Real Analysis Exchange

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ABSOLUTE CONTINUITY IN PARTIAL DIFFERENTIAL EQUATIONS

Abstract

In this note we study a function which frequently appears in partial differential equations. We prove that this function is absolutely continuous, hence it can be written as a definite integral. As a result we obtain some estimates regarding solutions of the Hamilton-Jacobi systems.

1 Introduction

Let H be a differential operator of order $m \in \mathbb{N}$ and let $f \in L^p(D)$ be a positive function, where $p \in (1, \infty)$ and D is a smooth bounded domain in $\mathbb{R}^n$. Consider the equation:

$$H(u) = f, \quad \text{in } D \tag{1}$$

A function $u \in W^{m,p}(D) \cap C(\overline{D})$ is called a strong solution of (1) provided that $H(u) = f$ almost everywhere (a. e.) in D. We assume the operator H satisfies the following condition:

$$\text{For any } u \in W^m(D) \text{ and } \gamma \in \mathbb{R}: \; H(u) = 0 \text{ a. e. in } E_\gamma := \{x \in D \mid u(x) = \gamma\} \tag{P}$$

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For a measurable function $h : D \to \mathbb{R}$, the distribution function of h, denoted $\lambda_h(\alpha)$, is defined as follows:

$$\lambda_h(\alpha) := |\{x \in D \mid h(x) \geq \alpha\}| \equiv |\{h \geq \alpha\}|, \quad (\forall \alpha \in \mathbb{R})$$

where $|\cdot|$ denotes the n-dimensional Lebesgue measure. Clearly λ_h is decreasing, and if h is continuous, then λ_h will be strictly decreasing. Moreover, in case the graph of h has no significant flat sections (i. e. $\forall \gamma \in \mathbb{R} : |\{h = \gamma\}| = 0$) then λ_h will be continuous. The decreasing rearrangement of h, denoted $h^*(s)$, is defined as follows:

$$\begin{cases} h^* : [0, |D|] \to \mathbb{R} \\ h^*(s) = \inf\{\alpha \mid \lambda_h(\alpha) \leq s\} \end{cases}$$

Note that when h is continuous and its graph has no significant flat sections then:

$$\lambda_h \circ h^*(s) = s \quad \text{and} \quad h^* \circ \lambda_h(\alpha) = \alpha.$$

We also need to recall some background from rearrangements of functions. Given $g_0 : D \subseteq \mathbb{R}^n \to \mathbb{R}$, the rearrangement class generated by g_0, denoted $\mathcal{R}(g_0)$, is the set of functions $g : D \to \mathbb{R}$ such that $\lambda_g(\alpha) = \lambda_{g_0}(\alpha)$, for every real α. In case $g_0 \in L^p(D)$ then $\mathcal{R}(g_0) \subseteq L^p(D)$, and $\forall g \in \mathcal{R}(g_0) : \|g\|_p = \|g_0\|_p$. The weak closure of $\mathcal{R}(g_0)$ in $L^p(D)$ is denoted as $\overline{\mathcal{R}(g_0)}$ which, unlike $\mathcal{R}(g_0)$, enjoys some nice properties and characterizations that are stated in the following lemma. For the proof and further reading see [3, 4, 5, 9]:

Lemma 1. *Let $g_0 \in L^p(D)$ be a non-negative function, and $\mathcal{R}(g_0)$ be the rearrangement class generated by g_0. Then:*

(1) $\overline{\mathcal{R}(g_0)}$ is convex, and weakly compact in $L^p(D)$.

(2) $\overline{\mathcal{R}(g_0)} = \overline{co\,(\mathcal{R}(g_0))}$, the closed convex hull of $\mathcal{R}(g_0)$.

(3) The following characterization stands:

$$\overline{\mathcal{R}(g_0)} = \Bigg\{ g \mid \forall s \in (0, |D|) : \int_0^s g^*(t)\,dt \leq \int_0^s g_0^*(t)\,dt, \text{ and } \int_0^{|D|} g^*(t)\,dt = \int_0^{|D|} g_0^*(t)\,dt \Bigg\}$$

The set of measure-preserving maps from D onto $[0, |D|]$ is a non-empty set (e. g. see [12, Chapter 11]) which will be denoted by $\mathcal{M}(D, [0, |D|])$. By a result attributed to Ryff [13], given $g : D \to \mathbb{R}$, there exists $\phi \in \mathcal{M}(D, [0, |D|])$ such that $g = g^* \circ \phi$ almost everywhere in D.

