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Research Article

Eighth-Order Iterative Methods without Derivatives for Solving Nonlinear Equations

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A new family of eighth-order derivative-free methods for solving nonlinear equations is presented. It is proved that these methods have the convergence order of eight. These new methods are derivative-free and only use four evaluations of the function per iteration. In fact, we have obtained the optimal order of convergence which supports the Kung and Traub conjecture. Kung and Traub conjectured that the multipoint iteration methods, without memory based on n evaluations, could achieve optimal convergence order 2^{n-1}. Thus, we present new derivative-free methods which agree with Kung and Traub conjecture for $n = 4$. Numerical comparisons are made to demonstrate the performance of the methods presented.

1. Introduction

Consider iterative methods for finding a simple root α of the nonlinear equation

$$f(x) = 0, \tag{1.1}$$

where $f : D \subset \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D, and it is sufficiently smooth in a neighbourhood of α. It is well known that the techniques to solve nonlinear equations have many applications in science and engineering. In this paper, a new family of three-point derivative-free methods of the optimal order eight is constructed by combining optimal two-step fourth-order methods and a modified third step. In order to obtain these new derivative-free methods, we replace derivatives with suitable approximations based on divided difference. In fact, it is well known that the various methods have been used in order to approximate the derivatives by the Newton interpolation, the Hermite interpolation, the Lagrange interpolation, and ration function [1, 2].

The prime motive of this study is to develop a class of very efficient three-step derivative-free methods for solving nonlinear equations. The eighth-order methods presented in this paper are derivative-free and only use four evaluations of the function per iteration. In fact, we have obtained the optimal order of convergence which supports the Kung and Traub conjecture. Kung and Traub conjectured that the multipoint iteration methods, without memory based on n evaluations, could achieve optimal convergence order 2^{n-1}. Thus, we present new derivative-free methods which agree with the Kung and Traub conjecture for $n = 4$. In addition, these new eighth-order derivative-free methods have an equivalent efficiency index to the established Kung and Traub eighth-order derivative-free method presented in [3]. Furthermore, the new eighth-order derivative-free methods have a better efficiency index than the three-step sixth-order derivative-free methods presented recently in [4, 5], and in view of this fact, the new methods are significantly better when compared with the established methods. Consequently, we have found that the new eighth-order derivative-free methods are consistent, stable, and convergent.

This paper is organised as follows. In Section 2 we construct the eighth-order methods that are free from derivatives and prove the important fact that the methods obtained preserve their convergence order. In Section 3 we will briefly state the established Kung and Traub method in order to compare the effectiveness of the new methods. Finally, in Section 4 we demonstrate the performance of each of the methods described.

2. Methods and Convergence Analysis

In this section we will define a new family of eighth-order derivative-free methods. In order to establish the order of convergence of these new methods, we state the three essential definitions.

Definition 2.1. Let $f(x)$ be a real function with a simple root α, and let $\{x_n\}$ be a sequence of real numbers that converge towards α. The order of convergence m is given by

$$\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^m} = \zeta \neq 0, \tag{2.1}$$

where ζ is the asymptotic error constant and $m \in \mathbb{R}^+$.

Definition 2.2. Suppose that x_{n-1}, x_n, and x_{n+1} are three successive iterations closer to the root α of (1.1). Then, the computational order of convergence [6] may be approximated by

$$\text{COC} \approx \frac{\ln\left|(x_{n+1}-\alpha)(x_n-\alpha)^{-1}\right|}{\ln\left|(x_n-\alpha)(x_{n-1}-\alpha)^{-1}\right|}, \tag{2.2}$$

where $n \in \mathbb{N}$.

Definition 2.3. Let β be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index [7, 8] and defined as

$$\mu^{1/\beta}, \tag{2.3}$$

where μ is the order of the method.

