$, $t \neq t'$:

$$\sum_{i \in S_t} \left(x_t^i + (\widetilde{d}_t^i - \lambda_t)^+ (1 - y_t^i) - s^i \right) + \sum_{i \in L_{t'}} \left(x_{t'}^i - \widetilde{d}_{t'}^i y_{t'}^i \right) \leq \widetilde{C}_t$$

For a given t and fixed $\lambda_t > 0$, we can solve the separation problem:

$$\max \sum_{i \in I} \left(\bar{x}_t^i + (\widetilde{d}_t^i - \lambda_t)^+ (1 - \bar{y}_t^i) - \bar{s}^i \right) u_i + \sum_{i \in I} \left(\bar{x}_{t'}^i - \widetilde{d}_{t'}^i \bar{y}_{t'}^i \right) v_i$$
$$\text{s.t.} \ \sum_{i \in I} \widetilde{d}_t^i u_i = \widetilde{C}_t + \lambda_t; \ \ v_i \leq u_i \ \ \forall i \in I; \ \ u_i, v_i \in \{0,1\} \ \ \forall i \in I$$

If the optimal value of the problem is strictly greater than $\widetilde{C}_t$, then a violated inequality is identified.

Next, we note that the separation procedures in the original space for the item-and-period-extended cover inequalities (15), for the period-extended partition inequalities (17), and for the item-and-period-extended partition inequalities (19) follow a very similar logic to the separation procedures of the period-extended cover inequalities in the original space (13). Therefore, we omit a detailed description here for the sake of brevity.

Finally, we introduce the separation procedure for two-period partition inequalities defined by (20). First, we rewrite it as follows:

$$\sum_{i \in S_t} \left(x_t^i + (\widetilde{d}_t^i - \xi_t)^+ (1 - y_t^i) - s^i \right) + \sum_{i \in T_t'} \left(x_t^i + (\widetilde{M}_t^i - \xi_t)^+ (1 - y_t^i) \right)$$
$$+ \sum_{i \in S_{t'}} \left(x_{t'}^i + (\widetilde{d}_{t'}^i - \xi_{t'})^+ (1 - y_{t'}^i) - s^i \right) + \sum_{i \in T_{t'}'} \left(x_{t'}^i + (\widetilde{M}_{t'}^i - \xi_{t'})^+ (1 - y_{t'}^i) \right)$$
$$+ \sum_{i \in S_t \cap S_{t'}} s^i \leq \widetilde{C}_t + \widetilde{C}_{t'}$$

For a given pair of t, t' as well as fixed $\xi_t > 0$ and $\xi_{t'} > 0$ values, we can define the following separation problem 2PPI:

$$\textbf{(2PPI)} \quad f_{\xi_t, \xi_{t'}} = \max \sum_{i \in I} \left(\bar{x}_t^i + (\widetilde{d}_t^i - \xi_t)^+ (1 - \bar{y}_t^i) - \bar{s}^i \right) u_t^i$$
$$+ \sum_{i \in I} \left(\bar{x}_t^i + (\widetilde{M}_t^i - \xi_t)^+ (1 - \bar{y}_t^i) \right) v_t^i + \sum_{i \in I} \left(\bar{x}_{t'}^i + (\widetilde{d}_{t'}^i - \xi_{t'})^+ (1 - \bar{y}_{t'}^i) - \bar{s}^i \right) u_{t'}^i$$
$$+ \sum_{i \in I} \left(\bar{x}_{t'}^i + (\widetilde{M}_{t'}^i - \xi_{t'})^+ (1 - \bar{y}_{t'}^i) \right) v_{t'}^i + \sum_{i \in I} \bar{s}^i z^i$$

s.t.
$$\sum_{i\in I} \widetilde{d}_t^i u_t^i + \sum_{i\in I} \widetilde{M}_t^i v_t^i = \widetilde{C}_t + \xi_t$$
$$\sum_{i\in I} \widetilde{d}_{t'}^i u_{t'}^i + \sum_{i\in I} \widetilde{M}_{t'}^i v_{t'}^i = \widetilde{C}_{t'} + \xi_{t'}$$
$$u_t^i + v_t^i \leq 1,\ \forall i \in I$$
$$u_{t'}^i + v_{t'}^i \leq 1,\ \forall i \in I$$
$$z^i \leq u_t^i,\ \forall i \in I$$
$$z^i \leq u_{t'}^i,\ \forall i \in I$$
$$u_t^i, v_t^i, z^i, u_{t'}^i, v_{t'}^i \in \{0,1\},\ \forall i \in I$$

