Nonlocal Kardar-Parisi-Zhang equation with spatially correlated noise

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The effects of spatially correlated noise on a phenomenological equation equivalent to a nonlocal version of the Kardar-Parisi-Zhang (KPZ) equation are studied via the dynamic renormalization group (DRG) techniques. The correlated noise coupled with the long ranged nature of interactions prove the existence of different phases in different regimes, giving rise to a range of roughness exponents defined by their corresponding critical dimensions. Finally self-consistent mode analysis is employed to compare the non-KPZ exponents obtained as a result of the long-range interactions with the DRG results. [S1063-651X(99)11507-1]

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Interest in nonequilibrium growth mechanisms in the formation of surfaces and interfaces, the description of directed polymers, bacterial growth, etc., and, recently, the protein folding problems, although they are apparently the representations of different physical processes, have all been encapsulated in one single nonlinear continuum equation, the much celebrated Kardar-Parisi-Zhang (KPZ) equation [1]. The notion of universality classes, defined by this standard KPZ equation [1], suggests the existence of a phase transition from the Edwards-Wilkinson (EW) class [2] to the nonlinear KPZ class above a particular critical dimension ($d>2$). However, although this stochastic equation has, by now, become a model of dynamic critical phenomena for a vast range of growth problems, the fact remains that the basic nonlinearity studied here is of a local nature; that is to say, the growth occurs along a continuously varying local normal.

Incorporating the long-ranged nature of interactions, which is necessary for a wide class of problems, eg., the long-ranged hydrodynamic interactions, proteins, colloids, etc., Mukherji and Bhattacharjee [3] developed a Langevin-type equation studying the effects of the long-ranged C_{LR} feature of an evolving surface, going beyond the local description of the KPZ nonlinearity for the case of white noise. There the approach essentially consisted of introducing a term in the basic Langevin equation capable of correlating each site of the growing surface with all other sites. The objective was the transformation of the local nonlinear term representing the lateral growth beyond the strict local description, such that the correlation length now becomes at least the system size. Still, the effects of the interaction of correlated colored noise with the KPZ or KPZ-type nonlinearity remain to be seen.

The results of nonwhite noise for the growth of rough interface has been generalized by Medina *et al.* [4]. In two remarkable papers, Chekhlov and Yakhot [5,6], and Hayot and Jayaprakash [7] have observed the effects of correlated noise for the one-dimensional Burgers equation. They explored the occurence of shocks as well as the large distance, long time statistics of the fluctuations. Working along this line Frey, Tauber, and Janssen [8] have also resorted to the long-ranged description of noise in their treatment of the scaling regimes and critical dimensions in the standard KPZ problem, as well as in the conserved case [9]. Actually, in all of these cases, the nature of the noise is determined from the fact that to maintain turbulence in the flow, energy has to be supplied at large length scales near the boundaries, and the consequent Kolmogorov-type dimensional argument brings about a spatial dependence in the noise correlation.

Starting with the nonlocal equation proposed in [3], we have gone one step further in putting forth the effects of a nonwhite spatially correlated noise akin to that used in [4], the objective being the analysis of special features of the interaction of the long-ranged nature of KPZ-type nonlinearity with the long-ranged spatial correlation in noise. This, we hope, will be a justified step in explaining the effects arising from the nonlocal nature of the flow field, which has been observed in certain growth models with quenched noise [10,11], where convincing evidence has been found that shows that the power-law noise goes along naturally with a certain class of quenched noise models [12]. Somewhat similar explanations have been put forward by Xin-Ya Lei, *et al.* [13] in explaining the crossover phenomenon occuring in the fluid flow experiments of Rubio *et al.* [14].

In the following analysis, use is made of dynamic renormalization group (DRG) techniques in arriving at the dynamic exponents, etc. We see that, even for $d<d_c$, both weak and strong noise interacting with both the local and nonlocal natures of the nonlinearity give a range of critical exponents spanning a four-dimensional space in terms of the dimensionless interaction strengths. Finally, we reconfirm our DRG results from the self-consistent mode power counting arguments in the line developed in [15,16].