2.1. The Eighth-Order Derivative-Free Method (RT)

We consider the iteration scheme of the form

$$y_n = x_n - \left(\frac{f(x_n)}{f'(x_n)}\right),$$
$$z_n = y_n - \left(\frac{f(y_n)}{f'(y_n)}\right), \tag{2.4}$$
$$x_{n+1} = z_n - \left(\frac{f(z_n)}{f'(z_n)}\right).$$

This scheme consists of three steps in which the Newton method is repeated. It is clear that formula (2.4) requires six evaluations per iteration and has an efficiency index of $8^{1/6} = 1.414$, which is the same as the classical Newton method. In fact, scheme (2.4) does not increase the computational efficiency. The purpose of this paper is to establish new derivative-free methods with optimal order; hence, we reduce the number of evaluations to four by using some suitable approximation of the derivatives. To derive higher efficiency index, we consider approximating the derivatives by divided difference method. Therefore, the derivatives in (2.4) are replaced by

$$f'(x_n) \approx f[w_n, x_n] = \frac{f(w_n) - f(x_n)}{w_n - x_n} = \frac{f(w_n) - f(x_n)}{f(x_n)},$$
$$f'(y_n) \approx \frac{f[x_n, y_n] f[w_n, y_n]}{f[w_n, x_n]}, \tag{2.5}$$
$$f'(z_n) \approx (f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n]).$$

Substituting (2.5) into (2.4), we get

$$w_n = x_n + \beta f(x_n),$$
$$y_n = x_n - \left(\frac{f(x_n)^2}{f(w_n) - f(x_n)}\right),$$
$$z_n = y_n - \left(\frac{f[w_n, x_n] f(y_n)}{f[x_n, y_n] f[w_n, y_n]}\right), \tag{2.6}$$
$$x_{n+1} = z_n - \left(\frac{f(z_n)}{(f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n])}\right).$$

The first step of formula (2.6) is the classical Steffensen second-order method [9], and the second step is the new fourth-order method. Furthermore, we have found that the third step does not produce an optimal order of convergence. Therefore, we have introduced two weight functions in the third step in order to achieve the desired eighth-order derivative-free method. The two weight functions are expressed as

$$\left(1-\frac{f(z_n)}{f(w_n)}\right)^{-1},$$
$$\left(1+\frac{2f(y_n)^3}{f(w_n)^2 f(x_n)}\right)^{-1}. \tag{2.7}$$

Then the iteration scheme (2.4) in its final form is given as

$$w_n = x_n + \beta f(x_n),$$
$$y_n = x_n - \left(\frac{f(x_n)^2}{f(w_n)-f(x_n)}\right),$$
$$z_n = y_n - \left(\frac{f[w_n,x_n]f(y_n)}{f[x_n,y_n]f[w_n,y_n]}\right),$$
$$x_{n+1} = z_n - \left(1-\frac{f(z_n)}{f(w_n)}\right)^{-1}\left(1+\frac{2f(y_n)^3}{f(w_n)^2 f(x_n)}\right)^{-1}\left(\frac{f(z_n)}{(f[y_n,z_n]-f[x_n,y_n]+f[x_n,z_n])}\right), \tag{2.8}$$

where $n\in\mathbb{N}$, $\beta\in\mathbb{R}^{+}$, provided that the denominators in (2.8) are not equal to zero.

Thus the scheme (2.8) defines a new family of multipoint methods with two weight functions. To obtain the solution of (1.1) by the new derivative-free methods, we must set a particular initial approximation x_0, ideally close to the simple root. In numerical mathematics it is very useful and essential to know the behaviour of an approximate method. Therefore, we will prove the order of convergence of the new eighth-order method.

Theorem 2.4. *Assume that the function $f : D\subset\mathbb{R}\to\mathbb{R}$ for an open interval D has a simple root $\alpha\in D$. Letting $f(x)$ be sufficiently smooth in the interval D and the initial approximation x_0 is sufficiently close to α, then the order of convergence of the new derivative-free method defined by* (2.8) *is eight.*

Proof. Let α be a simple root of $f(x)$, that is, $f(\alpha)=0$ and $f'(\alpha)\neq 0$, and the error is expressed as

$$e = x-\alpha. \tag{2.9}$$

Using the Taylor expansion, we have

$$f(x_n) = f(\alpha)+f'(\alpha)e_n+2^{-1}f''(\alpha)e_n^2+6^{-1}f'''(\alpha)e_n^3+24^{-1}f^{iv}(\alpha)e_n^4+\cdots. \tag{2.10}$$

Taking $f(\alpha)=0$ and simplifying, expression (2.10) becomes

$$f(x_n) = c_1e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+\cdots, \tag{2.11}$$

where $n \in \mathbb{N}$ and

$$c_k = \frac{f^{(k)}(\alpha)}{(k!)} \quad \text{for } k = 1, 2, 3, 4, \ldots. \tag{2.12}$$

