54 *Network Science* **6** (1): 54–70, 2018.
doi:10.1017/nws.2017.25

A new design principle of robust onion-like networks self-organized in growth

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Abstract

Today's economy, production activity, and our life are sustained by social and technological network infrastructures, while new threats of network attacks by destructing loops have been found recently in network science. We inversely take into account the weakness, and propose a new design principle for incrementally growing robust networks. The networks are self-organized by enhancing interwoven long loops. In particular, we consider the range-limited approximation of linking by intermediations in a few hops, and show the strong robustness in the growth without degrading efficiency of paths. Moreover, we demonstrate that the tolerance of connectivity is reformable even from extremely vulnerable real networks according to our proposed growing process with some investment. These results may indicate a prospective direction to the future growth of our network infrastructures.

Keywords: *coexistence of efficiency and robustness, onion-like structure, long-distance relations, interwoven loops, unselfish self-organization*

1 Introduction

Social and technological networks for communication, collaboration, trading, travel, or supply chain become more and more important, since their systems support our daily life and economy. The connections between nodes facilitate information deliveries, physical logistics, and energy supplies. Moreover, through some intermediations, the connections sometimes lead to new business chances, acquaintanceship, or remote control of the infrastructures efficiently. Some case studies in organization theory such as the rapid recovery of Toyota group's supply chain from a large fire accident of their subcontract plants (Nishiguchi & Beaudet, 1998; Nishiguchi, 2007), world-wide economic networks with expanding business chances by Wenzhou people in China (Nishiguchi, 2007), and the brain circulation system known as Silicon Valley (SV) model for developing innovational high-tech industry with market opportunities by immigrant engineers (Saxenian, 2007) have suggested the importance of *long-distance relations* for both robustness of connectivity and efficiency of path in a network. The established connections via intermediations probably work well for managing cross-border operations.

On the other hand, many social, technological, and biological infrastructural networks have a common scale-free (SF) structure (Barabási et al., 1999) generated by the selfish preferential attachment referred to as *rich-get-richer* rule in consciously/unconsciously considering efficiency of paths between two nodes connected within a few hops. The SF networks also have an extreme vulnerability against intentional attacks (Albert et al., 2000). However, in these several years by percolation analyses, it has been clarified that onion-like topological structure with positive degree–degree correlations gives the optimal robustness even for the attacks in SF networks (Schneider et al., 2011; Tanizawa et al., 2012). Based on a natural but unselfish rule, onion-like networks can be incrementally grown by applying cooperative partial copying and adding shortcut (Hayashi, 2014, 2016a) instead of the expensive whole rewiring (Wu & Holme, 2011) or hierarchically expanding outer ring (Sampaio Filho et al., 2015) for enhancing the positive degree–degree correlations. One of the drawback is that the robustness is weak in early stage of the growth (Hayashi, 2016a). While none of incremental generation of networks has been so far based on interwoven loops, new threats of network attacks by destructing loops have been found recently (Morone & Makse, 2015; Mugisha & Zhou, 2016). They give severer damage than the conventional intentional attacks (Albert et al., 2000), and can be easily performed. One is Collective Influence (CI) attack (Morone & Makse, 2015) considered for a global optimization to identifying the most influence nodes called influencers in information spreading. Another is Belief Propagation (BP) attack (Mugisha & Zhou, 2016) derived from a message-passing approximation algorithm rooted by the spin glass model in statistical physics for the Feedback Vertex Set (FVS) problem in belonging to NP-hard (Karp, 1972; Kempe et al., 2003).

Inversely taking into account the weakness caused by the CI and BP attacks, we propose a new design principle for generating robust onion-like networks in focusing on enhancing of long loops, whose key factor is long-distance relation inspired from the organization theory (Nishiguchi, 2007). Furthermore, we consider a practical approximation of the network generation with moderately long loops, which is based on range-limited intermediations for finding linked nodes without large costs or efforts in the growth.

2 Self-organized growing network by a pair of attachments

We propose a self-organized growing network by enhancing long loops. After explaining the basic model, we consider the realistic range-limited approximation in Section 2.1. We estimate the degree distribution in our proposed network in Section 2.2.

2.1 Basic model and the practical approximation

We explain a basic model of self-organized growing network by enhancing long loops. At each time step of growing, a new node is added and connects to existing nodes. As the connection rule for even number m links emanated from the new node, we introduce a pair of attachments referred to as random and long distance attachments (RLD-A) or preferential and long distance attachments (PLD-A). The difference of the connection rule from that in the well-known Barabási–Albert (BA) model (Barabási et al., 1999) is a pair of attachments with long distance attachment. As shown in Figure 1(a), the following pair of attachments is repeated in $m/2$ times at each time step.

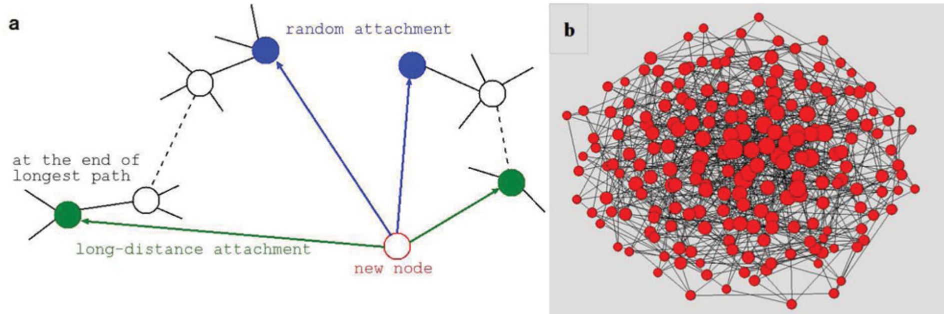

Fig. 1. Topological properties of the proposed networks. (a) In the case of $m = 4$, there are two pairs of attachments represented by green and blue lines from a new node added at each time step. The green node is at the end of the longest path represented by dashed-line in the shortest paths counted by hops from the blue node. The furthest node is easily findable by a labeling method. Once a link is generated, it is undirected. In PLD-A, the destination node of blue line is chosen with a probability proportional to its degree, instead of random selection in RLD-A. In MED, the destination node of green line is chosen in the $\mu + 1$th neighbors from the blue node, instead of the furthest node in RLD-A. (b) Example of onion-like structure by RLD-A for $m = 4$ at $N = 200$. The circle size of node is proportional to its degree. The structure is visualized at the positions as node degrees become smaller from core to peripheral. (Color online)

RLD-A: One of link destination is uniformly randomly chosen as encountering, and another link destination is the furthest node from the chosen node. When there are several candidates of the furthest with a same distance counted by hops, one of them is randomly selected. Some kind of randomness is useful to avoid fixed weak-points in the growth.

