COMBINATORICA
Bolyai Society – Springer-Verlag

Combinatorica 1–40

LOCAL CONVERGENCE OF RANDOM GRAPH COLORINGS†

AMIN COJA-OGHLAN*, CHARILAOS EFTHYMIOU**, NOR JAAFARI

Received January 26, 2015
Revised March 14, 2016

Let $\boldsymbol{G} = \boldsymbol{G}(n,m)$ be a random graph whose average degree $d = 2m/n$ is below the k-colorability threshold. If we sample a k-coloring $\boldsymbol{\sigma}$ of $\boldsymbol{G}$ uniformly at random, what can we say about the correlations between the colors assigned to vertices that are far apart? According to a prediction from statistical physics, for average degrees below the so-called *condensation threshold* $d_{k,\mathrm{cond}}$, the colors assigned to far away vertices are asymptotically independent [Krzakala et al.: Proc. National Academy of Sciences 2007]. We prove this conjecture for k exceeding a certain constant k_0. More generally, we investigate the joint distribution of the k-colorings that $\boldsymbol{\sigma}$ induces locally on the bounded-depth neighborhoods of any fixed number of vertices. In addition, we point out an implication on the *reconstruction problem.*

1. Introduction and results

Let $\boldsymbol{G}=\boldsymbol{G}(n,m)$ *denote the random graph on the vertex set* $[n]=\{1,\dots,n\}$ *with precisely* m *edges. Unless specified otherwise, we assume that* $m = m(n)=\lceil dn/2\rceil$ *for a fixed number* $d>0$. *As usual,* $\boldsymbol{G}(n,m)$ *has a property* $\mathcal{A}$ *"with high probability" ("w.h.p.") if* $\lim_{n\to\infty}\mathbb{P}[\boldsymbol{G}(n,m)\in\mathcal{A}]=1$.

Mathematics Subject Classification (2000): 05C80; 05C15

† An extended abstract version of this work appeared in the proceedings of RANDOM 2015.

* The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013)/ERC Grant Agreement n. 278857–PTCC.

** Research is supported by ARC GaTech.

1.1. Background and motivation

Going back to the seminal paper of Erdős and Rényi [22] that founded the theory of random graphs, the problem of coloring $\boldsymbol{G}(n,m)$ remains one of the longest-standing challenges in probabilistic combinatorics. Over the past half-century, efforts have been devoted to determining the likely value of the chromatic number $\chi(\boldsymbol{G}(n,m))$ [4,12,29,31] and its concentration [6,30,41] as well as to algorithmic problems such as constructing or sampling colorings of the random graph [3,16,18,19,24,26].

The single most tantalising feature of the random graph coloring problem is the interplay between local and global effects. *Locally* around almost any vertex the random graph is bipartite w.h.p. In fact, for any fixed average degree $d>0$ and for any fixed ω the depth-ω neighborhood of all but $o(n)$ vertices is just a tree w.h.p. Yet *globally* the chromatic number of the random graph may be large. Indeed, for any number $k\geq 3$ of colors there exists a *sharp threshold sequence* $d_{k\text{-col}}=d_{k\text{-col}}(n)$ such that for any fixed $\varepsilon>0$, $\boldsymbol{G}(n,m)$ is k-colorable w.h.p. if $2m/n<d_{k\text{-col}}(n)-\varepsilon$, whereas the random graphs fails to be k-colorable w.h.p. if $2m/n>d_{k\text{-col}}(n)+\varepsilon$ [1]. Whilst the thresholds $d_{k\text{-col}}$ are not known precisely, there are close upper and lower bounds. The best current ones read

$$\begin{aligned}d_{k,\mathrm{cond}}&=(2k-1)\ln k-2\ln 2+\delta_k\leq\liminf_{n\to\infty}d_{k\text{-col}}(n)\\&\leq\limsup_{n\to\infty}d_{k\text{-col}}(n)\leq(2k-1)\ln k-1+\varepsilon_k,\end{aligned}\tag{1}$$

where $\lim_{k\to\infty}\delta_k=\lim_{k\to\infty}\varepsilon_k=0$ [4,14,15]. To be precise, the lower bound in (1) is formally defined as

$$d_{k,\mathrm{cond}}=\inf\left\{d>0\colon\limsup_{n\to\infty}\mathbb{E}[Z_k(\boldsymbol{G}(n,m))^{1/n}]<k(1-1/k)^{d/2}\right\}.\tag{2}$$

This number, called the *condensation threshold* due to a connection with statistical physics [27], can be computed precisely for k exceeding a certain constant k_0 [8]. An asymptotic expansion yields the expression in (1).

The contrast between local and global effects was famously pointed out by Erdős, who produced $\boldsymbol{G}(n,m)$ as an example of a graph that simultaneously has a high chromatic number and a high girth [21]. The present paper aims at a more precise understanding of this collusion between short-range and long-range effects. For instance, do global effects entail "invisible" constraints on the colorings of the local neighborhoods such that certain "local" colorings do not extend to a coloring of the entire graph? And what correlations do typically exist between the colors of vertices at a large distance?

Perhaps the most natural way of formalising these questions is as follows. Let $k \geq 3$ be a number of colors, fix some number $\omega > 0$ and assume that $d < d_{k,\mathrm{cond}}$ so that $\boldsymbol{G} = \boldsymbol{G}(n,m)$ is k-colorable w.h.p. Moreover, pick a vertex v_0 and fix a k-coloring σ_0 of its depth-ω neighborhood. How many ways are there to extend σ_0 to a k-coloring of the entire graph, and how does this number depend on σ_0? Additionally, if we pick another vertex v_1 that is "far away" from v_0 and if we pick another k-coloring σ_1 of the depth-ω neighborhood of v_1, is there a k-coloring σ of $\boldsymbol{G}$ that simultaneously extends both σ_0 and σ_1? If so, how many such σ exist, and how does this depend on σ_0, σ_1?

The main result of this paper (Theorem 1.1 below) provides a very neat and accurate answer to these questions. It shows that w.h.p. all "local" k-colorings σ_0 extend to *asymptotically the same* number of k-colorings of the entire graph. Let us write $\mathcal{S}_k(G)$ for the set of all k-colorings of a graph G and let $Z_k(G) = |\mathcal{S}_k(G)|$ be the number of k-colorings. Moreover, let $\partial^\omega(G, v_0)$ be the depth-ω neighborhood of a vertex v_0 in G (i.e., the subgraph of G obtained by deleting all vertices at distance greater than ω from v_0). Then w.h.p. any k-coloring σ_0 of $\partial^\omega(\boldsymbol{G}, v_0)$ has

$$\frac{(1+o(1))Z_k(\boldsymbol{G})}{Z_k(\partial^\omega(\boldsymbol{G}, v_0))}$$

extensions to a k-coloring of $\boldsymbol{G}$. Moreover, if we pick another vertex v_1 at random and fix some k-coloring σ_1 of the depth-ω neighborhood of v_1, then w.h.p. the number of joint extensions of σ_0, σ_1 is

$$\frac{(1+o(1))Z_k(\boldsymbol{G})}{Z_k(\partial^\omega(\boldsymbol{G}, v_0))Z_k(\partial^\omega(\boldsymbol{G}, v_1))}.$$

In other words, if we choose a k-coloring $\boldsymbol{\sigma}$ uniformly at random, then the distribution of the k-coloring that $\boldsymbol{\sigma}$ induces on the subgraph $\partial^\omega(\boldsymbol{G}, v_0) \cup \partial^\omega(\boldsymbol{G}, v_1)$, which is a forest w.h.p., is asymptotically uniform. The same statement extends to any fixed number $v_0, \ldots, v_l$ of vertices.

This result, formally stated as Theorem 1.1/Corollary 1.2 below, is very much in line with and actually inspired by predictions from non-rigorous physics work. In fact, Corollary 1.3, a special case of Corollary 1.2, was conjectured explicitly in [27,44]. Moreover, also Theorem 1.1 and Corollary 1.2 hardly come as a surprise given the "replica symmetry breaking" picture drafted by physicists [25,27,44,32,33,34,35,36]. Furthermore, the results of this paper have a flavour of "decay of correlations" or "spatial mixing", a type of question that has been studied in prior work on sampling colorings of random graphs, e.g., [16,17,18,19,40]. However, we are not aware of a reference where Theorem 1.1 or Corollary 1.2 were conjecture explicitly.

1.2. Results

The appropriate formalism for describing the limiting behavior of the local structure of the random graph is the concept of *local weak convergence* [5,9]. The concrete installment of the formalism that we employ is reminiscent of that used in [11,38]. (Corollary 1.2 below provides a statement that is equivalent to the main result but that avoids the formalism of local weak convergence.)

Let $\mathfrak{G}$ be the set of all locally finite connected graphs whose vertex set is a countable subset of $\mathbb{R}$. Further, let $\mathfrak{G}_k$ be the set of all triples (G,v_0,σ) such that $G\in\mathfrak{G}$, $\sigma\colon V(G)\to[k]$ is a k-coloring of G and $v_0\in V(G)$ is a distinguished vertex that we call the *root*. We refer to (G,v_0,σ) as a *rooted k-colored graph.* If (G',v_0',σ') is another rooted k-colored graph, we call (G,v_0,σ) and (G',v_0',σ') *isomorphic* ($(G,v_0,\sigma)\cong(G',v_0',\sigma')$) if there is an isomorphism $\varphi\colon G\to G'$ such that $\varphi(v_0)=\varphi(v_0')$, $\sigma=\sigma'\circ\varphi$ and such that for any $v,w\in V(G)$ with $v<w$ we have $\varphi(v)<\varphi(w)$. Thus, φ preserves the root, the coloring and the order of the vertices (which are reals). Let $[G,v_0,\sigma]$ be the isomorphism class of (G,v_0,σ) and let $\mathcal{G}_k$ be the set of all isomorphism classes of rooted k-colored graphs.

For an integer $\omega\geq 0$ and $\Gamma\in\mathcal{G}_k$ we let $\partial^\omega\Gamma$ denote the isomorphism class of the rooted k-colored graph obtained from Γ by deleting all vertices whose distance from the root exceeds ω. Then any Γ, $\omega\geq 0$ give rise to a function

$$\mathcal{G}_k\to\{0,1\}\,,\qquad \Gamma'\mapsto\mathbf{1}\left\{\partial^\omega\Gamma'=\partial^\omega\Gamma\right\}. \tag{3}$$

We endow $\mathcal{G}_k$ with the coarsest topology that makes all of these functions continuous. Further, for $l\geq 1$ we equip $\mathcal{G}_k^l$ with the corresponding product topology. Additionally, the set $\mathcal{P}(\mathcal{G}_k^l)$ of probability measures on $\mathcal{G}_k^l$ carries the weak topology, as does the set $\mathcal{P}^2(\mathcal{G}_k^l)$ of all probability measures on $\mathcal{P}(\mathcal{G}_k^l)$. The spaces $\mathcal{G}_k^l,\mathcal{P}(\mathcal{G}_k^l),\mathcal{P}^2(\mathcal{G}_k^l)$ are Polish [5]. For $\Gamma\in\mathcal{G}_k$ we denote by $\delta_\Gamma\in\mathcal{P}(\mathcal{G}_k)$ the Dirac measure that puts mass one on Γ.

Let G be a finite k-colorable graph whose vertex set $V(G)$ is contained in $\mathbb{R}$ and let $v_1,\ldots,v_l\in V(G)$. Then we can define a probability measure on $\mathcal{G}_k^l$ as follows. Letting $G\|v$ denote the connected component of $v\in V(G)$ and $\sigma\|v$ the restriction of $\sigma\colon V(G)\to[k]$ to $G\|v$, we define

$$\lambda\left(G,v_1,\ldots,v_l\right)=\frac{1}{Z_k(G)}\sum_{\sigma\in\mathcal{S}_k(G)}\bigotimes_{i=1}^{l}\delta_{[G\|v_i,v_i,\sigma\|v_i]}\in\mathcal{P}(\mathcal{G}_k^l). \tag{4}$$

The idea is that $\lambda_{G,v_1,\ldots,v_l}$ captures the joint empirical distribution of colorings induced by a random coloring of G "locally" in the vicinity of the "roots" $v_1, \ldots, v_l$. Further, let

$$\boldsymbol{\lambda}^l_{n,m,k} = \frac{1}{n^l} \sum_{v_1,\ldots,v_l \in [n]} \mathbb{E}[\delta_{\lambda(\boldsymbol{G}(n,m),v_1,\ldots,v_l)} | \chi(\boldsymbol{G}(n,m)) \le k] \in \mathcal{P}^2(\mathcal{G}^l_k).$$

This measure captures the typical distribution of the local colorings in a random graph with l randomly chosen roots. We are going to determine the limit of $\boldsymbol{\lambda}^l_{n,m,k}$ as $n \to \infty$.

To characterise this limit, let $\boldsymbol{T}^*(d)$ be a (possibly infinite) random Galton-Watson tree rooted at a vertex v^*_0 with offspring distribution $\mathrm{Po}(d)$. We embed $\boldsymbol{T}^*(d)$ into $\mathbb{R}$ by independently mapping each vertex to a uniformly random point in $[0,1]$; with probability one, all vertices get mapped to distinct points. Let $\boldsymbol{T}(d) \in \mathfrak{G}$ signify the resulting random tree and let v_0 denote its root. For a number $\omega > 0$ we let $\partial^\omega \boldsymbol{T}(d)$ denote the (finite) rooted tree obtained from $\boldsymbol{T}(d)$ by removing all vertices at a distance greater than ω from v_0. Moreover, for $l \ge 1$ let $\boldsymbol{T}^1(d), \ldots, \boldsymbol{T}^l(d)$ be l independent copies of $\boldsymbol{T}(d)$ with roots $v^1_0, \ldots, v^l_0$ and set

$$(5) \qquad \begin{aligned} \boldsymbol{\vartheta}^l_{d,k}[\omega] &= \mathbb{E}\left[\delta_{\bigotimes_{i\in[l]} \lambda\left(\partial^\omega \boldsymbol{T}^i(d)\right)}\right] \in \mathcal{P}^2(\mathcal{G}^l_k), \text{ where} \\ \lambda\left(\partial^\omega \boldsymbol{T}^i(d)\right) &= \frac{1}{Z_k(\partial^\omega \boldsymbol{T}^i(d))} \sum_{\sigma \in \mathcal{S}_k(\partial^\omega \boldsymbol{T}^i(d))} \delta_{[\partial^\omega \boldsymbol{T}^i(d), v^i_0, \sigma]} \in \mathcal{P}(\mathcal{G}^l_k) \text{ (cf. (4))}. \end{aligned}$$

The sequence $(\boldsymbol{\vartheta}^l_{d,k}[\omega])_{\omega \ge 1}$ converges (see Appendix A) and we let

$$\boldsymbol{\vartheta}^l_{d,k} = \lim_{\omega \to \infty} \boldsymbol{\vartheta}^l_{d,k}[\omega].$$

Combinatorially, $\boldsymbol{\vartheta}^l_{d,k}$ corresponds to sampling l copies of the Galton-Watson tree $\boldsymbol{T}(d)$ independently. These trees are colored by assigning a random color to each of the l roots independently and proceeding down each tree by independently choosing a color for each vertex from the $k-1$ colors left unoccupied by the parent.

Theorem 1.1. *There is a number $k_0 > 0$ such that for all $k \ge k_0$, $d < d_{k,cond}$, $l > 0$ we have $\lim_{n\to\infty} \boldsymbol{\lambda}^l_{n,m,k} = \boldsymbol{\vartheta}^l_{d,k}$.*

Fix numbers $\omega \ge 1$, $l \ge 1$, choose a random graph $\boldsymbol{G} = \boldsymbol{G}(n,m)$ for some large enough n and choose vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l$ uniformly and independently at random. Then the depth-ω neighborhoods $\partial^\omega(\boldsymbol{G}, \boldsymbol{v}_1), \ldots, \partial^\omega(\boldsymbol{G}, \boldsymbol{v}_l)$ are pairwise disjoint and the union $\mathcal{F} = \partial^\omega(\boldsymbol{G}, \boldsymbol{v}_1) \cup \cdots \cup \partial^\omega(\boldsymbol{G}, \boldsymbol{v}_l)$ is a forest w.h.p. Moreover, the distance between any two trees in $\mathcal{F}$ is $\Omega(\ln n)$ w.h.p. Given that $\boldsymbol{G}$ is k-colorable, let $\boldsymbol{\sigma}$ be a random k-coloring of $\boldsymbol{G}$. Then $\boldsymbol{\sigma}$

induces a k-coloring of the forest $\mathcal{F}$. Theorem 1.1 implies that w.h.p. the distribution of the induced coloring is at a total variation distance $o(1)$ from the uniform distribution on the set of all k-colorings of $\mathcal{F}$. Formally, let us write $\mu_{k,G}$ for the probability distribution on $[k]^{V(G)}$ defined by

$$\mu_{k,G}(\sigma) = \mathbf{1}\{\sigma \in \mathcal{S}_k(G)\} Z_k(G)^{-1} \qquad (\sigma \in [k]^{V(G)}),$$

i.e., the uniform distribution on the set of k-colorings of the graph G. Moreover, for $U \subset V(G)$ let $\mu_{k,G|U}$ denote the projection of $\mu_{k,G}$ onto $[k]^U$, i.e.,

$$\mu_{k,G|U}(\sigma_0) = \mu_{k,G}\left(\left\{\sigma \in [k]^V : \forall u \in U : \sigma(u) = \sigma_0(u)\right\}\right) \qquad (\sigma_0 \in [k]^U).$$

If H is a subgraph of G, then we just write $\mu_{k,G|H}$ instead of $\mu_{k,G|V(H)}$. Let $\|\cdot\|_{\mathrm{TV}}$ denote the total variation norm.