We now introduce the function that is the main drive behind writing this note. To this end, we assume $u \in W^{m,p}(D) \cap C(\overline{D})$ is a strong solution of (1). We are interested in the function $\xi : [0, |D|] \to \mathbb{R}$ defined by:

$$\xi(s) = \int_{\{u \geq u^*(s)\}} f(x)\, dx. \tag{2}$$

Thanks to property **(P)** on page 1, and of course the fact that f is positive, the level sets $\{u = \gamma\}$ must have zero measure, hence ξ is well-defined. This function is frequently referred to in partial differential equations, particularly when one is interested in comparing the solution of a boundary value problem to that of a symmetrized problem, the latter being readily solved. There are many references in this regard, e. g. [2, 6, 14], to mention a few. In this note we prove that ξ is absolutely continuous, hence it can be represented by a definite integral of the form $\int_0^s F(\tau) d\tau$. Then, we will prove that the integrand F composed with any measure-preserving map $\phi \in \mathcal{M}(D, [0, |D|])$ belongs to $\overline{\mathcal{R}(f)}$. Using these two results we point out a couple of applications.

Throughout this paper we use some standard notations. For example, $W^{m,p}(D)$ and $W^m(D)$ denote the usual Sobolev spaces. The space $L^p(D)$ comprises functions whose p-th powers are integrable, and the norm in this space is defined by $\|f\|_p = \left(\int_D |f|^p\, dx\right)^{1/p}$. Moreover, $C(D)$ and $C(\overline{D})$ denote the spaces of continuous functions over D and its closure $\overline{D}$, respectively, and the corresponding norm is denoted by $\|\cdot\|_\infty$. The arrow "$\to$" indicates strong convergence, whilst "$\rightharpoonup$" indicates weak convergence in spaces under discussion.

2 Main results

Our first main result is the following:

Theorem 2. *The function ξ, as defined in (2), is absolutely continuous on* $[0, |D|]$.

PROOF. Let $\epsilon > 0$, and consider a finite sequence $\{(\alpha_i, \beta_i) \mid 1 \leq i \leq N\}$ of non-overlapping subintervals of $[0, |D|]$ such that $\sum_{i=1}^N (\beta_i - \alpha_i) < \delta$, where δ is a positive number to be determined later. By setting $t(\alpha_i) = u^*(\alpha_i)$ and $t(\beta_i) = u^*(\beta_i)$ we will have:

$$\sum_{i=1}^{N} |\xi(\beta_i) - \xi(\alpha_i)| = \sum_{i=1}^{N} \left| \int_{\{t(\beta_i) < u < t(\alpha_i)\}} f(x)\, dx \right| = \int_E f(x)\, dx, \tag{3}$$

where $E = \cup_{i=1}^{N}\{x : \; u^*(\beta_i) < u(x) < u^*(\alpha_i)\}$. By applying the Hölder inequality we obtain:

$$\int_E f(x)\, dx \le |E|^{\frac{1}{q}} \; \|f\|_p, \tag{4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Note that $|E| = \sum_{i=1}^{N}(\beta_i - \alpha_i)$. This, along with (3) and (4), will give the desired result, provided that $\delta < \left(\frac{\epsilon}{\|f\|_p}\right)^q$. □

Corollary 3. *The function ξ, as defined in (2), satisfies:*

$$\xi(s) = \int_0^s F(\tau)\, d\tau, \tag{5}$$

for some integrable function F.

Proof. By Theorem 2, ξ is absolutely continuous. Hence we can apply Corollary 14 in [12], together with the fact that $\xi(0) = 0$, to deduce

$$\xi(s) = \int_0^s \xi'(\tau)\, d\tau,$$

almost everywhere in $[0, |D|]$. So by setting $F(s) = \xi'(s)$, we get the desired result. □

We now state our second main result:

Theorem 4. *Let F be the function in Corollary 3 and $\phi \in \mathcal{M}(D, [0, |D|])$. Then $F \circ \phi \in \overline{\mathcal{R}(f)}$.*

Proof. Note that $\lambda_{F\circ\phi}(\alpha) = \lambda_F(\alpha)$, for every $\alpha \in \mathbb{R}$. Thus, $(F \circ \phi)^*(s) = F^*(s)$, for almost every $s \in [0, |D|]$. Hence, in view of item (3) of Lemma 1, it suffices to prove:

(i) $\int_0^{|D|} F^*(s)\, ds = \int_0^{|D|} f^*(s)\, ds$.

(ii) $\int_0^s F^*(t)\, dt \le \int_0^s f^*(t)\, dt, \quad \forall s \in (0, |D|)$.

Proving (i) is straightforward as

$$\begin{aligned}\int_0^{|D|} F^*(t)\, dt = \int_0^{|D|} F(t)\, dt = \xi(|D|)\\ = \int_{\{u \ge t(|D|)\}} f\, dx = \int_{\{u\ge 0\}} f\, dx = \int_D f\, dx = \int_0^{|D|} f^*(t)\, dt,\end{aligned}$$

where we have used Corollary 3.