Expanding the Taylor series of $f(w_n)$ and substituting $f(x_n)$ given by (2.11), we have

$$f(w_n) = c_1(1 + c_1\beta)e_n + \left(3\beta c_1 c_2 + \beta^2 c_1^2 c_2 + c_2\right)e_n^2 + \cdots. \tag{2.13}$$

Substituting (2.11) and (2.13) in expression (2.8) gives us

$$y_n - \alpha = x_n - \alpha - \left(\frac{\beta f(x_n)^2}{f(w_n) - f(x_n)}\right) = \left(\frac{c_2}{c_1}\right)(\beta c_1 + 1)e_n^2 + \cdots. \tag{2.14}$$

The expansion of $f(y_n)$ about α is given as

$$f(y_n) = \left[c_1(y_n - \alpha) + c_2(y_n - \alpha)^2 + c_3(y_n - \alpha)^3 + \cdots\right]. \tag{2.15}$$

Simplifying (2.15), we have

$$f(y_n) = c_2(c_1\beta + 1)e_n^2 + \left(\frac{\beta c_1^3 c_3 - 2c_2^2 + 3\beta c_1^2 c_3 + 2c_1 c_3 - \beta^2 c_1^2 c_2^2 - 2\beta c_1 c_2^2}{c_1}\right)e_n^3 + \cdots. \tag{2.16}$$

The expansion of the particular term used in (2.8) is given as

$$\begin{aligned}
f[w_n, x_n] &= \left(\frac{f(w_n) - f(x_n)}{w_n - x_n}\right) \\
&= c_1 + (2c_2 + \beta c_1 c_2)e_n + \left(3c_3 + 3\beta c_1 c_3 + \beta^2 c_1^2 c_3 + \beta c_2^2\right)e_n^2 + \cdots, \\
f[w_n, y_n] &= \left(\frac{f(w_n) - f(y_n)}{w_n - y_n}\right) \\
&= c_1 + (c_2 + \beta c_1 c_2)e_n + \left(\frac{\beta c_1^2 c_3 + c_2^2 + 2\beta c_1 c_2^2 + 2\beta c_1^2 c_3 + c_1^3 c_3}{c_1}\right)e_n^2 + \cdots, \\
f[x_n, y_n] &= \left(\frac{f(x_n) - f(y_n)}{x_n - y_n}\right) = c_1 + c_2 e_n + \left(\frac{c_1 c_3 + c_2^2 + \beta c_1 c_2^2}{c_1}\right)e_n^2 + \cdots, \\
\frac{f[w_n, x_n]}{f[x_n, y_n] f[w_n, y_n]} &= \frac{1}{c_1} + \left(\frac{\beta c_1^2 c_3 - 3c_2^2 - 3\beta c_1 c_2^2 + c_1 c_3}{c_1^3}\right)e_n^2 + \cdots.
\end{aligned} \tag{2.17}$$

Substituting appropriate expressions in (2.8), we obtain

$$z_n - \alpha = y_n - \alpha - \left(\frac{f[w_n, x_n] f(y_n)}{f[x_n, y_n] f[w_n, y_n]} \right). \tag{2.18}$$

The Taylor series expansion of $f(z_n)$ about α is given as

$$f(z_n) = \left[c_1(z_n - \alpha) + c_2(z_n - \alpha)^2 + c_3(z_n - \alpha)^3 + \cdots \right]. \tag{2.19}$$

Simplifying (2.18), we have

$$f(z_n) = \left(\frac{2c_2^3 - c_1c_2c_3 + 4\beta c_1 c_2^3 + 2\beta^2 c_1^2 c_2^3 - 2\beta c_1^2 c_2 c_3 - c_1^3 c_2 c_3}{c_1^2} \right) e_n^4 + \cdots . \tag{2.20}$$

In order to evaluate the essential terms of (2.8), we expand term by term

$$\begin{aligned} f[y_n, z_n] &= \left(\frac{f(y_n) - f(z_n)}{y_n - z_n} \right) = c_1 + \left(\frac{\beta c_1 c_2^2 + c_2^2}{c_1} \right) e_n^2 + \cdots , \\ f[x_n, z_n] &= \left(\frac{f(x_n) - f(z_n)}{x_n - z_n} \right) = c_1 + c_2 e_n + c_3 e_n^2 + \cdots . \end{aligned} \tag{2.21}$$