PLD-A: For the comparison with RLD-A, instead of uniformly random selection, one of the pair is a preferentially chosen node with a probability proportional to its degree (Barabási et al., 1999). Another link destination is the furthest node from the preferentially chosen node.

For attached even number m links, $m/2$ loops through the pair of nodes are created at each time step. The interwoven loops via new node are significant for $m \geqslant 4$ as shown in Sections 3 and 4. The minimum $m = 4$ is corresponded to the least effort of attachment linking to be strongly robust network in our growing method. Such connection rule in Figure 1(a) was not noticed because of the lack of emphasis on loops, but the importance of the part of long distance relation was covertly suggested in organization theory (Nishiguchi, 2007).

Moreover, since range-limited approach is useful for efficiently investigating global property of network such as influencer (Morone & Makse, 2015), centrality (Ercsey-Ravasz et al., 2012), or random percolation (Radicchi & Castellano, 2016), we apply it for generating robust networks. We consider a range-limited approximation of RLD-A as random and intermediated attachments (MED).

MED: Instead of the furthest node, we select a distant node to the extent of a few hops via intermediations from the randomly chosen pair node. Intermediations in one hop mean attachments to the 2nd neighbors of the randomly chosen node, intermediations in two hops mean ones to the 3rd neighbors, and intermediations

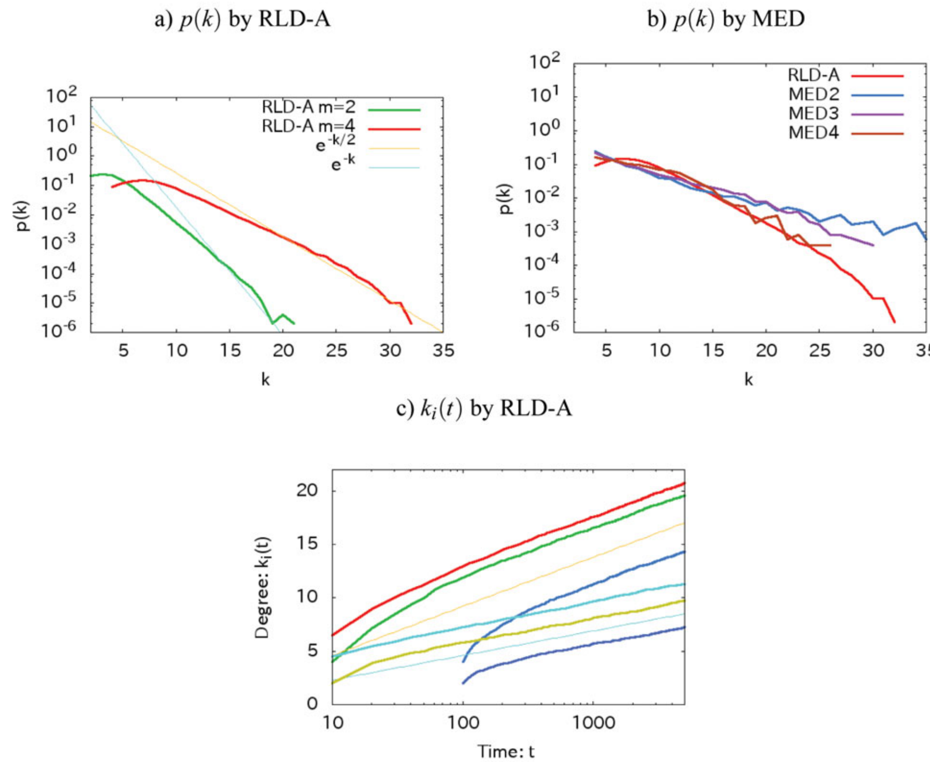

Fig. 2. Estimation of exponential tail of degree distribution. (a) Degree distribution $p(k)$ in the average over 100 samples of our networks at $N = 5,000$. Thin orange and cyan lines guide the exponential tails. (b) The cases of MED for $\mu = 2, 3, 4$. (c) Time course of degree $k_i(t)$ of node i in the average over 100 samples of our networks. The thick lines from top to bottom (red, green, light blue for $m = 4$, or cyan, yellow, blue for $m = 2$) denote $k_i(t)$ of node $i = 1, 10$, and 100 inserted at the birth times $t_i = i - m > 0$. Thin orange and cyan lines guide $O(\log(t))$. (Color online)

in μ hops mean ones to the $\mu + 1$th neighbors. When μ is small, the attachments to a few hops-th neighbors have reality without large connection costs or efforts.

If a same destination node in RLD-A, PLD-A, or MED is chosen, other selection of pair is tried due to the prohibition of multiple links between two nodes.

2.2 *Estimation of degree distribution*

We consider growing networks with a same condition of the total number $M = m(N - m) + m(m - 1)/2$ of links for size N: total number of nodes at time step $t = N - m$. As the initial configuration, we set a complete graph of $N_0 = m$ nodes and $M_0 = m(m-1)/2$ links at $t = 0$. Figure 1(b) shows onion-like structure in which older nodes form the core while younger nodes surround it. Figure 2(c) justifies that older nodes get more links. Moreover, we can derive exponential tails of degree distributions $p(k)$ by the asymptotic approximation (Hayashi, 2016b) as follows. The invariant ordering $k_n(t) < k_{n-1}(t) < \cdots < k_1(t)$ hold for large t in parallel curves in Figure 2(c). Since the time course of degree of node i follows $k_i(t) \sim \log(t)/\beta$ as a monotone increasing function of t with a constant $\beta > 0$, we obtain

$$\frac{t_i}{t} = \frac{e^{\beta k_i}}{e^{\beta k_i(t)}}$$

$$p(k_i(t) < k) = p\left(t_i > \frac{e^{\beta k_i}}{e^{\beta k}} t\right) = \left(1 - \frac{e^{\beta k_i}}{e^{\beta k}}\right) \frac{t}{N_0 + t}$$

$$p(k) = \frac{\partial p((k_i(t) < k)}{\partial k} \sim e^{-\beta k}$$

Indeed, the orange and cyan lines guide $\log(t)$ and $2\log(t)$ in Figure 2(c), the estimated same color lines of e^{-k} and $e^{-k/2}$ are fitting with the tails of $p(k)$ in Figure 2(a). The largest degree is bounded around 20–35 without heavy connection load as on hub nodes in SF structure of many real networks. Figure 2(b) shows that the range-limited cases of MED in $\mu = 2, 3, 4$ intermediations have slightly deviated but similar exponential tails of $p(k)$.

