Isomorphism theorems for classes of cyclically presented groups

Esamaldeen Mohamed and Gerald Williams

March 19, 2019

Abstract

We consider two multi-parameter classes of cyclically presented groups, introduced by Cavicchioli, Repovš, and Spaggiari, that contain many previously considered families of cyclically presented groups of interest both for their algebraic and for their topological properties. Building on results of Bardakov and Vesnin, O'Brien and the previously named authors, we prove theorems that establish isomorphisms of groups within these families.

Keywords: Cyclically presented group, isomorphism.
MSC: 20F05.

1 Introduction

The *cyclically presented group* $G_n(w)$ is defined by the *cyclic presentation*

$$\{x_0, \ldots, x_{n-1} \mid w, \theta(w), \ldots, \theta^{n-1}(w)\}$$

where $w = w(x_0, \ldots, x_{n-1})$ is a word in the free group F_n with generators $x_0, \ldots, x_{n-1}$ and $\theta : F_n \to F_n$ is the automorphism of F_n given by $\theta(x_i) = x_{i+1}$ for each $0 \leq i < n$ (subscripts mod n).

The first family of cyclically presented groups that we will consider is the following eight parameter family, introduced in [3], where it is shown that this family contains many classes of cyclically presented groups, previously considered in the literature, of interest both for their algebraic and for their topological properties. Let $n \geq 2$, $r, s \geq 0$, $0 \leq p, h < n$, and $l, k \in \mathbb{Z}$ (note that unlike in [3] we allow $k, l < 0$ and $r, s \in \{0, 1\}$) and define

$$G_n(h,k;p,q;r,s;l) = G_n\left(\left(\prod_{j=0}^{r-1} x_{jp}\right)^{l}\left(\prod_{j=0}^{s-1} x_{h+jq}\right)^{-k}\right) \qquad (1)$$

(where an empty product corresponds to the empty word). The groups $G_n(k,1;m,0;2,1;1)$ are the *groups of Fibonacci type* $G_n(m,k) = G_n(x_0x_mx_k^{-1})$ introduced in [11], and independently in [4]. In particular, the groups $H(n,m) = G_n(m,1) = G_n(1,1;m,0;2,1;1)$ are the *Gilbert-Howie groups* of [10] and the groups $F(2,n) = G_n(2,1;1,0;2,1;1)$ are the *Fibonacci groups* of [7]. Different values of the parameters of these cyclically presented groups may yield isomorphic groups. Our starting point is the following theorem:

Theorem 1 ([5, Theorem 2],[1, Theorem 1.1])**.** *Let $n \geq 2$, $1 \leq m,k,$ $m',k' < n$, $(n,m,k)=1$, and assume $(n,k')=1$. If $m'(m-k) \equiv mk' \bmod n$ then $G_n(m,k) \cong G_n(m',k')$.*

Setting $k = k' = 1$ in Theorem 1 recovers [10, Lemma 2.1]. A less concise version of Theorem 1, and without the hypothesis $(n,k')=1$, was asserted in [1, Theorem 1.1]; however, it was pointed out in [5] that the hypothesis $(n,k')=1$ is necessary (a counterexample of $G_6(1,3) \not\cong G_6(3,4)$ being provided), which led to the formulation of [5, Theorem 2] given above.

In this note we extend Theorem 1 to the class of groups $G_n(h,k;p,q;r,s;l)$. It is convenient to express our main result in terms of parameters A,B,A',B' where

$$A = h, B = A - p(r-1) + q(s-1), A' = h', B' = A' - p'(r-1) + q'(s-1).$$

Theorem A. *If $(n,h',p',q')=1$, $(n,B)=1$, $pA'+p'B \equiv 0 \bmod n$, and $qA'+q'B \equiv 0 \bmod n$, then $G_n(h,k;p,q;r,s;l) \cong G_n(h',k;p',q';r,s;l)$.*

We prove Theorem A in Section 2. The hypothesis $(n,h',p',q') = 1$, while necessary, is not strong: writing $d' = (n,h',p',q')$ we see that $G_n(h',k;p',q';r,s;l)$ is isomorphic to the free product of d' copies of $G_{n/d'}(h'/d',k;p'/d',q'/d';r,s;l)$ (see, for example, [8] or [15, Theorem 1]). As a corollary to Theorem A we obtain:

Corollary 2. *Let $n \geq 2$, $1 \leq m,k,$ $m',k' < n$, $(n,m',k')=1$, and assume $(n,m-k)=1$. If $m'(m-k) \equiv mk' \bmod n$ then $G_n(m,k) \cong G_n(m',k')$.*

Noting that if the congruence $m'(m-k) \equiv mk' \bmod n$ holds then $(n,m',k')=1$ and $(n,m-k)=1$ both hold if and only if $(n,m,k)=1$ and $(n,k')=1$ both hold, we obtain Theorem 1.

