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Fundamental groups and Euler characteristics of sphere-like digital images

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Abstract

The current paper focuses on fundamental groups and Euler characteristics of various digital models of the 2-dimensional sphere. For all models that we consider, we show that the fundamental groups are trivial, and compute the Euler characteristics (which are not always equal). We consider the connected sum of digital surfaces and investigate how this operation relates to the fundamental group and Euler characteristic. We also consider two related but different notions of a digital image having "no holes," and relate this to the triviality of the fundamental group.

Many of our results have origins in the paper [15] by S.-E. Han, which contains many errors. We correct these errors when possible, and leave some open questions. We also present some original results.

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Keywords: *digital topology; digital image; fundamental group; Euler characteristic.*

1. Introduction

A digital image is a graph that models an object in a Euclidean space. In digital topology we study properties of digital images analogous to the geometric and topological properties of the objects in Euclidean space that the images model. Among these properties are digital versions of the fundamental group

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and the Euler characteristic. The current paper focuses on fundamental groups and Euler characteristics of various digital models of the 2-dimensional sphere.

Most of our results were explored by Han in [15], where many errors appear. We correct almost all of these errors, leaving open some questions, and also obtain some new results. Many of the errors in [15] result from inattention to basepoint preservation in homotopies of loops. The difference between pointed and unpointed homotopy turns out to be complex, and must be carefully considered. This issue has been explored in [13] and [9], and we continue that work in this paper. In particular, Example 2.9 shows that contractibility does not imply pointed contractibility. Errors also appear in the discussion of Euler characteristics in [15]. We correct these, many of which seem due to simple counting mistakes.

2. Preliminaries

2.1. **Fundamentals of digital topology.** Much of this section is quoted or paraphrased from other papers in digital topology, such as [2, 3, 8].

We will assume familiarity with the topological theory of digital images. See, e.g., [1] for the standard definitions. All digital images X are assumed to carry their own adjacency relations (which may differ from one image to another). When we wish to emphasize the particular adjacency relation we write the image as (X, κ), where κ represents the adjacency relation.

Among the commonly used adjacencies are the c_u-adjacencies. Let $x, y \in \mathbb{Z}^n$, $x \neq y$. Let u be an integer, $1 \leq u \leq n$. We say x and y are c_u-adjacent if

- There are at most u indices i for which $|x_i - y_i| = 1$.
- For all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

We often label a c_u-adjacency by the number of points adjacent to a given point in $\mathbb{Z}^n$ using this adjacency. E.g.,

- In $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency.
- In $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency.
- In $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

Definition 2.1. A subset Y of a digital image (X, κ) is *κ-connected* [21], or *connected* when κ is understood, if for every pair of points $a, b \in Y$ there exists a sequence $P = \{y_i\}_{i=0}^m \subset Y$ such that $a = y_0$, $b = y_m$, and y_i and y_{i+1} are κ-adjacent for $0 \leq i < m$. P is then called a *path* from a to b in Y.

The following generalizes a definition of [21].

Definition 2.2 ([2])**.** Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is (κ, λ)-continuous if for every κ-connected $A \subset X$ we have that $f(A)$ is a λ-connected subset of Y.

When the adjacency relations are understood, we will simply say that f is *continuous*. Continuity can be reformulated in terms of adjacency of points:

Theorem 2.3 ([21, 2])**.** *A function $f : X \to Y$ is continuous if and only if, for any adjacent points $x, x' \in X$, the points $f(x)$ and $f(x')$ are equal or adjacent.* □

See also [11, 12], where similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings*.

It is perhaps unfortunate that "path" is also used with a meaning that is related to but distinct from the above. We will also use the following.

Definition 2.4 (see [17])**.** A $\kappa-$*path* in a digital image X is a $(2, \kappa)-$continuous function $f : [0, m]_{\mathbb{Z}} \to X$. If, further, $f(0) = f(m)$, we call f a *digital $\kappa-$loop*, and the point $f(0)$ is the *basepoint* of the loop f. If f is a constant function, it is called a *trivial loop*.