The starting point of our analysis is the equation

$$\frac{\partial h(\vec{r},t)}{\partial t}=\nu\nabla^2 h(\vec{r},t)+\frac{1}{2}\int d\vec{r}\,'v(\vec{r}\,')\vec{\nabla}h(\vec{r}+\vec{r}\,',t),\ \vec{\nabla}h(\vec{r}-\vec{r}\,',t)+\eta(\vec{r},t), \tag{1}$$

where ν is the diffusion constant and $\eta(\vec{r},t)$ is the noise defined by

$$\langle\eta(\vec{k},\omega)\eta(\vec{k}\,',\omega')\rangle=2D(\vec{k})\,\delta^d(\vec{k}+\vec{k}\,')\,\delta(\omega+\omega'). \tag{2}$$

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Again, going by the prescription of [3], the kernel $v(\vec{r})$ has a short-ranged part $\lambda_0 \delta^d(\vec{r})$ and a long-ranged part $r^{\rho-d}$. In Fourier space,

$$v(\vec{k})=\lambda_0+\lambda_\rho k^{-\rho}. \tag{3}$$

All standard KPZ results are expected for $\lambda_\rho=0$. But the nontrivial $\lambda_\rho\neq 0$ is the part that we are interested in.

The two exponents of interest, the roughness exponent α and the dynamic exponent z, go along with the two-point height correlation function in the hydrodynamic limit $(\vec{k},\omega)\to 0$,

$$\langle h(\vec{k},\omega)h(\vec{k}',\omega')\rangle \sim |\vec{k}|^{z-d-2\alpha}\delta^d(\vec{k}+\vec{k}')\,\delta(\omega+\omega')\times f\left(\frac{\omega}{|\vec{k}|^z}\right). \tag{4}$$

All information regarding the dynamic universality class of the phase will be contained in this α and z.

At $d=1$, the values of $\alpha(=\frac{1}{2})$ and $z(=\frac{3}{2})$ can be exactly determined. But at $d=2$, there is a transition from the Gaussian fixed point (EW), and the nonlinearity grows under rescaling. Simple scaling from $\vec{x}\to b\vec{x}$, $h\to b^\alpha h$, and $t\to b^z t$ shows that both the short $C_{\rm SR}$ and long-ranged $C_{\rm LR}$ contributions in the interaction kernel are relevant for $d<2$ (where by $C_{\rm SR}$ interaction we mean the standard KPZ-type nonlinearity, and the $C_{\rm LR}$ interaction implies a non-KPZ type $\vec{r}$ dependent part). Under this scale transformation the parameters of the equation change by $\nu\to b^{z-2}\nu,\lambda_0\to b^{\alpha+z-2}\lambda_0,\lambda_\rho\to b^{\alpha+z+\rho-2}\lambda_\rho$. If the noise strength $D(\vec{k})$ in the hydrodynamic limit $(\vec{k},\omega\to 0)$ is given by $D(\vec{k})=D_0+D_\sigma k^{-2\sigma}$, then $D_0\to b^{-d-2\alpha+z}D_0$ and $D_\sigma\to b^{-4\sigma-d-2\alpha+z}D_\sigma$. For $2<d<d_c=2+2\rho-4\sigma$, λ_ρ is relevant at the EW fixed point $(z=2)$ and, for $\rho>0$, a non-KPZ fixed spectrum should be the outcome. The following DRG analysis gives a horizon of unfounded results that shrink to the known results in $d=1$ and 2 as in [3] for $\sigma=0$ and, furthermore, the introduction of the nonlocal noise provides even more complexity in the interaction.

