Research Article

Krein Space Quantization of Casimir Effect for a Spherical Shell

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The Casimir stress on a spherical shell in de Sitter spacetime for a massless scalar field is calculated using Krein space quantization. In this method, the auxiliary negative frequency states have been utilized, the modes of which do not interact with the physical states and are not affected by the physical boundary conditions. These unphysical states just play the role of an automatic renormalization tool for the theory.

1. Introduction

The Casimir effect is a small attractive force acting between two parallel uncharged conducting plates and it is regarded as one of the most striking manifestation of vacuum fluctuations in quantum field theory. It is due to the quantum vacuum fluctuation of the field operator between two parallel plates. In other words, the Casimir effect can be viewed as the polarization of the vacuum by boundary conditions or geometry. The presence of reflecting boundaries alters the zero-point modes of a quantized field and results in the shifts in the vacuum expectation values of quantities quadratic in the field, such as the energy density and stresses.

In particular, vacuum forces arise acting on the constraining boundaries. The particular features of these forces depend on the nature of the quantum field, the type of spacetime manifold and its dimensionality, the boundary geometries, and the specific boundary conditions imposed on the field. Since the original work by Casimir in 1948 [1], many theoretical and experimental works have been done on this problem [2–14].

The time dependence of boundary conditions or geometries, the so-called dynamical Casimir effect, is also a new element which has to be taken into account. In particular, in [15]

the Casimir effect has been calculated for a massless scalar field satisfying Dirichlet boundary conditions on the spherical shell in de Sitter space. The Casimir stress is calculated for inside and outside of the shell with different backgrounds corresponding to different cosmological constants.

It should be remarked that the procedures of renormalization and regularization have been applied for eliminating the divergences appeared in physical quantities [16–20]. The Casimir effect has been also investigated for two parallel plates in Krein space quantization [21].

The historical background of Krein space quantization goes back to the covariant quantization of minimally coupled scaler field in de Sitter spacetime. It has been shown that the linear quantum gravity in the background field method is perturbatively nonrenormalizable and also there appears an infrared divergence. This infrared divergence does not manifest itself in the quadratic part of the effective action in the one-loop approximation. This means that the pathological behavior of the graviton propagator may be gauge dependent and so should not appear in an effective way as a physical quantity [22]. The infrared divergence which appears in the linear gravity in de Sitter space is the same as the minimally coupled scalar field in de Sitter space [23–27]. It is shown that one cannot construct a covariant quantization of the minimally coupled scalar field with only positive norm states [28]. It has been proved that the use of the two sets of solutions (positive and negative norm states) is an unavoidable feature if one wants to preserve causality (locality), covariance, and elimination of the infrared divergence in quantum field theory for the minimally coupled scalar field in de Sitter space [29, 30], that is, Krein space quantization.

The singular behavior of Green's function at short relative distances (ultraviolet divergence) or in the large relative distances (infrared divergence) leads to main divergences in the quantum field theory. It was conjectured that quantum metric fluctuations might smear out the singularities of Green's functions on the light cone, but it does not remove other ultraviolet divergences [31]. However, it has been shown that quantization in Krein space removes all ultraviolet divergences of quantum field theory (QFT) except the light cone singularity [32]. By using the Krein space quantization and the quantum metric fluctuations in the linear approximation, it has been showed that the infinities in the Green's function are disappeared [31, 33].

Quantization in Krein space instead of Hilbert space has some interesting features. For example, in this method, the vacuum energy becomes zero naturally, so the normal ordering would not be necessary [29, 32]. The auxiliary negative norm states, which are used in the Krein space quantization, play the regularization of the theory.

Applying the unphysical negative frequency states and defining the field operator in Krein space, we can calculate the gravitational pressure on a spherical shell yielding the standard result obtained.