Corollary 1.2. *There is a constant $k_0 > 0$ such that for any $k \geq k_0$, $d < d_{k,cond}$, $l \geq 1$, $\omega \geq 0$ we have*

$$\lim_{n\to\infty} \frac{1}{n^l} \sum_{v_1,\ldots,v_l \in [n]} \mathbb{E}\left\|\mu_{k,\boldsymbol{G}|\partial^\omega(\boldsymbol{G},v_1)\cup\cdots\cup\partial^\omega(\boldsymbol{G},v_l)} - \mu_{k,\partial^\omega(\boldsymbol{G},v_1)\cup\cdots\cup\partial^\omega(\boldsymbol{G},v_l)}\right\|_{\mathrm{TV}} = 0.$$

Since w.h.p. the pairwise distance of l randomly chosen vertices $v_1, \ldots, v_l$ in $\boldsymbol{G}$ is $\Omega(\ln n)$, we observe that w.h.p.

$$\mu_{k,\partial^\omega(\boldsymbol{G},v_1)\cup\cdots\cup\partial^\omega(\boldsymbol{G},v_l)} = \bigotimes_{i\in[l]} \mu_{k,\partial^\omega(\boldsymbol{G},v_i)}.$$

With very little work it can be verified that Corollary 1.2 is actually equivalent to Theorem 1.1. Setting $\omega = 0$ in Corollary 1.2 yields the following statement, which is of interest in its own right.

Corollary 1.3. *There is a number $k_0 > 0$ such that for all $k \geq k_0$, $d < d_{k,cond}$ and any integer $l > 0$ we have*

$$\lim_{n\to\infty} \frac{1}{n^l} \sum_{v_1,\ldots,v_l \in [n]} \mathbb{E}\left\|\mu_{k,\boldsymbol{G}|\{v_1,\ldots,v_l\}} - \bigotimes_{i\in[l]} \mu_{k,\boldsymbol{G}|\{v_i\}}\right\|_{\mathrm{TV}} = 0. \tag{6}$$

By the symmetry of the colors, $\mu_{k,\boldsymbol{G}|\{v\}}$ is just the uniform distribution on $[k]$ for every vertex v. Hence, Corollary 1.3 states that for $d < d_{k,\mathrm{cond}}$ w.h.p. in the random graph $\boldsymbol{G}$ for randomly chosen vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l$ the following is true: if we choose a k-coloring $\boldsymbol{\sigma}$ of $\boldsymbol{G}$ at random, then $(\boldsymbol{\sigma}(\boldsymbol{v}_1), \ldots, \boldsymbol{\sigma}(\boldsymbol{v}_l)) \in [k]^l$ is asymptotically uniformly distributed. Prior results of Montanari and

Gershenfeld [23] and of Montanari, Restrepo and Tetali [39] imply that (6) holds for $d < 2(k-1)\ln(k-1)$, about an additive $\ln k$ below $d_{k,\text{cond}}$.

The above results and their proofs are inspired by ideas from statistical physics. More specifically, physicists have developed a non-rigorous but analytic technique, the so-called "cavity method" [32], which has led to various conjectures on the random graph coloring problem. These include a prediction as to the precise value of $d_{k,\text{cond}}$ for any $k \geq 3$ [44] as well as a conjecture as to the precise value of the k-colorability threshold $d_{k\text{-col}}$ [28]. While the latter formula is complicated, asymptotically we expect that $d_{k\text{-col}} = (2k-1)\ln k - 1 + \varepsilon_k$, where $\lim_{k\to\infty} \varepsilon_k = 0$. According to this conjecture, the upper bound in (1) is asymptotically tight and $d_{k\text{-col}}$ is strictly greater than $d_{k,\text{cond}}$. Furthermore, according to the physics considerations (6) holds for any $k \geq 3$ and any $d < d_{k,\text{cond}}$ [27]. Corollary 1.3 verifies this conjecture for $k \geq k_0$. By contrast, according to the physics predictions, (6) does *not* hold for $d_{k,\text{cond}} < d < d_{k\text{-col}}$. As (6) is the special case of $\omega = 0$ of Theorem 1.1 (resp. Corollary 1.2), the conjecture implies that neither of these extend to $d > d_{k,\text{cond}}$. In other words, the physics picture suggests that Theorem 1.1, Corollary 1.2 and Corollary 1.3 are *optimal*, except that the assumption $k \geq k_0$ can possibly be replaced by $k \geq 3$.

Remark 1.4. The assumption $k \geq k_0$ comes from the corresponding assumption in [8,15], which give no numerical clue as to the value of k_0. Thus, the issue is that the condensation threshold is not known for small k. Yet for any $k \geq 3$ the proofs of Theorem 1.1 goes through for $d < 2(k-1)\ln(k-1)$, which is the degree up to which the second moment argument from [4] succeeds.

1.3. An application

Suppose we draw a k-coloring $\boldsymbol{\sigma}$ of $\boldsymbol{G}$ at random and consider some vertex v. What is the effect of the assignment of v to the assignment of the rest of the vertices in the graph? Of course, $\boldsymbol{\sigma}$ assigns to the neighbors of v colors that are distinct to that of v. More generally, it seems reasonable to expect that for any *fixed* "radius" ω the colors assigned at v influences the color assignment of the vertices at distance ω from v. But do these correlations persist as $\omega \to \infty$? This is the core question of the so-called "reconstruction problem". The reconstruction problem has received considerable attention in the context of random constraint satisfaction problems in general and in random graph coloring in particular [27,23,39,20,10,42]. To illustrate the use of Theorem 1.1 we will show how it readily implies the result on the reconstruction problem for random graph coloring from [39].

The reconstruction problem considers the effect of the color assignment of a vertex v to the vertices at distance ω in a random coloring of $\boldsymbol{G}$ as $\omega \to \infty$ ("point-to-set correlation"). Equivalently, the problem can be formulated by considering the effect on the coloring of v by a "typical coloring" of the vertices at distance ω from v as $\omega \to \infty$. We formally state the problem by considering the second approach.

Assume that G is a finite k-colorable graph. For $v \in V(G)$ and a subset $\emptyset \neq \mathcal{R} \subset \mathcal{S}_k(G)$ let $\mu_{k,G|v}(\cdot|\mathcal{R})$ be the probability distribution on $[k]$ defined by

$$\mu_{k,G|v}(i|\mathcal{R}) = \frac{1}{|\mathcal{R}|} \sum_{\sigma \in \mathcal{R}} \mathbf{1}\{\sigma(v) = i\},$$

i.e., the distribution of the color of v in a random coloring $\sigma \in \mathcal{R}$. For $v \in V(G)$, $\omega \geq 1$ and $\sigma_0 \in \mathcal{S}_k(G)$ let

$$\mathcal{R}_{k,G}(v,\omega,\sigma_0) = \left\{\sigma \in \mathcal{S}_k(G) \colon \forall u \in V(G) \setminus \partial^{\omega-1}(G,v) \colon \sigma(u) = \sigma_0(u)\right\}.$$

Thus, $\mathcal{R}_{k,G}(v,\omega,\sigma_0)$ contains all k-colorings that coincide with σ_0 on vertices whose distance from v is *at least* ω. Moreover, let

$$\mathrm{bias}_{k,G}(v,\omega,\sigma_0) = \frac{1}{2} \sum_{i \in [k]} \left| \mu_{k,G|v}(i|\mathcal{R}_{k,G}(v,\omega,\sigma_0)) - \frac{1}{k} \right|,$$
$$\mathrm{bias}_{k,G}(v,\omega) = \frac{1}{Z_k(G)} \sum_{\sigma_0 \in \mathcal{S}_k(G)} \mathrm{bias}_{k,G}(v,\omega,\sigma_0).$$

Clearly, for symmetry reasons, if we draw a k-coloring $\boldsymbol{\sigma} \in \mathcal{S}_k(G)$ uniformly at random, then $\boldsymbol{\sigma}(v)$ is uniformly distributed over $[k]$. What $\mathrm{bias}_{k,G}(v,\omega,\sigma_0)$ measures is how much conditioning on the event $\boldsymbol{\sigma} \in \mathcal{R}_{k,G}(v,\omega,\sigma_0)$ biases the color of v. Accordingly, $\mathrm{bias}_{k,G}(v,\omega)$ measures the bias induced by a *random* "boundary condition" σ_0. We say that we have *non-reconstruction* occurs for the k-colorings of $\boldsymbol{G}(n,m)$ if

$$\lim_{\omega \to \infty} \lim_{n \to \infty} \frac{1}{n} \sum_{v \in [n]} \mathbb{E}[\mathrm{bias}_{k,\boldsymbol{G}(n,m)}(v,\omega)] = 0.$$

Otherwise, we have *reconstruction.*

Analogously, recalling that $\boldsymbol{T}(d)$ is the Galton-Watson tree rooted at v_0, we say that *tree non-reconstruction* occurs at d if

$$\lim_{\omega \to \infty} \mathbb{E}[\mathrm{bias}_{k,\partial^{\omega}\boldsymbol{T}(d)}(v_0,\omega)] = 0.$$

Otherwise, *tree reconstruction* occurs.

Corollary 1.5. *There is a number* $k_0 > 0$ *such that for all* $k \geq k_0$ *and* $d < d_{k,\mathrm{cond}}$ *the following is true.*

(7) *Reconstruction occurs in* $\boldsymbol{G}(n,m) \Leftrightarrow$ *tree reconstruction occurs at* d*.*

Montanari, Restrepo and Tetali [39] proved (7) for $d < 2(k-1)\ln(k-1)$, about an additive $\ln k$ below $d_{k,\mathrm{cond}}$. This gap could be plugged by invoking recent results on the geometry of the set of k-colorings [7,14,37]. However, we shall see that Corollary 1.5 is actually an immediate consequence of Theorem 1.1.

The point of Corollary 1.5 is that it reduces the reconstruction problem on a combinatorially extremely intricate object, namely the random graph $\boldsymbol{G}(n,m)$, to the same problem on a much simpler structure, namely the Galton-Watson tree $\boldsymbol{T}(d)$. That said, the reconstruction problem on $\boldsymbol{T}(d)$ is far from trivial. The best current bounds show that there exists a sequence $(\delta_k)_k \to 0$ such that non-reconstruction holds in $\boldsymbol{T}(d)$ if $d < (1-\delta_k)k\ln k$ while reconstruction occurs if $d > (1+\delta_k)k\ln k$ [20].

1.4. Preliminaries and notation

For $\sigma : [n] \to [k]$ let

$$\mathcal{F}(\sigma) = \sum_{i=1}^{k} \binom{|\sigma^{-1}(i)|}{2} \tag{8}$$

be the number of edges of the complete graph on n vertices that are monochromatic under σ. Similarly, for two maps $\sigma, \tau : [n] \to [k]$ let $\mathcal{F}(\sigma,\tau)$ be the number of edges of the complete graph that are monochromatic under either σ or τ. If we define the *overlap* of $\sigma, \tau : [n] \to [k]$ as the $k \times k$ matrix $\rho(\sigma,\tau)$ with entries

$$\rho_{ij}(\sigma,\tau) = \frac{1}{n}\left|\sigma^{-1}(i) \cap \tau^{-1}(j)\right|,$$

then by inclusion/exclusion we have

$$\mathcal{F}(\sigma,\tau) = \mathcal{F}(\sigma) + \mathcal{F}(\tau) - \sum_{i,j\in[k]} \binom{n\rho_{ij}(\sigma,\tau)}{2}. \tag{9}$$

We can view $\rho(\sigma,\tau)$ as a distribution on $[k]\times[k]$. Throughout the paper we let $\bar\rho = (\bar\rho_{ij})_{i,j\in[k]}$ be the matrix with entries $\bar\rho_{ij} = k^{-2}$ for all i,j, viz., the uniform distribution on $[k]\times[k]$.

Let G be a k-colorable graph. By $\boldsymbol{\sigma}^{k,G}, \boldsymbol{\sigma}_1^{k,G}, \boldsymbol{\sigma}_2^{k,G}, \ldots \in \mathcal{S}_k(G)$ we denote independent uniform samples from $\mathcal{S}_k(G)$. Where G, k are apparent from the context, we omit the superscript. Moreover, if $X : \mathcal{S}_k(G) \to \mathbb{R}$, we write

$$\langle X(\boldsymbol{\sigma}) \rangle_{G,k} = \frac{1}{Z_k(G)} \sum_{\sigma \in \mathcal{S}_k(G)} X(\sigma).$$

More generally, if $X : \mathcal{S}_k(G)^l \to \mathbb{R}$, then

$$\langle X(\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_l) \rangle_{G,k} = \frac{1}{Z_k(G)^l} \sum_{\sigma_1, \ldots, \sigma_l \in \mathcal{S}_k(G)} X(\sigma_1, \ldots, \sigma_l).$$

We omit the subscript G and/or k where it is apparent from the context.

Thus, the symbol $\langle \cdot \rangle_{G,k}$ refers to the average over randomly chosen k-colorings of a *fixed* graph G. By contrast, the standard notation $\mathbb{E}[\cdot]$, $\mathbb{P}[\cdot]$ will be used to indicate that the expectation/probability is taken over the choice of the random graph. Unless specified otherwise, we use the standard O-notation to refer to the limit $n \to \infty$. Throughout the paper, we tacitly assume that n is sufficiently large for our various estimates to hold.

By a *rooted graph* we mean a graph G together with a distinguished vertex v, the *root*. The vertex set is always assumed to be a subset of $\mathbb{R}$. If $\omega \geq 0$ is an integer, then $\partial^\omega(G, v)$ signifies the subgraph of G obtained by removing all vertices at distance greater than ω from v (including those vertices of G that are not reachable from v), rooted at v. An *isomorphism* between two rooted graphs (G, v), (G', v') is an isomorphism $G \to G'$ of the underlying graphs that maps v to v' and that preserves the order of the vertices (which is why we insist that they be reals).

For a finite or countable set $\mathcal{X}$ we denote by $\mathcal{P}(\mathcal{X})$ the set of all probability distributions on $\mathcal{X}$, which we identify with the set of all maps $p : \mathcal{X} \to [0,1]$ such that $\sum_{x \in \mathcal{X}} p(x) = 1$. Furthermore, if $N > 0$ is an integer, then $\mathcal{P}_N(\mathcal{X})$ is the set of all $p \in \mathcal{P}(\mathcal{X})$ such that $Np(x)$ is an integer for every $x \in \mathcal{X}$. With the convention that $0 \ln 0 = 0$, we denote the entropy of $p \in \mathcal{P}(\mathcal{X})$ by

$$H(p) = -\sum_{x \in \mathcal{X}} p(x) \ln p(x).$$

Finally, we need the following inequality.

Lemma 1.6 **([43]).** *Let $X_1, \ldots, X_N$ be independent random variables with values in a finite set Λ. Assume that $f : \Lambda^N \to \mathbb{R}$ is a function, that $\Gamma \subset \Lambda^N$*

is an event and that $c, c' > 0$ are numbers such that the following is true.

(10) *If $x, x' \in \Lambda^N$ are such that there is $k \in [N]$ with $x_i = x'_i$ for all $i \neq k$, then* $|f(x) - f(x')| \leq \begin{cases} c & \text{if } x \in \Gamma, \\ c' & \text{if } x \notin \Gamma. \end{cases}$

Then for any $\gamma \in (0,1]$ and any $t > 0$ we have

$$\mathbb{P}\left[|f(X_1,\ldots,X_N) - \mathbb{E}[f(X_1,\ldots,X_N)]| > t\right]$$
$$\leq 2\exp\left(-\frac{t^2}{2N(c+\gamma(c'-c))^2}\right) + \frac{2N}{\gamma}\,\mathbb{P}\left[(X_1,\ldots,X_N) \notin \Gamma\right].$$

2. Outline

None of the arguments in the present paper are particularly difficult. It is rather that a combination of several relatively simple ingredients proves quite powerful. In this section we give an overview of the various pieces and their interplay. In a nutshell, we are going to prove Theorem 1.1 and its corollaries by studying random pairs $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ of k-colorings of the random graph $\boldsymbol{G}$. Specifically, we are going to show that for any fixed integer $\omega \geq 0$, any fixed rooted tree T and any two k-colorings τ_1, τ_2 of T the number of vertices v such that $\partial^\omega(\boldsymbol{G}, v, \boldsymbol{\sigma}_1) \cong (T, \tau_1)$ and $\partial^\omega(\boldsymbol{G}, v, \boldsymbol{\sigma}_2) \cong (T, \tau_2)$ equals $n\,\mathbb{P}[\partial^\omega \boldsymbol{T}(d) \cong T]\, Z_k(T)^{-2} + o(n)$ w.h.p. What might be called a subtle double-counting argument then yields Theorem 1.1. This proof strategy can be viewed as a generalisation of the arguments from [23,39], which are based on studying the "vertex overlap" $\rho(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ of two random k-colorings of $\boldsymbol{G}$ rather than the aforementioned "tree overlaps".

2.1. The number of k-colorings

In order to study random pairs $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ of k-colorings we employ a concentration result for the total number $Z_k(\boldsymbol{G})$ of k-colorings.

Theorem 2.1 ([7]). *There is $k_0 > 0$ such that for all $k \geq k_0$ and all $d < d_{k,cond}$ we have*

$$\lim_{\omega\to\infty}\lim_{n\to\infty} \mathbb{P}\left[|\ln Z_k(\boldsymbol{G}) - \ln \mathbb{E}[Z_k(\boldsymbol{G})]| \leq \omega\right] = 1.$$

To put Theorem 2.1 to work we recall the formula for the first moment of the number of k-colorings.

Lemma 2.2. *For any* $d>0$, $k\geq 3$ *we have* $\mathbb{E}[Z_k(\boldsymbol{G})]=\Theta(k^n(1-1/k)^m)$.

Although Lemma 2.2 is folklore, let us briefly comment on how the expression comes about. For any $\sigma\colon[n]\to[k]$,

$$\mathbb{P}\left[\sigma\in\mathcal{S}_k(\boldsymbol{G})\right]=\binom{\binom{n}{2}-\mathcal{F}(\sigma)}{m}\Big/\binom{\binom{n}{2}}{m}. \tag{11}$$

By convexity we have $\mathcal{F}(\sigma)\geq\frac{1}{k}\binom{n}{2}-n$ for all σ. In combination with (11) and the linearity of expectation, this implies that $\mathbb{E}[Z_k(\boldsymbol{G}(n,m))]=O(k^n(1-1/k)^m)$. Conversely, there are $\Omega(k^n)$ maps $\sigma\colon[n]\to[k]$ such that $\left|\frac{n}{k}-|\sigma^{-1}(i)|\right|\leq\sqrt{n}$ for all i, and $\mathcal{F}(\sigma)/\binom{n}{2}=1/k+O(1/n)$ for all such σ. Hence, $\mathbb{E}[Z_k(\boldsymbol{G})]=\Omega(k^n(1-1/k)^m)$.