3 Strong robustness and the small-world effect

For our proposed networks, we investigate the robustness index

$$R \stackrel{\text{def}}{=} \frac{1}{N} \sum_{q=1/N}^{1} S(q)$$

where $S(q)$ denotes the number of nodes included in the giant component (GC as the largest cluster) after removing qN nodes, q is a fraction of removed nodes by High Degree Adaptive (HDA), CI for $l = 3$ layer (Morone & Makse, 2015), and BP (Mugisha & Zhou, 2016) attacks. As in appendix or Morone & Makse (2015) and Mugisha & Zhou (2016), the highest value of $CI_l(i)$ in Equation (A4) or q_i^0 in Equations (A5)–(A9) to be removed is recalculated after each node removal. Note that the maximum $R \geqslant 0$ is 0.5 in general. The following results are insensitive for varying values of inverse temperature $x = 7$ and 100 rounds of the message-passing (Mugisha & Zhou, 2016), and there is no difference for $l \geqslant 3$ in CI attacks. Figure 3(a) shows that our networks by RLD-A for $m = 4$ have strong robustness $R > 0.3$ even in the early stage of growth, while Figure 3(b) shows that R is lower in the conventional SF networks by BA model. The networks by PLD-A show the intermediate R values. In Figure 3(c) for $m = 2$, these lines fall in overall, but it is invariant that the ordering of damage by attacks is BP > CI_3 > HDA whose differences are very small. Each value of R is almost constant in the growing at least from the initial complete graph. In the range-limited cases of MED in $\mu = 2, 3, 4$ intermediations, we obtain $0.31 < R < 0.35$ and $0.28 < R < 0.34$ against HDA and BP attacks, respectively. Figure 4(a) and (b) shows the relative size $S(q)/N$ with the sudden breakdowns by BP attacks (blue lines) as mentioned in Mugisha & Zhou (2016). Each of the robustness in Figure 4(a) and (b) for $m = 4$ is improved from the corresponding one in Figure 4(c) for $m = 2$, although larger m requires more links.

We also investigate the assortativity $-1 \leqslant r \leqslant 1$ as the Pearson correlation coefficient for degrees (Newman, 2002).

$$r \stackrel{\text{def}}{=} \frac{4M \sum_e (k_e k'_e) - \left[\sum_e (k_e + k'_e)\right]^2}{2M \sum_e (k_e^2 + k'^2_e) - \left[\sum_e (k_e + k'_e)\right]^2}$$

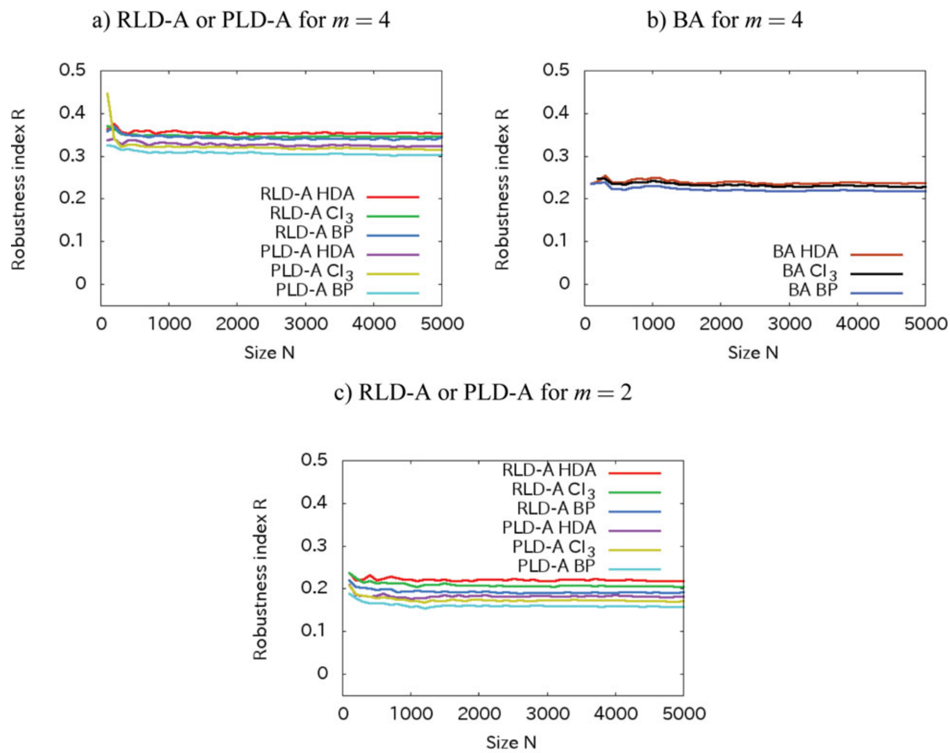

Fig. 3. Robustness in the growing networks. Robustness index R against HDA, CI_3, and BP attacks vs size N in the networks by (a) RLD-A or PLD-A, (b) BA model for $m = 4$, and (c) RLD-A or PLD-A for $m = 2$. (Color online)

where k_e and k'_e denote degrees at both end-nodes of link e and M is the total number of links. Figure 5(a) shows that our networks by RLD-A for $m = 4$ (red line) have high assortativity $r > 0.2$ as similar to the copying model (Hayashi, 2014, 2016a). However PLD-A is insufficient to create strong correlations. Figure 5(b) shows that the range-limited cases of MED in $\mu = 3, 4$ (purple and cyan lines) are close to the case of RLD-A (red line). Although there is no clear criteria for the value of r in order to be an onion-like network with necessary positive degree-degree correlations, too large r is unsuitable (Tanizawa et al., 2012). We do not discuss the optimally robust onion structure, but concern about incrementally growing proper good onion-like networks self-organized by natural and reasonable attachments. From Figures 3–5, our networks by RLD-A and MED in $\mu = 3, 4$ for $m = 4$ have onion-like structure with both high R and r, but other cases are not. Moreover, they have efficient small-world property (Watts & Strogatz, 1998): the average shortest path length is $O(\log(N))$ as shown in Figure 6, even though half links in RLD-A or MED are created by random attachment without intention to be efficiency. In the growing from the initial complete graph, the number $\mu \approx 3$ of intermediations is at the similar level of the average path length. On average, the length of simple one-round loop (as shown in Figure 1(a), it consists of the path between blue and green nodes + the corresponding blue and green links) generated by the pair of RLD-A or MED becomes short and inexpensive as $O(\log(N))$.