Collecting the above terms,

$$\begin{aligned} \psi &= \left[f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n] = \right]^{-1} = \frac{1}{c_1} + \left(\frac{c_2c_3 + \beta c_1 c_2 c_3}{c_1^3} \right) e_n^3 + \cdots , \\ \omega &= \left(1 - \frac{f(z_n)}{f(w_n)} \right)^{-1} = 1 - \left(\frac{\beta c_1^2 c_2 c_3 - 2\beta c_1 c_2^3 + c_1 c_2 c_3 - 2c_2^3}{c_1^3} \right) e_n^3 + \cdots , \\ \xi &= \left(1 + \frac{2f(y_n)^3}{f(w_n)^2 f(x_n)} \right)^{-1} = 1 - \left(\frac{2\beta c_1 c_2^3 + 2c_2^3}{c_1^3} \right) e_n^3 + \cdots , \\ \omega\xi &= 1 - \left(\frac{\beta c_1 c_2 c_3 + c_2 c_3}{c_1^2} \right) e_n^3 + \cdots , \\ \psi\omega\xi &= \frac{1}{c_1} + \left(\frac{-\beta^2 c_1^2 c_2^4 + 6\beta c_1^2 c_2^2 c_3 + 3\beta^2 c_1^3 c_2^2 c_3 + \beta c_1 c_2^4 - 2\beta c_1^3 c_2 c_4 + 2c_2^4 - c_1^2 c_2 c_4 + 3c_1 c_2^2 c_3 - \beta^2 c_1^4 c_2 c_4}{c_1^5} \right) \\ &\quad \times e_n^4 + \cdots . \end{aligned} \tag{2.22}$$

Substituting appropriate expressions in (2.8), we obtain

$$e_{n+1} = z_n - \alpha - \psi\omega\xi f(z_n). \tag{2.23}$$

Simplifying (2.23), we obtain the error equation

$$\begin{aligned} e_{n+1} = -c_1^{-7}\Big[&30\beta^2 c_1^2 c_2^2 - 2c_1^2 c_2^4 c_4 + 26\beta c_1 c_2^7 + 14\beta^3 c_1^3 c_2^7 - 2c_1^2 c_2^3 c_3^2 + 9\beta^2 c_1^3 c_2^5 c_3 - 12\beta^2 c_1^4 c_2^4 c_4 \\ &- 8\beta^3 c_1^5 c_2^4 c_4 + 3\beta c_1^2 c_2^4 c_3 - 8\beta c_1^3 c_2^4 c_4 + 9\beta^3 c_1^4 c_2^5 c_3 - 12\beta^2 c_1^4 c_2^3 c_3^2 - 8\beta c_1^3 c_2^3 c_3^2 \\ &- 8\beta^3 c_1^3 c_2^3 c_3^2 + 8c_2^7 + 2\beta^4 c_1^4 c_2^7 + c_1^3 c_2^2 c_3 c_4 + \beta^4 c_1^7 c_2^2 c_3 c_4 - 2\beta^4 c_1^6 c_2^4 c_4 + 4\beta c_1^4 c_2^2 c_3 c_4 \\ &+6\beta^2 c_1^5 c_2^2 c_3 c_4 + 4\beta^3 c_1^5 c_2^2 c_3 c_4 - 2\beta^4 c_1^6 c_2^3 c_3^2 + 3\beta^4 c_1^5 c_2^5 c_3\Big] e_n^8. \end{aligned} \tag{2.24}$$

Expression (2.24) establishes the asymptotic error constant for the eighth order of convergence for the new eighth-order derivative-free method defined by (2.8). □

2.2. Method 2: Liu 1

The second of three-step eighth-order derivative-free method is constructed by combining the two-step fourth-order method presented by Liu et al. [2], and the third step is developed to achieve the eighth order. As before, we have introduced two weight functions in the third step in order to achieve the desired eighth-order method. In this particular case the two weight functions are expressed as

$$\begin{gathered} \left(1 - \frac{f(z_n)}{f(w_n)}\right)^{-1}, \\ \left(1 + \frac{2f(y_n)^3}{f(w_n)^2 f(x_n)}\right)^{-1}. \end{gathered} \tag{2.25}$$

Then the iteration scheme based on Liu et al. method is given as

$$z_n = y_n - \left[\frac{f(y_n)}{(f[x_n, y_n] + f[w_n, y_n] - f[w_n, x_n])}\right], \tag{2.26}$$

$$x_{n+1} = z_n - \left[1 - \frac{f(z_n)}{f(w_n)}\right]^{-1}\left[1 + \frac{f(y_n)^3}{f(x_n) f(z_n)^2}\right]^{-1}\left[\frac{f(z_n)}{(f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n])}\right], \tag{2.27}$$

where w_n, y_n are given in (2.8) and x_0 is the initial approximation provided that the denominators of (2.26)-(2.27) are not equal to zero.