Due to the Galilean invariance of Eq. (1), λ_0 is not renormalized. Since the RG transformation is analytic in nature, λ_ρ is also not renormalized; only the Galilean identity is modified in this case ($2-\rho$ instead of 2),

$$\alpha+z=2-\rho, \tag{5}$$

where $\rho=0$ for λ_0 flow. From the above considerations, we get the following flow equations for ν, D's, and λ's,

$$\frac{d\nu}{dl}=\nu\left[z-2-K_d\frac{D(1)v(2)v(1)}{\nu^3}\;\frac{d-2+f(1)+3g(1)}{4d}\right], \tag{6}$$

$$\frac{dD(k)}{dl}=D(k)[z-2\alpha-d-f(k)]+K_d\frac{D^2(1)}{4\nu^3}\;v^2(2), \tag{7}$$

$$\frac{d\lambda_x}{dl}=\lambda_x\;[\alpha+z-2+x], \tag{8}$$

where $x=0$ or ρ, respectively. Here $f(q)=[\partial lnD(k)]/\partial lnk|_{k=q}$ and $g(q)=[\partial ln v(k)]/\partial lnk|_{k=q}$, $K_d=S_d/(2\pi)^d$, where S_d represents the d-dimensional surface of a unit $d+1$-dimensional sphere.

In terms of the dimensionless interaction strengths $U_{0,\sigma}^2=K_d(\lambda_0^2 D_\sigma)/\nu^3$, where short-ranged interaction couples with long-ranged noise (the Medina *et al.* zone), and similar other parameters $U_{\rho,\sigma}^2=K_d(\lambda_\rho^2 D_\sigma)/\nu^3$, $U_{0,0}^2=K_d(\lambda_0^2 D_0)/\nu^3$ (ordinary KPZ case), $U_{\rho,0}^2=K_d(\lambda_\rho^2 D_0)/\nu^3$ (the SM and SMB zone), the flow equations can be combined to give

$$\begin{aligned}\frac{dU_{0,\sigma}}{dl}=&\left[\frac{2-d+2\sigma}{2}\right]U_{0,\sigma}+3\left[\frac{d-2-2\sigma}{8d}\right]U_{0,\sigma}^3+\frac{U_{0,\sigma}}{8d}\\&\times[3(d-2)U_{0,0}^2+3.2^{-\rho}(d-2-3\rho)U_{\rho,0}^2\\&+3(1+2^{-\rho})(d-2)U_{0,0}U_{\rho,0}+3.2^{-\rho}\\&\times(d-2-2\sigma-3\rho)U_{\rho,\sigma}^2+3(1+2^{-\rho})\\&\times(d-2-2\sigma)U_{0,\sigma}U_{\rho,\sigma}],\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{dU_{\rho,\sigma}}{dl}=&\left[\frac{2-d+2\sigma+2\rho}{2}\right]U_{\rho,\sigma}+3.2^{-\rho}\left[\frac{d-2-2\sigma-3\rho}{8d}\right]\\&\times U_{\rho,\sigma}^3+\frac{U_{\rho,\sigma}}{8d}[3(d-2)U_{0,0}^2+3.2^{-\rho}\\&\times(d-2-3\rho)U_{\rho,0}^2+3(1+2^{-\rho})(d-2)U_{0,0}U_{\rho,0}\\&+3(d-2-2\sigma)U_{0,\sigma}^2+3(1+2^{-\rho})\\&\times(d-2-2\sigma)U_{0,\sigma}U_{\rho,\sigma}],\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{dU_{0,0}}{dl}=&\left[\frac{2-d}{2}\right]U_{0,0}+\left[\frac{2d-3}{4d}\right]U_{0,0}^3+\frac{U_{0,0}}{8d}[2^{-\rho}\{(3+2^{-\rho})d\\&-6-9\rho\}U_{\rho,0}^2+\{3(1+2^{-\rho})(d-2)+d.2^{-\rho+1}\}\\&\times U_{0,0}U_{\rho,0}+3(d-2-2\sigma)U_{0,\sigma}^2+3.2^{-\rho}\\&\times(d-2-2\sigma-3\rho)U_{\rho,\sigma}^2+3(1+2^{-\rho})\\&\times(d-2-2\sigma)U_{0,\sigma}U_{\rho,\sigma}],\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{dU_{\rho,0}}{dl}=&\left[\frac{2-d+2\rho}{2}\right]U_{\rho,0}+\left[\frac{(3+2^{-\rho})d-6-9\rho}{8d}\right]U_{\rho,0}^3\\&+\frac{U_{\rho,0}}{8d}[(4d-6)U_{0,0}^2\\&+\{3(1+2^{-\rho})(d-2)+d.2^{-\rho+1}\}\\&\times U_{0,0}U_{\rho,0}+3(d-2-2\sigma)U_{0,\sigma}^2\\&+3.2^{-\rho}(d-2-2\sigma-3\rho)U_{\rho,\sigma}^2+3(1+2^{-\rho})\\&\times(d-2-2\sigma)U_{0,\sigma}U_{\rho,\sigma}].\end{aligned} \tag{12}$$