We now record some further consequences of Theorem A concerning other cyclically presented groups that have arisen in the literature. The *cyclically presented groups with length three positive relators* are the groups $\Gamma_n(k,l) = G_n(x_0x_kx_l)$ $(0 \leq k,l < n)$ considered in [6, Section 4],[9],[12],[2]. These are both the groups $G_n(k,-1;0,l-k;1,2;1)$ and the groups $G_n(l,-1;k,0;2,1;1)$. Applying Theorem A to these expressions gives:

Corollary 3. *(a) If $(n,k',l')=1,(n,l)=1$ and $k'k\equiv l'l$ mod n then $\Gamma_n(k,l)\cong\Gamma_n(k',l')$.*

(b) If $(n,k',l')=1,(n,l-k)=1$ and $k'l\equiv k(k'-l')$ mod n then $\Gamma_n(k,l)\cong\Gamma_n(k',l')$.

The *Prishchepov groups* $P(r,n,k,s,q)=$

$$G_n((x_0x_q\ldots x_{(r-1)q})(x_{(k-1)}x_{(k-1)+q}\ldots x_{(k-1)+(s-1)q})^{-1})$$

$(n\geq 2, r,s\geq 1, 0\leq k,q<n)$, introduced in [13] and studied further in [15], are the groups $G_n(k-1,1;q,q;r,s;1)$. Applying Theorem A to these we obtain:

Corollary 4. *If $n\geq 2,(n,k'-1,q')=1$, $(n,(k-1)-q(r-s))=1$, and $q'(k-1)+q(k'-1)\equiv qq'(r-s)$ mod n, then $P(r,n,k,s,q)\cong P(r,n,k',s,q')$.*

The *generalized Prischepov groups* $\tilde{P}(r,n,k,s,p,q)=$

$$G_n((x_0x_q\ldots x_{(r-1)q})(x_kx_{k+p}\ldots x_{k+(s-1)p})^{-1})$$

$(n\geq 2, r,s\geq 1, 0\leq k,p,q<n)$, studied in [14], are the groups $G_n(k,1;q,p;r,s;1)$. (An unfortunate conflict of notation arises: $P(r,n,k,s,q)=\tilde{P}(r,n,k-1,s,q,q)$, rather than the more desirable $P(r,n,k,s,q)=\tilde{P}(r,n,k,s,q,q)$.) Applying Theorem A we obtain:

Corollary 5. *Suppose $n\geq 2$, $(n,k',p',q')=1$, $(n,k+p(s-1)-q(r-1))=1$, $qk'+q'k\equiv q'(q(r-1)-p(s-1))$ mod n and $pk'+p'k\equiv p'(q(r-1)-p(s-1))$ mod n. Then $\tilde{P}(r,n,k,s,p,q)\cong\tilde{P}(r,n,k',s,p',q')$.*

Another class of groups fitting within the family $G_n(h,k;p,q;r,s;l)$ was considered in [2]. Given $f\in\mathbb{Z}$ and $r\geq 0$ let $\Lambda(r,f)=\prod_{i=0}^{r-1}x_{if}$ be the positive word of length r whose first letter is x_0 and whose *step size* is f. We call any shift of $\Lambda(r,f)$ or its inverse an *f-block* (of *block length* r) and we say that a cyclically presented group G is of *type* $\mathfrak{F}$ if there is some f such that $G\cong G_n(w)$ where w is the product of two f-blocks. Thus a group is of type $\mathfrak{F}$ if it is isomorphic to

$$G_n\left(\left(\prod_{i=0}^{r_1-1}x_{if}\right)\left(\prod_{i=0}^{r_2-1}x_{a+if}\right)^{\epsilon}\right) \tag{2}$$

for some $r_1,r_2\geq 0$, $0\leq f,a<n$, $\epsilon=\pm 1$. (See the proof of [2, Lemma 1].) Of course the case $\epsilon=-1$ gives the Prishchepov groups (which contain the groups of Fibonacci type) and the case $\{r,s\}=\{2,1\}$, $\epsilon=1$ gives the cyclically presented groups with length three positive relators. The group of type $\mathfrak{F}$ given at (2) is the group $G_n(a,-\epsilon;f,f;r_1,r_2;1)$. Applying Theorem A to this we obtain:

Corollary 6. *Suppose* $(n,a',f')=1$, $(n,a-f(r_1-r_2))=1$, $f'a+fa'\equiv ff'(r_1-r_2)$ *mod* n. *Then*

$$G_n\left(\left(\prod_{i=0}^{r_1-1}x_{if}\right)\left(\prod_{i=0}^{r_2-1}x_{a+if}\right)^{\epsilon}\right)\cong G_n\left(\left(\prod_{i=0}^{r_1-1}x_{if'}\right)\left(\prod_{i=0}^{r_2-1}x_{a'+if'}\right)^{\epsilon}\right).$$

In [2] groups of type $\mathfrak{F}$ that satisfy a certain condition relating the step size, block lengths, and sign, were called *groups of type* $\mathfrak{M}$. Specifically, the *cyclically presented group in the class* $\mathfrak{M}$ *defined by the parameters* (r,n,s,f,A) where $r\geq 0$, $s\in\mathbb{Z}$, $0\leq f,A<n$, and $f(r-s)\equiv 0$ mod n is the group $G_n(w)$ where

$$w=\begin{cases}\left(\prod_{i=0}^{r-1}x_{if}\right)\left(\prod_{i=0}^{s-1}x_{A+if}\right)^{-1} & \text{if } s\geq 0,\\ \left(\prod_{i=0}^{r-1}x_{if}\right)\left(\prod_{i=0}^{|s|-1}x_{A+(r+i)f}\right) & \text{if } s\leq 0\end{cases}$$

(see equation (6) of [2]). Thus the group in the class $\mathfrak{M}$, defined by the parameters (r,n,s,f,A), where $r\geq 0$, $s\in\mathbb{Z}$ and $f(r-s)\equiv 0$ mod n is the group

$$G=\begin{cases}G_n(A,1;f,f;r,s;1) & \text{if } s\geq 0,\\ G_n(A+rf,-1;f,f;r,|s|;1) & \text{if } s\leq 0.\end{cases}$$

Applying Theorem A to these groups we have the following:

Corollary 7. *(a) Suppose* $r,s\geq 0$, $f(r-s)\equiv 0$ *mod* n, $f'(r-s)\equiv 0$ *mod* n. *If* $(n,A',f')=1$, $(n,A)=1$ *and* $f'A+fA'\equiv 0$ *mod* n *then the groups in the class* $\mathfrak{M}$*, defined by the parameters* (r,n,s,f,A)*, and by the parameters* (r,n,s,f',A') *are isomorphic.*

(b) Suppose $r\geq 0$, $s\leq 0$, $f(r-s)\equiv 0$ *mod* n, $f'(r-s)\equiv 0$ *mod* n. *If* $(n,A',f')=1$, $(n,A+|s|f)=1$ *and* $f'A+fA'\equiv 0$ *mod* n *then the groups in the class* $\mathfrak{M}$*, defined by the parameters* (r,n,s,f,A)*, and by the parameters* (r,n,s,f',A') *are isomorphic.*

We now turn to a second family of cyclically presented groups introduced in [6]. For $(a,b,r,s)\in\mathbb{Z}^4$, $n\geq 2$, $0\leq m,k,h<n$ let

$$G_n^{(a,b,r,s)}(m,k,h)=G_n(x_0^a x_k^b x_{h+m}^a(x_h^r x_m^r)^{-s}). \tag{3}$$

It was shown in [6] that this family also contains many classes of cyclically presented groups previously considered in the literature. In particular, $G_n^{(1,1,2,1)}(m,k,0)$ are the groups of Fibonacci type $G_n(m,k)$ and $G_n^{(1,1,0,0)}(0,k,l)$ are the cyclically presented groups with length three positive relators $\Gamma_n(k,l)$. We have the following theorem, a generalization of [1, Theorem 1.1], which establishes isomorphisms of groups within this class:

Theorem 8 ([6, Theorem 2.6])**.** *Suppose* $0 \leq m, k, h < n$*,* $(n, k-h-m) = 1$*,* $(n, m', k', h') = 1$ *and that* $\rho = (n, k-h-m)$ *divides* k' *and there exist positive integers* $\alpha, \beta, \gamma, \delta$ *such that*

$$\alpha + \beta(k-h-m) \equiv 1 - m \bmod n, \tag{4}$$

$$\gamma + \delta(k-h-m) \equiv 1 - h \bmod n, \tag{5}$$

$$\alpha + \beta k' \equiv 1 + m' \bmod n, \tag{6}$$

$$\gamma + \delta k' \equiv 1 + h' \bmod n, \tag{7}$$

where $1 \leq \alpha, \gamma \leq \rho$ *and* $1 \leq \beta, \delta \leq n/\rho$*. Then for all* $(a, b, r, s) \in \mathbb{Z}^4$ $G_n^{(a,b,r,s)}(m, k, h) \cong G_n^{(a,b,r,s)}(m', k', h')$*.*

Note that the hypothesis $(n, k-h-m) = 1$ was omitted from the original statement [6, Theorem 2.6] and is necessary: just as the pair of groups $G_6(1,3)$, $G_6(3,4)$ provided a counterexample to [1, Theorem 1.1], the same pair of groups $G_6^{(1,1,2,1)}(1,3,0) = G_6(1,3)$, $G_6^{(1,1,2,1)}(3,4,0) = G_6(3,4)$ provide a counterexample to the original statement [6, Theorem 2.6]. In the same way that the (necessary) imposition of the hypothesis $(n, k') = 1$ leads to the more concise formulation of [1, Theorem 1.1] given in Theorem 1, the (necessary) imposition of the hypothesis $(n, k-h-m) = 1$ in Theorem 8 leads to the following formulation:

Corollary B. *Suppose* $0 \leq m, k, h < n$*,* $(n, k-h-m) = 1$*,* $(n, m', k', h') = 1$*,* $m'(h+m-k) \equiv mk'$*,* $h'(h+m-k) \equiv hk'$*. Then for all* $(a, b, r, s) \in \mathbb{Z}^4$ $G_n^{(a,b,r,s)}(m, k, h) \cong G_n^{(a,b,r,s)}(m', k', h')$*.*

Proof. Suppose $d = (n, k') > 1$. Since d divides k', it must also divide mk' and hk' and hence $m'(h+m-k)$ and $h'(h+m-k)$. But since $(n, k-h-m) = 1$, d divides m' and h', and so d divides $(n, m', k', h') = 1$, a contradiction. Thus $(n, k') = 1$. Writing $(k')^{-1}$ for the multiplicative inverse of k' mod n, let $\beta = m'(k')^{-1}$ mod n, $\delta = h'(k')^{-1}$ mod n. In the notation of Theorem 8, $\rho = 1$ and hence $\alpha = \gamma = 1$. The hypotheses imply that the congruences (4)–(7) hold, so the result follows. $\square$

Putting $(a, b, r, s) = (1, 1, 2, 1)$ and $h = h' = 0$ in Corollary B yields Corollary 2, and hence Theorem 1. Since there was a missing hypothesis in the original statement of Theorem 8 we include a direct proof of Corollary B (based on the proof of [6, Theorem 2.6]) in Section 3.

2 Proof of Theorem A

Suppose $d = (n, A') > 1$. Since d divides A', it must also divide pA' and qA', and hence $p'B$ and $q'B$. But, since $(n, B) = 1$, d divides p' and q' so d divides $(n, A', p', q') = (n, h', p', q') = 1$, a contradiction. Thus $(n, A') = 1$.