If $f_{\xi_t,\xi_{t'}} > \widetilde{C}_t + \widetilde{C}_{t'}$ holds, then the inequality is violated and hence the cutting plane is added to the formulation.

6. Computational Results

In this section, we present numerical results indicating the strength of the various cuts proposed earlier. We note that our primary aim here is not necessarily to build a practically efficient computational framework, which is addressed in a companion paper (Akartunalı et al., 2017), but instead to exhaustively generate all violated inequalities by exact separation to measure their practical strength and effectiveness. All the separation algorithms and mathematical models are implemented and executed using the Mosel language of FICO® Xpress Optimization Suite (Mosel version 3.6.0, Xpress-MP v7.7) on a PC with Intel® Core i5 3.10GHz processor and 4GB RAM, where all possible two-period relaxations, both consecutive and non-consecutive, were considered.

In order to test the effectiveness of the cuts proposed, we have generated 240 random test instances in total, which we describe in detail next. First of all, we note that exact separation is computationally expensive, causing issues with available memory or prohibitively long times when the problem size became bigger than $|T| = 12$ and $|I| = 10$, so that we set the maximum size to these values. We also note that even with this maximum size, computational times can be extensive. On the other hand, we have set the minimum size to $|T| = 2$ and $|I| = 3$, in order to capture the simplest problem with the two-period, multi-item structure. We have varied $|T|$ and $|I|$ values with intervals growing exponentially, in order to capture the variety created by the fact that the problem complexity grows exponentially

(rather than using equal length intervals), resulting in 16 different $|T|, |I|$ combinations. On the other hand, we have considered low, medium and high demand variability for a good mix of problems, randomly generating d_t^i parameters in the intervals of $[10, 20]$, $[10, 40]$ and $[10, 60]$, respectively. This results in 48 combinations, where for each combination, we have generated 5 test instances. The capacities in each period are generated as a random variable from the interval $[0.8 \times |I| \times mid_t, 1.2 \times |I| \times mid_t]$, where mid_t indicates the median demand in that interval. Finally, we note that the holding costs h_t^i are randomly generated from the interval $[0.1, 1]$ and the setup costs f_t^i takes a value of $\{1, 10, 50\}$, each with probability of $\frac{1}{3}$, in order to generate a good mix of low and high setup cost items (and in between).

Next, we present the computational results for low, medium and high demand variability, in Tables 1, 2 and 3, respectively. Since the exact separation procedures are significantly time-consuming for our proposed inequalities, we generated only one round of violated cuts and added to the formulation. In each table, the first column indicates the combination $|T|, |I|$, followed by the columns indicating average **I**nitial **G**ap, **G**ap **C**losed by **F**low **C**over inequalities only (i.e., only (extended) flow cover inequalities are generated), and the percentage **G**ap **C**losed for 5 instances with all the violated cuts generated. Note that the initial gap is based on the strengthened LP relaxation with all violated (ℓ, S) inequalities added a priori, which are known to be very effective in practice for multi-item problems, see, e.g., Akartunalı and Miller (2012). In the remainder of the tables, the columns indicate the total number of cuts generated of each type for 5 instances, in the following order: **C**over (8), **I**tem-extended **C**over (9), **P**eriod-extended **C**over (13), **I**tem-and-**P**eriod-extended **C**over (15), **P**artition (10), **I**tem-extended **P**artition (11), **P**eriod-extended **P**artition (17), **I**tem-and-**P**eriod-extended **P**artition (19), **T**wo-**P**eriod **P**artition (20), and **F**low **C**over (7). We note that for biggest instances of $d_t^i \in [10, 40], [10, 60]$, the separation procedure for two-period partition inequalities took prohibitive times and therefore they were removed from the framework, which are indicated by $\star$ in the tables.