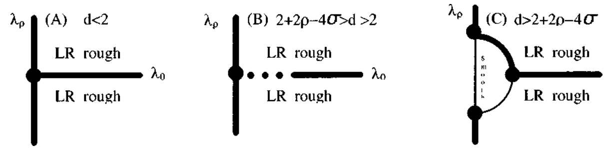

FIG. 1. λ_ρ vs λ_0 phase diagram. The solid lines along x axis represent $C_{\rm LR}$ phases, while the dotted line in B shows a smooth phase.

Now let us define two sets of parameters: $R_0 = U_{0,0}/U_{\rho,0}$, $R_\sigma = U_{0,\sigma}/U_{\rho,\sigma}$ and $S_0 = U_{0,0}/U_{0,\sigma}$, $S_\rho = U_{\rho,0}/U_{\rho,\sigma}$.

We see that $dR_y/dl = -\rho R_y$, where $y=0$, σ and consequently, this rules out any off-axial fixed point in the R_0, R_σ parameter space (except for the trivial $\sigma=0$ case).

In the $U_{0,0}$, $U_{\rho,0}$ plane, the axial fixed points are given by

$$C_{\rm SR}-N_{\rm SR} \equiv U^{*2}{}_{\rho,0}=0,$$

$$U^{*2}{}_{0,0}=\frac{2d(d-2)}{2d-3},$$

$$\alpha+z=2, \tag{13}$$

$$C_{\rm LR}-N_{\rm SR} \equiv U^{*2}_{0,0}=0,$$

$$U^{*2}_{\rho,0}=\frac{4(d-2-2\rho)}{2^{-\rho}\{(3+2^{-\rho})d-6-9\rho\}},$$

$$\alpha+z=2-\rho, \tag{14}$$

where $N_{\rm SR}$ and $N_{\rm LR}$ represent the short- and long-ranged part in the noise spectrum, respectively. The first set ($C_{\rm SR} - N_{\rm SR}$) gives the well-known KPZ fixed point with $z = 3/2$, $\alpha = 1/2$ for $d=1$. But the second set ($C_{\rm LR} - N_{\rm SR}$) gives the non-KPZ behavior and the results exactly match [3] in this $U_{0,0}$, $U_{\rho,0}$ plane:

$$z|_{U^{*2}_{0,0}=0}=2+\frac{(d-2-2\rho)(d-2-3\rho)}{[(3+2^{-\rho})-6-9\rho]},$$

$$\alpha|_{U^{*2}_{0,0}=0}=-\rho-\frac{(d-2-2\rho)(d-2-3\rho)}{[(3+2^{-\rho})-6-9\rho]}. \tag{15}$$

In diagram B of Fig. 1, the dotted line gives an unstable zone between $d=(9\rho+6)/(3+2^{-\rho})$ to $d=2+2\rho$. Above the critical dimension, diagram C shows a smooth phase. For $\rho=0$, all of the $C_{\rm LR}$ fixed points go over to the $C_{\rm SR}$ ones.