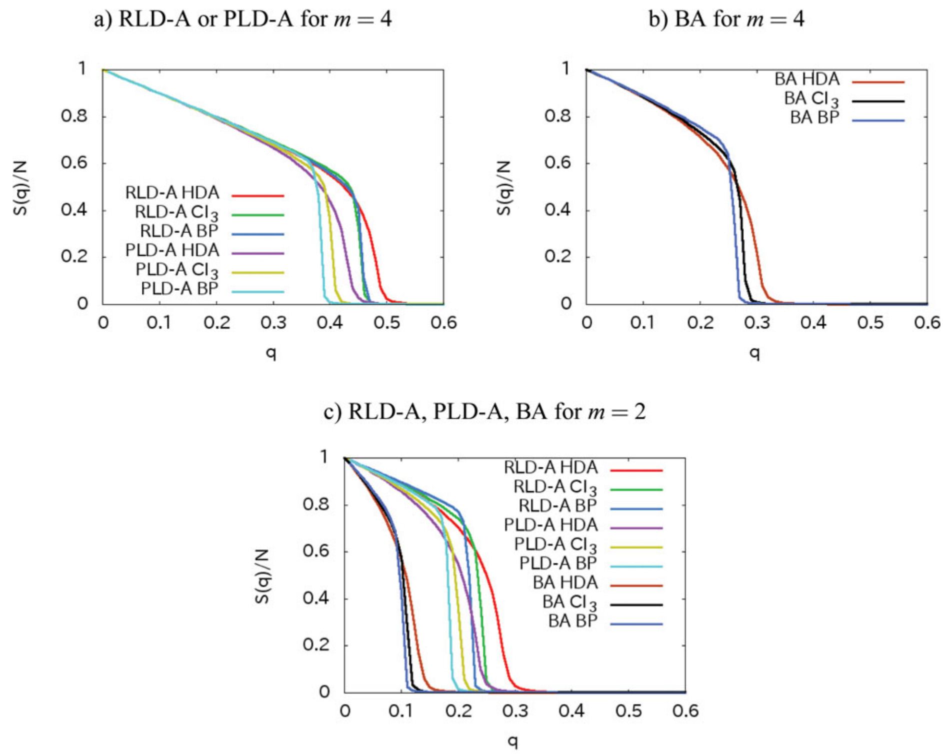

Fig. 4. Relative size $S(q)/N$ vs fraction q of removed nodes in the networks by (a) RLD-A, PLD-A, (b) BA for $m = 4$, and (c) RLD-A, PLD-A, BA for $m = 2$ at $N = 5,000$. The red curves against HDA attacks are gradually decreased, while blue ones against BP attacks are suddenly dropped. The yellow-green or black curves against CI_3 attacks are the intermediate. Note that R is defined by the area under the line of $S(q)/N$. (Color online)

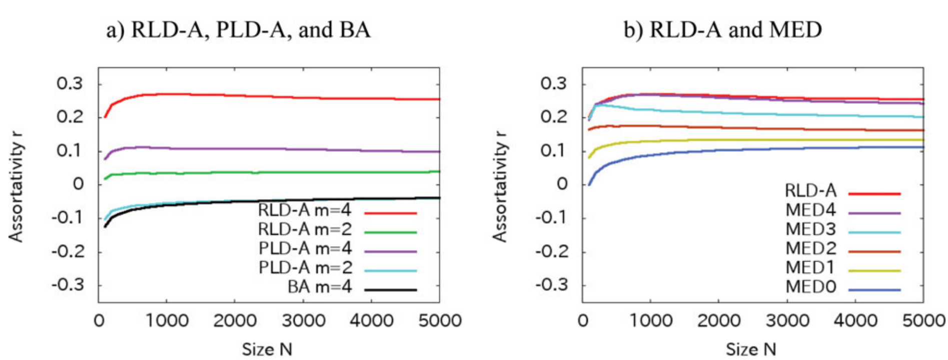

Fig. 5. Degree–degree correlations in the growing networks. Assortativity r as the measure of correlations for size N in comparison with the networks by (a) RLD-A, PLD-A, and BA model for $m = 2$ or $m = 4$, (b) RLD-A and MED0–4 for $m = 4$. MED0–4 denote the case of MED for $\mu = 0, 1, 2, 3, 4$. (Color online)

Table 1. Basic data of the real networks after converting from each of them to an undirected graph without multiple links.

Network	N_0	M_0	$\langle k \rangle = 2M_0/N_0$	Average path length	Diameter
Facebook	1,899	13,838	14.573	3.055	8
USair	1,574	17,215	21.874	3.115	8
USpower	4,941	6,594	2.669	18.989	46

USair and USpower are abbreviations of US Airport Network and US Power Grid, respectively. The average path length and diameter are defined by the averaged length of the shortest paths with the minimum number of hops between two nodes and the longest length in a network.

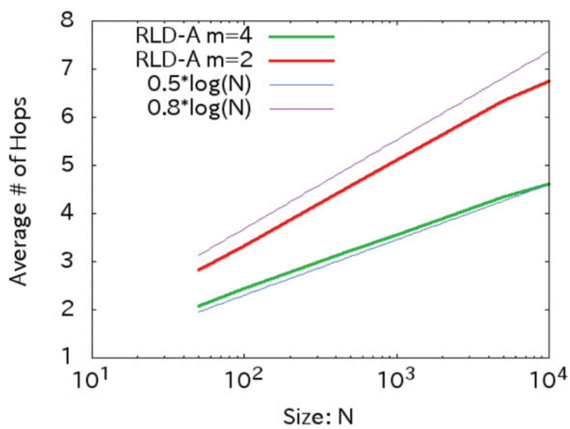

Fig. 6. Average path length on the shortest paths counted by hops in our growing networks by RLD-A. The purple and blue thin lines guide $O(\log(N))$ as the small-world effect. These results are averaged over 100 samples. (Color online)

4 Virtual test for prospective growth of real networks

As a virtual test for exploring future design of networks, we study the robustness of our model in growing to onion-like structure from the initial configuration of real networks[1] in Table 1. For these social and technological networks, long distance connections will be somewhat required in order to seek solution strategies to undeveloped relationship or inconvenience, and realizable by intermediation or investment (e.g., for low-cost carrier or innovation of power transmission) in a trade-off between the benefit and the cost.