Writing $(A')^{-1}$ for the multiplicative inverse of A' mod n, let $\sigma = p'(A')^{-1}$, $\tau = q'(A')^{-1}$, so

$$\sigma A' \equiv p' \bmod n, \tag{8}$$
$$\tau A' \equiv q' \bmod n. \tag{9}$$

Further $\sigma B \equiv p'(A')^{-1}B \equiv (A')^{-1}(p'B) \equiv (A')^{-1}(-pA') \equiv -p \bmod n$; similarly $\tau B \equiv -q \bmod n$, i.e.

$$\sigma B \equiv -p \bmod n, \tag{10}$$
$$\tau B \equiv -q \bmod n. \tag{11}$$

Let $G = G_n(h,k;p,q;r,s;l)$ as defined at (1). Inverting the relators gives

$$G = G_n\left(\left(\prod_{j=0}^{r-1} x^{-1}_{(r-1)p-jp}\right)^l \left(\prod_{j=0}^{s-1} x^{-1}_{(s-1)q+h-jq}\right)^{-k}\right).$$

Replacing each generator by its inverse and then subtracting $(r-1)p$ from all subscripts gives

$$\begin{aligned} G &= G_n\left(\left(\prod_{j=0}^{r-1} x_{-jp}\right)^l \left(\prod_{j=0}^{s-1} x_{-(r-1)p+(s-1)q+h-jq}\right)^{-k}\right) \\ &= G_n\left(\left(\prod_{j=0}^{r-1} x_{-jp}\right)^l \left(\prod_{j=0}^{s-1} x_{B-jq}\right)^{-k}\right) \\ &= \left\langle\ x_0,\ldots,x_{n-1} \ \mid\ \left(\textstyle\prod_{j=0}^{r-1} x_{i-jp}\right)^l \left(\textstyle\prod_{j=0}^{s-1} x_{i+B-jq}\right)^{-k}\ (0 \le i < n)\ \right\rangle. \end{aligned}$$

Now $(n,B)=1$ implies that $\{i \mid 0 \le i < n\} = \{\alpha B \mid 0 \le \alpha < n\}$ so

$$\begin{aligned} G &= \left\langle\ x_0,\ldots,x_{n-1} \ \mid\ \left(\textstyle\prod_{j=0}^{r-1} x_{\alpha B-jp}\right)^l \left(\textstyle\prod_{j=0}^{s-1} x_{\alpha B+B-jq}\right)^{-k}\ (0 \le \alpha < n)\ \right\rangle \\ &= \left\langle\ x_0,\ldots,x_{n-1} \ \mid\ \left(\textstyle\prod_{j=0}^{r-1} x_{\alpha B+j\sigma B}\right)^l \left(\textstyle\prod_{j=0}^{s-1} x_{\alpha B+B+j\tau B}\ \right)^{-k} (0 \le \alpha < n)\ \right\rangle \end{aligned}$$

by (10),(11), so

$$G = \left\langle\ x_0,\ldots,x_{n-1} \ \mid\ \left(\textstyle\prod_{j=0}^{r-1} x_{(\alpha+j\sigma)B}\right)^l \left(\textstyle\prod_{j=0}^{s-1} x_{(\alpha+1+j\tau)B}\ \right)^{-k} (0 \le \alpha < n)\ \right\rangle.$$

Noting that $(n, A') = 1$ and $(n, B) = 1$ we may adjoin generators $\{y_0, \ldots, y_{n-1}\}$ as follows:

$$G = \left\langle \begin{array}{l} x_0, \ldots, x_{n-1}, \\ y_0, \ldots, y_{n-1} \end{array} \left| \begin{array}{l} \left(\prod_{j=0}^{r-1} x_{(\alpha+j\sigma)B}\right)^l \left(\prod_{j=0}^{s-1} x_{(\alpha+1+j\tau)B}\right)^{-k} \\ y_{\alpha A'} = x_{\alpha B} \ (0 \le \alpha < n) \end{array} \right. \right\rangle.$$

Eliminating generators $x_0, \ldots, x_{n-1}$ then gives

$$\begin{aligned} G &= \left\langle \ y_0, \ldots, y_{n-1} \ \mid \ \left(\prod_{j=0}^{r-1} y_{(\alpha+j\sigma)A'}\right)^l \left(\prod_{j=0}^{s-1} y_{(\alpha+1+j\tau)A'}\right)^{-k} \ (0 \le \alpha < n) \ \right\rangle \\ &= \left\langle \ y_0, \ldots, y_{n-1} \ \mid \ \left(\prod_{j=0}^{r-1} y_{\alpha A'+j(\sigma A')}\right)^l \left(\prod_{j=0}^{s-1} y_{\alpha A'+A'+j(\tau A')}\right)^{-k} \ (0 \le \alpha < n) \ \right\rangle \\ &= \left\langle \ y_0, \ldots, y_{n-1} \ \mid \ \left(\prod_{j=0}^{r-1} y_{\alpha A'+jp'}\right)^l \left(\prod_{j=0}^{s-1} y_{\alpha A'+A'+jq'}\right)^{-k} \ (0 \le \alpha < n) \ \right\rangle \end{aligned}$$

