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Research Article

Probabilistic Solution of Rational Difference Equations System with Random Parameters

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We study the periodicity of the solutions of the rational difference equations system of type $x_n = a/y_{n-p}$, $y_n = b/x_{n+p-2}$ $(p \geq 1)$, and then we propose new *exact* procedure to find the probability density function of the solution, where a, b, $x_0 = N$ and $y_0 = M$ are independent random variables.

1. Introduction

Stochastic systems of difference equations usually appear in the investigation of systems with discrete time or in the numerical solution of systems with continuous time. A lot of difference systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, and sudden environment changes. Recently, there has been great interest in studying difference equation systems. One of the reasons for this is the necessity for some techniques that can be used in investigating equations arising in mathematical models describing real-life situations in population biology, economics, probability theory, genetics, psychology, and so forth. There are many papers related to the difference equations system; for example, Çinar [1] studied the solutions of the system of difference equations:

$$x_{n+1} = \frac{1}{y_n}, \qquad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}. \tag{1.1}$$

Papaschinopoulos and Schinas [2] studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of nonlinear difference equations:

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \qquad y_{n+1} = A + \frac{x_n}{y_{n-q}}, \qquad n = 0, 1, \ldots, p, q. \tag{1.2}$$

Özban [3] studied the positive solutions of the system of rational difference equations:

$$x_n = \frac{a}{y_{n-3}}, \qquad y_n = \frac{b y_{n-3}}{x_{n-q} y_{n-q}}. \tag{1.3}$$

Clark and Kulenović [4] investigated the global asymptotic stability:

$$x_{n+1} = \frac{x_n}{a + c y_n}, \qquad y_{n+1} = \frac{y_n}{b + d x_n}. \tag{1.4}$$

Yang and Xu [5] propose a method to deal with the mean square exponential stability of impulsive stochastic difference equations.

In this paper, we investigate the analytic solution of the stochastic difference equations system:

$$x_n = \frac{a}{y_{n-p}}, \qquad y_n = \frac{b}{x_{n+p-2}}, \tag{1.5}$$

where a, b, $x_0 = N$ and $y_0 = M$ are independent random variables.

2. Periodicity of the Solutions

In this section, we study the periodicity of the solutions of system (1.5).

Theorem 2.1. *All solutions of* (1.5) *are periodic with period 2.*

Proof. One has the following:

$$\begin{aligned} x_{n+2} &= \frac{a}{y_{n-p+2}} = \frac{a}{b/x_n} = \frac{a}{b} x_n, \\ y_{n+2} &= \frac{b}{x_{n+p}} = \frac{b}{a/y_n} = \frac{b}{a} y_n. \end{aligned} \tag{2.1}$$

Therefore, the proof was completed. □

Theorem 2.2. *For $p = 1$, all solutions of* (1.5) *are*

$$x_{2n+1} = \frac{a^{n+1}}{b^n M}, \qquad y_{2n+1} = \frac{b^{n+1}}{a^n N},$$
$$x_{2n+2} = \frac{a^{n+1} N}{b^{n+1}}, \qquad y_{2n+2} = \frac{b^{n+1} M}{a^{n+1}}. \tag{2.2}$$

Proof. By induction, suppose the result holds for $n-1$:

$$x_{2n-1} = \frac{a^n}{b^{n-1} M}, \qquad y_{2n-1} = \frac{b^n}{a^{n-1} N},$$
$$x_{2n} = \frac{a^n N}{b^n}, \qquad y_{2n} = \frac{b^n M}{a^n}. \tag{2.3}$$

For n,

$$x_{2n+1} = \frac{a}{y_{2n}} = \frac{a^{n+1}}{b^n M}, \qquad y_{2n+1} = \frac{b}{x_{2n}} = \frac{b^{n+1}}{a^n N},$$
$$x_{2n+2} = \frac{a}{y_{2n+1}} = \frac{a^{n+1} N}{b^{n+1}}, \qquad y_{2n+2} = \frac{b}{x_{2n+1}} = \frac{b^{n+1} M}{a^{n+1}}. \tag{2.4}$$

Hence, the proof is completed by induction. □

3. Analytic Stochastic Solutions

In this section, we develop an analytic technique to find the stochastic solutions of (cf. Theorem 2.2)

$$x_{2n+1} = \frac{a^{n+1}}{b^n M}, \qquad y_{2n+1} = \frac{b^{n+1}}{a^n N},$$
$$x_{2n+2} = \frac{a^{n+1} N}{b^{n+1}}, \qquad y_{2n+2} = \frac{b^{n+1} M}{a^{n+1}}. \tag{3.1}$$

The solution of a stochastic system of difference equations is obtained when evaluating the statistical characteristic of the solution process like the mean, standard deviation, high-order moments, and the most important characteristic, "the probability density function" (pdf). Our proposed technique is based on the transformation of random variables to get the pdf of x_n and y_n.