2. Scalar Casimir Effect for a Sphere in de Sitter Space

The Casimir force due to fluctuations of a free massless scalar field satisfying Dirichlet boundary conditions on a spherical shell in Minkowski space time has been studied in [34]. Doing the calculations in Krein space, the two-point Green's function $G_K(x, t; x', t')$ is defined

as the vacuum expectation value of the time-ordered product of two fields (K and T stand for quantities in Krein space and the time-ordered product, resp.)

$$G_K(x, t; x', t') \equiv -i\langle 0|T\Phi_K(x, t)\Phi_K(x', t')|0\rangle, \tag{2.1}$$

$$\Phi_K(x, t) = \sum_{\vec{k}} \left[\left(a_{\vec{k}} + b_{\vec{k}}^\dagger \right) u_p(k, x) + \left(a_{\vec{k}}^\dagger + b_{\vec{k}} \right) u_n(k, x) \right]. \tag{2.2}$$

The operators $a^\dagger(\vec{k})$ and $a(\vec{k})$ create and destroy, respectively, the mode $u_p(k, x)$ with positive energy ($k^0 = \omega_{\vec{k}}$), which may be considered as the operators of creation and annihilation of a particle and the operators $b^\dagger(\vec{k})$ and $b(\vec{k})$ create and destroy, respectively, the mode $u_n(k, x)$ with negative energy ($-k^0 = -\omega_{\vec{k}}$), which may be considered as the operators of creation and annihilation of an "antiparticle" in the inverse time direction. The two sets of modes do not affect each other and in the standard QFT, the negative energy states are eliminated in the quantum field operators, which are the origin of the appearance of divergence. On the contrary, the divergence disappears by taking these states into account.

The two-point Green's function has to satisfy the Dirichlet boundary conditions on the shell:

$$G_K(x, t; x', t')\big|_{|x|=a} = 0, \tag{2.3}$$

where a is radius of the spherical shell. The stress-energy tensor in Krein space $T_K^{\mu\nu}(x, t)$ is given by

$$T_K^{\mu\nu}(x, t) \equiv \partial^\mu \Phi_K(x, t)\partial^\nu \Phi_K(x, t)$$
$$- \frac{1}{2}\eta^{\mu\nu}\partial_\lambda \Phi_K(x, t)\partial^\lambda \Phi_K(x, t). \tag{2.4}$$

The radial Casimir force per unit area F/A on the sphere, called Casimir stress, is obtained from the radial-radial component of the vacuum expectation value of the stress-energy tensor:

$$\frac{F}{A} = \langle 0|T_{\text{in}}^{rr} - T_{\text{out}}^{rr}|0\rangle\big|_{r=a}. \tag{2.5}$$

Taking into account the relation (2.1) between the vacuum expectation value of the stress-energy tensor $T_K^{\mu\nu}(x, t)$ and the Green's function at equal times $G_K(x, t; x', t)$, we obtain

$$\frac{F}{A} = \frac{i}{2}\left[\frac{\partial}{\partial r}\frac{\partial}{\partial r'}G_K(x, t; x', t)_{\text{in}} \right.$$
$$\left. - \frac{\partial}{\partial r}\frac{\partial}{\partial r'}G_K(x, t; x', t)_{\text{out}} \right]\Bigg|_{x=x', |x|=a}. \tag{2.6}$$

One may use of the above flat space calculation in de Sitter space time by taking the de Sitter metric in conformally flat form

$$ds^2 = \Omega(\eta)\left[d\eta^2 - \sum_{i=1}^{3}(dx^i)^2\right], \tag{2.7}$$

where $\Omega(\eta) = \alpha/\eta$ and η is the conformal time

$$-\infty < \eta < 0. \tag{2.8}$$

Assuming a canonical quantization of the scalar field in Krein space, the conformally transformed quantized scalar field in de Sitter spacetime is given by

$$\overline{\Phi}_K(x,\eta) = \sum_k\left[\left(a_{\vec{k}} + b_{\vec{k}}^\dagger\right)\overline{u}_k(\eta,x) + \left(a_{\vec{k}}^\dagger + b_{\vec{k}}\right)\overline{u}_k^*(\eta,x)\right], \tag{2.9}$$

where $a_k^\dagger$ and a_k are creation and annihilation operators, respectively, and the vacuum states associated with the physical modes $\overline{u}_k$ defined by $a_k|\overline{0}\rangle = 0$ are called conformal vacuum. Given the flat space Green's function (2.1), we obtain