This network design in growth is different task from healing or recovering by rewirings e.g., between second neighbors (Gallos & Fefferman, 2015; Park & Hahn, 2016) in almost constant numbers of nodes and links for a damaged network by earthquakes or terrorist attacks, etc. Because we focus on a structural change of network from almost uncorrelated SF to onion-like without hubs in the growth rather than topologically partial changes by rewirings. We also investigate the dependence of the initial network structure that is not a complete graph and the

[1] http://toreopsahl.com/datasets/

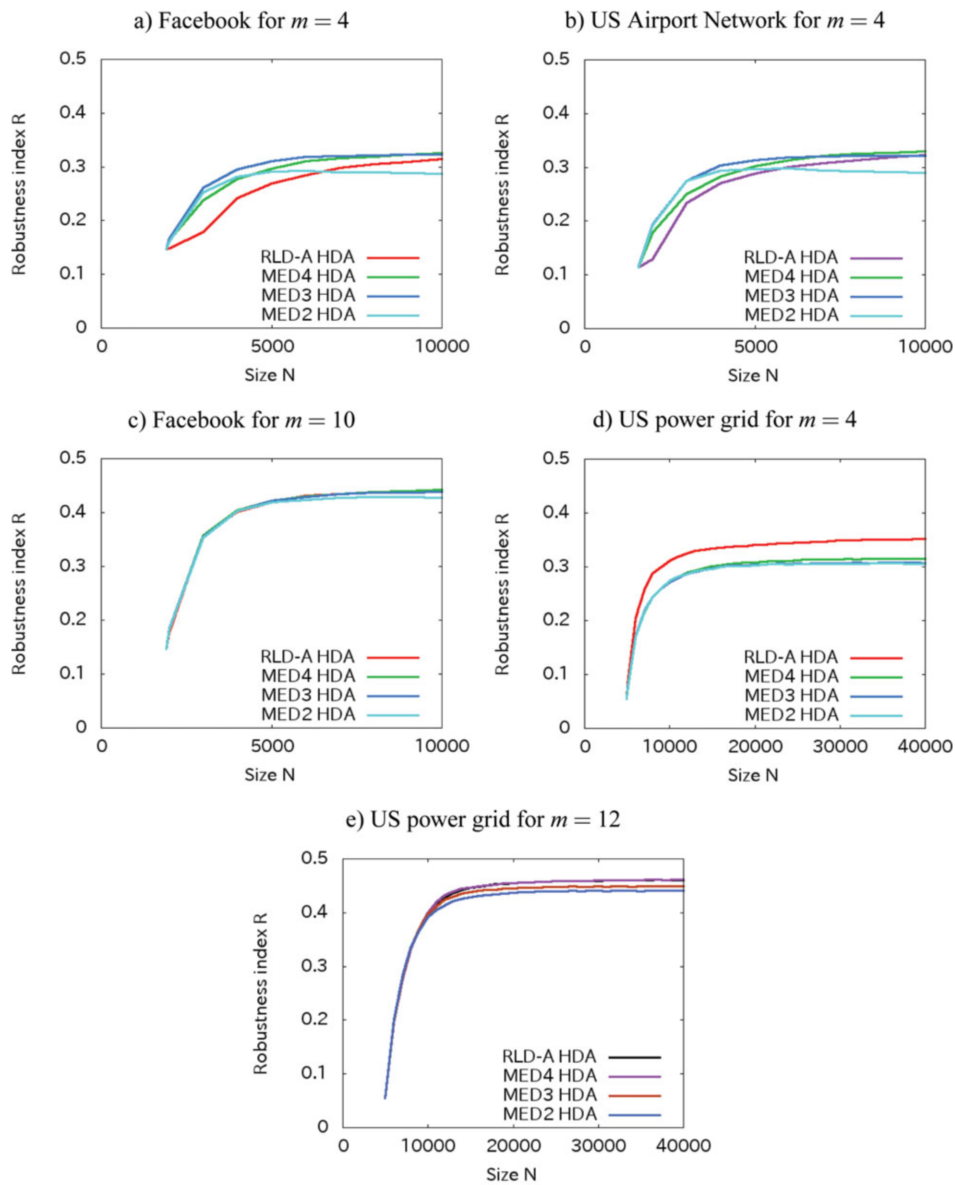

Fig. 7. Drastically improved robustness against HDA attacks in growing networks from the initial configuration of real networks. Robustness index R vs N in the growing networks from the initial (a) Facebook for $m = 4$, (b) US Airport Network for $m = 4$, (c) Facebook for $m = 10$, (d) US power grid for $m = 4$, and (e) US power grid for $m = 12$. (Color online)

initial size N_0 on the robustness and degree–degree correlations for our growing method. Of course, some investments may be required for the growing network in larger size than the initial real one; however, the virtual test will give a prospective insight.

In the following, the cases of intermediately destructive CI_3 attacks and not very effective PLD-A are omitted to simplified the discussion. Figures 7(a) and (b) and 8(a) and (b) show that high robustness against both HDA and BP attacks