As the results in Tables 1-3 indicate, the cuts can close on average more than 25% of the initial gap. As one could naturally expect, the average gap closed by the cuts deteriorates when either the number of items or the number of periods is increased, where this deterioration seems more sensitive to the increase in the numbers of items than to the increase in the numbers of periods. When the number of items increases, the problem resembles more the structure of an uncapacitated problem, the convex hull of which can be effectively described by the (ℓ, S) inequalities and hence there is little room for improvement by other cuts. This

Table 1: Average closed gaps and number of cuts generated of each type for test problems with $d_t^i \in [10, 20]$.

				# Cuts generated									
$\|T\|, \|I\|$	**IG**	**GCFC**	**GC**	**C**	**IC**	**PC**	**IPC**	**P**	**IP**	**PP**	**IPP**	**TPP**	**FC**
2,3	18.01	52.88	52.88	5	0	0	0	0	0	0	0	0	5
2,4	18.03	45.11	50.52	5	0	0	0	0	0	0	0	0	5
2,6	9.85	48.30	48.30	7	0	0	0	0	0	0	0	0	7
2,10	6.01	34.52	34.52	2	0	0	0	0	0	0	0	0	2
3,3	17.40	10.34	38.79	5	0	0	0	10	0	0	0	0	4
3,4	14.24	0	28.99	1	0	0	0	9	0	0	0	0	0
3,6	12.65	10.27	27.68	5	2	0	0	12	1	0	0	0	5
3,10	10.06	0	15.60	0	0	0	0	20	4	0	0	0	2
6,3	19.39	8.53	25.90	8	3	4	0	31	8	3	0	22	7
6,4	15.35	0.82	20.37	6	0	1	0	26	4	1	1	16	2
6,6	8.93	1.63	16.62	6	4	0	0	16	13	0	0	14	7
6,10	8.40	4.41	7.87	4	2	0	0	22	11	0	0	8	6
12,3	16.47	3.05	18.25	13	6	14	0	51	13	23	7	99	11
12,4	13.81	0.73	17.21	8	4	6	2	64	23	12	7	98	6
12,6	15.22	0	18.28	10	0	1	0	46	13	0	0	108	0
12,10	7.46	0	5.88	5	1	0	0	144	20	0	0	204	2
Ave=	**13.08**	**13.79**	**26.84**	**5.63**	**1.38**	**1.63**	**0.13**	**28.19**	**6.88**	**2.44**	**0.94**	**35.56**	**4.44**

can be indeed consistently observed from the average initial gaps for all the problems with 10 items. On the other hand, as the number of periods increases, the problem becomes further away from the "ideal" two-period problem, for which these cuts are originally derived. However, we note that when all instances with 10 items are taken out, even the average gap closed for the instances with 12 periods is 23.05%, which is a substantial improvement with only one round of added cuts, and also only slightly lower than 24.48%, the average gap closed for the instances with 6 periods and 3/4/6 items. As we will discuss in Section 7, the experimentation with 24 period problems, albeit using a heuristic approach, indicate similar gap improvements compared to instances with 12 periods.

The results also indicate which types of inequalities are more inherent for different sizes of problems. A small number of cover inequalities seem to close substantial gaps for the 2-period problems almost only on their own, and the number of these inequalities do not vary much as the problem size gets bigger. On the other hand, the number of partition inequalities and two-period partition inequalities generated increases significantly as the number of items and periods increase, making both types of inequalities most often generated inequalities in our framework, hence also pointing to where an computationally efficient framework can focus on. We also note our observation from the computational tests that adding two-period partition inequalities on top of cover and partition inequalities and their variants is not very effective in closing the integrality gap, although this might also be the consequence of a single round of separation. Another interesting aspect the results point at is the fact that

Table 2: Average closed gaps and number of cuts generated of each type for test problems with $d_t^i \in [10, 40]$.