Plugging the asymptotic formula (1) for $d_{k,\mathrm{cond}}$ into Lemma 2.2, we find that $\mathbb{E}[Z_k(\boldsymbol{G})]=\exp(\Omega(n))$ for $k>k_0$ and $d<d_{k,\mathrm{cond}}$. Hence, Theorem 2.1 establishes a remarkably strong form of concentration: the random variable $\ln Z_k(\boldsymbol{G})$, typically of order n, has *bounded* fluctuations.

As pointed out in [7], Theorem 2.1 gives us a handle on the experiment of first generating a random graph $\boldsymbol{G}$ and then sampling a *single* k-coloring $\boldsymbol{\sigma}$ of $\boldsymbol{G}$ uniformly at random. Namely, the distribution of the pair $(\boldsymbol{G},\boldsymbol{\sigma})$ can be approximated by a much simpler probability distribution, the so-called "planted model". Indeed, the approximation enabled by Theorem 2.1 is much more accurate than the one previously established in [2]. However, by itself even the result from [7] is not powerful enough to derive Theorem 1.1 (cf. also the discussion in [11]). Instead, we are going to have to cope with the experiment of sampling a random *pair* $(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)$ of colorings of $\boldsymbol{G}$.

2.2. Planting replicas

To this end, we consider a probability distribution $\pi^{\mathrm{pr}}_{n,m,k}$ on triples (G,σ_1,σ_2) such that G is a graph on $[n]$ with m edges and $\sigma_1,\sigma_2\in\mathcal{S}_k(G)$: the *planted replica model* is induced by the following experiment.

PR1: Sample a pair $(\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)$ of maps $[n]\to[k]$ uniformly at random from the set of all pairs such that

$$\mathcal{F}(\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)\leq\binom{n}{2}-m.$$

PR2: Choose a graph $\hat{\boldsymbol{G}}$ on $[n]$ with precisely m edges uniformly at random, subject to the condition that both $\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2$ are proper k-colorings.

We define

$$\pi^{\mathrm{pr}}_{n,m,k}(G,\sigma_1,\sigma_2)=\mathbb{P}\left[(\hat{\boldsymbol{G}},\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)=(G,\sigma_1,\sigma_2)\right].$$

It is easy to bring the known techniques from the theory of random graphs to bear on the planted replica model. Indeed, the conditioning in **PR1** is harmless because $\mathbb{E}[\mathcal{F}(\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)]\sim(2/k-1/k^2)\binom{n}{2}$ while $m=O(n)$. Hence, by the Chernoff bound we have $\mathcal{F}(\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)\leq\binom{n}{2}-m$ w.h.p. Moreover, **PR2** just means that we draw m random edges out of the $\binom{n}{2}-\mathcal{F}(\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)$ edges of the complete graph that are bichromatic under both $\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2$. In particular,

$$\pi^{\mathrm{pr}}_{n,m,k}(G,\sigma_1,\sigma_2)$$
$$=\frac{1}{\left|\left\{(\tau_1,\tau_2)\in[k]^n\times[k]^n:\mathcal{F}(\tau_1,\tau_2)\leq\binom{n}{2}-m\right\}\right|}\binom{\binom{n}{2}-\mathcal{F}(\sigma_1,\sigma_2)}{m}^{-1}.$$

By contrast, the experiment of first choosing a random graph $\boldsymbol{G}$ and then sampling two k-colorings $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2$ uniformly at random seems far less amenable. Formally, the *random replica model* $\pi^{\mathrm{rr}}_{n,m,k}$ is the probability distribution on triples (G,σ_1,σ_2) induced by the following experiment.

RR1: Choose a random graph $\boldsymbol{G}=\boldsymbol{G}(n,m)$ subject to the condition that $\boldsymbol{G}$ is k-colorable.
RR2: Sample two colorings $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2$ of $\boldsymbol{G}$ uniformly and independently.

Thus, the random replica model is defined by the formula

$$\begin{aligned}\pi^{\mathrm{rr}}_{n,m,k}(G,\sigma_1,\sigma_2)&=\mathbb{P}\left[(\boldsymbol{G},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)=(G,\sigma_1,\sigma_2)\right]\\&=\left[\binom{\binom{n}{2}}{m}\mathbb{P}\left[\chi(\boldsymbol{G})\leq k\right]Z_k(G)^2\right]^{-1}.\end{aligned}\tag{12}$$

If $d<d_{k,\mathrm{cond}}$, then $\boldsymbol{G}$ is k-colorable w.h.p. and thus the conditioning in **RR1** is innocent. But this is far from true of the experiment described in **RR2**. For instance, we have no idea as to how one might implement **RR2** efficiently for d anywhere near $d_{k,\mathrm{cond}}$. In fact, the best current algorithms for finding a single k-coloring of $\boldsymbol{G}$, let alone a random pair, stop working for degrees d about a factor of two below $d_{k,\mathrm{cond}}$ (cf. [2]).

Yet for $d<d_{k,\mathrm{cond}}$, the "difficult" random replica model can be studied by means of the "simple" planted replica model. More precisely, recall that a sequence $(\mu_n)_n$ of probability measures is *contiguous* with respect to another sequence $(\nu_n)_n$ if μ_n,ν_n are defined on the same ground set for all n and if for any sequence $(\mathcal{A}_n)_n$ of events such that $\lim_{n\to\infty}\nu_n(\mathcal{A}_n)=0$ we have $\lim_{n\to\infty}\mu_n(\mathcal{A}_n)=0$.

Proposition 2.3. *There is k_0 such that for all $k \geq k_0, d < d_{k,\mathrm{cond}}$, $\pi^{\mathrm{rr}}_{n,m,k}$ is contiguous with respect to $\pi^{\mathrm{pr}}_{n,m,k}$.*

The proof of Proposition 2.3, based on Theorem 2.1, can be found in Section 3. Apart from the concentration of $Z_k(\boldsymbol{G}(n,m))$, the proof involves a study of the overlap of two randomly chosen colorings of $\boldsymbol{G}(n,m)$. The overlap was studied in prior work on reconstruction [23,39] in the case that $d < 2(k-1)\ln(k-1)$ via the second moment argument of Achlioptas and Naor [4]. To extend the study of the overlap to the whole range $d \in (0, d_{k,\mathrm{cond}})$, we harness insights from the recent improvements [8,15] of [4].

2.3. Tree overlaps

As mentioned initially, we aim to understand the "dicolored neighborhood statistics" of the random graph $\boldsymbol{G}$, i.e., the colorings that two random k-colorings $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ of $\boldsymbol{G}$ induce on the neighborhoods $\partial^\omega(\boldsymbol{G}, v)$, $v \in [n]$, for a fixed radius ω. Formally, if ϑa is a rooted tree, $\tau_1, \tau_2 \in \mathcal{S}_k(\vartheta a)$, $\omega \geq 0$ and if G is a k-colorable graph on $[n]$ and $\sigma_1, \sigma_2 \in \mathcal{S}_k(G)$, we define

$$
\begin{aligned}
& Q_{\vartheta a, \tau_1, \tau_2, \omega}(G, \sigma_1, \sigma_2) \\
(13) \qquad & = \frac{1}{n} \sum_{v \in [n]} \mathbf{1}\left\{\partial^\omega\left(G, v, \sigma_1\right) \cong (\vartheta a, \tau_1)\right\} \cdot \mathbf{1}\left\{\partial^\omega\left(G, v, \sigma_2\right) \cong (\vartheta a, \tau_2)\right\}.
\end{aligned}
$$

In words, this is the probability that the depth-ω neighborhood of a random vertex v is isomorphic to ϑa and that the coloring of this neighborhood induced by σ_j coincides with τ_j for $j = 1, 2$. We are going to study the random variables $Q_{\vartheta a, \tau_1, \tau_2, \omega}(\boldsymbol{G}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ on the random replica model by way of the planted replica model. Set

$$
q_{\vartheta a, \omega} = Z_k(\vartheta a)^{-2}\, \mathbb{P}\left[\partial^\omega \boldsymbol{T}(d) \cong \vartheta a\right].
$$

Proposition 2.4. *Let $k \geq 3$ and $d > 0$. Let ϑa be a rooted tree, $\tau_1, \tau_2 \in \mathcal{S}_k(\vartheta a)$ and $\omega \geq 0$. Then for $(\hat{\boldsymbol{G}}, \hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2)$ chosen from $\pi^{\mathrm{pr}}_{n,m,k}$ we have*

$$
Q_{\vartheta a, \tau_1, \tau_2, \omega}(\hat{\boldsymbol{G}}, \hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2) \stackrel{n \to \infty}{\longrightarrow} q_{\vartheta a, \omega} \qquad \textit{in probability.}
$$

Although the proof is based on standard techniques, it requires a bit of work, see Section 4.

2.4. Double-counting and local weak convergence

Combining Proposition 2.4 with a double-counting argument that generalises an elegant idea from [23], we obtain the following.

Proposition 2.5. *There is k_0 such that for any $k \geq k_0$ and $d < d_{k,\mathrm{cond}}$ the following is true. Let $\omega \geq 0$, let $\vartheta a_1, \ldots, \vartheta a_l$ be rooted trees and let $\tau_1 \in \mathcal{S}_k(\vartheta a_1), \ldots, \tau_l \in \mathcal{S}_k(\vartheta a_l)$. Let*

$$X_n = X_n(\vartheta a_1, \tau_1, \ldots, \vartheta a_l, \tau_l) = \sum_{v_1,\ldots,v_l \in [n]} \left\langle \prod_{i=1}^{l} \mathbf{1}\left\{\partial^\omega(\boldsymbol{G}, v_i, \boldsymbol{\sigma}) \cong (\vartheta a_i, \tau_i)\right\} \right\rangle_{\boldsymbol{G}}.$$

Then

$$n^{-l} X_n \stackrel{n\to\infty}{\longrightarrow} \prod_{i=1}^{l} \mathbb{P}\left[\partial^\omega \boldsymbol{T}(d) \cong (\vartheta a_i, \tau_i)\right] \qquad \text{in probability.}$$

In words, suppose we first choose a (k-colorable) random graph $\boldsymbol{G}$, a random k-coloring $\boldsymbol{\sigma}$ of $\boldsymbol{G}$ and l random vertices $v_1, \ldots, v_l$ of $\boldsymbol{G}$. Then $n^{-l}X_n$ is the probability that for each $i \in [l]$ the depth-ω neighborhood $\partial^\omega(\boldsymbol{G}, v_i)$ is isomorphic to ϑa_i and that the colorings that $\boldsymbol{\sigma}$ induces on $\partial^\omega(\boldsymbol{G}, v_i)$ coincide with τ_i for all i.

To get an idea of how Proposition 2.5 follows from Proposition 2.4, let us consider the simplest case. If $l = 2$, $\omega = 0$, then we aim to show that w.h.p. for all but $o(n^2)$ vertex pairs v_1, v_2, the random pair $(\boldsymbol{\sigma}(v_1), \boldsymbol{\sigma}(v_2)) \in [k]^2$ is asymptotically uniformly distributed. (The following computations are along the lines of [23], which actually deals with the case $l \geq 2$ and $\omega = 0$.) Thus, fix any two colors $i_1, i_2 \in [k]$. We write

$$\begin{aligned}
&\sum_{v_1,v_2} \mathbb{E}\left[\left\langle \mathbf{1}\{\boldsymbol{\sigma}(v_1) = i_1, \boldsymbol{\sigma}(v_2) = i_2\} - k^{-2} \right\rangle_{\boldsymbol{G}}^2\right] \\
&\quad = \sum_{v_1,v_2} \mathbb{E}\left[\left\langle (\mathbf{1}\{\boldsymbol{\sigma}(v_1) = i_1\} - k^{-1})(\mathbf{1}\{\boldsymbol{\sigma}(v_2) = i_2\} - k^{-1}) \right\rangle_{\boldsymbol{G}}^2\right] \\
&\quad = \sum_{v_1,v_2} \mathbb{E}\left\langle \prod_{j=1}^{2} \left((\mathbf{1}\{\boldsymbol{\sigma}_j(v_1) = i_1\} - k^{-1})(\mathbf{1}\{\boldsymbol{\sigma}_j(v_2) = i_2\} - k^{-1})\right) \right\rangle_{\boldsymbol{G}}.
\end{aligned}$$

The first equality sign holds because $\langle \mathbf{1}\{\boldsymbol{\sigma}(v_1) = i_1\}\rangle_{\boldsymbol{G}} = \langle \mathbf{1}\{\boldsymbol{\sigma}(v_2) = i_2\}\rangle_{\boldsymbol{G}} = k^{-1}$ by symmetry. The second one is due to the independence of $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$.

Combining both these observations, we see that the last expression equals

$$\begin{aligned}\mathbb{E}\sum_{v_1,v_2}&\left\langle \mathbf{1}\{\boldsymbol\sigma_1(v_1)=\boldsymbol\sigma_2(v_1)=i_1,\boldsymbol\sigma_1(v_2)=\boldsymbol\sigma_2(v_2)=i_2\}-k^{-4}\right\rangle_{\boldsymbol G}\\ &-2k^{-1}\left\langle \mathbf{1}\{\boldsymbol\sigma_1(v_1)=\boldsymbol\sigma_2(v_1)=i_1,\boldsymbol\sigma_1(v_2)=i_2\}-k^{-3}\right\rangle_{\boldsymbol G}\\ &-2k^{-1}\left\langle \mathbf{1}\{\boldsymbol\sigma_1(v_1)=i_1,\boldsymbol\sigma_1(v_2)=\boldsymbol\sigma_2(v_2)=i_2\}-k^{-3}\right\rangle_{\boldsymbol G}\\ &+4k^{-2}\left\langle \mathbf{1}\{\boldsymbol\sigma_1(v_1)=i_1,\boldsymbol\sigma_1(v_2)=i_2\}-k^{-2}\right\rangle_{\boldsymbol G}.\end{aligned}\tag{14}$$

Further, applying Proposition 2.4 to the trees τ_1,τ_2 consisting of the root only and invoking Proposition 2.3, we conclude that (14) is $o(n^2)$. In fact, according to Proposition 2.4 w.h.p. in $(\hat{\boldsymbol G},\hat{\boldsymbol\sigma}_1,\hat{\boldsymbol\sigma}_2)$ there are $(1+o(1))k^{-2}n$ vertices v_1 such that $\hat{\boldsymbol\sigma}_1(v_1)=\hat{\boldsymbol\sigma}_2(v_1)=i_1$ w.h.p. Similarly, there are $(1+o(1))k^{-2}n$ vertices v_2 with $\hat{\boldsymbol\sigma}_1(v_2)=\hat{\boldsymbol\sigma}_2(v_2)=i_2$ w.h.p. Hence, by Proposition 2.3 the same is true of the triple $(\boldsymbol G,\boldsymbol\sigma_1,\boldsymbol\sigma_2)$ w.h.p. and therefore

$$\mathbb{E}\left\langle \textstyle\sum_{v_1,v_2}\mathbf{1}\{\boldsymbol\sigma_1(v_1)=\boldsymbol\sigma_2(v_1)=i_1,\boldsymbol\sigma_1(v_2)=\boldsymbol\sigma_2(v_2)=i_2\}\right\rangle_{\boldsymbol G}\sim n^2k^{-4}.$$

A similar argument applies to the other three terms. Hence, retracing our steps, we obtain

$$\sum_{v_1,v_2}\mathbb{E}\left[\left\langle \mathbf{1}\{\boldsymbol\sigma(v_1)=i_1,\boldsymbol\sigma(v_2)=i_2\}-k^{-2}\right\rangle_{\boldsymbol G}^2\right]=o(n^2).$$

Thus, by Markov's inequality and Cauchy-Schwarz, w.h.p. we have $|\langle \mathbf{1}\{\boldsymbol\sigma(v_1)=i_1,\boldsymbol\sigma(v_2)=i_2\}\rangle_{\boldsymbol G}-k^{-2}|=o(1)$ for all but $o(n^2)$ pairs (v_1,v_2). Finally, taking a union bound over $i_1,i_2\in[k]$, we conclude that w.h.p. for all but $o(n^2)$ vertex pairs the distribution of $(\boldsymbol\sigma(v_1),\boldsymbol\sigma(v_2))$ is within $o(1)$ of the uniform distribution in total variation.

The full proof of Proposition 2.5 can be found in Section 5. Theorem 1.1 follows rather immediately from Proposition 2.5; all that is required is unraveling the construction of the topology.

Proof of Theorem 1.1 (assuming Proposition 2.5). As $\mathcal{P}^2(\mathcal{G}_k^l)$ carries the weak topology, we need to show that for any continuous $f\colon\mathcal{P}(\mathcal{G}_k^l)\to\mathbb{R}$ with a compact support,

$$\lim_{n\to\infty}\int f\,\mathrm{d}\boldsymbol\lambda_{n,m,k}^l=\int f\,\mathrm{d}\boldsymbol\vartheta_{d,k}^l.\tag{15}$$

Thus, let $\varepsilon>0$. Since $\boldsymbol\vartheta_{d,k}^l=\lim_{\omega\to\infty}\boldsymbol\vartheta_{d,k}^l[\omega]$, we have

$$\begin{aligned}\int f\,\mathrm{d}\boldsymbol\vartheta_{d,k}^l&=\lim_{\omega\to\infty}\int f\,\mathrm{d}\boldsymbol\vartheta_{d,k}^l[\omega]\\ &=\lim_{\omega\to\infty}\mathbb{E}\int f\,\mathrm{d}\delta_{\bigotimes_{i\in[l]}\lambda_{\partial^\omega\boldsymbol T^i(d)}}=\lim_{\omega\to\infty}\mathbb{E}f\left(\textstyle\bigotimes_{i\in[l]}\lambda_{\partial^\omega\boldsymbol T^i(d)}\right).\end{aligned}$$

Hence, there is $\omega_0 = \omega_0(\varepsilon)$ such that for $\omega > \omega_0$ we have

$$\left| \int f \, \mathrm{d}\, \boldsymbol{\vartheta}^l_{d,k} - \mathbb{E} f \left(\bigotimes_{i \in [l]} \lambda_{\partial^\omega \boldsymbol{T}^i(d)} \right) \right| < \varepsilon. \tag{16}$$

Furthermore, the topology of $\mathcal{G}_k$ is generated by the functions (3). Because f has a compact support, this implies that there is $\omega_1 = \omega_1(\varepsilon)$ such that for any $\omega > \omega_1(\varepsilon)$ and all $\Gamma_1, \ldots, \Gamma_l \in \mathcal{G}_k$ we have

$$\left| f \left(\bigotimes_{i \in [l]} \delta_{\Gamma_i} \right) - f \left(\bigotimes_{i \in [l]} \delta_{\partial^\omega \Gamma_i} \right) \right| < \varepsilon. \tag{17}$$

Hence, pick some $\omega > \omega_0 + \omega_1$ and assume that $n > n_0(\varepsilon, \omega)$ is large enough.

Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l$ denote vertices of $\boldsymbol{G}$ that are chosen independently and uniformly at random. By the linearity of expectation and the definitions of $\boldsymbol{\lambda}^l_{n,m,k}$ and $\lambda_{\boldsymbol{G}, v_1, \ldots, v_l}$,

$$\begin{aligned} \int f \, \mathrm{d}\, \boldsymbol{\lambda}^l_{n,d,k} &= \mathbb{E} \int f \, \mathrm{d}\, \delta_{\lambda_{\boldsymbol{G}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_l}} = \mathbb{E} f(\lambda_{\boldsymbol{G}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_l}) \\ &= \mathbb{E} \left\langle f(\textstyle\bigotimes_{i \in [l]} \delta_{[\boldsymbol{G} \| \boldsymbol{v}_i, \boldsymbol{v}_i, \boldsymbol{\sigma} \| \boldsymbol{v}_i]}) \right\rangle . \end{aligned}$$

Consequently, (17) yields

$$\left| \int f \, \mathrm{d}\, \boldsymbol{\lambda}^l_{n,d,k} - \mathbb{E} \left\langle f(\textstyle\bigotimes_{i \in [l]} \delta_{\partial^\omega [\boldsymbol{G} \| \boldsymbol{v}_i, \boldsymbol{v}_i, \boldsymbol{\sigma} \| \boldsymbol{v}_i]}) \right\rangle \right| < \varepsilon. \tag{18}$$

Hence, we need to compare $\mathbb{E} \left\langle f(\bigotimes_{i \in [l]} \delta_{\partial^\omega [\boldsymbol{G} \| \boldsymbol{v}_i, \boldsymbol{v}_i, \boldsymbol{\sigma} \| \boldsymbol{v}_i]}) \right\rangle$ and $\mathbb{E} f \left(\bigotimes_{i \in [l]} \lambda_{\partial^\omega \boldsymbol{T}^i(d)} \right)$.

Because the tree structure of $\boldsymbol{T}(d)$ stems from a Galton-Watson branching process, there exist a finite number of pairwise non-isomorphic rooted trees $\vartheta a_1, \ldots, \vartheta a_h$ together with k-colorings $\tau_1 \in \mathcal{S}_k(\vartheta a_1), \ldots, \tau_h \in \mathcal{S}_k(\vartheta a_h)$ such that with $p_i = \mathbb{P}[\partial^\omega \boldsymbol{T}(d) \cong (\vartheta a_i, \tau_i)]$ we have

$$\sum_{i \in [h]} p_i > 1 - \varepsilon. \tag{19}$$

Further, Proposition 2.5 implies that for n large enough and any $i_1, \ldots, i_l \in [h]$ we have

$$\mathbb{E} \left| \left\langle \prod_{i=1}^{l} \mathbf{1} \left\{ \partial^\omega [\boldsymbol{G} \| \boldsymbol{v}_i, \boldsymbol{v}_i, \boldsymbol{\sigma} \| \boldsymbol{v}_i] \cong (\vartheta a_{h_i}, \tau_{h_i}) \right\} \right\rangle - \prod_{i \in [l]} p_{h_i} \right| < \varepsilon, \tag{20}$$

with the expectation taken jointly over $\boldsymbol{G}$ and $\boldsymbol{v}_1,\dots\boldsymbol{v}_l$. Combining (17), (19) and (20), we conclude that

$$\left|\mathbb{E}\left\langle f(\bigotimes_{i\in[l]}\delta_{\partial^\omega[\boldsymbol{G}\|\boldsymbol{v}_i,\boldsymbol{v}_i,\boldsymbol{\sigma}\|\boldsymbol{v}_i]})\right\rangle - \mathbb{E}f\left(\bigotimes_{i\in[l]}\lambda_{\partial^\omega \boldsymbol{T}^i(d)}\right)\right| < 3l\,\|f\|_\infty\,\varepsilon. \tag{21}$$

Finally, (15) follows from (16), (18) and (21). ∎

Proof of Corollary 1.2. While it is not difficult to derive Corollary 1.2 from Theorem 1.1, Corollary 1.2 is actually immediate from Proposition 2.5. ∎

Proof of Corollary 1.3. Corollary 1.3 is simply the special case of setting $\omega=0$ in Corollary 1.2. ∎

Proof of Corollary 1.5. For integer $\omega\geq 0$, consider the quantities $\frac{1}{n}\sum_{v\in[n]}\mathbb{E}[\mathrm{bias}_{k,\boldsymbol{G}(n,m)}(v,\omega)]$ and $\mathbb{E}[\mathrm{bias}_{k,\partial^\omega \boldsymbol{T}(d)}(v_0,\omega)]$. The corollary follows by showing that

$$\left|\frac{1}{n}\sum_{v\in[n]}\mathbb{E}[\mathrm{bias}_{k,\boldsymbol{G}(n,m)}(v,\omega)] - \mathbb{E}[\mathrm{bias}_{k,\partial^\omega \boldsymbol{T}(d)}(v_0,\omega)]\right| = o(1). \tag{22}$$

Let us call $\mathcal{A}$, the quantity on the l.h.s. of the above equality. With $\boldsymbol{G}=\boldsymbol{G}(n,m)$ it holds that

$$\begin{aligned}\mathcal{A} \leq & \left|\frac{1}{n}\sum_{v\in[n]}\left(\mathbb{E}[\mathrm{bias}_{k,\boldsymbol{G}}(v,\omega)] - \mathbb{E}[\mathrm{bias}_{k,\partial^\omega(\boldsymbol{G},v)}(v,\omega)]\right)\right| \\ &+\left|\frac{1}{n}\sum_{v\in[n]}\mathbb{E}[\mathrm{bias}_{k,\partial^\omega(\boldsymbol{G},v)}(v,\omega)] - \mathbb{E}[\mathrm{bias}_{k,\partial^\omega \boldsymbol{T}(d)}(v_0,\omega)]\right|.\end{aligned} \tag{23}$$

We observe that, for any v-rooted $G\in\mathfrak{G}$ and ω it holds that $\mathrm{bias}_{k,G}(v,\omega)\in[0,1]$. Then, by using Corollary 1.2 where $l=1$ (i.e., weak convergence) we get that

$$\left|\frac{1}{n}\sum_{v\in[n]}\left(\mathbb{E}[\mathrm{bias}_{k,\boldsymbol{G}}(v,\omega)] - \mathbb{E}[\mathrm{bias}_{k,\partial^\omega(\boldsymbol{G},v)}(v,\omega)]\right)\right| = o(1). \tag{24}$$

For bounding the second quantity we use the following observation: The above implies that

$$\begin{aligned}&\left|\frac{1}{n}\sum_{v\in[n]}\mathbb{E}[\mathrm{bias}_{k,\partial^\omega(\boldsymbol{G},v)}(v,\omega)] - \mathbb{E}[\mathrm{bias}_{k,\partial^\omega \boldsymbol{T}(d)}(v_0,\omega)]\right| \\ &\qquad\leq \mathbb{P}\left[\partial^\omega(\boldsymbol{G},\boldsymbol{v}^*)\not\cong\partial^\omega \boldsymbol{T}(d)\right]\cdot\max_{\vartheta a}\{\mathrm{bias}_{k,\vartheta a}(v,\omega)\},\end{aligned} \tag{25}$$

where $\boldsymbol{v}^*$ is a randomly chosen vertex of $\boldsymbol{G}$. The probability term $\mathbb{P}[\partial^\omega(\boldsymbol{G},\boldsymbol{v}^*)\not\cong\partial^\omega\boldsymbol{T}(d)]$ is w.r.t. any coupling of $\partial^\omega(\boldsymbol{G},\boldsymbol{v}^*)$ and $\partial^\omega\boldsymbol{T}(d)$. Also, the maximum index ϑa varies over all trees with at most n vertices and with at most ω levels.

Using the standard graph exploration process to obtain the depth-ω neighborhood of $\boldsymbol{v}^*$, there is a coupling of $\partial^\omega(\boldsymbol{G}(n,m),\boldsymbol{v}^*)$ and $\partial^\omega\boldsymbol{T}(d)$, where $d=2m/n$, such that

$$\mathbb{P}\left[\partial^\omega(\boldsymbol{G}(n,m),\boldsymbol{v})\cong\partial^\omega\boldsymbol{T}(d)\right]=1-o(1). \tag{26}$$

Plugging (26) into (25) we get that

$$\left|\frac{1}{n}\sum_{v\in[n]}\mathbb{E}[\mathrm{bias}_{k,\partial^\omega(\boldsymbol{G},v)}(v,\omega)]-\mathbb{E}[\mathrm{bias}_{k,\partial^\omega\boldsymbol{T}(d)}(v_0,\omega)]\right|=o(1), \tag{27}$$

since it always holds that $\mathrm{bias}_{k,\vartheta a}(v,\omega)\in[0,1]$. From (24) and (27), we get that $\mathcal{A}=o(1)$, i.e., (22) is true. The corollary follows. ∎

Remark 2.6. Alternatively, one could deduce Corollary 1.5 from [23, Theorem 1.4] and Lemma 3.1 below.

2.5. Concluding remarks

What bits of the proof strategy fall apart beyond $d_{k,\mathrm{cond}}$? Of course, first and foremost we do not currently know that the k-colorability threshold exceeds $d_{k,\mathrm{cond}}$ at this time. But even if it does, Theorem 2.1 breaks down for all $d>d_{k,\mathrm{cond}}$. In fact, we have $\ln Z_k(\boldsymbol{G})<\ln\mathbb{E}[Z_k(\boldsymbol{G})]-\Omega(n)$ w.h.p. for $d>d_{k,\mathrm{cond}}$ [8]. In effect, the contiguity statement Proposition 2.3 collapses as well. (To be clear, it is not just that the proof collapses, but the statement itself is provably false [8, Proposition 2.2].) Hence, although Proposition 2.4 holds even for $d>d_{k,\mathrm{cond}}$, we cannot transfer the result to the random replica model anymore.

According to physics considerations, the deeper reason behind these issues is that the "shape" of the set of k-colorings changes at the condensation point. For $d<d_{k,\mathrm{cond}}$ the set of k-colorings (provably) decomposes into tiny well-separated "clusters" that each carry mass $\exp(-\Omega(n))$ under the measure $\mu_{k,\boldsymbol{G}}$ w.h.p. [2,37]. However, for $d_{k,\mathrm{cond}}<d<d_{k\text{-col}}$ a bounded number of clusters are expected to contain a $1–o(1)$ fraction of the probability mass [27]. Hence, two randomly chosen k-colorings have a non-vanishing probability of belonging to the same cluster. In effect, the "vertex overlap" $\rho(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)$ is not concentrated about $\bar\rho$ anymore, and thus even the weak decorrelation

property (6) fails to hold. A detailed derivation of the physics predictions can be found in [32].

Nonetheless, the argument that we developed for $d<d_{k,\mathrm{cond}}$ is reasonably generic, i.e., it does not depend much on the particulars of the k-colorability problem. We expect it to extend to alike "random constraint satisfaction problems" (say, with a similar "vertex overlap" behavior) up to their respective condensation thresholds. A natural class to think of are the binary problems studied in [39]. Another candidate might be the hardcore model, which was studied in [11] by a somewhat different approach.

3. The planted replica model

Throughout this section we assume that $k\geq k_0$ for some large enough constant k_0.

In this section we prove Proposition 2.3. A key step is to establish the following fact about the "vertex overlap".

Lemma 3.1. *Assume that $d < d_{k,cond}$ and let $\omega = \omega(n)$ be such that $\lim_{n\to\infty}\omega(n)=\infty$. Then*

$$\lim_{n\to\infty} \mathbb{E}\left\langle \mathbf{1}\left\{\|\rho(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)-\bar\rho\|_2 > \sqrt{\omega/n}\right\}\right\rangle_{\boldsymbol{G}} = 0.$$

Thus, with high probability over the choice of the random graph $\boldsymbol{G}$ (the outer $\mathbb{E}$) and over the choice of a pair of k-colorings (the inner $\langle\,\cdot\,\rangle_{\boldsymbol{G}}$) the ℓ_2-distance of the overlap from $\bar\rho$ is bounded by $\sqrt{\omega/n}$. The $d<2(k-1)\ln(k-1)$ case of Lemma 3.1 was previously proved in [39] by way of the second moment analysis from [4]. As it turns out, the regime $2(k-1)\ln(k-1)<d<d_{k,\mathrm{cond}}$ requires a somewhat more sophisticated argument. In any case, our proof of Lemma 3.1 below includes the case $d<2(k-1)\ln(k-1)$, which does not add much to the argument.

The proof of Lemma 3.1 depends upon a few facts from prior work. For $\alpha=(\alpha_1,\ldots,\alpha_k)\in\mathcal{P}_n([k])$ we let $Z_\alpha(\boldsymbol{G})$ be the number of k-colorings σ of $\boldsymbol{G}$ such that $|\sigma^{-1}(i)|=\alpha_i n$ for all $i\in[k]$. Conversely, for a map $\sigma\colon [n]\to[k]$ let $\alpha(\sigma)=n^{-1}(|\sigma^{-1}(i)|)_{i\in[k]}\in\mathcal{P}_n([k])$. Additionally, let $\bar\alpha=k^{-1}\mathbf{1}=(1/k,\ldots,1/k)$.

Lemma 3.2 ([7, Lemma 3.1]). *Let $\varphi(\alpha)=H(\alpha)+\frac{d}{2}\ln\left(1-\|\alpha\|_2^2\right)$. Then*
$\mathbb{E}[Z_\alpha(\boldsymbol{G})]=O(\exp(n\varphi(\alpha)))$ uniformly for all $\alpha\in\mathcal{P}_n([k])$,
$\mathbb{E}[Z_\alpha(\boldsymbol{G})]=\Theta(n^{(1-k)/2})\exp(n\varphi(\alpha))$ uniformly for all $\alpha\in\mathcal{P}_n([k])$ such that $\|\alpha-\bar\alpha\|_2\leq k^{-3}$.

We need Lemma 3.2 to derive the following claim; the case $d<2(k-1)\ln(k-1)$ was known previously [39].

Claim 3.3. *Suppose that $d < d_{k,\mathrm{cond}}$ and that $\omega = \omega(n)$ is such that $\lim_{n\to\infty}\omega(n)=\infty$ but $\omega=o(n)$. Then w.h.p. $\boldsymbol{G}$ is such that*

$$\left\langle \mathbf{1}\left\{\|\alpha(\boldsymbol{\sigma})-\bar\alpha\|_2 > \sqrt{\omega/n}\right\}\right\rangle_{\boldsymbol{G}} \leq \exp(-\Omega(\omega)).$$

Proof. We combine Theorem 2.1 with a standard "first moment" estimate similar to the proof of [39, Lemma 5.4]. The entropy function $\alpha\in\mathcal{P}([k])\mapsto H(\alpha)=-\sum_{i=1}^k\alpha_i\ln\alpha_i$ is concave and attains its global maximum at $\bar\alpha$. In fact, the Hessian of $\alpha\mapsto H(\alpha)$ satisfies $D^2H(\alpha)\preceq -2\,\mathrm{id}$. Moreover, since $\alpha\mapsto\|\alpha\|_2^2$ is convex, $\alpha\mapsto\frac{d}{2}\ln(1-\|\alpha\|_2^2)$ is concave and attains is global maximum at $\bar\alpha$ as well. Hence, letting φ denote the function from Lemma 3.2, we find $D^2\varphi(\alpha)\preceq-2\,\mathrm{id}$. Therefore, we obtain from Lemma 3.2 that

(28)
$$\mathbb{E}[Z_\alpha(\boldsymbol{G})] \leq \exp\left(n(\varphi(\bar\alpha)-\|\alpha-\bar\alpha\|_2^2)\right)\cdot\begin{cases}O(1) & \text{if } \|\alpha-\bar\alpha\|_2>1/\ln n,\\ O(n^{(1-k)/2}) & \text{otherwise.}\end{cases}$$

Further, letting

$$Z'(\boldsymbol{G})=\sum_{\alpha\in\mathcal{P}_n([k]):\ \|\alpha-\bar\alpha\|_2>\sqrt{\omega/n}} Z_\alpha(\boldsymbol{G})$$

and treating the cases $\omega\leq\ln^2 n$ and $\omega\geq\ln^2 n$ separately, we obtain from (28) that

$$\mathbb{E}[Z'(\boldsymbol{G})]\leq\exp(-\Omega(\omega))\exp(n\varphi(\bar\alpha)). \tag{29}$$

Since Lemma 2.2 shows that $\mathbb{E}[Z_k(\boldsymbol{G})]=\Theta(k^n(1-1/k)^m)=\exp(n\varphi(\bar\alpha))$, (29) yields $\mathbb{E}[Z'(\boldsymbol{G})]=\exp(-\Omega(\omega))\mathbb{E}[Z_k(\boldsymbol{G})]$. Hence, by Markov's inequality

$$\mathbb{P}\left[Z'(\boldsymbol{G})\leq\exp(-\Omega(\omega))\mathbb{E}[Z_k(\boldsymbol{G})]\right]\geq 1-\exp(-\Omega(\omega)). \tag{30}$$