In the $U_{0,\sigma}$, $U_{\rho,\sigma}$ plane defined by R_σ, there also are only two sets of axial fixed points:

$$C_{\rm SR}-N_{\rm LR} \equiv \; U^{*2}_{\rho,\sigma}=0, \; U^{*2}_{0,\sigma}=\frac{4d}{3}, \tag{16}$$

$$C_{\rm LR}-N_{\rm LR} \equiv \; U^{*2}_{0,\sigma}=0, U^{*2}_{\rho,\sigma}=\frac{4d(2-d+2\sigma+2\rho)}{3.2^{-\rho}(2+2\sigma+3\rho-d)}. \tag{17}$$

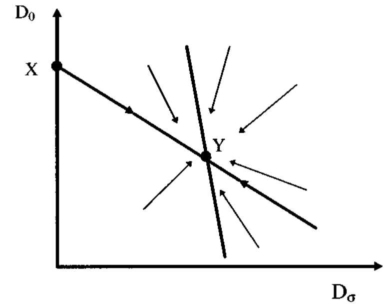

FIG. 2. D_0 vs D_σ flow in the $U_{0,0}$ vs $U_{0,\sigma}$ plane. X and Y are the axial and off-axial fixed points, respectively.

In this case, the phase diagrams are exactly identical to Fig. 1, only the critical dimension is now modified to $d_c = 2+2\rho-4\sigma$ and the unstable zone now lies between $d = 2+2\sigma+2\rho$ and $d=2+2\sigma+3\rho$ in this plane:

$$z|_{U^{*2}_{0,\sigma}=0}=\frac{1}{3}\;(4+d-2\sigma-2\rho),$$

$$\alpha|_{U^{*2}_{0,\sigma}=0}=-\rho+\frac{1}{3}\;(2-d+2\sigma+2\rho). \tag{18}$$

However, a completely different qualitative behavior is observed with S_0 and S_σ. The S_0 flow equation reads $dS_0/dl = S_0(-\sigma+\frac{1}{8}U^2_{0,0}+2^{-2\rho}U^2_{\rho,0}+2^{-\rho}/4U_{0,0}U_{\rho,0}$ and, as such, off-axial fixed points exist in this case for the four-dimensional space of U's. But in the $U_{0,0}$, $U_{0,\sigma}$ plane, we get only axial fixed points where now $U_{0,0}$, $U_{\rho,0}=$const. Here the axial fixed points are

$$C_{\rm SR}-N_{\rm SR} \equiv U^{*2}_{0,\sigma}=0, \; U^{*2}_{0,0}=\frac{2d(d-2)}{2d-3}, \tag{19}$$

$$C_{\rm SR}-N_{\rm LR} \equiv U^{*2}_{0,0}=0, \; U^{*2}_{0,\sigma}=\frac{4d}{3}. \tag{20}$$

This gives

$$z|_{U^{*2}_{0,0}=0}=\frac{1}{3}\;(d+4-2\sigma),$$

$$\alpha|_{U^{*2}_{0,0}=0}=\frac{1}{3}\;(2-d-2\sigma). \tag{21}$$

The results exactly match the Medina *et al.* predictions, and the phase diagram is given by Fig. 2, X and Y are the axial and off-axial fixed points, respectively. The point to be noted here is that, unlike the previous two cases, there is no unstable excluded region in the non-KPZ case.

Although similar arguments as above apply for the S_ρ flow, the most spectacular results are seen in the $U_{\rho,0}$, $U_{\rho,\sigma}$ plane where the fixed points are given by

$$C_{\rm LR}-N_{\rm SR} \equiv U^{*2}_{\rho,\sigma}=0,$$

$$U^{*2}_{\rho,0}=\frac{4d(2-d+2\rho)}{2^{-\rho}[6+9\rho-(3+2^{-\rho})]}, \tag{22}$$

$$C_{\rm LR}-N_{\rm LR} \equiv U^{*2}_{\rho,0}=0,$$

$$U^{*2}_{\rho,\sigma}=\frac{4d(2-d+2\sigma+2\rho)}{3.2^{-\rho}(2+2\sigma+3\rho-d)}. \tag{23}$$