$$\overline{G}_K = -i\left\langle\overline{0}\left|T\overline{\Phi}_K(x,\eta)\overline{\Phi}_K(x',\eta')\right|\overline{0}\right\rangle = \Omega^{-1}(\eta)\Omega^{-1}(\eta')G_K, \tag{2.10}$$

where $\overline{\Phi}_K(x,\eta) = \Omega^{-1}(\eta)\Phi_K(x,\eta)$ has been used. Therefore, using (2.5), (2.6), and (2.10), we obtain the total stress on the sphere in de Sitter spacetime and using Krein space quantization as

$$\frac{\overline{F}}{A} = \frac{\eta^2}{\alpha^2}\frac{F}{A}, \tag{2.11}$$

in accordance with the standard result.

3. Spherical Shell with Different Vacua in Krein Space Quantization

We assume different vacua inside and outside, corresponding to different α_{in} and α_{out} for the Lorentzian metric (2.7) and use the following relation for the stress on the shell due to boundary conditions in flat spacetime [8]:

$$\frac{F}{A} = \frac{-1}{4\pi a^2}\frac{\partial E}{\partial a}, \tag{3.1}$$

where the Casimir energy E is the sum of Casimir energies E_{in} and E_{out} for inside and outside of the shell. The corresponding relation in de Sitter spacetime and applying Krein method leads to the following result [15]:

$$\frac{\overline{F}}{A} = \frac{-1}{4\pi a^2}\frac{\partial \overline{E}}{\partial a} = \frac{\eta^2}{8\pi a^4}\left(\frac{c_1}{\alpha_{\text{in}}^2} + \frac{c_2}{\alpha_{\text{out}}^2}\right), \tag{3.2}$$

where we have used the automatically renormalized total zero-point energy in Krein Space

$$\overline{E} = \frac{\eta^2}{2a}\left(\frac{c_1}{\alpha_{\text{in}}^2} + \frac{c_2}{\alpha_{\text{out}}^2}\right), \tag{3.3}$$

in which $c_1 = 0.008873$, $c_2 = -0.003234$.

Now, we obtain the pure effect of vacuum polarization due to the gravitational field without any boundary conditions in Euclidean (outside) region with the following metric:

$$ds^2 = -\Omega(\eta)\left[d\eta^2 + \sum_{\iota=1}^{3}(dx^\iota)^2\right]. \tag{3.4}$$

To this end, we calculate the automatically renormalized stress tensor in Krein space quantization for the massless scalar field in de Sitter spacetime with Euclidean signature. One may use [2]

$$\left\langle 0\left|T_\mu^\nu[g_{kl}]\right|0\right\rangle\Big|_K = \left(\frac{\tilde{g}}{g}\right)^{1/2}\left\langle 0\left|T_\mu^\nu[\tilde{g}_{kl}]\right|0\right\rangle\Big|_K$$
$$- \frac{1}{2880\pi^2}\left[\frac{1}{6}\,^{(1)}H_\mu^\nu - \,^{(3)}H_\mu^\nu\right], \tag{3.5}$$

where $\tilde{g}_{kl}$ is the flat Euclidean metric for which $\langle 0|T_\mu^\nu[\tilde{g}_{kl}]|0\rangle|_K = 0$, and

$$^{(1)}H_\mu^\nu = 0,$$
$$^{(3)}H_\mu^\nu = \frac{3}{\alpha^4}\delta_\mu^\nu. \tag{3.6}$$

We then obtain

$$\left\langle 0\left|T_\mu^\nu[g_{kl}]\right|0\right\rangle\Big|_K = \frac{1}{960\pi^2\alpha^4}\delta_\mu^\nu, \tag{3.7}$$

which is exactly the same result for the Lorentzian case [2]. Therefore, the corresponding effective radial pressures for the Euclidean (outside) and Lorentzian (inside) regions with α_{out} and α_{in}, due to pure effect of gravitational vacuum polarization without any boundary condition, are given, respectively, by

$$P_{\text{out}}^E = -\left\langle 0|T_r^r[g_{kl}]|0\right\rangle\Big|_K = -\frac{1}{960\pi^2\alpha_{\text{out}}^4},$$
$$P_{\text{in}}^L = -\left\langle 0|T_r^r[g_{kl}]|0\right\rangle\Big|_K = -\frac{1}{960\pi^2\alpha_{\text{in}}^4}. \tag{3.8}$$