Finally, since $\left\langle\|\alpha(\boldsymbol{\sigma})-\bar\alpha\|_2>\sqrt{\omega/n}\right\rangle_{\boldsymbol{G}} = Z'(\boldsymbol{G})/Z_k(\boldsymbol{G})$ and because $Z_k(\boldsymbol{G})\geq\mathbb{E}[Z_k]/\omega$ w.h.p. by Theorem 2.1, the assertion follows from (30). ∎

With respect to pairs of colorings, (9) yields (cf. [7, Fact 5.4])

$$\mathbb{P}[\sigma,\tau\in\mathcal{S}_k(\boldsymbol{G})]=\binom{\binom{n}{2}-\mathcal{F}(\sigma,\tau)}{m}\Big/\binom{\binom{n}{2}}{m}$$

(31)
$$=O\left(\left[1-\sum_{i\in[k]}\left[\sum_{j\in[k]}\rho_{ij}(\sigma,\tau)\right]^2+\left[\sum_{j\in[k]}\rho_{ji}(\sigma,\tau)\right]^2+\|\rho(\sigma,\tau)\|_2^2\right]^m\right).$$

For $\rho \in \mathcal{P}_n([k]^2)$ let $Z_\rho^{\otimes}(\boldsymbol{G})$ be the number of pairs $\sigma_1, \sigma_2 \in \mathcal{S}_k(\boldsymbol{G})$ with overlap $\rho(\sigma_1,\sigma_2)=\rho$. Finally, let

$$\mathcal{R}_{n,k}(\omega) = \left\{\rho \in \mathcal{P}_n([k]^2) \colon \forall i \in [k] : \|\rho_{i\cdot} - \bar{\alpha}\|_2\,, \|\rho_{\cdot i} - \bar{\alpha}\|_2 \le \sqrt{\omega/n}\right\}, \tag{32}$$

and

$$f(\rho) = H(\rho) + \frac{d}{2}\ln(1 - 2/k + \|\rho\|_2^2). \tag{33}$$

Lemma 3.4 ([4]). *Assume that $\omega=\omega(n)\to\infty$ but $\omega=o(n)$. For all $d>0$ we have*
$\mathbb{E}[Z_\rho^{\otimes}(\boldsymbol{G})] = O(n^{(1-k^2)/2})\exp(nf(\rho))$ *uniformly for all $\rho \in \mathcal{R}_{n,k}(\omega)$ s.t. $\|\rho-\bar{\rho}\|_\infty \le k^{-3}$,*
$\mathbb{E}[Z_\rho^{\otimes}(\boldsymbol{G})]=O(\exp(nf(\rho)))$ *uniformly for all $\rho\in\mathcal{R}_{n,k}(\omega)$.*
Moreover, if $d<2(k-1)\ln(k-1)$, then for any $\eta>0$ there exists $\delta>0$ such that

$$f(\rho) < f(\bar{\rho}) - \delta \qquad \text{for all } \rho \in \mathcal{R}_{n,k}(\omega) \text{ such that } \|\rho - \bar{\rho}\|_2 > \eta. \tag{34}$$

The bound (34) applies for $d<2(k-1)\ln(k-1)$, about $\ln k$ below $d_{k,\mathrm{cond}}$. To bridge the gap, let $\kappa = 1 - \ln^{20} k/k$ and call $\rho \in \mathcal{P}_n([k]^2)$ *separable* if $k\rho_{ij} \notin (0.51, \kappa)$ for all $i,j\in[k]$. Moreover, $\sigma\in\mathcal{S}_k(\boldsymbol{G})$ is *separable* if $\rho(\sigma,\tau)$ is separable for all $\tau\in\mathcal{S}_k(\boldsymbol{G})$. Otherwise, we call σ *inseparable*. Further, ρ is *s-stable* if there are precisely s entries such that $k\rho_{ij}\ge\kappa$.

Lemma 3.5 ([15]). *There is k_0 such that for all $k > k_0$ and all $2(k-1)\ln(k-1)\le d\le 2k\ln k$ the following is true.*

1. *Let $\tilde{Z}_k(\boldsymbol{G})=|\{\sigma\in\mathcal{S}_k(\boldsymbol{G})\colon \sigma \text{ is inseparable}\}|$. Then*
$$\mathbb{E}[\tilde{Z}_k(\boldsymbol{G})] \le \exp(-\Omega(n))\mathbb{E}[Z_k(\boldsymbol{G})].$$
2. *Let $1\le s\le k-1$. Then $f(\rho)<f(\bar{\rho})-\Omega(1)$ uniformly for all s-stable ρ.*
3. *For any $\eta>0$ there is $\delta>0$ such that*
$$\sup\{f(\rho)\colon \rho \text{ is 0-stable and } \|\rho-\bar{\rho}\|_2 > \eta\} < f(\bar{\rho}) - \delta.$$

Lemma 3.5 omits the k-stable case. To deal with it, we introduce

$$\mathcal{C}(G,\sigma) = \{\tau \in \mathcal{S}_k(G)\colon \rho(\sigma,\tau) \text{ is } k\text{-stable}\}. \tag{35}$$

Lemma 3.6 ([8]). *There exist k_0 such that for all $k\ge k_0$, $2(k-1)\ln(k-1)\le d< d_{k,cond}$ we have*

$$\lim_{n\to\infty}\mathbb{P}\left[\langle|\mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})|\rangle_{\boldsymbol{G},k} \le \exp(-\Omega(n))\mathbb{E}\left[Z_k(\boldsymbol{G})\right]\right] = 1.$$

Proof of Lemma 3.1. Let $\omega = \omega(n)$ be any sequence such that $\lim_{n\to\infty} \omega(n) = \infty$ but $\omega(n) = o(\ln n)$. Set

$$\begin{aligned} \Lambda_1 &= \sum_{\sigma_1, \sigma_2 \in \mathcal{S}_k(\boldsymbol{G})} \mathbf{1}\left\{\max\{\|\alpha(\sigma_1) - \bar{\alpha}\|_2, \|\alpha(\sigma_2) - \bar{\alpha}\|_2\} > \sqrt{\omega/n}\right\} \\ &\le 2Z_k(\boldsymbol{G})^2 \left\langle \|\alpha(\boldsymbol{\sigma}) - \bar{\alpha}\|_2 > \sqrt{\omega/n} \right\rangle_{\boldsymbol{G}}. \end{aligned}$$

Then Claim 3.3 implies that

$$\mathbb{P}\left[\Lambda_1 \le \exp(-\Omega(\omega)) Z_k(\boldsymbol{G})^2\right] = 1 - o(1). \tag{36}$$

Moreover, let $\mathcal{S}_k'(\boldsymbol{G})$ be the set of all $\sigma \in \mathcal{S}_k(\boldsymbol{G})$ such that $\|\alpha(\sigma) - \bar{\alpha}\|_2 \le \sqrt{\omega/n}$ and define

$$\Lambda = \sum_{\sigma_1, \sigma_2 \in \mathcal{S}_k'(\boldsymbol{G})} \|\rho(\sigma_1, \sigma_2) - \bar{\rho}\|_2 .$$

Further, let $\eta > 0$ be a small but n-independent number and let

$$\begin{aligned} \Lambda_2 &= \sum_{\sigma_1, \sigma_2 \in \mathcal{S}_k'(\boldsymbol{G})} \mathbf{1}\{\|\rho(\sigma_1, \sigma_2) - \bar{\rho}\|_2 \le \eta\} \|\rho(\sigma_1, \sigma_2) - \bar{\rho}\|_2 , \\ \Lambda_3 &= \sum_{\sigma_1, \sigma_2 \in \mathcal{S}_k'(\boldsymbol{G})} \mathbf{1}\{\|\rho(\sigma_1, \sigma_2) - \bar{\rho}\|_2 > \eta\} . \end{aligned}$$

Since $\|\rho(\sigma_1, \sigma_2) - \bar{\rho}\|_2 \le 2$ for all σ_1, σ_2, we have

$$\Lambda \le 4(\Lambda_2 + \Lambda_3). \tag{37}$$

We are going to establish in the following that

$$\mathbb{E}[\Lambda_2]/\mathbb{E}[Z_k(\boldsymbol{G})]^2 \le O(n^{-1/2}), \tag{38}$$

$$\mathbb{P}[\Lambda_3/\mathbb{E}[Z_k(\boldsymbol{G})^2] \le \exp(-\Omega(n))] = 1 - o(1). \tag{39}$$

Plugging (38) and (39) into (37), we find

$$\mathbb{P}\left[\Lambda \le \sqrt{\omega/n}\, \mathbb{E}[Z_k(\boldsymbol{G})]^2\right] = 1 - o(1). \tag{40}$$

Since $Z_k(\boldsymbol{G}) \ge \omega^{-1/4} \mathbb{E}[Z_k(\boldsymbol{G})]$ w.h.p. by Theorem 2.1, (40) implies that

$$\mathbb{P}\left[\Lambda/Z_k(\boldsymbol{G})^2 \le \omega/\sqrt{n}\right] = 1 - o(1). \tag{41}$$

Finally, the assertion follows from (36) and (41).

To estimate Λ_2, we let f denote the function from Lemma 3.4. Observe that $Df(\bar\rho)=0$, because $\bar\rho$ maximizes the entropy and minimizes the ℓ_2-norm. Further, a straightforward calculation reveals that for any $i,j,i',j'\in[k]$, $(i,j)\neq(i',j')$,

$$\frac{\partial^2 f(\rho)}{\partial\rho_{ij}^2}=-\frac{1}{\rho_{ij}}+\frac{d}{1-2/k+\|\rho\|_2^2}-\frac{2d\rho_{ij}^2}{(1-2/k+\|\rho\|_2^2)^2},$$
$$\frac{\partial^2 f(\rho)}{\partial\rho_{ij}\partial\rho_{i'j'}}=-\frac{2d\rho_{ij}\rho_{i'j'}}{(1-2/k+\|\rho\|_2^2)^2}.$$

Consequently, choosing, say, $\eta<k^{-4}$, ensures that the Hessian satisfies

$$D^2 f(\rho)\preceq -2\,\mathrm{id}\qquad\text{for all }\rho\text{ such that }\|\rho-\bar\rho\|_2^2\le\eta. \tag{42}$$

Therefore, Lemma 3.4 yields

$$\begin{aligned}\mathbb{E}[\Lambda_2]&\le\sum_{\rho\in\mathcal{R}_{n,k}(\eta)}\|\rho-\bar\rho\|_2\,\mathbb{E}[Z_\rho^{\otimes}(\boldsymbol{G})]\\&\le O(n^{(1-k^2)/2})\exp(nf(\bar\rho))\sum_{\rho\in\mathcal{R}_{n,k}(\eta)}\|\rho-\bar\rho\|_2\exp(n(f(\rho)-f(\bar\rho)))\\&\le O(n^{(1-k^2)/2})\exp(nf(\bar\rho))\sum_{\rho\in\mathcal{R}_{n,k}(\eta)}\|\rho-\bar\rho\|_2\exp(-nk^{-2}\,\|\rho-\bar\rho\|^2)\\&\qquad\qquad\text{[by (42)]}.\end{aligned}\tag{43}$$

Further, since $\rho_{kk}=1-\sum_{(i,j)\neq(k,k)}\rho_{ij}$ for any $\rho\in\mathcal{R}_{n,k}(\eta)$, substituting $x=\sqrt{n}(\rho-\bar\rho)$ in (43) yields

$$\begin{aligned}\mathbb{E}[\Lambda_2]&\le O(\exp(nf(\bar\rho)))\int_{\mathbb{R}^{k^2-1}}\frac{\|x\|_2}{\sqrt{n}}\exp(-k^{-2}\,\|x\|_2^2)dx\\&=O(n^{-1/2})\exp(nf(\bar\rho)).\end{aligned}\tag{44}$$

Since $f(\bar\rho)=2\ln k+d\ln(1-1/k)$, Lemma 2.2 yields

$$\exp(nf(\bar\rho))\le O(\mathbb{E}[Z_k(\boldsymbol{G})]^2).\tag{45}$$

Therefore, (44) entails (38).

To bound Λ_3, we consider two separate cases. The first case is that $d\le 2(k-1)\ln(k-1)$. Then Lemma 3.4 and (45) yield

$$\mathbb{E}[\Lambda_3]\le\exp(nf(\bar\rho)-\Omega(n))\le\exp(-\Omega(n))\mathbb{E}[Z_k(\boldsymbol{G})]^2.\tag{46}$$

The second case is that $2(k-1)\ln(k-1)\leq d<d_{k,\mathrm{cond}}$. We introduce

$$\begin{aligned}
\Lambda_{31} &= \sum_{\sigma_1,\sigma_2\in\mathcal{S}'_k(\boldsymbol{G})} \mathbf{1}\left\{\sigma_1 \text{ fails to be separable}\right\},\\
\Lambda_{32} &= \sum_{\sigma_1,\sigma_2\in\mathcal{S}'_k(\boldsymbol{G})} \mathbf{1}\left\{\rho(\sigma_1,\sigma_2) \text{ is } s\text{-stable for some } 1\leq s\leq k\right\},\\
\Lambda_{33} &= \sum_{\sigma_1,\sigma_2} \mathbf{1}\left\{\rho(\sigma_1,\sigma_2) \text{ is 0-stable and } \left\|\rho(\sigma_1,\sigma_2)-\bar{\rho}\right\|_2>\eta\right\},\\
\Lambda_{34} &= \sum_{\sigma_1,\sigma_2\in\mathcal{S}'_k(\boldsymbol{G})} \mathbf{1}\left\{\rho(\sigma_1,\sigma_2) \text{ is } k\text{-stable}\right\},
\end{aligned}$$

so that

$$\Lambda_3\leq \Lambda_{31}+\Lambda_{32}+\Lambda_{33}+\Lambda_{34}. \tag{47}$$

By the first part of Lemma 3.5 and Markov's inequality,

$$\mathbb{P}\left[\Lambda_{31}\leq \exp(-\Omega(n))Z_k(\boldsymbol{G})\mathbb{E}[Z_k(\boldsymbol{G})]\right]=1-o(1). \tag{48}$$

Further, combining Lemma 3.4 with the second part of Lemma 3.5, we obtain

$$\mathbb{P}\left[\Lambda_{32}\leq \exp(nf(\bar{\rho})-\Omega(n))\right]=1-o(1). \tag{49}$$

Additionally, Lemma 3.4 and the third part of Lemma 3.5 yield

$$\mathbb{P}\left[\Lambda_{33}\leq \exp(nf(\bar{\rho})-\Omega(n))\right]=1-o(1). \tag{50}$$

Moreover, Lemma 3.6 entails that

$$\mathbb{P}\left[\Lambda_{34}\leq \exp(-\Omega(n))Z_k(\boldsymbol{G})\mathbb{E}[Z_k(\boldsymbol{G})]\right]=1-o(1). \tag{51}$$

Finally, combining (48)–(51) with (45) and (47) and using Markov's inequality, we obtain (39). ∎

Lemma 3.1 puts us in a position to prove Proposition 2.3 by extending the argument that was used to "plant" single k-colorings in [7, Section 2] to the current setting of "planting" pairs of k-colorings.

Proof of Proposition 2.3. Assume for contradiction that $(\mathcal{A}'_n)_{n\geq1}$ is a sequence of events such that for some fixed number $\varepsilon>0$ we have

$$\lim_{n\to\infty}\pi^{\mathrm{pr}}_{n,m,k}\left[\mathcal{A}'_n\right]=0 \quad\text{while}\quad \limsup_{n\to\infty}\pi^{\mathrm{rr}}_{n,m,k}\left[\mathcal{A}'_n\right]>2\varepsilon. \tag{52}$$

Let $\omega(n) = \ln\ln 1/\pi^{\mathrm{pr}}_{n,m,k}[\mathcal{A}'_n]$. Then $\omega = \omega(n) \to \infty$. Let $\mathcal{B}_n$ be the set of all pairs (σ_1, σ_2) of maps $[n] \to [k]$ such that $\|\rho(\sigma_1,\sigma_2) - \bar\rho\|_2 \leq \sqrt{\omega/n}$ and define

$$\mathcal{A}_n = \left\{(G, \sigma_1, \sigma_2) \in \mathcal{A}'_n \colon (\sigma_1, \sigma_2) \in \mathcal{B}_n\right\}.$$

Then Lemma 3.1 and (52) imply that

$$\lim_{n\to\infty} \pi^{\mathrm{pr}}_{n,m,k}[\mathcal{A}_n] = 0 \quad \text{while} \quad \limsup_{n\to\infty} \pi^{\mathrm{rr}}_{n,m,k}[\mathcal{A}_n] > \varepsilon. \tag{53}$$

Furthermore,

$$\ln\ln\left(1/\pi^{\mathrm{pr}}_{n,m,k}[\mathcal{A}_n]\right) \geq (1 + o(1))\omega(n) \to \infty. \tag{54}$$

For $\sigma_1, \sigma_2 \colon [n] \to [k]$ let $\boldsymbol{G}(n, m|\sigma_1, \sigma_2)$ be the random graph $\boldsymbol{G}(n,m)$ conditional on the event that σ_1, σ_2 are k-colorings. That is, $\boldsymbol{G}(n, m|\sigma_1, \sigma_2)$ consists of m random edges that are bichromatic under σ_1, σ_2. Then

(55)

$$\begin{aligned}
&\mathbb{E}[Z_k(\boldsymbol{G}(n,m))^2 \langle \mathbf{1}\{(\boldsymbol{G}(n,m), \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) \in \mathcal{A}_n\}\rangle] \\
&= \sum_{(\sigma_1,\sigma_2)\in\mathcal{B}_n} \mathbb{P}[\sigma_1,\sigma_2 \in \mathcal{S}_k(\boldsymbol{G}(n,m)), (\boldsymbol{G}(n,m), \sigma_1, \sigma_2) \in \mathcal{A}_n] \\
&= \sum_{(\sigma_1,\sigma_2)\in\mathcal{B}_n} \mathbb{P}[(\boldsymbol{G}(n,m), \sigma_1,\sigma_2) \in \mathcal{A}_n | \sigma_1,\sigma_2 \in \mathcal{S}_k(\boldsymbol{G}(n,m))]\mathbb{P}[\sigma_1,\sigma_2 \in \mathcal{S}_k(\boldsymbol{G}(n,m))]
\end{aligned}$$