Theorem 2.5. *Assume that the function f is sufficiently differentiable and f has a simple root $\alpha \in D$. If the initial approximation x_0 is sufficiently close to α, then the method defined by* (2.27) *converges to α with eighth order.*

Proof. Using appropriate expressions in the proof of Theorem 2.4 and substituting them into (2.27), we obtain the asymptotic error constant

$$
\begin{aligned}
e_{n+1} = -c_1^{-7}\Big[&12\beta^2 c_1^2 c_2^7 - c_1^2 c_2^4 c_4 + 10\beta c_1 c_2^7 - c_1 c_2^5 c_3 + 6\beta^3 c_1^3 c_2^7 - 2c_1^2 c_2^3 c_3^2 - 6\beta^2 c_1^4 c_2^4 c_4 \\
&- 4\beta^3 c_1^5 c_2^4 c_4 - 2\beta c_1^2 c_2^5 c_3 - 4\beta c_1^3 c_2^4 c_4 + 2\beta^3 c_1^4 c_2^5 c_3 - 12\beta^2 c_1^4 c_2^3 c_3^2 - 8\beta c_1^3 c_2^3 c_3^2 \\
&+ 3c_2^7 + \beta^4 c_1^4 c_2^7 + c_1^3 c_2^2 c_3 c_4 + \beta^4 c_1^7 c_2^2 c_3 c_4 - \beta^4 c_1^6 c_2^4 c_4 + 4\beta c_1^4 c_2^2 c_3 c_4 + 6\beta^2 c_1^5 c_2^2 c_3 c_4 \\
&+ 4\beta^3 c_1^6 c_2^2 c_3 c_4 - 2\beta^4 c_1^6 c_2^3 c_3^2 + \beta^4 c_1^5 c_2^5 c_3 - 8\beta^3 c_1^5 c_2^3 c_3^2\Big] e_n^8.
\end{aligned}
\tag{2.28}
$$

Expression (2.28) establishes the asymptotic error constant for the eighth order of convergence for the new eighth-order derivative-free method defined by (2.27). □

2.3. *Method 3: Liu 2*

The third of three-step eighth-order derivative-free method is constructed by combining the two-step fourth-order method presented by Liu et al. [2], and the third step is developed to achieve the eighth-order. As before, we have introduced two weight functions in the third step in order to achieve the desired eighth-order method. In this particular case the two weight functions are expressed as

$$
\left(1 - \frac{f(z_n)}{f(w_n)}\right)^{-1}, \tag{2.29}
$$

$$
\left(1 + \frac{2f(y_n)^3}{f(w_n)^2 f(x_n)}\right)^{-1}. \tag{2.30}
$$

Then the iteration scheme based on Liu et al. method is given as

$$
z_n = y_n - \left[\frac{f(y_n)\left(f[x_n, y_n] - f[w_n, y_n] + f[w_n, x_n]\right)}{\left(f[x_n, y_n]\right)^2}\right], \tag{2.31}
$$

$$
\begin{aligned}
x_{n+1} = z_n - &\left[1 - \left(\frac{f(z_n)}{f(w_n)}\right)\right]^{-1}\left[1 - \left(\frac{f(y_n)}{f(z_n)}\right)^3\right]\left[1 + \frac{f(y_n)^3}{f(x_n) f(z_n)^2}\right]^{-1} \\
&\times \left[\frac{f(z_n)}{\left(f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n]\right)}\right],
\end{aligned}
\tag{2.32}
$$

where w_n, y_n are given in (2.8) and x_0 is the initial approximation provided that the denominators of (2.31)-(2.32) are not equal to zero.