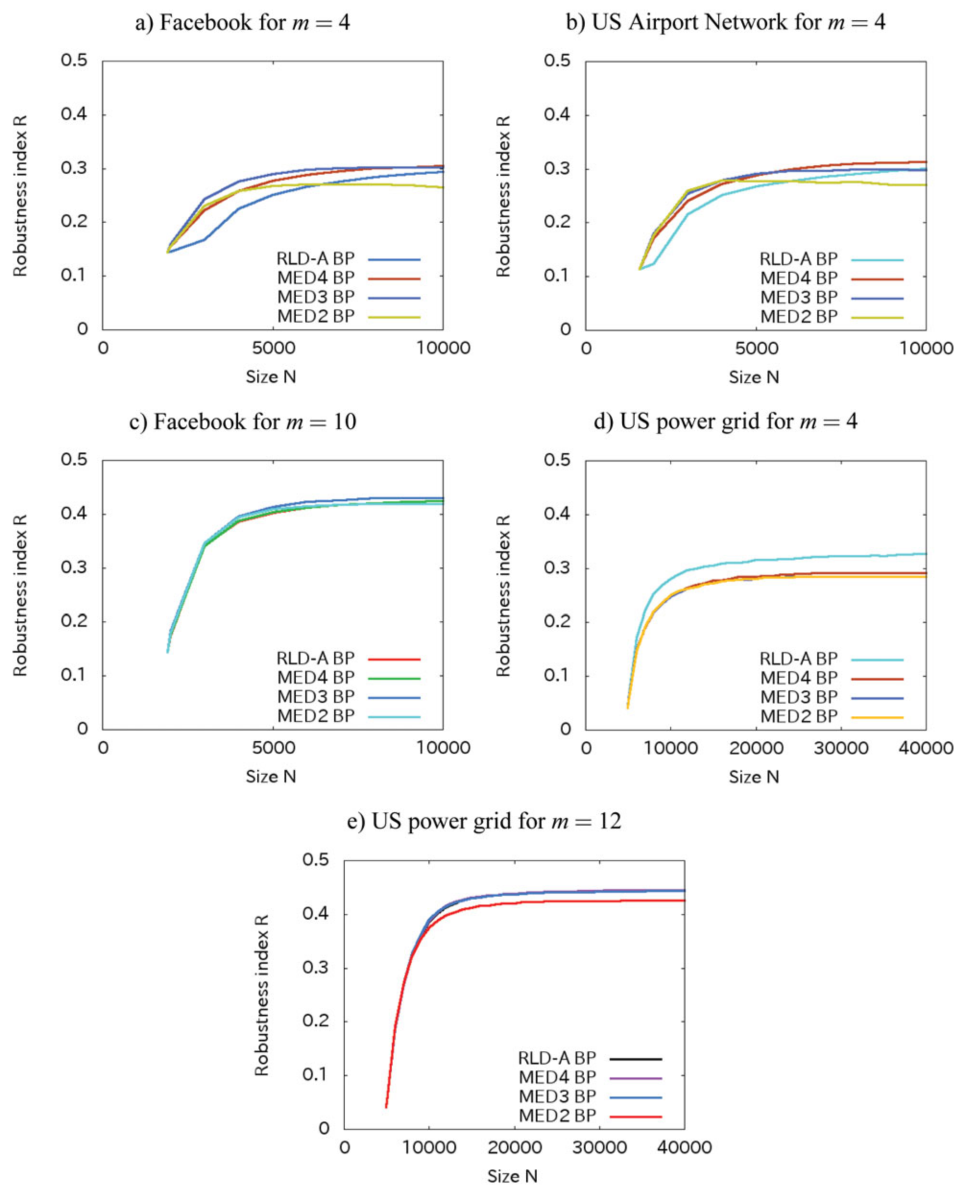

Fig. 8. Drastically improved robustness against BP attacks in growing networks from the initial configuration of real networks. Robustness index R vs N in the growing networks from the initial (a) Facebook for $m = 4$, (b) US Airport Network for $m = 4$, (c) Facebook for $m = 10$, (d) US power grid for $m = 4$, and (e) US power grid for $m = 4$ and $m = 12$. (Color online)

is obtained with increasing to $R > 0.3$ from $R \approx 0.1$ in initial vulnerable real networks of Facebook and USair. In Figures 7(c) and (e) and 8(c) and (e), the cases of $m = 10$ and $m = 12$ are investigated for checking the emergence of onion-like structure with high assortativity $r > 0.2$. We remark that some range-limited cases of MED in $\mu = 3$ intermediations (blue lines) have higher R than the cases of RLD-A with the attachments to the furthest nodes, but the effect is weak in the

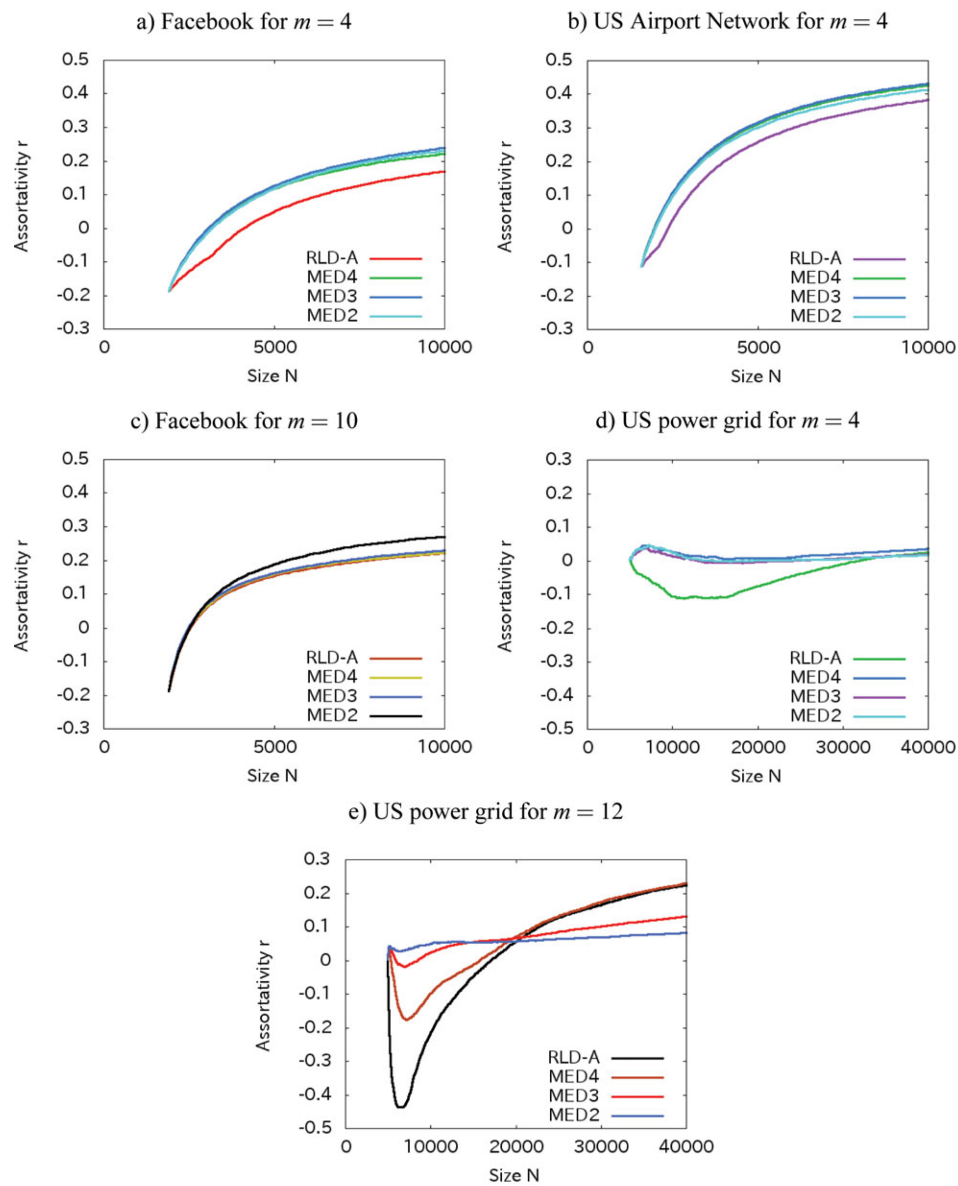

Fig. 9. Assortativity r vs N in growing networks from the initial (a) Facebook for $m = 4$, (b) US Airport Network for $m = 4$, (c) Facebook for $m = 10$, (d) US Power Grid for $m = 4$, and (e) US Power Grid for $m = 12$. (Color online)

cases of MED in $\mu = 2$ intermediations. If we do not insist onion-like networks, USpower can be already grown with high robustness before around $N \approx 10,000$ of double size of the initial as shown in Figures 7(d) and (e) and 8(d) and (e). It suggests a possibility for incrementally growing other robust networks with $r < 0$ or $r \approx 0$ (see Figure 9(d) and (e)) instead of onion-like networks. Figure 9(a)–(c) shows the increase of degree–degree correlations in increasing values of r in the growth of Facebook or USair. However, a different behavior of late increasing after