				# Cuts generated									
$\|T\|, \|I\|$	**IG**	**GCFC**	**GC**	**C**	**IC**	**PC**	**IPC**	**P**	**IP**	**PP**	**IPP**	**TPP**	**FC**
2,3	11.15	59.49	59.49	5	0	0	0	0	0	0	0	0	5
2,4	16.63	41.02	42.27	7	2	0	0	1	0	0	0	0	9
2,6	10.19	46.62	46.62	7	0	0	0	0	0	0	0	0	7
2,10	5.88	8.39	8.39	3	0	0	0	0	0	0	0	0	3
3,3	20.20	11.50	40.74	5	0	0	0	10	0	0	0	7	3
3,4	10.15	0	30.43	4	0	0	0	12	0	0	0	2	0
3,6	10.67	14.65	25.30	7	1	0	0	14	4	0	0	0	7
3,10	4.79	0	29.51	5	0	0	0	41	11	0	0	0	0
6,3	11.23	4.12	29.55	12	2	3	0	28	6	4	0	42	5
6,4	12.28	3.01	21.11	7	2	4	0	34	13	1	0	46	5
6,6	9.09	0	17.81	3	0	2	0	36	20	1	2	41	0
6,10	7.06	1.58	17.35	7	1	0	0	50	12	3	0	47	4
12,3	10.52	4.79	38.95	25	1	20	3	72	10	38	14	119	5
12,4	13.82	0.41	25.39	16	1	7	0	45	10	10	2	60	5
12,6	6.30	0.09	17.65	10	0	7	0	77	12	10	3	$\star$	2
12,10	5.21	0	2.32	4	0	0	1	366	22	2	1	$\star$	0
Ave=	**10.32**	**12.23**	**28.31**	**7.94**	**0.63**	**2.69**	**0.25**	**49.13**	**7.5**	**4.31**	**1.38**	**22.75**	**3.25**

the number of item- and period-extended versions of the cuts remain small compared to the "simple" versions of these cuts.

Table 3: Average closed gaps and number of cuts generated of each type for test problems with $d_t^i \in [10, 60]$.

				# Cuts generated									
$\|T\|, \|I\|$	**IG**	**GCFC**	**GC**	**C**	**IC**	**PC**	**IPC**	**P**	**IP**	**PP**	**IPP**	**TPP**	**FC**
2,3	14.61	44.77	44.77	5	0	0	0	1	0	0	0	0	5
2,4	16.73	26.01	37.21	7	2	0	0	2	0	0	0	0	9
2,6	9.42	33.50	33.50	8	0	0	0	0	0	0	0	0	8
2,10	7.01	33.81	33.81	3	0	0	0	0	0	0	0	0	3
3,3	20.26	16.65	41.64	8	0	0	0	16	0	0	0	13	5
3,4	8.80	14.67	38.37	6	0	0	0	11	3	0	0	4	3
3,6	10.92	11.34	31.77	6	0	0	0	19	5	0	0	0	4
3,10	6.82	2.01	27.28	10	0	0	0	53	7	0	0	0	1
6,3	11.98	12.43	33.30	15	1	7	0	31	3	10	1	45	7
6,4	7.79	1.42	31.72	15	1	2	0	38	3	4	0	50	5
6,6	6.16	0	23.98	4	0	2	0	32	19	6	4	59	0
6,10	2.03	0	16.33	1	0	1	0	55	28	0	0	76	2
12,3	11.20	1.11	35.84	19	1	22	4	79	6	22	5	114	3
12,4	12.43	0.02	20.05	15	3	11	1	73	14	19	2	7	1
12,6	8.69	0.06	14.13	9	0	4	1	60	21	9	4	$\star$	1
12,10	5.99	0	7.18	9	0	1	0	614	16	2	0	$\star$	2
Ave=	**10.03**	**12.36**	**29.43**	**8.75**	**0.5**	**3.13**	**0.38**	**67.75**	**7.81**	**4.5**	**1**	**27.44**	**3.69**

Finally, we make a remark on the effect of the proposed cuts when the demand variability changes. As the tables clearly indicate, the cuts (in particular partition inequalities and its variants) are more often violated when the demand variability increases: these are also the instances when our cuts make more of an impact for the amount of the gap closed. This makes intuitive sense that partition inequalities are more flexible than covers and hence a

higher demand variability will be able to generate more violated inequalities of this type.