Here for both the sets we have unstable regions bounded by $2+2\rho>d>(6+9\rho)/(3+2^{-\rho})$ $(C_{\rm LR}-N_{\rm SR})$ and $2+2\sigma+2\rho<d<2+2\sigma+3\rho$ for $\rho,\sigma>0$ $(C_{\rm LR}-N_{\rm LR})$. Also, both of these fixed points give non-KPZ results:

$$z|_{U^{*2}_{\rho,0}=0}=\frac{1}{3}\ (d+4-2\sigma-2\rho) \tag{24}$$

and

$$z|_{U^{*2}_{\rho,\sigma}=0}=2+\frac{(d-2-2\rho)(d-2-3\rho)}{(3+2^{-\rho})d-6-9\rho} \tag{25}$$

and, as such, the phase diagram in $\lambda_0,\ \lambda_\rho,\ D_0,\ D_\sigma$ is actually on a four-dimensional space with both axial and non-axial fixed points.

Now use is made of self-consistent mode analysis (exact in the spherical limit) to generate the non-KPZ exponents in this complex space, where both of the $N_{\rm SR}-N_{\rm LR}$ noises are interacting with the $C_{\rm SR}-C_{\rm LR}$ nonlinearity.

Starting with the Dyson equation, given by $G^{-1}(\vec{k},\omega)=-i\omega+\nu k^2+\Sigma(\vec{k},\omega)$, where $\Sigma(\vec{k},\omega)$ is the self-energy term and following exactly the scheme adapted in [15] (with the same scaling ansatz for $\Sigma(\vec{k},\omega)$ and $D(\vec{k},\omega)$), in the limit $\omega\to 0$, taking $D(\vec{k},0)=D_\sigma\ k^{-2\sigma}$ and $v(\vec{k})=\lambda_\rho\ k^{-\rho}$, simple power counting gives

$$z=\frac{1}{3}\ (d+4-2\sigma-2\rho),$$

$$\alpha=-\rho+\frac{1}{3}\ (2-d+2\sigma+2\rho). \tag{26}$$

These values of z and α are seen to tally very well with our DRG derivations in the $U_{0,0},\ U_{\rho,0},\ U_{0,\sigma},\ U_{\rho,\sigma}$ space where, in either of the two planes, the non-KPZ values coincide with Eq. (24). However, the $U_{\rho,0},\ U_{\rho,\sigma}$ plane is special in that it provides two axial non-KPZ values, only one of which appears in the self-consistent results. The other value is actually an offshoot of [3] where the relevant $C_{\rm LR}$ nonlinearity is interacting with the $N_{\rm SR}$ part of the noise, where both the non-KPZ values of z (and α) exist, although no rough-rough phase transition apparently takes place.

In summary, we have started with a simple phenomenological equation, which incorporates a nonlocal term in its interaction spectrum and, while coupling with spatially correlated noise, develops a set of dynamic and growth exponents, which contain a non-KPZ part. As the $C_{\rm SR}$ part in the spectrum becomes unstable, these non-KPZ domains surface, and a completely different critical behavior comes into existence. With negative values for the long- and short-ranged nonlinearities, the phase diagrams are modified, with the $C_{\rm LR}$ roughness now giving way to $C_{\rm SR}$ roughness without the appearance of any excluded instability in the phases. The only exception appears in the case of the special plane already discussed. Also, it would be worthwhile to mention that although the nonlocal contribution in the nonlinearity never generates its short-ranged counterpart, the nonlocal part in the noise spectrum develops a white noise. We reconfirm all of these DRG observations from a self-consistent technique and arrive at the same set of non-KPZ exponents when the noise strength remains nonrenormalized. Finally, comparisons with established results [3,4], which constitute only parts of our whole domain, provide expected results.

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