4. Transformation of Random Variables Technique (TRVT)

Definition 4.1. Let X be a continuous random variable with generic probability density function $f(x)$ defined over the support $c_1 < x < c_2$. And, let $Y = u(X)$ be an *invertible* function of X with inverse function $X = v(Y)$. Then, using the transformation of random variable technique (TRVT) defined by Kadry and Younes [6], Kadry [7], El-Tawil et al. [8], and Kadry and Smaili [9], the probability density function of Y is

$$\mathrm{pdf}(y) = \mathrm{pdf}(v(y)) \cdot \left|v'(y)\right| \tag{4.1}$$

defined over the support $u(c_1) < y < u(c_2)$.

To simplify our technique, let us consider the following stochastic equation:

$$z = \frac{y^{\beta}}{x^{\alpha}} \quad (\alpha, \beta \in Z), \tag{4.2}$$

where x and y are two independent random variables. To find the pdf of z, firstly we linearize the denominator, and then we find the pdf of the product of two random variables:

we suppose $\delta = 1/x^{\alpha}$, and then we apply the TRVT technique to get the pdf of δ,

$$f_{\delta}(\delta) = -\frac{1}{\delta^2 \alpha} \sqrt[\alpha]{\delta^{-1+\alpha}} f_x(x). \tag{4.3}$$

Once the linearization step has been done, it is required to find the pdf of the product of two random variables. To do that, we developed the following theorem.

Theorem 4.2. *Let X be a continuous random variable with distribution function $f(X)$ which is defined on the interval $[a, b]$, where $0 < a < b < \infty$. Similarly, let Y be a random variable of the continuous type with distribution function $g(Y)$ which is defined on the interval $[c, d]$, where $0 < c < d < \infty$. The pdf of $z = XY, h(z)$ is obtained as follows.*

Case 1 ($ad < bc$). One has

$$h(z) = \begin{cases} \displaystyle\int_{a}^{z/c} g\left(\frac{z}{X}\right) f(X) \frac{1}{X} dx, & ac < z < ad, \\ \displaystyle\int_{z/d}^{z/c} g\left(\frac{z}{X}\right) f(X) \frac{1}{X} dx, & ad < z < bc, \\ \displaystyle\int_{z/d}^{b} g\left(\frac{z}{X}\right) f(X) \frac{1}{X} dx. & bc < z < bd. \end{cases} \tag{4.4}$$

Case 2 ($ad = bc$). One has

$$h(z) = \begin{cases} \displaystyle\int_{a}^{z/c} g\left(\frac{z}{X}\right) f(X) \frac{1}{X} dx, & ac < z < ad, \\ \displaystyle\int_{z/d}^{b} g\left(\frac{z}{X}\right) f(X) \frac{1}{X} dx, & ad < z < bd. \end{cases} \tag{4.5}$$

Case 3 ($ad > bc$). One has

$$h(z) = \begin{cases} \displaystyle\int_a^{z/c} g\left(\frac{z}{X}\right) f(X) \frac{1}{X} dx, & ac < z < bc, \\ \displaystyle\int_a^{b} g\left(\frac{z}{X}\right) f(X) \frac{1}{X} dx, & bc < z < ad, \\ \displaystyle\int_{z/d}^{b} g\left(\frac{z}{X}\right) f(X) \frac{1}{X} dx, & ad < z < bd. \end{cases} \tag{4.6}$$

Proof. Only the case of $ad < bc$ is considered. The other cases are proven analogously. Using [kadry], the transformation $w = X$ and $z = XY$ is bijective. The Jacobian of this transformation becomes

$$J = \begin{vmatrix} 1 & 0 \\ -\dfrac{z}{w^2} & \dfrac{1}{w} \end{vmatrix} = \frac{1}{w}. \tag{4.7}$$

The joint pdf of w and z is

$$f_{W,Z}(w, z) = f(w) f\left(\frac{z}{w}\right) \frac{1}{|w|}. \tag{4.8}$$

Integrating with respect to w over the appropriate intervals and replacing w with X in the final result yields the marginal pdf of z:

$$h(z) = \begin{cases} \displaystyle\int_a^{z/c} g\left(\frac{z}{X}\right) f(X) \frac{1}{X} dx, & ac < z < ad, \\ \displaystyle\int_{z/d}^{z/c} g\left(\frac{z}{X}\right) f(X) \frac{1}{X} dx, & ad < z < bc, \\ \displaystyle\int_{z/d}^{b} g\left(\frac{z}{X}\right) f(X) \frac{1}{X} dx, & bc < z < bd. \end{cases} \tag{4.9}$$

□

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