(56)

$$= \sum_{(\sigma_1,\sigma_2)\in\mathcal{B}_n} \mathbb{P}[(\boldsymbol{G}(n,m|\sigma_1,\sigma_2), \sigma_1, \sigma_2) \in \mathcal{A}_n] \cdot \mathbb{P}[\sigma_1, \sigma_2 \in \mathcal{S}_k(\boldsymbol{G}(n,m))].$$

Letting $q_n = \max\{\mathbb{P}[\sigma_1, \sigma_2 \in \mathcal{S}_k(\boldsymbol{G}(n,m))] : (\sigma_1,\sigma_2) \in \mathcal{B}_n\}$, we obtain from (56) and the definition **PR1–PR2** of the planted replica model that

$$\begin{aligned}
&\mathbb{E}[Z_k(\boldsymbol{G}(n,m))^2 \langle \mathbf{1}\{(\boldsymbol{G}(n,m), \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) \in \mathcal{A}_n\}\rangle] \\
&\leq q_n \sum_{(\sigma_1,\sigma_2)\in\mathcal{B}_n} \mathbb{P}[(\boldsymbol{G}(n,m|\sigma_1,\sigma_2), \sigma_1, \sigma_2) \in \mathcal{A}_n] \leq k^{2n} q_n \pi^{\mathrm{pr}}_{n,m,k}[\mathcal{A}_n].
\end{aligned} \tag{57}$$

Furthermore, since $\mathcal{F}(\sigma_1), \mathcal{F}(\sigma_2) \geq \frac{1}{k}\binom{n}{2} - n$, (31) implies

$$\begin{aligned}
&\frac{1}{n}\ln \mathbb{P}[\sigma_1, \sigma_2 \in \mathcal{S}_k(\boldsymbol{G}(n,m))] \\
&\qquad \leq \frac{d}{2}\ln\left(1 - \frac{2}{k} + \|\rho(\sigma_1,\sigma_2)\|_2^2\right) + O(1/n) \\
&\qquad = d\ln(1 - 1/k) + O(\omega/n) \qquad \text{for all } (\sigma_1, \sigma_2) \in \mathcal{B}_n.
\end{aligned}$$

Hence, $q_n \le (1-1/k)^{2m}\exp(O(\omega))$. Plugging this bound into (57) and setting $\bar z = \mathbb{E}[Z_k(\boldsymbol{G}(n,m))]$, we see that

$$\begin{aligned}\mathbb{E}[Z_k(\boldsymbol{G}(n,m))^2 \langle \mathbf{1}\{(\boldsymbol{G}(n,m),\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)\in\mathcal{A}_n\}\rangle]\\ \le k^{2n}(1-1/k)^{2m}\exp(O(\omega))\pi^{\mathrm{pr}}_{n,m,k}[\mathcal{A}_n] = \bar z^2\exp(O(\omega))\pi^{\mathrm{pr}}_{n,m,k}[\mathcal{A}_n].\end{aligned} \tag{58}$$

On the other hand, if $\pi^{\mathrm{rr}}_{n,m,k}[\mathcal{A}_n] > \varepsilon$, then Theorem 2.1 implies that

$$\pi^{\mathrm{rr}}_{n,m,k}[\mathcal{A}_n \cap \{Z_k(\boldsymbol{G}(n,m)) \ge \bar z/\omega\}] > \varepsilon/2.$$

Hence, (12) yields

$$\mathbb{E}[Z_k(\boldsymbol{G}(n,m))^2 \langle \mathbf{1}\{(\boldsymbol{G}(n,m),\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)\in\mathcal{A}_n\}\rangle] \ge \frac{\varepsilon}{2}\left(\frac{\bar z}{\omega}\right)^2. \tag{59}$$

But due to (54), (59) contradicts (58). ∎

4. Analysis of the planted replica model

In this section we assume that $k \ge 3$ and that $d > 0$.

In this section we prove Proposition 2.4, which asserts that in the planted replica model, the distribution of the "dicoloring" that $\hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2$ induce in the depth-ω neighborhood of a random vertex v converges to the uniform distribution on the tree that the depth-ω neighborhood of v induces. The proof is by extension of an argument from [8] for the "standard" planted model (with a single coloring). More specifically, it is going to be convenient to work with the following *binomial* version $\pi^{\mathrm{pr}}_{n,p,k}$ of the planted replica model, where $p \in (0,1)$.

PR1' sample two maps $\hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2 : [n] \to [k]$ independently and uniformly at random.

PR2' generate a random graph $\tilde{\boldsymbol{G}}$ by including each of the $\binom{n}{2} - \mathcal{F}(\hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2)$ edges that are bichromatic under both $\hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2$ with probability p independently.

The distributions $\pi^{\mathrm{pr}}_{n,m,k}$, $\pi^{\mathrm{pr}}_{n,p,k}$ are related as follows.

Lemma 4.1. *Let $p = m/\left(\binom{n}{2}(1-1/k)^2\right)$. For any event $\mathcal{E}$ we have $\pi^{\mathrm{pr}}_{n,m,k}[\mathcal{E}] \le O(\sqrt{n})\pi^{\mathrm{pr}}_{n,p,k}[\mathcal{E}] + o(1)$.*

Proof. Let $\mathcal{B}$ be the event that $\|\rho(\hat{\boldsymbol\sigma}_1,\hat{\boldsymbol\sigma}_2)-\bar\rho\|_2^2\leq n^{-1}\ln\ln n$. Since $\hat{\boldsymbol\sigma}_1,\hat{\boldsymbol\sigma}_2$ are chosen uniformly and independently, the Chernoff bound yields

$$\pi^{\mathrm{pr}}_{n,p,k}\left[\mathcal{B}\right],\pi^{\mathrm{pr}}_{n,m,k}\left[\mathcal{B}\right]=1-o(1). \tag{60}$$

Furthermore, given that $\mathcal{B}$ occurs we obtain $\mathcal{F}(\hat{\boldsymbol\sigma}_1,\hat{\boldsymbol\sigma}_2)=(2/k-1/k^2)\binom{n}{2}+o(n^{3/2})$. Therefore, Stirling's formula implies that the event $\mathcal{A}$ that the graph $\tilde{\boldsymbol G}$ has precisely m edges satisfies

$$\pi^{\mathrm{pr}}_{n,p,k}\left[\mathcal{A}|\mathcal{B}\right]=\Omega(n^{-1/2}). \tag{61}$$

By construction, $\pi^{\mathrm{pr}}_{n,p,k}$ given $\mathcal{A}\cap\mathcal{B}$ is identical to $\pi^{\mathrm{pr}}_{n,m,k}$ given $\mathcal{B}$. Consequently, (60) and (61) yield

$$\begin{aligned}\pi^{\mathrm{pr}}_{n,m,k}\left[\mathcal{E}\right]&\leq\pi^{\mathrm{pr}}_{n,m,k}\left[\mathcal{E}|\mathcal{B}\right]+o(1)=\pi^{\mathrm{pr}}_{n,p,k}\left[\mathcal{E}|\mathcal{A},\mathcal{B}\right]+o(1)\\&\leq O(\sqrt{n})\pi^{\mathrm{pr}}_{n,p,k}\left[\mathcal{E}\right]+o(1),\end{aligned}$$

as desired. ∎

The following proofs are based on a simple observation. Given the colorings $\hat{\boldsymbol\sigma}_1,\hat{\boldsymbol\sigma}_2$, we can construct $\tilde{\boldsymbol G}$ as follows. First, we simply insert each of the $\binom{n}{2}$ edges of the complete graph on $[n]$ with probability p independently. The result of this is, clearly, the Erdős-Rényi random graph $\boldsymbol G(n,p)$. Then, we "reject" (i.e., remove) each edge of this graph that joins two vertices that have the same color under either $\hat{\boldsymbol\sigma}_1$ or $\hat{\boldsymbol\sigma}_2$.

Lemma 4.2. *Let $\omega=\lceil\ln\ln n\rceil$ and assume that $p=O(1/n)$.*

1. *Let $\mathcal{K}(G)$ be the total number of vertices v of the graph G such that $\partial^\omega(G,v)$ contains a cycle. Then*
$$\pi^{\mathrm{pr}}_{n,p,k}\left[\mathcal{K}(\tilde{\boldsymbol G})>n^{2/3}\right]=o(n^{-1/2}).$$
2. *Let $\mathcal{L}$ be the event that there is a vertex v such that $\partial^\omega(\tilde{\boldsymbol G},v)$ contains more than $n^{0.1}$ vertices. Then*
$$\pi^{\mathrm{pr}}_{n,p,k}\left[\mathcal{L}\right]\leq\exp(-\Omega(\ln^2 n)).$$

Proof. Obtain the random graph $\boldsymbol G'$ from $\tilde{\boldsymbol G}$ by adding every edge that is monochromatic under either $\hat{\boldsymbol\sigma}_1,\hat{\boldsymbol\sigma}_2$ with probability $p=m/\left(\binom{n}{2}(1-1/k)^2\right)$ independently. Then $\boldsymbol G'$ has the same distribution as the standard binomial random graph $\boldsymbol G(n,p)$. Since $\mathcal{K}(\tilde{\boldsymbol G})\leq\mathcal{K}(\boldsymbol G')$, the first assertion follows from the well-known fact that $\mathbb{E}[\mathcal{K}(\boldsymbol G(n,p))]\leq n^{o(1)}$ and Markov's inequality. A similar argument yields the second assertion. ∎

Lemma 4.3. *Let ϑa be a rooted tree, let $\tau_1,\tau_2\in\mathcal{S}_k(\vartheta a)$ and let $\omega\geq 0$. Then*

$$\pi^{\mathrm{pr}}_{n,p,k}\left[\left|Q_{\vartheta a,\tau_1,\tau_2,\omega}(\tilde{\boldsymbol{G}},\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)-\mathbb{E}[Q_{\vartheta a,\tau_1,\tau_2,\omega}(\tilde{\boldsymbol{G}},\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)]\right|>n^{-1/3}\right] \leq \exp(-\Omega(\ln^2 n)).$$

Proof. The proof is based on Lemma 1.6. To apply it, we view $(\tilde{\boldsymbol{G}},\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)$ as chosen from a product space $X_2,\ldots,X_N$ with $N=2n$ where $X_v\in[k]^2$ is uniformly distributed for $v\in[n]$ and where X_{n+v} is a 0/1 vector of length $v-1$ whose components are independent $\mathrm{Be}(p)$ variables for $v\in[n]$. Namely, X_v with $v\in[n]$ represents the color pair $(\hat{\boldsymbol{\sigma}}_1(v),\hat{\boldsymbol{\sigma}}_2(v))$, and X_{n+v} for $v\in[n]$ indicates to which vertices $w<v$ with $\hat{\boldsymbol{\sigma}}_1(w)\neq\hat{\boldsymbol{\sigma}}_1(v)$, $\hat{\boldsymbol{\sigma}}_2(w)\neq\hat{\boldsymbol{\sigma}}_2(v)$ vertex v is adjacent ("vertex exposure").

Define random variables $S_v=S_v(\tilde{\boldsymbol{G}},\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)$ and S by letting

$$S_v=\mathbf{1}\left\{\partial^\omega\left(\tilde{\boldsymbol{G}},v,\hat{\boldsymbol{\sigma}}_1\right)\cong(\vartheta a,\tau_1)\right\}\cdot\mathbf{1}\left\{\partial^\omega\left(\tilde{\boldsymbol{G}},v,\hat{\boldsymbol{\sigma}}_2\right)\cong(\vartheta a,\tau_2)\right\},$$
$$S=\frac{1}{n}\sum_{v\in[n]}S_v.$$

Then by (13) we have

$$Q_{\vartheta a,\tau_1,\tau_2,\omega}=S. \tag{62}$$

Further, set $\lambda=n^{0.01}$ and let Γ be the event that $|\partial^\omega\left(\tilde{\boldsymbol{G}},v\right)|\leq\lambda$ for all vertices v. Then by Lemma 4.2 we have

$$\mathbb{P}\left[\Gamma\right]\geq 1-\exp(-\Omega(\ln^2 n)). \tag{63}$$

Furthermore, let $\boldsymbol{G}'$ be the graph obtained from $\tilde{\boldsymbol{G}}$ by removing all edges e that are incident with a vertex v such that $|\partial^\omega\left(\tilde{\boldsymbol{G}},v\right)|>\lambda$ and let

$$S'_v=\mathbf{1}\left\{\partial^\omega\left(\boldsymbol{G}',v,\hat{\boldsymbol{\sigma}}_2\right)\cong(\vartheta a,\tau_1)\right\}\cdot\mathbf{1}\left\{\partial^\omega\left(\boldsymbol{G}',v,\hat{\boldsymbol{\sigma}}_2\right)\cong(\vartheta a,\tau_2)\right\},$$
$$S'=\frac{1}{n}\sum_{v\in[n]}S'_v.$$

If Γ occurs, then $S=S'$. Hence, (63) implies that

$$\mathbb{E}[S']=\mathbb{E}[S]+o(1). \tag{64}$$

The random variable S' satisfies (10) with $c=\lambda$ and $c'=n$. Indeed, altering either the colors of one vertex u or its set of neighbors can only affect those vertices v that are at distance at most ω from u, and in $\boldsymbol{G}'$

there are no more than λ such vertices. Thus, Lemma 1.6 applied with, say, $t=n^{2/3}$ and $\gamma=1/n$ and (63) yield

$$\mathbb{P}\left[|S'-\mathbb{E}[S']|>t\right]\leq\exp(-\Omega(\ln^2 n)). \tag{65}$$

Finally, the assertion follows from (62), (64) and (65). ∎

To proceed, we need the following concept. A *k-dicolored graph* $(G,v_0,\sigma_1,\sigma_2)$ consists of a k-colorable graph G with $V(G)\subset\mathbb{R}$, a root $v_0\in V(G)$ and two k-colorings $\sigma_1,\sigma_2\colon V(G)\to[k]$. We call two k-dicolored graphs $(G,v_0,\sigma_1,\sigma_2)$, $(G',v_0',\sigma_1',\sigma_2')$ *isomorphic* if there is an isomorphism $\pi\colon G\to G'$ such that $\pi(v_0)=v_0'$ and $\sigma_1=\sigma_1'\circ\pi$, $\sigma_2=\sigma_2'\circ\pi$ and such that for any $v,u\in V(G)$ such that $v<u$ we have $\pi(v)<\pi(u)$.

Lemma 4.4. *Let ϑa be a rooted tree, let $\tau_1,\tau_2\in\mathcal{S}_k(\vartheta a)$ and let $\omega\geq 0$. Then*

$$\mathbb{E}\left[Q_{\vartheta a,\tau_1,\tau_2,\omega}(\tilde{\boldsymbol{G}})\right]=q_{\vartheta a,\omega}+o(1). \tag{66}$$

Proof. Recall that $\boldsymbol{T}(d)$ is the (possibly infinite) Galton-Watson tree rooted at v_0. Let $\boldsymbol{\tau}_1,\boldsymbol{\tau}_2$ denote two k-colorings of $\partial^\omega\boldsymbol{T}(d)$ chosen uniformly at random. In addition, let $\boldsymbol{v}^*\in[n]$ denote a uniformly random vertex of $\tilde{\boldsymbol{G}}$. To establish (66) it suffices to construct a coupling of the random dicolored tree $(\boldsymbol{T}(d),v_0,\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)$ and the random graph $\partial^\omega(\tilde{\boldsymbol{G}},\boldsymbol{v}^*,\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)$ such that

$$\mathbb{P}\left[\partial^\omega(\tilde{\boldsymbol{G}},\boldsymbol{v}^*,\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)\cong(\boldsymbol{T}(d),v_0,\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)\right]=1-o(1). \tag{67}$$

To this end, let $(u(i))_{i\in[n]}$ be a family of independent random variables such that $u(i)$ is uniformly distributed over the interval $((i-1)/n,i/n)$ for each $i\in[n]$.

The construction of this coupling is based on the principle of deferred decisions. More specifically, we are going to view the exploration of the depth-ω neighborhood of $\boldsymbol{v}^*$ in the random graph $\tilde{\boldsymbol{G}}$ as a random process, reminiscent of the standard breadth-first search process for the exploration of the connected components of the random graph. The colors of the individual vertices and their neighbors are revealed in the course of the exploration process. The result of the exploration process will be a dicolored tree $(\hat{\boldsymbol{T}},u(\boldsymbol{v}^*),\hat{\boldsymbol{\tau}}_1,\hat{\boldsymbol{\tau}}_1)$ whose vertex set is contained in $[0,1]$. This tree is isomorphic to $\partial^\omega(\tilde{\boldsymbol{G}},\boldsymbol{v}^*,\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)$ w.h.p. Furthermore, the distribution of the tree is at total variance distance $o(1)$ from that of $(\boldsymbol{T}(d),v_0,\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)$.

Throughout the exploration process, every vertex is marked either *dead*, *alive*, *rejected* or *unborn*. The semantics of the marks is similar to the one in the usual “branching process” argument for the component exploration in the random graph: vertices whose neighbors have been explored are "dead", vertices that have been reached but whose neighbors have not yet been inspected are "alive", and vertices that the process has not yet discovered are "unborn". The additional mark "rejected" is necessary because we reveal the colors of the vertices as we explore them. More specifically, as we explore the neighbors of an alive v vertex, we insert a "candidate edge" between the alive vertex and *every* unborn vertex with probability p independently. If upon revealing the colors of the "candidate neighbor" w of v we find a conflict (i.e., $\hat{\boldsymbol{\sigma}}_1(v)=\hat{\boldsymbol{\sigma}}_1(w)$ or $\hat{\boldsymbol{\sigma}}_2(v)=\hat{\boldsymbol{\sigma}}_2(w)$), we "reject" w and the "candidate edge" $\{v,w\}$ is discarded. Additionally, we will maintain for each vertex v a number $D(v)\in[0,\infty]$; the intention is that $D(v)$ is the distance from the root $\boldsymbol{v}^*$ in the part of the graph that has been explored so far. The formal description of the process is as follows.