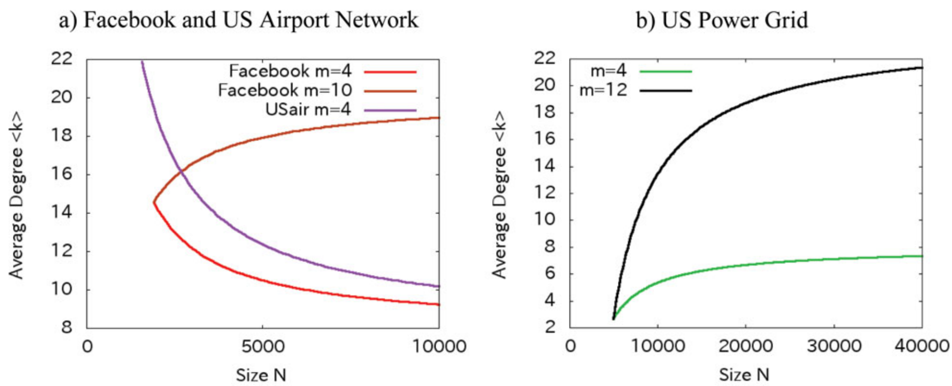

Fig. 10. Average degree $\langle k \rangle$ vs N in the growing networks from the initial (a) Facebook for $m = 4$ and $m = 10$ and US Airport Network for $m = 4$, (b) US Power Grid for $m = 4$ and $m = 12$. (Color online)

decreasing to negative correlations is obtained in USpower as shown in Figure 9(d) and (e). Some sort of structural change occurred in the growth. The reason may be from that USpower includes chain-like parts that induce large diameter, while Facebook and USair are compact with small diameter as shown in Table 1. We note that the average path length is monotonously increasing to 4.13 and 4.21 at $N = 10,000$ in Facebook and USair, but decreasing to 4.97 at $N = 40{,}000$ in USpower. These distributions of path lengths are bell-shaped with the peak around the average length for each size N. In addition, the average path length in USpower is substantially greater than the number μ of intermediations at least in the early stage of the growth as shown in Table 1.

From Figures 7–9, an onion-like structure with both high R and r emerges by RLD-A or MED for $\mu \geqslant 3$ intermediations in the growing from Facebook for $m = 10$, USair for $m = 4$, and USpower for $m = 12$. The growth to onion-like networks needs more steps as the initial size N_0 is larger compared to Facebook or USair and USpower. We remark that the effect of degree–degree correlations on the robustness works well in the early stage of the growth with structural change from real networks; however, the robustness index R is saturated in $N > 6,000$ in Figures 7(a)–(c) and 8(a)–(c) and in $N > 20,000$ in Figures 7(d) and (e) and 8(d) and (e) for increasing assortativity r with the increase of correlations in Figure 9. It is also interesting that the robustness in Figures 7 and 8 is improved in spite of decreasing average degree $\langle k \rangle$ in the growing from Facebook and USair for $m = 4$ (red and purple lines) as shown in Figure 10(a). Note that the average degree approaches to $2 \times m$ for $N \to \infty$ in Figure 10.

5 Conclusion

We have proposed a second method for incrementally growing strongly robust onion-like networks self-organized by more natural and reasonable pair of attachments than the copying model (Hayashi, 2014, 2016a). In addition, it becomes robust even in the early stage of the growth, and there is no huge hub whose largest degree is bounded. Since random attachments make an exponential degree distribution

(Barabási et al., 1999), the random ones are dominant in the tail for high degrees, while another intermediated attachments mainly work for low degrees and the positive correlations among them. In a virtual test for the growing from real networks with extreme vulnerability, we have shown that the proposed growing networks have reformable robustness to be future prospective infrastructures. It is also expected that the range-limited intermediations in a few hops reduce the Euclidean distances of links embedded on a space.

We emphasize the emergence of robust onion-like networks in enhancing moderately long loops by range-limited MED in a few hops without both degrading efficiency of paths and large connection costs or efforts. We should remember that the coexistence of robustness and efficiency has not been realized in many real networks, and the threat against attacks (Morone & Makse, 2015; Mugisha & Zhou, 2016) is never decreased rather increased more and more, unless the dependence on selfish preferential attachment (Barabási et al., 1999) is changed by ourselves. Therefore, our study suggests that we should discontinue the dependence on selfish rule and develop the potential of distant connections for the half of links, which may mean necessary investment for highly reliable connectivity in our network infrastructures even against intelligent attacks.

Acknowledgments

The author expresses appreciation to anonymous reviewers and the website noted in footnote for their valuable comments and using dataset of real networks. This research was supported in part by JSPS KAKENHI under Grant number JP.17H01729.

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Appendix

We briefly summarize the theoretical background of CI and BP.

For CI, according to Morone & Makse (2015), we consider the optimal immunization to prevent disease or information spreading. This is mapped onto the breakdown problem as minimizing the size of GC of a network. In the GC, after removing a fraction q of nodes, there are two states represented by the quantity $n_i = 1$ or $n_i = 0$: node i exists or not. The fraction q is denoted by $q = 1 - \frac{1}{N}\sum_{i=1}^{N} n_i$.

The probability $v_{i\to j}$ for information spreading is computed in self-consistency through the following message passing equations

$$v_{i\to j} = n_i \left[1 - \Pi_{k\in\partial i\setminus j}(1 - v_{k\to i})\right] \quad \text{(A1)}$$

where ∂i denotes node i's set of connecting neighbor nodes, and $\partial i \backslash j$ is the node subset obtained by removing node j from ∂i. Since the stability condition at the fixed point of origin for the iterated map of Equation (A1) is given by that the largest eigenvalue $\lambda(n;q)$ of the Jacobian matrix

$$\left. \frac{\partial v_{i \to j}}{\partial v_{k \to l}} \right|_{v_{i \to j}=0} = n_i B_{k \to l, i \to j}$$

is less than one, to find the minimum set of immunization nodes is equivalent to select the minimum set of removed nodes until satisfying $\lambda(n;q) = 1$. Here, $B_{k \to l, i \to j}$ is an element of $2M \times 2M$ non-backtracking (NB) matrix (Hashimoto, 1989) for the network. At the critical $\lambda(n;q_c) = 1$, the GC is broken.