Theorem 2.6. *Assume that the function f is sufficiently differentiable and f has a simple root $\alpha \in D$. If the initial approximation x_0 is sufficiently close to α, then the method defined by* (2.32) *converges to α with eighth order.*

Proof. Using appropriate expressions in the proof of Theorem 2.4 and substituting them into (2.32), we obtain the asymptotic error constant

$$
\begin{aligned}
e_{n+1} = -c_1^{-7}\Big[&\beta^4 c_1^7 c_2^2 c_3 c_4 - \beta^4 c_1^6 c_2^4 c_4 - 2\beta^4 c_1^6 c_2^3 c_3^2 + \beta^4 c_1^5 c_2^5 c_3 + \beta^4 c_1^4 c_2^7 + 4\beta^3 c_1^6 c_2^2 c_3 c_4 \\
&- 5\beta^3 c_1^5 c_2^4 c_4 + 3\beta^3 c_1^4 c_2^5 c_3 + 8\beta^3 c_1^3 c_2^7 + 6\beta^2 c_1^5 c_2^2 c_3 c_4 - 12\beta^2 c_1^4 c_2^3 c_3^2 - 9\beta^2 c_1^4 c_2^4 c_4 \\
&+ 3\beta^2 c_1^3 c_2^5 c_3 + 21\beta^2 c_1^2 c_2^7 + 4\beta c_1^4 c_2^2 c_3 c_4 - 7\beta c_1^3 c_2^4 c_4 - 8\beta c_1^3 c_2^3 c_3^2 + \beta c_1^2 c_2^5 c_3 \\
&+22\beta c_1 c_2^7 - 2c_1^2 c_2^3 c_3^2 - 2c_1^2 c_2^4 c_4 + 8c_2^7 + c_1^3 c_2^2 c_3 c_4 - 8\beta^3 c_1^5 c_2^3 c_3^2\Big] e_n^8.
\end{aligned}
\tag{2.33}
$$

Expression (2.33) establishes the asymptotic error constant for the eighth order of convergence for the new eighth-order derivative-free method defined by (2.32). □

2.4. Method 4: SKK

The fourth of three-step eighth-order derivative-free method is constructed by combining the two-point fourth-order method presented by Khattri and Agarwal [10], and the third point is developed to achieve the eighth order. Here also, we have introduced two weight functions in the third step in order to achieve the desired eighth-order method. In this particular case the two weight functions are expressed as

$$
\begin{gathered}
\left(1 - \frac{f(z_n)}{f(w_n)}\right)^{-1}, \\
\left(1 + \frac{3f(y_n)^3}{f(w_n)^2 f(x_n)}\right)^{-1}.
\end{gathered}
\tag{2.34}
$$

Then the iteration scheme based on the Khattri and Agarwal method is given as

$$
w_n = x_n - \beta f(x_n), \tag{2.35}
$$

$$
y_n = x_n - \frac{f(x_n)^2}{f(x_n) - f(w_n)}, \tag{2.36}
$$

$$
z_n = y_n - \left(\frac{f(x_n) f(y_n)}{f(x_n) - f(w_n)}\right)\left[1 + \frac{f(y_n)}{f(x_n)} + \left(\frac{f(y_n)}{f(x_n)}\right)^2 + \frac{f(y_n)}{f(w_n)} + \left(\frac{f(y_n)}{f(w_n)}\right)^2\right], \tag{2.37}
$$

$$
x_{n+1} = z_n - \left[\frac{f(w_n)}{f(w_n) - f(z_n)}\right]\left[1 + \frac{3f(y_n)^3}{f(x_n) f(z_n)^2}\right]^{-1}\left[\frac{f(z_n)}{(f[y_n, z_n] - f[x_n, y_n] + f[x_n, z_n])}\right], \tag{2.38}
$$

where w_n, y_n are given in (2.8) and x_0 is the initial approximation provided that the denominators of (2.35)–(2.37) are not equal to zero.

Theorem 2.7. *Assume that the function f is sufficiently differentiable and f has a simple root $\alpha \in D$. If the initial approximation x_0 is sufficiently close to α, then the method defined by* (2.38) *converges to α with eighth order.*