To prove (ii), we consider the following steps:

Step 1. Let $\mathcal{U}$ be an open subset of $(0, |D|)$. Then, we can write $\mathcal{U} = \cup_{i=1}^{\infty}(A_i, B_i)$, where (A_i, B_i) are mutually disjoint. Hence,

$$\begin{aligned}\int_{\mathcal{U}} F(\tau)d\tau &= \sum_{i=1}^{\infty}\int_{A_i}^{B_i} F(\tau)d\tau = \sum_{i=1}^{\infty}\left(\int_0^{B_i} F(\tau)d\tau - \int_0^{A_i} F(\tau)d\tau\right)\\ &= \sum_{i=1}^{\infty}\left(\int_{\{u\geq t(B_i)\}} f\,dx - \int_{\{u\geq t(A_i)\}} f\,dx\right) = \sum_{i=1}^{\infty}\int_{\{t(B_i)\leq u<t(A_i)\}} f\,dx\\ &= \int_{\bigcup\{t(B_i)\leq u<t(A_i)\}} f\,dx \leq \int_0^{|\bigcup\{t(B_i)\leq u<t(A_i)\}|} f^*(s)\,ds\\ &= \int_0^{\sum(B_i-A_i)} f^*(s)\,ds = \int_0^{|\mathcal{U}|} f^*(s)\,ds.\end{aligned}$$

Step 2. Let $\mathcal{V}$ be a measurable subset of $(0, |D|)$, and $\epsilon > 0$. By Theorem 3.6 in [15], there exists an open set G containing $\mathcal{V}$ such that $|G \setminus \mathcal{V}| < \epsilon$. Whence

$$\begin{aligned}\int_{\mathcal{V}} F(t)\,dt \leq \int_G F(t)\,dt &\leq \int_0^{|G|} f^*(s)\,ds\\ &= \int_0^{|\mathcal{V}|} f^*(s)\,ds + \int_{|\mathcal{V}|}^{|G|} f^*(s)\,ds \qquad (6)\\ &\leq \int_0^{|\mathcal{V}|} f^*(s)\,ds + \|f\|_p(|G| - |\mathcal{V}|)^{1/q},\end{aligned}$$

where we have used Step 1, and Hölder's inequality. Since $|G| - |\mathcal{V}| = |G \setminus \mathcal{V}| < \epsilon$, from (6) we infer

$$\int_{\mathcal{V}} F(t)\,dt \leq \int_0^{|\mathcal{V}|} f^*(s)\,ds + \epsilon^{1/q}\|f\|_p. \qquad (7)$$

Since ϵ is arbitrary, (7) implies

$$\int_{\mathcal{V}} F(t)\,dt \leq \int_0^{|\mathcal{V}|} f^*(s)\,ds.$$

Step 3. We recall the following maximization from [1]:

$$\sup_{\{\omega\subseteq[0,|D|]:|\omega|=\gamma\}}\int_{\omega} F(t)\,dt = \int_0^{|\omega|} F^*(s)\,ds.$$

Now, fix $s \in (0, |D|)$, and apply Step 3 to obtain

$$\sup_{\{\omega\subseteq[0,|D|]:|\omega|=s\}}\int_{\omega} F(t)\,dt = \int_0^{s} F^*(t)\,dt. \qquad (8)$$

On the other hand, from Step 2, we have:

$$\int_{\omega} F(t)\,dt \leq \int_0^{|\omega|} f^*(s)\,ds. \tag{9}$$

From (8) and (9) we deduce

$$\int_0^s F^*(t)\,dt \leq \int_0^s f^*(t)\,dt,$$

as desired. □

Corollary 5. *Suppose the hypotheses of Theorem 4 hold. Then, there exists a sequence of functions* $\{F_n\}$ *such that* $F_n^*(s) = f^*(s)$ *and* $F_n \rightharpoonup F$ *in* $L^p(0, |D|)$.

Proof. By Ryff's result, $f = f^* \circ \phi$, for some $\phi \in \mathcal{M}(D, [0, |D|])$. From Theorem 4, we infer $F \circ \phi \in \overline{\mathcal{R}(f)}$. So, there exists a sequence $\{f_n\} \subseteq \mathcal{R}(f)$ such that $f_n \rightharpoonup F \circ \phi$ in $L^p(D)$. Therefore, $f_n \circ \phi^{-1} \rightharpoonup F$ in $L^p(0, |D|)$. Clearly, $\lambda_{f_n \circ \phi^{-1}}(\alpha) = \lambda_f(\alpha)$, so $(f_n \circ \phi^{-1})^*(s) = f^*(s)$. This completes the proof. □

3 Applications

In this section we will present a couple of applications of the results of the previous section. Throughout we will assume the extra condition $f \in C(\overline{D})$. Let us consider the following Hamilton-Jacobi system:

$$\begin{cases} |\nabla u| = f(x), & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{10}$$

Lemma 6. *The system (10) has a strong positive solution* $u \in W^{1,\infty}(D)$.