EX1: Initially, $\boldsymbol{v}^*$ is alive, $D(\boldsymbol{v}^*)=0$, and all other vertices $v\neq\boldsymbol{v}^*$ are unborn and $D(v)=\infty$. Choose a pair of colors $(\hat{\boldsymbol{\sigma}}_1(\boldsymbol{v}^*),\hat{\boldsymbol{\sigma}}_2(\boldsymbol{v}^*))\in[k]^2$ uniformly at random. Let $\hat{\boldsymbol{T}}$ be the tree consisting of the root vertex $u(\boldsymbol{v}^*)$ only and let $\hat{\boldsymbol{\tau}}_h(u(\boldsymbol{v}^*))=\hat{\boldsymbol{\sigma}}_h(\boldsymbol{v}^*)$ for $h=1,2$.

EX2: While there is an alive vertex y such that $D(y)<\omega$, let v be the least such vertex. For each vertex w that is either rejected or unborn let $a_{vw}=\mathrm{Be}(p)$; the random variables a_{vw} are mutually independent. For each unborn vertex w such that $a_{vw}=1$ choose a pair $(\hat{\boldsymbol{\sigma}}_1(w),\hat{\boldsymbol{\sigma}}_2(w))\in[k]^2$ independently and uniformly at random and set $D(w)=D(v)+1$. Extend the tree $\hat{\boldsymbol{T}}$ by adding the vertex $u(w)$ and the edge $\{u(v),u(w)\}$ and by setting $\hat{\boldsymbol{\tau}}_1(u(w))=\hat{\boldsymbol{\sigma}}_1(w)$, $\hat{\boldsymbol{\tau}}_2(u(w))=\hat{\boldsymbol{\sigma}}_2(w)$ for every unborn w such that $a_{vw}=1$, $\hat{\boldsymbol{\sigma}}_1(v)\neq\hat{\boldsymbol{\sigma}}_1(w)$ and $\hat{\boldsymbol{\sigma}}_2(v)\neq\hat{\boldsymbol{\sigma}}_2(w)$. Finally, declare the vertex v dead, declare all w with $a_{vw}=1$ and $\hat{\boldsymbol{\sigma}}_1(v)\neq\hat{\boldsymbol{\sigma}}_1(w)$ and $\hat{\boldsymbol{\sigma}}_2(v)\neq\hat{\boldsymbol{\sigma}}_2(w)$ alive, and declare all other w with $a_{vw}=1$ rejected.

The process stops once there is no alive vertex y such that $D(y)<\omega$ anymore, at which point we have got a tree $\hat{\boldsymbol{T}}$ that is embedded into $[0,1]$.

Let $\mathcal{A}$ be the event that $\partial^\omega(\hat{\boldsymbol{G}},\boldsymbol{v}^*)$ is an acyclic subgraph that contains no more than $n^{0.1}$ vertices. Furthermore, let $\mathcal{R}$ be the event that in **EX2** it never occurs that $a_{vw}=1$ for a rejected vertex w. Then Lemma 4.2 implies that $\mathbb{P}[\mathcal{A}]=1-o(1)$. Moreover, since $p=O(1/n)$ we have $\mathbb{P}[\mathcal{R}|\mathcal{A}]=1-O(n^{-0.8})=1-o(1)$, whence $\mathbb{P}[\mathcal{A}\cap\mathcal{R}]=1-o(1)$. Further, given that $\mathcal{A}\cap\mathcal{R}$ occurs, $\partial^\omega(\hat{\boldsymbol{G}},\boldsymbol{v}^*,\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)$ is isomorphic to $(\hat{\boldsymbol{T}},u(\boldsymbol{v}^*),\hat{\boldsymbol{\tau}}_1,\hat{\boldsymbol{\tau}}_2)$. Thus,

$$\mathbb{P}\left[\partial^\omega(\hat{\boldsymbol{G}},\boldsymbol{v}^*,\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2)\cong(\hat{\boldsymbol{T}},u(\boldsymbol{v}^*),\hat{\boldsymbol{\tau}}_1,\hat{\boldsymbol{\tau}}_2)\right]=1-o(1). \tag{68}$$

Further, if $\mathcal{A}\cap\mathcal{R}$ occurs, then whenever **EX2** processes an alive vertex v with $D(v)<\omega$, the number of unborn neighbors of v of every color combination (s_1,s_2) such that $s_1\neq\hat{\boldsymbol{\sigma}}(v)$, $s_2\neq\hat{\boldsymbol{\sigma}}(v)$ is a binomial random variable whose mean lies in the interval $[(n-n^{0.1})p/k^2, np/k^2]$. The total variation distance of this binomial distribution and the Poisson distribution $\mathrm{Po}(d/(k-1)^2)$, which is precisely the distribution of the number of children colored (s_1,s_2) in the dicolored Galton-Watson tree, is $O(n^{-0.9})$ by the choice of p. In addition, let $\mathcal{B}$ be the event that each interval $((i-1)/n, i/n)$ for $i=1,\ldots,n$ contains at most one vertex of the tree $\partial^\omega\boldsymbol{T}(d)$. Then $\mathbb{P}[\mathcal{B}]=1-o(1)$ and given $\mathcal{A}\cap\mathcal{R}$ and $\mathcal{B}$, there is a coupling of $(\hat{\boldsymbol{T}}, u(\boldsymbol{v}^*), \hat{\boldsymbol{\tau}}_1, \hat{\boldsymbol{\tau}}_2)$ and $\partial^\omega(\boldsymbol{T}(d), v_0, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ such that

$$\mathbb{P}\left[\partial^\omega(\boldsymbol{T}(d), v_0, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2) = (\hat{\boldsymbol{T}}, u(\boldsymbol{v}^*), \hat{\boldsymbol{\tau}}_1, \hat{\boldsymbol{\tau}}_2)\right] = 1-o(1). \tag{69}$$

Finally, (67) follows from (68) and (69). ∎

Corollary 4.5. *Let ϑa be a rooted tree, let $\tau_1,\tau_2\in\mathcal{S}_k(\vartheta a)$ and let $\omega\geq 0$. Moreover, let $p=m/(\binom{n}{2}(1-1/k)^2)$. Then*

$$\lim_{\varepsilon\searrow 0}\lim_{n\to\infty}\sqrt{n}\cdot\pi^{\mathrm{pr}}_{n,p,k}\left[|Q_{\vartheta a,\tau_1,\tau_2,\omega} - q_{\vartheta a,\tau_1,\tau_2,\omega}|>\varepsilon\right]=0. \tag{70}$$

Proof. This follows by combining Lemmas 4.3 and 4.4. ∎

Finally, Proposition 2.4 is immediate from Lemma 4.1 and Corollary 4.5.

5. Establishing local weak convergence

Throughout this section we assume that $k\geq k_0$ for some large enough constant k_0 and that $d<d_{k,cond}$.

In this section we prove Proposition 2.5. The purpose of Propositions 2.3 and 2.4 was to facilitate the proof of the following fact.

Lemma 5.1. *Let $\vartheta a_1,\ldots,\vartheta a_l$ be rooted trees and let $\tau_1\in\mathcal{S}_k(\vartheta a_1),\ldots,\tau_l\in\mathcal{S}_k(\vartheta a_l)$. Then*

$$\begin{aligned}Q &= Q(\vartheta a_1,\tau_1,\ldots,\vartheta a_l,\tau_l)\\ &= \frac{1}{n^l}\sum_{v_1,\ldots,v_l\in[n]}\left\langle\prod_{i=1}^l\prod_{j=1}^2\left(\mathbf{1}\left\{\partial^\omega(\boldsymbol{G},v_i,\boldsymbol{\sigma}_j)\cong(\vartheta a_i,\tau_i)\right\}-Z_k(\vartheta a_i)^{-1}\right)\right\rangle_{\boldsymbol{G}}\\ &\qquad\qquad\cdot\prod_{i=1}^l\mathbf{1}\left\{\partial^\omega(\boldsymbol{G},v_i)\cong\vartheta a_i\right\}\end{aligned}$$

converges to 0 *in probability.*

Proof. Let $t_i(G,v,\sigma) = \mathbf{1}\{\partial^\omega(G,v,\sigma) \cong (\vartheta a_i, \tau_i)\}$, $z_i = Z_k(\vartheta a_i)$ and $z_J = \prod_{i \in J} z_i$ for $J \subset [l]$. Moreover, let $V_i(G)$ be the set of all vertices v of G such that $\partial^\omega(G,v) \cong \vartheta a_i$, let $\bar{n}_i = n\mathbb{P}[\partial^\omega \boldsymbol{T}(d) \cong \vartheta a_i]$ and for $J \subset [l]$ let $\bar{n} = \prod_{i \in J} \bar{n}_i$. Then

$$Q = n^{-l} \sum_{v_1 \in V_1(\boldsymbol{G}), \dots, v_l \in V_l(\boldsymbol{G})} \left\langle \prod_{i=1}^{l} \prod_{j=1}^{2} \left(t_i(\boldsymbol{G}, v_i, \boldsymbol{\sigma}_j) - z_i^{-1} \right) \right\rangle_{\boldsymbol{G}}.$$

We estimate this quantity by way of the planted replica model. In fact, by Proposition 2.3 it suffices to prove that

$$\hat{Q} = n^{-l} \sum_{v_1 \in V_1(\hat{\boldsymbol{G}}), \dots, v_l \in V_l(\hat{\boldsymbol{G}})} \prod_{i=1}^{l} \prod_{j=1}^{2} \left(t_i(\hat{\boldsymbol{G}}, v_i, \hat{\boldsymbol{\sigma}}_j) - z_i^{-1} \right)$$

converges to 0 in probability. To show this, we decompose $\hat{Q}$ as follows: letting $(J_1, \dots, J_4)$ range over all decompositions of $[l]$ into pairwise disjoint sets, we write

$$\begin{aligned} n^l \hat{Q} = & \sum_{v_1 \in V_1(\hat{\boldsymbol{G}}), \dots, v_l \in V_l(\hat{\boldsymbol{G}})} \prod_{i=1}^{l} \Big(t_i(\hat{\boldsymbol{G}}, v_i, \hat{\boldsymbol{\sigma}}_1) t_i(\hat{\boldsymbol{G}}, v_i, \hat{\boldsymbol{\sigma}}_2) \\ & \qquad - t_i(\hat{\boldsymbol{G}}, v_i, \hat{\boldsymbol{\sigma}}_1) z_i^{-1} - z_i^{-1} t_i(\hat{\boldsymbol{G}}, v_i, \hat{\boldsymbol{\sigma}}_2) + z_i^{-2} \Big) \\ = & \sum_{v_1 \in V_1(\hat{\boldsymbol{G}}), \dots, v_l \in V_l(\hat{\boldsymbol{G}})} \sum_{J_1, \dots, J_4} \frac{(-1)^{|J_2|+|J_3|}}{z_{J_2 \cup J_3} z_{J_4}^2} \prod_{i \in J_1} t_i(\hat{\boldsymbol{G}}, v_i, \hat{\boldsymbol{\sigma}}_1) t_i(\hat{\boldsymbol{G}}, v_i, \hat{\boldsymbol{\sigma}}_2) \\ & \qquad \cdot \prod_{i \in J_2} t_i(\hat{\boldsymbol{G}}, v_i, \hat{\boldsymbol{\sigma}}_1) \cdot \prod_{i \in J_3} t_i(\hat{\boldsymbol{G}}, v_i, \hat{\boldsymbol{\sigma}}_2). \end{aligned}$$

Hence, letting

$$\begin{aligned} \hat{N}_{J_1,J_2,J_3} = & \sum_{v_1 \in V_1(\hat{\boldsymbol{G}}), \dots, v_l \in V_l(\hat{\boldsymbol{G}})} \prod_{i \in J_1} t_i(\hat{\boldsymbol{G}}, v_i, \hat{\boldsymbol{\sigma}}_1) t_i(\hat{\boldsymbol{G}}, v_i, \boldsymbol{\sigma}_2) \\ & \qquad \cdot \prod_{i \in J_2} t_i(\hat{\boldsymbol{G}}, v_i, \hat{\boldsymbol{\sigma}}_1) \cdot \prod_{i \in J_3} t_i(\hat{\boldsymbol{G}}, v_i, \hat{\boldsymbol{\sigma}}_2), \end{aligned}$$

we have

$$\hat{Q} = \sum_{J_1, \dots, J_4} \frac{(-1)^{|J_2|+|J_3|}}{z_{J_2 \cup J_3} z_{J_4}^2} \cdot \frac{\hat{N}_{J_1,J_2,J_3}}{n^l}.$$

Therefore, it suffices to prove that

$$\frac{\hat{N}_{J_1,J_2,J_3}}{n^l} \to \frac{\bar{n}}{n^l} z_{J_1}^{-2} z_{J_2\cup J_3}^{-1} \qquad \text{in probability} \tag{71}$$

for every decomposition $J_1,\ldots,J_4$. But (71) follows from Proposition 2.4. Indeed, observe that $\hat{N}_{J_1,J_2,J_3}$ is nothing but the number of tuples $(v_1,\ldots,v_l)$ with the following properties.

1. For every $i\in J_1$ we have $\partial^\omega(\hat{\boldsymbol{G}},v_i,\hat{\boldsymbol{\sigma}}_j)\cong(\vartheta a_i,\tau_i)$ for $j=1,2$.
2. For every $i\in J_2$ we have $\partial^\omega(\hat{\boldsymbol{G}},v_i,\hat{\boldsymbol{\sigma}}_1)\cong(\vartheta a_i,\tau_i)$.
3. For every $i\in J_3$ we have $\partial^\omega(\hat{\boldsymbol{G}},v_i,\hat{\boldsymbol{\sigma}}_2)\cong(\vartheta a_i,\tau_i)$.
4. For every $i\in J_4$ we have $\partial^\omega(\hat{\boldsymbol{G}},v_i)\cong\vartheta a_i$.

Proposition 2.4 shows explicitly that for every $i \in J_1$ the number of vertices v_i that satisfy (1) is $(1+o(1))\bar{n}_i z_i^{-2}$ w.h.p. Moreover, marginalising $\hat{\boldsymbol{\sigma}}_2$ in Proposition 2.4 we see that the asymptotic number of v_i that satisfy (2) is $(1+o(1))\bar{n}_i z_i^{-1}$ w.h.p. A similar argument applies to (3). Finally, marginalising both $\hat{\boldsymbol{\sigma}}_1,\hat{\boldsymbol{\sigma}}_2$ we conclude that the number of v_i that satisfy (4) is $(1+o(1))\bar{n}_i$ w.h.p. ∎

We complete the proof of Proposition 2.5 by generalising the elegant argument that was used in [23, Proposition 3.2] to establish a statement similar to the $\omega=0$ case of Proposition 2.5.

Lemma 5.2. *Let $\vartheta a_1,\ldots,\vartheta a_l$ be rooted trees, let $\tau_1 \in \mathcal{S}_k(\vartheta a_1),\ldots,\tau_l \in \mathcal{S}_k(\vartheta a_l)$ and let $\omega\geq 0$ be an integer. There exists a sequence $\varepsilon=\varepsilon(n)=o(1)$ such that for every $\emptyset \neq J \subset [l]$ the following is true. For a graph G let $\mathcal{X}_{\vartheta a_1,\ldots,\vartheta a_l}(G,J,\omega)$ be the set of all vertex sequences $u_1,\ldots,u_l$ such that $\partial^\omega(G,u_i)\cong\vartheta a_i$ while*

$$\left|\left\langle \prod_{i\in J} \mathbf{1}\left\{\partial^\omega(G,u_i,\boldsymbol{\sigma})\cong(\vartheta a_i,\tau_i)\right\} - \frac{1}{Z_k(\vartheta a_i)}\right\rangle_G\right| > \varepsilon.$$

Then $|\mathcal{X}_{\vartheta a_1,\ldots,\vartheta a_l}(\boldsymbol{G},J,\omega)|\leq\varepsilon n^l$ w.h.p.

Proof. Let $t_i(v,\sigma)=\mathbf{1}\{\partial^\omega(\boldsymbol{G},v,\sigma)\cong(\vartheta a_i,\tau_i)\}$ and $z_i=Z_k(\vartheta a_i)$ for the sake of brevity. By Lemma 5.1 there exists $\varepsilon=\varepsilon(n)=o(1)$ such that w.h.p. for all $J\subset[l]$ we have $Q_J=Q((\vartheta a_i,\tau_i)_{i\in J})\leq\varepsilon^3$. Hence, recalling that $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2$ are independently chosen k-colorings, we obtain w.h.p.

$$\begin{aligned}
&\frac{\varepsilon^2}{n^l}|\mathcal{X}_{\vartheta a_1,\ldots,\vartheta a_l}(\boldsymbol{G},J,\omega)| \\
&\leq \frac{1}{n^l}\sum_{u_1,\ldots,u_l\in[n]}\left\langle\prod_{i\in J}(t_i(u_i,\boldsymbol{\sigma})-z_i^{-1})\right\rangle_{\boldsymbol{G}}^2\prod_{i\in J}\mathbf{1}\left\{\partial^\omega(\boldsymbol{G},u_i)\cong\vartheta a_i\right\} \\
&=\frac{1}{n^l}\sum_{u_1,\ldots,u_l\in[n]}\left\langle\prod_{i\in J}\left[(t_i(u_i,\boldsymbol{\sigma}_1)-z_i^{-1})(t_i(u_i,\boldsymbol{\sigma}_2)-z_i^{-1})\right]\right\rangle_{\boldsymbol{G}} \\
&\qquad\qquad\cdot\prod_{i\in J}\mathbf{1}\left\{\partial^\omega(\boldsymbol{G},u_i)\cong\vartheta a_i\right\}=Q_J\leq\varepsilon^3,
\end{aligned}$$

as desired. ∎

Corollary 5.3. *Let $\omega\geq 0$ be an integer, let $\vartheta a_1,\ldots,\vartheta a_l$ be rooted trees, let $\tau_1\in\mathcal{S}_k(\vartheta a_1),\ldots,\tau_l\in\mathcal{S}_k(\vartheta a_l)$ and let $\delta>0$. For a graph G let $Y(G)$ be the number of vertex sequences $v_1,\ldots,v_l$ such that $\partial^\omega(G,v_i)\cong\partial^\omega\vartheta a_i$ while*

$$\left|\left\langle\prod_{i\in[l]}\mathbf{1}\left\{\partial^\omega(G,v_i,\boldsymbol{\sigma})\cong(\vartheta a_i,\tau_i)\right\}\right\rangle_G-\prod_{i\in[l]}\frac{1}{Z_k(\vartheta a_i)}\right|>\delta. \tag{72}$$

Then $n^{-l}Y(\boldsymbol{G})$ converges to 0 in probability.