7. A Heuristic Separation Approach

As the exact separation procedures discussed in Section 5 are significantly time-consuming (and even prohibitive with respect to two-period partition inequalities for largest instances tested in Section 6), we cannot expect to generate all cuts within acceptable computational times, in particular for large instances. This motivated us to further analyze the nature of the violated inequalities identified by the exact separation framework, in order to develop a simple but effective heuristic approach for substantially reducing the computational times required by the separation process.

First of all, we conducted an extensive computational experiment focusing primarily on the cover inequalities (8) and partition inequalities (10), as their numbers and effectiveness played a significant role in the gaps closed for the instances tested in the previous section. In order to achieve unbiased results, we generated 80 new random instances with $d_t^i \in [10, 60]$ and analyzed the violated inequalities identified. This analysis indicated that majority of the violated inequalities of cover and partition inequalities are generated from the two-period subproblems consisting of two consecutive periods rather than two non-consecutive periods. This provides a significant potential for reduction in the number of separation problems solved (e.g., in a 12 period problem, there are 66 possible combinations of two periods, whereas the consecutive two periods are only limited to 11 combinations), and unsurprisingly, for these 80 instances, we observed on average 70% improvement in computational times by generating consecutive two-period subproblems compared to all two-period subproblems. Most importantly, using consecutive two-period subproblems, the closed gaps obtained were on average 95% of the closed gaps obtained by generating cuts from all two-period subproblems. These observations motivated us to generate the violated inequalities from the consecutive two-period subproblems only.

In addition, we performed an analysis on the two-period partition inequalities identified for a randomly selected set of 12 period instances from Section 6. As discussed earlier, the exact separation of these inequalities requires substantially more computational time compared to other types of inequalities due to the sheer number of possible combinations of ξ_t and $\xi_{t'}$ values in the separation problem (and hence they were not generated for the largest instances tested in Section 6), and in addition, their effectiveness for closing the duality gaps

were observed to be rather limited. Therefore, we limited our analysis here to previously used instances rather than newly generated instances. First of all, similar to our observation for other types of inequalities, we observed that most of the violated two-period partition inequalities were generated from consecutive two-period subproblems. We also observed that most of the violated two-period partition inequalities occurred in the centre of the planning horizon, rather than earlier or later periods. Our preliminary analysis suggested to focus on the two-period subproblems from the central 1/3 of the horizon, so, e.g., for a 12-period problem, periods 5 to 8 to be used for the in the heuristic separation. Although we could not observe any particular pattern for the potentially more effective value ranges of $\xi_{t'}$, we made the observation that ξ_t values often coincide with the ξ values of the partition inequalities (10) from period t. This motivated us for the simple but significantly time saving approach as presented in Algorithm 1.

It is noteworthy to remark that even though we are still solving the exact separation problems of Section 5, this heuristic algorithm significantly reduces the number of problems to be solved, and therefore, a substantial reduction in computational times required can be achieved. Thanks to this advantage and based on our preliminary tests, we decided to run 3 rounds of separation with this heuristic approach, as opposed to the single round we performed with the exact separation approach.

Input : A fractional solution $(\bar{x}, \bar{y}, \bar{s})$; a two-period subproblem X^{2PL} for t, t' in the centre of the planning horizon
Output: A violated two-period partition inequality
for $\xi_t > 0$ **do**
  if $\xi_t \in \{\bar{\xi}_t \mid$ *there exists violated partition inequality for* $\bar{\xi}_t\}$ **then**
    for $\xi_{t'} > 0$ **do**
      Solve the maximization problem 2PPI
      if $f_{\xi_t,\xi_{t'}} > \widetilde{C}_t + \widetilde{C}_{t'}$ *and* $S_t \cap S_{t'}$ **then**
        Add the violated cut (20)
      end
    end
  end
end

Algorithm 1: Heuristic separation algorithm for two-period partition inequalities (20)

The results are presented in Table 4, which follows the same notation and similar structure used in Section 6, with the addition of specifying the gaps closed with the EXAct and HEUristic approaches for the sake of comparison. First of all, when we compare the aver-

age performance of the heuristic approach with the performance of the exact approach for the instances with 12 periods, we note that it is very competitive, and even often superior, thanks to the advantage of 3 rounds of separation and therefore being able to identify further inequalities which could not be identified by the single round of the exact separation approach. As expected, computational times are significantly improved, for example, instances with $(NT = 12, NI = 10)$ and $d_t^i \in [10, 60]$ took on average 7 days using the exact approach whereas the heuristic approach took on average only 3.5 hours for the same instances.