Proof. From [10] we know that the system (10) has a strong solution $u \in W^{1,\infty}(D)$. Replacing u by $|u| \in W^{1,\infty}(D)$ if necessary, taking into account that $|\nabla(|u|)| = |\nabla u|$, we can assume u is non-negative. On the other hand, since f is positive, we can apply Lemma 7.7 in [7], to ensure that the level sets $\{u = \gamma\}$ have zero measure. Thus, u is essentially positive, as desired. □

Remark 1. For f and u as in Lemma 6, the function:

$$\xi(s) = \int_{\{u \geq t\}} f(x)\,dx, \quad \text{(where } s = \lambda_u(t)\text{)},$$

is well defined. As a result, the function F from Corollary 3 is also well defined. Moreover, the conclusions of Theorem 2 and Theorem 4 hold.

Our first application is as follows:

Theorem 7. *Let $u \in W^{1,\infty}(D)$ be a strong positive solution of the Hamilton-Jacobi system (10) and let v be the unique solution of the following system:*

$$\begin{cases} |\nabla Z| = F(\omega_n |x|^n), & \text{in } B \\ Z = 0, & \text{on } \partial B, \end{cases} \tag{11}$$

in which:

- *B is the ball centred at the origin with radius $(|D|/\omega_n)^{1/n}$, and ω_n indicates the volume of the unit n-dimensional ball.*
- *The function F is as in Corollary 3, which is well defined by Remark 1.*

Also, let $u^\sharp(x) \equiv u^*(\omega_n|x|^n)$, which in the literature is referred to as the Schwarz symmetrization of u. Then, $u^\sharp(x) \leq v(x)$, for $x \in B$.

PROOF. The proof is a consequence of Corollary 3, along the same lines as in the proof of Lemma 2.2 in [6]. □

Example 1. Choosing $f(x) = 1$ in Theorem 7 yields $F(t) = 1$. Thus, the conclusion of Theorem 7 states:

$$u^\sharp(x) \leq v(x) = R - |x|, \qquad x \in B,$$

where $R = (|D|/\omega_n)^{1/n}$. This estimate can be obtained directly as follows:

$$\begin{aligned} \lambda_u(t) &= \int_{\{u \geq t\}} dx = \int_{\{u \geq t\}} |\nabla u|\, dx \\ &= \int_t^{\|u\|_\infty} \left(\int_{\{u=\tau\}} dH^{n-1} \right) d\tau = \int_t^{\|u\|_\infty} P(\{u \geq \tau\})\, d\tau, \end{aligned} \tag{12}$$

where we have used the co-area formula (e. g. see [11]). Here, $P(E)$ stands for the perimeter of E in the sense of De Giorgi. By differentiating (12), and applying the classical Isoperimetric Inequality (e. g. see [8]), we derive:

$$\lambda_u'(t) = -P(\{u \geq t\}) \leq -n\omega_n^{\frac{1}{n}} \lambda_u^{1-\frac{1}{n}}(t).$$

Thus, we obtain:

$$1 \leq -\frac{\lambda_u'(t)}{n\omega_n^{\frac{1}{n}} \lambda_u^{1-\frac{1}{n}}(t)}. \tag{13}$$

Integrating (13) from 0 to t leads to:

$$\begin{aligned} t \le -\frac{1}{n\omega_n^{1/n}} \int_0^t \frac{\lambda_u'(\tau)}{\lambda_u^{1-\frac{1}{n}}(\tau)}\, d\tau &= -\frac{1}{n\omega_n^{1/n}} \int_{|D|}^{\lambda_u(t)} \frac{ds}{s^{1-\frac{1}{n}}} \\ &= \frac{1}{\omega_n^{1/n}}(|D|^{1/n} - \lambda_u^{1/n}(t)) = R - \left(\frac{\lambda_u(t)}{\omega_n}\right)^{1/n}. \end{aligned} \tag{14}$$

By letting $t = u^*(\omega_n |x|^n)$ in (14), and recalling $\lambda_u(u^*(\omega_n |x|^n)) = \omega_n |x|^n$, we obtain $u^\sharp(x) \le R - |x|$ for $x \in B$, as expected.

The second application is stated in the following Theorem:

Theorem 8. *Let u be as in Theorem 7. Then*

$$\|u\|_\infty \le C|D|^{1/n}\|f\|_\infty.$$

PROOF. The proof is a consequence of Corollary 5, along the same lines as in the proof of Corollary 2.1 in [6]. □

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