Other terminology we use includes the following. Given a digital image $(X, \kappa) \subset \mathbb{Z}^n$ and $x \in X$, the set of points adjacent to $x \in \mathbb{Z}^n$, the neighborhood of x in $\mathbb{Z}^n$, and the boundary of X in $\mathbb{Z}^n$ are, respectively,

$$N_\kappa(x) = \{y \in \mathbb{Z}^n \mid y \text{ is } \kappa\text{-adjacent to } x\},$$
$$N_\kappa^*(x) = N_\kappa(x) \cup \{x\},$$

and

$$\delta_\kappa(X) = \{y \in X \mid N_\kappa(y) \setminus X \neq \varnothing\}.$$

2.2. **Digital homotopy.** Material appearing in this section is largely quoted or paraphrased from other papers in digital topology. See, e.g., [10].

A homotopy between continuous functions may be thought of as a continuous deformation of one of the functions into the other over a finite time period.

Definition 2.5 ([2]; see also [17])**.** Let X and Y be digital images. Let $f, g : X \to Y$ be (κ, κ')-continuous functions. Suppose there is a positive integer m and a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

- for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by
$$F_x(t) \;=\; F(x, t) \text{ for all } t \in [0, m]_{\mathbb{Z}}$$
is $(2, \kappa')-$continuous. That is, $F_x(t)$ is a path in Y.
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$ defined by
$$F_t(x) \;=\; F(x, t) \text{ for all } x \in X$$
is $(\kappa, \kappa')-$continuous.

Then F is a digital $(\kappa, \kappa')-$homotopy between f and g, and f and g are digitally $(\kappa, \kappa')-$homotopic in Y. If for some $x \in X$ we have $F(x, t) = F(x, 0)$ for all $t \in [0, m]_{\mathbb{Z}}$, we say F holds x fixed, and F is a pointed homotopy.

We denote a pair of homotopic functions as described above by $f \simeq_{\kappa,\kappa'} g$. When the adjacency relations κ and κ' are understood in context, we say f and g are *digitally homotopic* to abbreviate "digitally $(\kappa, \kappa')-$homotopic in Y," and write $f \simeq g$.

Proposition 2.6 ([17, 2]). *Digital homotopy is an equivalence relation among digitally continuous functions $f : X \to Y$.*

Definition 2.7 ([3]). Let $f : X \to Y$ be a (κ, κ')-continuous function and let $g : Y \to X$ be a (κ', κ)-continuous function such that

$$f \circ g \simeq_{\kappa',\kappa'} 1_X \text{ and } g \circ f \simeq_{\kappa,\kappa} 1_Y.$$

Then we say X and Y have the same (κ, κ')-homotopy type and that X and Y are (κ, κ')-homotopy equivalent, denoted $X \simeq_{\kappa,\kappa'} Y$ or as $X \simeq Y$ when κ and κ' are understood. If for some $x_0 \in X$ and $y_0 \in Y$ we have $f(x_0) = y_0$, $g(y_0) = x_0$, and there exists a homotopy between $f \circ g$ and 1_X that holds x_0 fixed, and a homotopy between $g \circ f$ and 1_Y that holds y_0 fixed, we say (X, x_0, κ) and (Y, y_0, κ') are pointed homotopy equivalent and that (X, x_0) and (Y, y_0) have the same pointed homotopy type, denoted $(X, x_0) \simeq_{\kappa,\kappa'} (Y, y_0)$ or as $(X, x_0) \simeq (Y, y_0)$ when κ and κ' are understood.

It is easily seen, from Proposition 2.6, that having the same homotopy type (respectively, the same pointed homotopy type) is an equivalence relation among digital images (respectively, among pointed digital images).

For $p \in Y$, we denote by $\overline{p}$ the constant function $\overline{p} : X \to Y$ defined by $\overline{p}(x) = p$ for all $x \in X$.

Definition 2.8. A digital image (X, κ) is κ-contractible [17, 1] if its identity map is (κ, κ)-homotopic to a constant function $\overline{p}$ for some $p \in X$. If the homotopy of the contraction holds p fixed, we say (X, p, κ) is pointed κ-contractible.

When κ is understood, we speak of *contractibility* for short. It is easily seen that X is contractible if and only if X has the homotopy type of a one-point digital image.

The following is the first example in the literature of a digital image that is contractible but is not pointed contractible.

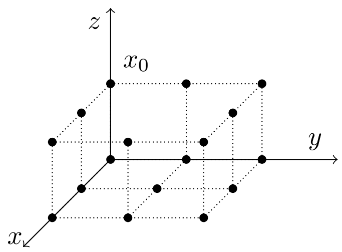

FIGURE 1. The image X discussed in Example 2.9. All coordinates in this paper are ordered according to the axes in this Figure.