using (8),(9). Now $(n, A') = 1$ implies that $\{\alpha A' \mid 0 \le \alpha < n\} = \{i \mid 0 \le i < n\}$ so

$$\begin{aligned} G &= \left\langle \ y_0, \ldots, y_{n-1} \ \mid \ \left(\prod_{j=0}^{r-1} y_{i+jp'}\right)^l \left(\prod_{j=0}^{s-1} y_{i+A'+jq'}\right)^{-k} \ (0 \le i < n) \ \right\rangle \\ &= \left\langle \ y_0, \ldots, y_{n-1} \ \mid \ \left(\prod_{j=0}^{r-1} y_{i+jp'}\right)^l \left(\prod_{j=0}^{s-1} y_{i+h'+jq'}\right)^{-k} \ (0 \le i < n) \ \right\rangle \\ &= G_n(h', k; p', q'; r, s; l) \end{aligned}$$

as required.

3 Proof of Corollary B

As in our earlier proof of Corollary B, $(n, k') = 1$. Again, let $\beta = m'(k')^{-1}$, $\delta = h'(k')^{-1} \bmod n$. Then $\beta(k-h-m) \equiv m'(k')^{-1}(k-h-m) \equiv (k')^{-1}m'(k-h-m) \equiv -(k')^{-1}mk' \equiv -m \bmod n$; similarly $\delta(k-h-m) \equiv -h$, i.e.

$$\beta(k-h-m) \equiv -m \bmod n, \tag{12}$$
$$\delta(k-h-m) \equiv -h \bmod n. \tag{13}$$

Let $G = G_n^{(a,b,r,s)}(m, k, h)$ as defined at (3). Replacing each generator by its inverse, inverting the relators and subtracting $(h+m)$ from the subscripts gives

$$G = \left\langle \ x_i \ \mid \ x_i^a x_{i+k-h-m}^b x_{i-h-m}^a = (x_{i-h}^r x_{i-m}^r)^s \ (0 \le i < n) \ \right\rangle.$$

Since $(k-h-m,n)=1$ the set $\{i \mid 0 \leq i < n\} = \{t(k-h-m) \mid 0 \leq t < n\}$ so

$$G = \left\langle \; x_{t(k-h-m)} \;\middle|\; \begin{array}{c} x_{t(k-h-m)}^a x_{t(k-h-m)+k-h-m}^b x_{t(k-h-m)-h-m}^a = \\ (x_{t(k-h-m)-h}^r x_{t(k-h-m)-m}^r)^s \; (0 \leq t < n) \end{array} \right\rangle$$

and using (12),(13) we get

$$G = \left\langle \; x_{t(k-h-m)} \;\middle|\; \begin{array}{c} x_{t(k-h-m)}^a x_{(t+1)(k-h-m)}^b x_{(t+\beta+\delta)(k-h-m)}^a = \\ (x_{(t+\delta)(k-h-m)}^r x_{(t+\beta)(k-h-m)}^r)^s \; (0 \leq t < n) \end{array} \right\rangle.$$

Noting that $(n,k')=1$ and $(n,k-h-m)=1$ we may adjoin generators $\{y_0,\ldots,y_{n-1}\} = \{y_{tk'} \mid (0 \leq t < n)\}$ as follows:

$$G = \left\langle \begin{array}{l} x_{t(k-h-m)}, \\ y_{tk'} \end{array} \;\middle|\; \begin{array}{l} x_{t(k-h-m)}^a x_{(t+1)(k-h-m)}^b x_{(t+\beta+\delta)(k-h-m)}^a = \\ \quad (x_{(t+\delta)(k-h-m)}^r x_{(t+\beta)(k-h-m)}^r)^s, \\ y_{tk'} = x_{t(k-h-m)} \; (0 \leq t < n) \end{array} \right\rangle.$$