Table 4: Average closed gaps using the heuristic procedure (HEU) vs. using the exact separation (EXA).

	$d_t^i \in [10, 20]$			$d_t^i \in [10, 40]$			$d_t^i \in [10, 60]$		
		GC			**GC**			**GC**	
$\|T\|, \|I\|$	**IG**	EXA	HEU	**IG**	EXA	HEU	**IG**	EXA	HEU
12,3	16.47	18.25	25.12	10.52	38.95	37.34	11.20	35.84	36.84
12,4	13.81	17.21	17.87	13.82	25.39	25.03	12.43	20.05	20.61
12,6	15.22	18.28	17.74	6.30	17.65	19.23	8.69	14.13	16.51
12,10	7.46	5.88	7.30	5.21	2.32	4.39	5.99	7.18	7.38
24,3	21.51	-	31.94	17.15	-	25.77	9.60	-	44.33
24,4	16.13	-	22.58	13.59	-	16.53	11.39	-	30.01
24,6	12.84	-	13.09	8.64	-	8.27	7.86	-	5.58
24,10	7.84	-	2.08	6.65	-	3.99	5.49	-	4.32
Ave=	13.91	-	17.22	10.24	-	17.56	9.08	-	20.69

In order to further test our heuristic approach on larger instances, we have also generated instances with 24 periods, using the same random generation procedure described in Section 6. As computational times significantly increase even in case of our heuristic, we note that the largest and most challenging instances in this set (i.e., instances with $(NT = 24, NI = 10)$ and $d_t^i \in [10, 60]$) took on average 1.5 days, which is still superior to the times taken by the exact separation for instances with 12 periods. We present our results in Table 4 indicate, where, for a given number of items, the average gap closure of the heuristic for these instances is not significantly different from its average performance on the instances with 12 periods. It is also noteworthy to remark that on average over all these 120 instances with 12 and 24 periods, gaps are closed around 18%. We conclude that in addition to the substantial gains in computational times and ability to experiment with larger instances, the simple heuristic framework achieves successful closure of gaps.

8. Conclusions

In this paper, we investigated a two-period subproblem of the big-bucket lot-sizing problem from a theoretical perspective. In particular, we have identified various families of valid inequalities for a relaxation of this subproblem in the special case of zero setup times, described their facet-defining properties, and we have also mapped and extended these inequalities to the original space of the two-period subproblem. The computational results indicated significant potential for improving lower bounds, and we are currently investigating this thoroughly in a companion study in two immediate directions: i) understanding polyhedral characteristics of the general case with non-zero setup times and identifying further valid inequalities, and ii) designing a branch-and-cut framework with routines generating cutting planes of both zero and non-zero setup time settings in realistic times for multi-item lot-sizing problems.

The theoretical results we presented in this paper can be extended to other MIP problems thanks to the commonality of the mixed integer sets inherent in different problems. We have already noted various studies on the polyhedron of the PIR_0, the single node fixed charge set, which is a common mixed integer set in various MIP problems. On the other hand, the structure of PIR_1 poses different challenges and opportunities, and it is worth investigating further its link to other mixed integer sets. Finally, there is also immediate interest in investigating if and how our understanding of the two-period subproblems can be further extended to more sophisticated subproblems, e.g., a k-period subproblem. As Van Vyve and Wolsey (2005) observed in their framework, even limiting it to the values of $k = 3$ and $k = 4$, there is substantial potential to develop a thorough understanding of the complex lot-sizing problems, which we plan to study in the future.

Acknowledgement. The research presented in this paper is supported by the EPSRC grant EP/L000911/1, entitled "Multi-Item Production Planning: Theory, Computation and Practice".

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