Moreover, according to Power Method, we have

$$\lambda(n;q) = \lim_{l \to \infty} \left[\frac{|\mathbf{w}_l(n)|}{|\mathbf{w}_0|} \right]^{1/l} \tag{A2}$$

where $\mathbf{w}_l(n)$ denotes the vector at the lth iterations by multiplying the Jacobian matrix for an initial arbitrary nonzero vector $\mathbf{w}_0$. The elements of $2M$ dimensional vectors $\mathbf{w}_l(n)$ and $\mathbf{w}_0$ are for bidirectional $i \to j$ and $j \to i$ links. The denominator $|\mathbf{w}_0|$ in the right-hand side of Equation (A2) is constant and independent of the set $\{n_i\}$. The numerator is approximated by the following expression corresponding to $2l$-body problem:

$$|\mathbf{w}_l(n)|^2 \approx \sum_{i=1}^{N} (k_i - 1) \sum_{j \in \partial Ball(i, 2l-1)} \left(\Pi_{k \in P_{2l-1}(i,j)} n_k \right) (k_j - 1) \tag{A3}$$

where $P_{2l-1}(i,j)$ is the set of nodes belonging to the shortest path of length $2l - 1$ connecting i and j nodes, $\partial Ball(i,l)$ denotes the set of nodes on the frontier of the ball with radius l hops from node i, k_i and k_j denotes the degrees of nodes i and j. We remark the factor $\Pi_k n_k = 1$ in the parenthesis in the right-hand side of Equation (A3) when all nodes on the shortest path are not absent, the path is connected. The above approximation can be extended to the case of even length path to $\partial Ball(i, 2l)$. Thus, for $i = 1, 2, \ldots, N$, the expression in the right-hand side of Equation (A3) gives

$$CI_l(i) \stackrel{\text{def}}{=} (k_i - 1) \sum_{j \in \partial Ball(i,l)} (k_j - 1) \tag{A4}$$

This approximation of CI in l hops is categorized in range-limited approach. Thus, the scalable algorithm for calculating $CI_l(i)$ is based on the minimization of the energy of a many-body system, which is equivalent to find the principal part of connectivity as the minimal set of nodes that minimize the largest eigenvalue $\lambda(n,q)$. The critical transition of the eigenvalue of the NB matrix from one to zero is caused as the network that consists of a single loop is changed to a tree by a node removal eventually. Once it becomes a tree, fragmentation to large components occurs by any node removal. When a network includes more than one loops, the eigenvalue is greater than one. Thus, it has been pointed out that *the best attack strategy is to destroy the loops* (Morone & Makse, 2015)(Supplementary Information). Note that the NB matrix is intrinsic for enumerating the number of loops on a length basis in Zeta function of graphs (Hashimoto, 1989).

For BP, according to Zhou (2013), it is assumed that nodes $j \in \partial i$ are mutually independent of each other when node i is removed. Such approximated tree-like graph is called as cavity graph. We consider the marginal probability $q_i^{A_i}$ for the state A_i of node i. Since A_i represent the index of root node of i, it is influenced by the neighboring nodes in the cavity graph after removing node i denoted by $\backslash i$. Based on the product of independent marginal probability $q_{j\to i}^{A_j}$ for the state A_j, we consider the joint probability

$$\mathscr{P}_{\backslash i}(A_j : j \in \partial i) \approx \Pi_{j\in\partial i} q_{j\to i}^{A_j}$$

In the cavity graph, if all nodes $j \in \partial i$ are either empty ($A_j = 0$) or roots ($A_j = j$), the added node i can be a root ($A_i = i$). There are the following exclusive states.

1. $A_i = 0$: i is empty (removed). Since i is unnecessary as a root, it belongs to FVS.
2. $A_i = i$: i becomes its own root.
   The state $A_j = j$ of $j \in \partial i$ is changeable to $A_j = i$ when node i is added.
3. $A_i = k$: one node $k \in \partial i$ becomes the root of i when it is added, if k is occupied and all other $j \in \partial i$ are either empty or roots.

The corresponding probabilities to the above states are

$$q_i^0 \stackrel{\text{def}}{=} \frac{1}{z_i} \tag{A5}$$

$$q_i^i \stackrel{\text{def}}{=} \frac{e^x \Pi_{j\in\partial i(t)} \left[q_{j\to i}^0 + q_{j\to i}^j\right]}{z_i}$$

$$q_i^k \stackrel{\text{def}}{=} \frac{e^x \frac{(1-q_{k\to i}^0)}{q_{k\to i}^0 + q_{k\to i}^k} \Pi_{j\in\partial i(t)} \left[q_{j\to i}^0 + q_{j\to i}^j\right]}{z_i}$$

$$q_{i\to j}^0 = \frac{1}{z_{i\to j}(t)} \tag{A6}$$

$$q_{i\to j}^i = \frac{e^x \Pi_{k\in\partial i(t)\backslash j} \left[q_{k\to i}^0 + q_{k\to i}^k\right]}{z_{i\to j}(t)} \tag{A7}$$

where $\partial i(t)$ denotes node i's set of connecting neighbor nodes at time t, and $x > 0$ is a parameter of inverse temperature. We have the normalization constants

$$z_i \stackrel{\text{def}}{=} 1 + e^x \left[1 + \sum_{k\in\partial i(t)} \frac{1 - q_{k\to i}^0}{q_{k\to i}^0 + q_{k\to i}^k}\right] \Pi_{j\in\partial i(t)} \left[q_{j\to i}^0 + q_{j\to i}^j\right] \tag{A8}$$

$$z_{i\to j}(t) \stackrel{\text{def}}{=} 1 + e^x \Pi_{k\in\partial i(t)\backslash j} \left[q_{k\to i}^0 + q_{k\to i}^k\right] \times \left[1 + \sum_{l\in\partial i(t)\backslash j} \frac{1 - q_{l\to i}^0}{q_{l\to i}^0 + q_{l\to i}^l}\right] \tag{A9}$$

to be satisfied for any i and $i \to j$ as

$$q_i^0 + q_i^i + \sum_{k\in\partial i} q_i^k = 1$$

$$q_{i\to j}^0 + q_{i\to j}^i + \sum_{k\in\partial i} q_{i\to j}^k = 1$$

In BP attacks with sudden breakdown (Mugisha & Zhou, 2016), which gives severer damages than CI and hub attacks, the node deletion process also focus on the destroy of loops, since *the FVS of graph G is a subset of nodes such that if all the nodes of this set and the attached links are removed from G, the remaining graph will have no loops* (Zhou, 2013). A node with the highest q_i^0 is chosen as the removed target at each time step t that consists of a number of rounds by the updating calculations of Equations (A5)–(A9) in order of random permutation of nodes 1–N.

On the other hand in computer science, to find a suitable node for removing and inserting into FVS, the largest node of its degree $deg(v)$ minus the number $comp(G-v)$ of connecting components formed by removing the node v is recursively chosen in an approximation algorithm (Vazirani, 2001). When several links connect to a component, pairs of these links form loops through v. Then, the number of them (precisely the number -1) is not decreased from its degree. The most subtracted case is only one link connects to a component, then $deg(v) - comp(G - v)$ becomes 0. Thus, $deg(v) - comp(G - v)$ is considered as a characteristic index to delete loops as many as possible. However, this heuristic approximation method requires large computation.