Proof. Using appropriate expressions in the proof of Theorem 2.4 and substituting them into (2.38), we obtain the asymptotic error constant

$$\begin{aligned} e_{n+1} = -c_1^{-7}\Big[&8\beta c_1^3c_2^3c_3^2 + \beta^4 c_1^5c_2^5c_3 - 2\beta^4 c_1^6c_2^3c_3^2 + 15c_2^7 + 60\beta^2 c_1^2c_2^7 - 4\beta c_1^4c_2^2c_3c_4 + 6\beta^2 c_1^5c_2^2c_3c_4 \\ &+ c_1^3c_2^2c_3c_4 - 4\beta^3 c_1^6c_2^2c_3c_4 + 15\beta^4 c_1^4c_2^7 + \beta^4 c_1^7c_2^2c_3c_4 - 3\beta^4 c_1^6c_2^4c_4 - 8\beta c_1^2c_2^5c_3 - 3c_1^2c_2^4c_4 \\ &+ 12\beta c_1^3c_2^4c_4 - 18\beta^2 c_1^4c_2^4c_4 + 12\beta^3 c_1^5c_2^4c_4 - 2c_1^2c_2^3c_3^2 + c_1c_2^5c_3 - 48\beta c_1c_2^7 - 39\beta^3 c_1^3c_2^7 \\ &+\beta^5 c_1^6c_2^5c_3 - 3\beta^5 c_1^5c_2^7 - 11\beta^3 c_1^4c_2^5c_3 + 8\beta^3 c_1^5c_2^3c_3^2 + 16\beta c_1^3c_2^5c_3 - 12\beta^2 c_1^4c_2^3c_3^2\Big]e_n^8. \end{aligned} \tag{2.39}$$

Expression (2.38) establishes the asymptotic error constant for the eighth order of convergence for the new eighth-order derivative-free method defined by (2.38). □

3. The Kung-Traub Eighth-Order Derivative-Free Method

The classical Kung-Traub eighth-order derivative-free method considered is given in [3]. Since this method is well established, we will state the essential expressions used in order to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new eighth-order derivative-free methods. The Kung-Traub method is given as

$$\begin{aligned} z_n &= y_n - \left(\frac{f(x_n)f(w_n)}{f(y_n)-f(x_n)}\right)\left[\frac{1}{f[w,x]} - \frac{1}{f[w,y]}\right], \\ x_{n+1} &= z_n - \left(\frac{f(w_n)f(x_n)f(y_n)}{f(z_n)-f(x_n)}\right) \\ &\quad\times\left\{\left(\frac{1}{f(z_n)-f(w_n)}\right)\left[\frac{1}{f[y,z]} - \frac{1}{f[w,y]}\right] - \left(\frac{1}{f(y_n)-f(x_n)}\right)\left[\frac{1}{f[w,y]} - \frac{1}{f[w,x]}\right]\right\}, \end{aligned} \tag{3.1}$$

where w_n, y_n are given in (2.8) and x_0 is the initial approximation provided that the denominators of (3.1) are not equal to zero.

4. Application of the New Derivative-Free Iterative Methods

To demonstrate the performance of the new eighth-order methods, we take ten particular nonlinear equations. We will determine the consistency and stability of results by examining the convergence of the new derivative-free iterative methods. The findings are generalised by illustrating the effectiveness of the eighth-order methods for determining the simple root

Table 1: Test functions and their roots.

Function	Root
$f_1(x) = (x-2)(x^{10}+x+1)\exp(-x-1)$,	$\alpha = 2$
$f_2(x) = \exp(-x^2+x+2) - \cos(x+1) + x^3 + 1$,	$\alpha = -1$
$f_3(x) = \sin(x)^2 - x^2 + 1$,	$\alpha = 1.40449165\cdots$
$f_4(x) = e^{-x} - \cos(x)$,	$\alpha = -0.666273126\cdots$
$f_5(x) = \ln(x^2+x+2) - x + 1$,	$\alpha = 4.15259074\cdots$
$f_6(x) = \exp(x^2+7x-30) - 1$,	$\alpha = 1.40449165\cdots$
$f_7(x) = \cos(x)^2 - \dfrac{x}{5}$	$\alpha = 1.08598268\cdots$
$f_8(x) = \sin(x) - \dfrac{x}{2}$,	$\alpha = 0$
$f_9(x) = x^{10} - x^3 - x + 1$	$\alpha = 0.591448093\cdots$
$f_{10}(x) = \sin(x) - x + 1$	$\alpha = 2.63066415\cdots$

Table 2: Comparison of various iterative methods.