Eliminating generators $x_{t(k-h-m)}$ $(0 \leq t < n)$ gives

$$\begin{aligned} G &= \langle \; y_{tk'} \mid y_{tk'}^a y_{(t+1)k'}^b y_{(t+\beta+\delta)k'}^a = (y_{(t+\delta)k'}^r y_{(t+\beta)k'}^r)^s \; (0 \leq t < n) \; \rangle \\ &= \langle \; y_{tk'} \mid y_{tk'}^a y_{tk'+k'}^b y_{tk'+\beta k'+\delta k'}^a = (y_{tk'+\delta k'}^r y_{tk'+\beta k'}^r)^s \; (0 \leq t < n) \; \rangle \\ &= \langle \; y_{tk'} \mid y_{tk'}^a y_{tk'+k'}^b y_{tk'+m'+h'}^a = (y_{tk'+h'}^r y_{tk'+m'}^r)^s \; (0 \leq t < n) \; \rangle \\ &= \langle \; y_i \mid y_i^a y_{i+k'}^b y_{i+m'+h'}^a = (y_{i+h'}^r y_{i+m'}^r)^s \; (0 \leq i < n) \; \rangle \\ &= G_n^{(a,b,r,s)}(m',k',h') \end{aligned}$$

as required.

References

[1] V. G. Bardakov and A. Yu. Vesnin. A generalization of Fibonacci groups. *Algebra Logika*, 42(2):131–160, 2003. Translation in *Algebra Logic* 42(2), 73–91, 2003.

[2] William A. Bogley and Gerald Williams. Coherence, subgroup separability, and metacyclic structures for a class of cyclically presented groups. *J. Algebra*, 480:266–297, 2017.

[3] A. Cavicchioli, D. Repovš, and F. Spaggiari. Topological properties of cyclically presented groups. *J. Knot Theory Ramifications*, 12(2):243–268, 2003.

[4] Alberto Cavicchioli, Friedrich Hegenbarth, and Dušan Repovš. On manifold spines and cyclic presentations of groups. In *Knot theory*, volume 42, pages 49–56. Warszawa: Polish Academy of Sciences, Institute of Mathematics, Banach Cent. Publ., 1998.

[5] Alberto Cavicchioli, E.A. O'Brien, and Fulvia Spaggiari. On some questions about a family of cyclically presented groups. *J. Algebra*, 320(11):4063–4072, 2008.

[6] Alberto Cavicchioli, Dušan Repovš, and Fulvia Spaggiari. Families of group presentations related to topology. *J. Algebra*, 286(1):41–56, 2005.

[7] J. Conway. Advanced problem 5327. *Am. Math. Mon.*, 72(8):915, 1965.

[8] M. Edjvet. On irreducible cyclic presentations. *J. Group Theory*, 6(2):261–270, 2003.

[9] Martin Edjvet and Gerald Williams. The cyclically presented groups with relators $x_i x_{i+k} x_{i+l}$. *Groups, Geom., Dyn.*, 4(4):759–775, 2010.

[10] N.D. Gilbert and J. Howie. LOG groups and cyclically presented groups. *J. Algebra*, 174(1):118–131, 1995.

[11] D.L. Johnson and H. Mawdesley. Some groups of Fibonacci type. *J. Aust. Math. Soc.*, 20:199–204, 1975.

[12] Esmaldeen Mohamed and Gerald Williams. An investigation into the cyclically presented groups with length three positive relators. arXiv:1806.06821, 26 pages, 2018.

[13] M.I. Prishchepov. Asphericity, atoricity and symmetrically presented groups. *Comm. Algebra*, 23(13):5095–5117, 1995.

[14] Agnese Ilaria Telloni. Combinatorics of a class of groups with cyclic presentation. *Discrete Math.*, 310(22):3072–3079, 2010.

[15] Gerald Williams. Largeness and SQ-universality of cyclically presented groups. *Int. J. Algebra and Computation*, 22(4):19 pages, 2012.

Department of Mathematics, Faculty of Education, Omar-Almukhtar University, Albyda, Libya.

E-mail address, essamadeen@yahoo.com

Department of Mathematical Sciences, University of Essex, Wivenhoe Park, Colchester, Essex CO4 3SQ, UK.
E-mail address, Gerald.Williams@essex.ac.uk