The corresponding gravitational pressure on the spherical shell in Krein space quantization is then given by

$$P_G = P_{in}^L - P_{out}^E = -\frac{1}{960\pi^2}\left(\frac{1}{\alpha_{in}^4} - \frac{1}{\alpha_{out}^4}\right). \tag{3.9}$$

4. Conclusion

The Casimir effect has been studied for spherical bubbles with different vacua inside and outside, corresponding to de Sitter metrics. The metrics inside and outside are taken by Lorentzian and Euclidean, respectively. The case of different vacua in a Lorentzian de Sitter spacetime has already been studied in [15]. In the present work, we have shown that the same results could be attained using Krein space quantization. Presence of the unphysical negative frequency states plays the role of an automatic renormalization tool.

Appendix

In this appendix, we review the elementary facts about Krein space quantization. As we know, the origin of divergences in standard quantum field theory lies in the singularity of the Green's function. The divergence appears in the imaginary part of the Feynman propagator, and the real part is convergent [35]:

$$G_F^P(x,x') = -\frac{1}{8\pi}\delta(\sigma_0)$$
$$+ \frac{m^2}{8\pi}\theta(\sigma_0)\left[\frac{J_1\left(\sqrt{2m^2\sigma_0}\right) - iN_1\left(\sqrt{2m^2\sigma_0}\right)}{\sqrt{2m^2\sigma_0}}\right] \tag{A.1}$$
$$- \frac{im^2}{4\pi^2}\theta(-\sigma_0)\frac{K_1\left(\sqrt{2m^2(-\sigma_0)}\right)}{\sqrt{2m^2(-\sigma_0)}}.$$

Consideration of negative frequency states removes singularity of the Green's function with exception of delta function singularity:

$$G_T(x,x') = -\frac{1}{8\pi}\delta(\sigma_0)$$
$$+ \frac{m^2}{8\pi}\theta(\sigma_0)\frac{J_1\left(\sqrt{2m^2\sigma_0}\right)}{\sqrt{2m^2\sigma_0}}, \quad \sigma_0 \geq 0. \tag{A.2}$$

However, considering the quantum metric fluctuations removes the latter singularity:

$$\langle G_T(x, x') \rangle = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2\langle \sigma_1^2 \rangle}} \exp\left(-\frac{\sigma_0^2}{2\langle \sigma_1^2 \rangle} \right)$$
$$+ \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1\left(\sqrt{2m^2\sigma_0} \right)}{\sqrt{2m^2\sigma_0}}, \tag{A.3}$$

where $\langle \sigma_1^2 \rangle$ is related to the density of gravitons. When $\sigma_0 = 0$, due to the metric quantum fluctuation $\langle \sigma_1^2 \rangle \neq 0$, and we have

$$\langle G_T(0) \rangle = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2\langle \sigma_1^2 \rangle}} + \frac{m^2}{16\pi}. \tag{A.4}$$

By using the Fourier transformation, we obtain [36]

$$\left\langle \widetilde{G}_T(p) \right\rangle = \widetilde{G}_T(p) + PP \frac{m^2}{p^2(p^2 - m^2)}. \tag{A.5}$$

However, in the one-loop approximation, the contribution of delta function is negligible and the Green's function in Krein space quantization appearing in the transition amplitude is

$$\left\langle \widetilde{G}_T(p) \right\rangle\Big|_{\text{one-loop}} \equiv \widetilde{G}_T(p)\Big|_{\text{one-loop}} \equiv PP \frac{m^2}{p^2(p^2 - m^2)}. \tag{A.6}$$
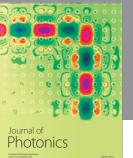
Acknowledgment

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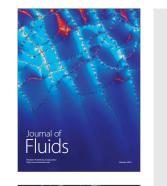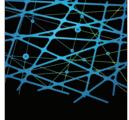[36] A. Refaei and M. V. Takook, "Scalar effective action in Krein space quantization," *Modern Physics Letters A*, vol. 26, no. 1, pp. 31–41, 2011.