Proof. Let $z_i=Z_k(\partial^\omega\vartheta a_i)$ for the sake of brevity. Let $\mathcal{E}_{\vartheta a_1,\ldots,\vartheta a_l}$ be the set of all l-tuples $(v_1,\ldots,v_l)$ of distinct vertices such that $\partial^\omega(\boldsymbol{G},v_i)\cong\vartheta a_i$ for all $i\in[l]$. Moreover, with the notation of Lemma 5.2 let

$$\mathcal{X}_{\vartheta a_1,\ldots,\vartheta a_l}=\bigcup_{\emptyset\neq J\subset[l]}\mathcal{X}_{\vartheta a_1,\ldots,\vartheta a_l}(\boldsymbol{G},J,\omega)$$

and set $\mathcal{Y}_{\vartheta a_1,\ldots,\vartheta a_l}=\mathcal{E}_{\vartheta a_1,\ldots,\vartheta a_l}\setminus\mathcal{X}_{\vartheta a_1,\ldots,\vartheta a_l}$. With $\varepsilon=\varepsilon(n)=o(1)$ from Lemma 5.2, we are going to show that for each $J\subset[l]$ there exists an (n-independent) number C_J such that

$$\left|\left\langle\prod_{i\in J}\mathbf{1}\left\{\partial^\omega(\boldsymbol{G},v_i,\boldsymbol{\sigma})\cong(\vartheta a_i,\tau_i)\right\}\right\rangle_{\boldsymbol{G}}-\prod_{i\in J}z_i^{-1}\right|\leq C_J\varepsilon^{1/2} \tag{73}$$

for all $(v_1,\ldots,v_l)\in\mathcal{Y}_{\vartheta a_1,\ldots,\vartheta a_l}$.

Since $|\mathcal{X}_{\vartheta a_1,\ldots,\vartheta a_l}|=o(n^l)$ w.h.p. by Lemma 5.2, the assertion follows from (73) by setting $J=[l]$.

The proof of (73) is by induction on $|J|$. In the case $J = \emptyset$ there is nothing to show as both products are empty. As for the inductive step, set $t_i = \mathbf{1}\{\partial^\omega(\boldsymbol{G}, v_i, \boldsymbol{\sigma}) \cong (\vartheta a_i, \tau_i)\}$ for the sake of brevity. Then

(74)
$$\begin{aligned}\left\langle \prod_{i\in J} t_i - z_i^{-1}\right\rangle_{\boldsymbol{G}} &= \sum_{I\subset J}(-1)^{|I|}\prod_{i\in I} z_i^{-1}\left\langle \prod_{i\in J\setminus I} t_i\right\rangle_{\boldsymbol{G}} \\ &= \left\langle \prod_{i\in J} t_i - \prod_{i\in J} z_i^{-1}\right\rangle_{\boldsymbol{G}} + \prod_{i\in J} z_i^{-1} + \sum_{\emptyset\neq I\subset J}(-1)^{|I|}\prod_{i\in I} z_i^{-1}\left\langle \prod_{i\in J\setminus I} t_i\right\rangle_{\boldsymbol{G}} \\ &= \left\langle \prod_{i\in J} t_i - \prod_{i\in J} z_i^{-1}\right\rangle_{\boldsymbol{G}} + \sum_{\emptyset\neq I\subset J}(-1)^{|I|}\prod_{i\in I} z_i^{-1}\left[\left\langle \prod_{i\in J\setminus I} t_i\right\rangle_{\boldsymbol{G}} - \prod_{i\in J\setminus I} z_i^{-1}\right].\end{aligned}$$

By the induction hypothesis, for all $\emptyset \neq I \subset J$ we have

$$\left|\left\langle \prod_{i\in J\setminus I} t_i\right\rangle_{\boldsymbol{G}} - \prod_{i\in J\setminus I} z_i^{-1}\right| \leq C_I \varepsilon^{1/2}.$$

Hence, by the triangle inequality

$$\begin{aligned}&\left|\sum_{\emptyset\neq I\subset J}(-1)^{|I|}\prod_{i\in I} z_i^{-1}\left[\left\langle \prod_{i\in J\setminus I} t_i\right\rangle_{\boldsymbol{G}} - \prod_{i\in J\setminus I} z_i^{-1}\right]\right| \\ &\qquad \leq \sum_{\emptyset\neq I\subset J}\left|\left\langle \prod_{i\in J\setminus I} t_i\right\rangle_{\boldsymbol{G}} - \prod_{i\in J\setminus I} z_i^{-1}\right| \leq \varepsilon^{1/2}\sum_{\emptyset\neq I\subset J} C_I.\end{aligned} \tag{75}$$

Set $C_J = 2(1 + \sum_{\emptyset\neq I\subset J} C_I)$. Combining (74) and (75), we obtain

$$\left|\left\langle \prod_{i\in J} t_i - z_i^{-1}\right\rangle_{\boldsymbol{G}} - \left\langle \prod_{i\in J} t_i - \prod_{i\in J} z_i^{-1}\right\rangle_{\boldsymbol{G}}\right| \leq C_J \varepsilon^{1/2}/2. \tag{76}$$

Since $(v_1, \ldots, v_l) \notin \mathcal{X}_{\vartheta a_1, \ldots, \vartheta a_l}$, we have $\left|\langle \prod_{i\in J} t_i - z_i^{-1}\rangle_{\boldsymbol{G}}\right| \leq \varepsilon$. Plugging this bound into (76) yields (73). ∎

Proof of Proposition 2.5. Let $\mathcal{U} = \mathcal{U}(\boldsymbol{G})$ be the set of all tuples $(v_1, \ldots, v_l) \in [n]^l$ such that $\partial^\omega(\boldsymbol{G}, v_i) \cong \vartheta a_i$ for all $i \in [l]$. Since the random graph converges locally to the Galton-Watson tree [13], w.h.p. we have

$$n^{-l}|\mathcal{U}| = o(1) + \prod_{i\in[l]} \mathbb{P}\left[\partial^\omega \boldsymbol{T}(d) \cong \vartheta a_i\right]. \tag{77}$$

(Alternatively, (77) follows from Propositions 2.3 and 2.4 by marginalising $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$.) The assertion follows by combining (77) with Corollary 5.3. ∎

Acknowledgement. We thank Ralph Neininger for helpful discussions and the anonymous referees for their very careful reading.

References

[1] D. Achlioptas and E. Friedgut: A sharp threshold for k-colorability, *Random Struct. Algorithms* **14** (1999), 63–70.

[2] D. Achlioptas and A. Coja-Oghlan: Algorithmic barriers from phase transitions, *Proc. 49th FOCS* (2008), 793–802.

[3] D. Achlioptas and M. Molloy: The analysis of a list-coloring algorithm on a random graph, *Proc. 38th FOCS* (1997), 204–212.

[4] D. Achlioptas and A. Naor: The two possible values of the chromatic number of a random graph, *Annals of Mathematics* **162** (2005), 1333–1349.

[5] D. Aldous and J. Steele: The objective method: probabilistic combinatorial optimization and local weak convergence (2003), in: *Probability on discrete structures* (H. Kesten (ed.)), Springer 2004.

[6] N. Alon and M. Krivelevich: The concentration of the chromatic number of random graphs, *Combinatorica* **17** (1997), 303–313.

[7] V. Bapst, A. Coja-Oghlan and C. Efthymiou: Planting colourings silently, arXiv:1411.0610 (2014).

[8] V. Bapst, A. Coja-Oghlan, S. Hetterich, F. Rassmann and Dan Vilenchik: The condensation phase transition in random graph coloring, *Communications in Mathematical Physics* **341** (2016), 543–606.

[9] I. Benjamini and O. Schramm: Recurrence of distributional limits of finite planar graphs, *Electronic J. Probab.* **6** (2001), 1–13.

[10] N. Bhatnagar, J. Vera, E. Vigoda and D. Weitz: Reconstruction for Colorings on Trees, *SIAM Journal on Discrete Mathematics.* **25** (2011), 809–826.

[11] N. Bhatnagar, A. Sly and P. Tetali: Decay of correlations for the hardcore model on the d-regular random graph, *Electronic Journal of Probability* **21** (2016).

[12] B. Bollobás: The chromatic number of random graphs, *Combinatorica* **8** (1988), 49–55.

[13] C. Bordenave and P. Caputo: Large deviations of empirical neighborhood distribution in sparse random graphs, *Probability Theory and Related Fields* **163** (2015), 149–222.

[14] A. Coja-Oghlan: Upper-bounding the k-colorability threshold by counting covers, *Electronic Journal of Combinatorics* **20** (2013), P32.

[15] A. Coja-Oghlan and D. Vilenchik: The chromatic number of random graphs for most average degrees, *International Mathematics Research Notices*, (2015): rnv333.

[16] M. E. Dyer and A. M. Frieze: Randomly coloring random graphs, *Random Struct. Algorithms* **36** (2010), 251–272.

[17] M. Dyer, A. Flaxman, A. Frieze and E. Vigoda: Randomly coloring sparse random graphs with fewer colors than the maximum degree, *Random Structures and Algorithms* **29** (2006), 450–465.

[18] C. Efthymiou: MCMC sampling colourings and independent sets of $G(n, d/n)$ near uniqueness threshold, *Proc. 25th SODA* (2014), 305–316.
[19] C. Efthymiou: Switching colouring of $G(n, d/n)$ for sampling up to Gibbs uniqueness threshold, *Proc. 22nd ESA* (2014), 371–381.
[20] C. Efthymiou: Reconstruction/non-reconstruction thresholds for colourings of general Galton-Watson trees, *Proc. 19th RANDOM* (2015), 756–774.
[21] P. Erdős: Graph theory and probability, *Canad. J. Math.* **11** (1959), 34–38.
[22] P. Erdős and A. Rényi: On the evolution of random graphs, *Magayar Tud. Akad. Mat. Kutato Int. Kozl.* **5** (1960), 17–61.
[23] A. Gerschenfeld and A. Montanari: Reconstruction for models on random graphs, *Proc. 48th FOCS* (2007), 194–204.
[24] G. Grimmett and C. McDiarmid: On colouring random graphs, *Mathematical Proceedings of the Cambridge Philosophical Society* **77** (1975), 313–324.
[25] D. A. Johnston and P. Plecháč: Equivalence of ferromagnetic spin models on trees and random graphs, *J. Phys. A* **31** (1998), 475-482.
[26] M. Krivelevich and B. Sudakov: Coloring random graphs, *Information Processing Letters* **67** (1998), 71–74.
[27] F. Krzakala, A. Montanari, F. Ricci-Tersenghi, G. Semerjian and L. Zdeborova: Gibbs states and the set of solutions of random constraint satisfaction problems, *Proc. National Academy of Sciences* **104** (2007), 10318–10323.
[28] F. Krzakala, A. Pagnani and M. Weigt: Threshold values, stability analysis and high-q asymptotics for the coloring problem on random graphs, *Phys. Rev. E* **70** (2004), 046705.
[29] T. Łuczak: The chromatic number of random graphs, *Combinatorica* **11** (1991), 45–54.
[30] T. Łuczak: A note on the sharp concentration of the chromatic number of random graphs, *Combinatorica* **11** (1991), 295–297.
[31] D. Matula: Expose-and-merge exploration and the chromatic number of a random graph, *Combinatorica* **7** (1987), 275–284.
[32] M. Mézard and A. Montanari: *Information, physics and computation*, Oxford University Press 2009.
[33] M. Mézard and G. Parisi: The Bethe lattice spin glass revisited, *Eur. Phys. J. B* **20** (2001), 217–233.
[34] M. Mézard and G. Parisi: The cavity method at zero temperature, *Journal of Statistical Physics* **111** (2003), 1–34.
[35] M. Mézard, M. Palassini and O. Rivoire: Landscape of solutions in constraint satisfaction problems, *Phys. Rev. E* **95** (2005), 200202.
[36] M. Mézard, G. Parisi and R. Zecchina: Analytic and algorithmic solution of random satisfiability problems, *Science* **297** (2002), 812–815.
[37] M. Molloy: The freezing threshold for k-colourings of a random graph, *Proc. 43rd STOC* (2012), 921–930.
[38] A. Montanari, E. Mossel and A. Sly: The weak limit of Ising models on locally tree-like graphs, *Probab. Theory Relat. Fields* **152** (2012), 31–51.
[39] A. Montanari, R. Restrepo and P. Tetali: Reconstruction and clustering in random constraint satisfaction problems, *SIAM J. Discrete Math.* **25** (2011), 771–808.
[40] E. Mossel and A. Sly: Gibbs rapidly samples colorings of $G(n, d/n)$, *Probability Theory and Related Fields* **148** (2010), 37–69.

[41] E. Shamir and J. Spencer: Sharp concentration of the chromatic number of random graphs $G(n,p)$, *Combinatorica* **7** (1987), 121–129.
[42] A. Sly: Reconstruction of random colourings, *Comm. Math. Phys.* **288** (2009), 943–961.
[43] L. Warnke: On the method of typical bounded differences, *Combinatorics, Probability and Computing*, **25** (2016) 269–299.
[44] L. Zdeborová and F. Krzakala: Phase transitions in the coloring of random graphs, *Phys. Rev. E* **76** (2007), 031131.

A. Convergence of $\boldsymbol{\vartheta}^{l}_{d,k}[\omega]$

We use a standard argument to prove that the sequence defined in (5) converges.

Lemma A.1. *The sequence* $(\boldsymbol{\vartheta}^{l}_{d,k}[\omega])_{\omega\geq 1}$ *converges for any* $d>0, k\geq 3, l>0$.

Proof. The space $\mathcal{P}^2(\mathcal{G}^l_k)$ is Polish and thus complete. Therefore, it suffices to prove that $(\boldsymbol{\vartheta}^{l}_{d,k}[\omega])_{\omega\geq 1}$ is a Cauchy sequence. As $\mathcal{P}^2(\mathcal{G}^l_k)$ is endowed with the weak topology, this amounts to proving that for any bounded continuous function $f:\mathcal{P}(\mathcal{G}^l_k)\to\mathbb{R}$ with a compact support and any $\varepsilon>0$ there exists integer $N=N(\varepsilon)\geq 0$ such that

$$\left|\int f\,\mathrm{d}\,\boldsymbol{\vartheta}^{l}_{d,k}[\omega_1]-\int f\,\mathrm{d}\,\boldsymbol{\vartheta}^{l}_{d,k}[\omega_2]\right|<\varepsilon \qquad \text{if } \omega_1,\omega_2\geq N. \tag{78}$$

By the definition of $\boldsymbol{\vartheta}^{l}_{d,k}$,

$$\int f\,\mathrm{d}\,\boldsymbol{\vartheta}^{l}_{d,k}[\omega]=\mathbb{E}\int f\,\mathrm{d}\,\delta_{\bigotimes_{i\in[l]}\lambda\left(\partial^\omega \boldsymbol{T}^i(d)\right)}=\mathbb{E}f\left(\bigotimes\nolimits_{i\in[l]}\lambda_{\partial^\omega \boldsymbol{T}^i(d)}\right). \tag{79}$$

Hence, to prove (78) if suffices to show that for any $\varepsilon>0$ there exists $N>0$ such that

$$\mathbb{E}\left|f\left(\bigotimes\nolimits_{i\in[l]}\lambda_{\partial^{\omega_1}\boldsymbol{T}^i(d)}\right)-f\left(\bigotimes\nolimits_{i\in[l]}\lambda_{\partial^{\omega_2}\boldsymbol{T}^i(d)}\right)\right|<\varepsilon \text{ for all } \omega_1,\omega_2\geq N. \tag{80}$$

To establish (80), we observe that the sequence $\lambda_{\partial^\omega T}$ converges as $\omega\to\infty$ for any locally finite rooted tree T. Indeed, $(\lambda_{\partial^\omega T})_\omega$ is a sequence in the space $\mathcal{P}(\mathcal{G}_k)$, which, equipped with the weak topology, is Polish. Hence, it suffices to prove that for any continuous function $g:\mathcal{G}_k\to\mathbb{R}$ with a compact support the sequence $\left(\int g\,\mathrm{d}\,\lambda_{\partial^\omega T}\right)_\omega$ converges. Indeed, because the topology of $\mathcal{G}_k$ is generated by the functions of the form (3), it suffices to verify that that for any $\Gamma\in\mathcal{G}_k$ and any $\omega_0\geq 0$ the sequence $\left(\int g_{\Gamma,\omega_0}\,\mathrm{d}\,\lambda_{\partial^\omega T}\right)_\omega$ converges, where

$$g_{\Gamma,\omega_0}:\mathcal{G}_k\to\{0,1\}, \qquad \Gamma'\mapsto\mathbf{1}\left\{\partial^{\omega_0}\Gamma=\partial^{\omega_0}\Gamma'\right\}.$$

But this last convergence statement holds simply because the construction of $\lambda_{\partial^\omega T}$ ensures that

$$\int g_{\Gamma,\omega_0}\,\mathrm{d}\,\lambda_{\partial^\omega T} = \int g_{\Gamma,\omega_0}\,\mathrm{d}\,\lambda_{\partial^{\omega_0} T} \qquad \text{for all } \omega > \omega_0.$$

Finally, because $\lim_{\omega\to\infty}\lambda_{\partial^\omega T}$ exists for any T, (80) follows from the fact that the continuous function f has a compact support. ∎

Amin Coja-Oghlan, Nor Jaafari

Goethe University
Mathematics Institute
10 Robert Mayer St.
Frankfurt 60325, Germany
{acoghlan,jaafari}@math.uni-frankfurt.de

Charilaos Efthymiou

Goethe University
Mathematics Institute
10 Robert Mayer St.
Frankfurt 60325, Germany
efthymiou@gmail.com