Functions	K-T	Liu1	Liu2	RT	SKK
$f_1(x), x_0 = 1.9$	$0.199e-37$	$0.121e-66$	$0.108e-66$	$0.114e-75$	—
$f_2(x), x_0 = -1.5$	$0.166e-158$	$0.832e-207$	$0.490e-282$	$0.128e-156$	$0.145e-100$
$f_3(x), x_0 = 1.6$	$0.172e-258$	$0.629e-417$	$0.204e-393$	$0.498e-297$	$0.308e-285$
$f_4(x), x_0 = 0.125$	$0.166e-1385$	$0.107e-1133$	$0.868e-800$	$0.566e-1060$	$0.609e-220$
$f_5(x), x_0 = 4.4$	$0.115e-927$	$0.350e-1017$	$0.268e-1022$	$0.471e-1025$	$0.639e-791$
$f_6(x), x_0 = 3.1$	$0.118e-32$	$0.581e-59$	$0.236e-58$	$0.224e-53$	$0.191e-12$
$f_7(x), x_0 = 0.75$	$0.756e-568$	$0.119e-571$	$0.333e-607$	$0.522e-557$	$0.367e-335$
$f_8(x), x_0 = 0.5$	$0.260e-317$	$0.651e-344$	$0.205e-301$	$0.298e-285$	$0.253e-685$
$f_9(x), x_0 = 0.5$	$0.293e-416$	$0.713e-433$	$0.347e-448$	$0.644e-423$	$0.535e-283$
$f_{10}(x), x_0 = 2.5$	$0.687e-739$	$0.439e-696$	$0.981e-698$	$0.121e-694$	$0.770e-643$

of a nonlinear equation. Consequently, we will give estimates of the approximate solution produced by the eighth-order methods and list the errors obtained by each of the methods. The numerical computations listed in the tables were performed on an algebraic system called Maple. In fact, the errors displayed are of absolute value, and insignificant approximations by the various methods have been omitted in Tables 1, 2, and 3.

Remark 4.1. The family of three-step methods requires four function evaluations and has the order of convergence eight. Therefore, this family is of optimal order and supports the Kung-Traub conjecture [3]. To determine the efficiency index of these new derivative-free methods, we will use Definition 2.3. Hence, the efficiency index of the eighth-order derivative-free methods given is $\sqrt[4]{8} \approx 1.682$.

Remark 4.2. The test functions and their exact root α are displayed in Table 1. The differences between the root α and the approximation x_n for test functions with initial approximation x_0 are displayed in Table 2. In fact, x_n is calculated by using the same total number of function evaluations (TNFEs) for all methods. Here, the TNFE for all the methods is 12. Furthermore, the computational order of convergence (COC) is displayed in Table 3.

Table 3: COC of various iterative methods.

Functions	K-T	Liu1	Liu2	RT	SKK
$f_1(x), x_0 = 1.9$	7.5146	7.5602	7.5593	7.5078	—
$f_2(x), x_0 = -1.5$	8.0021	7.9707	8.0000	8.0026	7.8899
$f_3(x), x_0 = 1.6$	8.0000	8.0000	8.0000	8.0000	7.9847
$f_4(x), x_0 = 0.125$	12.000	11.000	10.000	11.000	8.0001
$f_5(x), x_0 = 4.4$	8.0000	8.0000	8.0000	8.0000	8.0000
$f_6(x), x_0 = 3.1$	7.7384	7.5837	7.5694	7.4350	—
$f_7(x), x_0 = 0.75$	8.0000	8.0000	8.0000	8.0000	7.9889
$f_8(x), x_0 = 0.5$	10.973	10.977	10.970	10.967	10.994
$f_9(x), x_0 = 0.5$	8.0000	8.0000	8.0000	8.0000	7.9845
$f_{10}(x), x_0 = 2.5$	8.0000	8.0000	8.0000	8.0000	8.0000

5. Remarks and Conclusion

We have demonstrated the performance of a new family of eighth-order derivative-free methods. Convergence analysis proves that the new methods preserve their order of convergence. There are two major advantages of the eighth-order derivative-free methods. Firstly, we do not have to evaluate the derivative of the functions; therefore they are especially efficient where the computational cost of the derivative is expensive, and secondly we have established a higher order of convergence method than the existing derivative-free methods [4, 5]. We have examined the effectiveness of the new derivative-free methods by showing the accuracy of the simple root of a nonlinear equation. The main purpose of demonstrating the new eighth-order derivative-free methods for many different types of nonlinear equations was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, and the consistency of the results and to determine the efficiency of the new iterative method. In addition, it should be noted that like all other iterative methods, the new methods have their own domain of validity and in certain circumstances should not be used.

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