Example 2.9. Let $X = ([0, 2]_{\mathbb{Z}}^2 \times [0, 1]_{\mathbb{Z}}) \setminus \{(1, 1, 1)\}$ (see Figure 1). Let $x_0 = (0, 0, 1) \in X$. Then X is 6-contractible, but (X, x_0) is not pointed 6-contractible.

Proof. We show X is 6-contractible as follows. Let $H : X \times [0,5]_{\mathbb{Z}} \to X$ be defined by

$$H(x,0) = x; \quad H(a,b,c,1) = (a,b,0);$$
$$H(a,b,c,t) = (a, \max\{0, b+1-t\}, 0) \text{ for } t \in \{2,3\};$$
$$H(a,b,c,t) = (\max\{0, a+3-t\}, 0, 0) \text{ for } t \in \{4,5\}.$$

It is easy to see that H is a 6-contraction of X.

Let

$$P = ([0,2]_{\mathbb{Z}}^2 \setminus \{(1,1)\}) \times \{1\} \subset X.$$

Let $K : X \times [0,m]_{\mathbb{Z}} \to X$ be a 6-contraction of X. As a simple closed curve of more than 4 points, P is not contractible [6], so there exist $(a,b,1) \in P$ and some t_0 such that $K(a,b,1,t_0) \in [0,2]_{\mathbb{Z}} \times \{0\}$. Since the induced function K_{t_0} is 6-continuous, we must have $K_{t_0}(P) \subset [0,2]_{\mathbb{Z}} \times \{0\}$ (note this argument is suggested by the notion of "path-pulling" homotopy discussed in [13]). In particular, $K_{t_0}(x_0) \in [0,2]_{\mathbb{Z}} \times \{0\}$, so K is not a pointed contraction. Since K is an arbitrary contraction of X, it follows that (X, x_0) is not pointed contractible. □

Definition 2.10. A continuous function $f : X \to Y$ is *nullhomotopic* in Y if f is homotopic in Y to a constant function. A pointed continuous function $f : (X, x_0) \to (Y, y_0)$ is *pointed nullhomotopic* in Y if f is pointed homotopic in Y to the constant function $\overline{y_0}$. A subset Y of X is *nullhomotopic* in X if the inclusion map $i : Y \to X$ is nullhomotopic in X. A pointed subset (Y, x_0) of (X, x_0) is *pointed nullhomotopic* in X if the inclusion map $i : (Y, x_0) \to (X, x_0)$ is pointed nullhomotopic in X.

2.3. **Digital Loops.** Material in this section is largely quoted or paraphrased from [4].

If f and g are digital $\kappa-$paths in X such that g starts where f ends, the *product* (see [17]) of f and g, written $f \cdot g$, is, intuitively, the $\kappa-$path obtained by following f by g. Formally, if $f : [0, m_1]_{\mathbb{Z}} \to X$, $g : [0, m_2]_{\mathbb{Z}} \to X$, and $f(m_1) = g(0)$, then $(f \cdot g) : [0, m_1 + m_2]_{\mathbb{Z}} \to X$ is defined by

$$(f \cdot g)(t) = \begin{cases} f(t) & \text{if } t \in [0, m_1]_{\mathbb{Z}}; \\ g(t - m_1) & \text{if } t \in [m_1, m_1 + m_2]_{\mathbb{Z}}. \end{cases}$$

It is undesirable to restrict homotopy classes of loops to loops defined on the same digital interval. The following notion of *trivial extension* allows a loop to "stretch" and remain in the same pointed homotopy class. Intuitively, f' is a trivial extension of f if f' follows the same path as f, but more slowly, with pauses for rest (subintervals of the domain on which f' is constant).

Definition 2.11 ([2])**.** Let f and f' be $\kappa-$paths in a digital image X. We say f' is a trivial extension of f if there are sets of $\kappa-$paths $\{f_1, f_2, \ldots, f_k\}$ and $\{F_1, F_2, \ldots, F_p\}$ in X such that

(1) $k \le p$;
(2) $f = f_1 \cdot f_2 \cdot \ldots \cdot f_k$;
(3) $f' = F_1 \cdot F_2 \cdot \ldots \cdot F_p$; and

(4) there are indices $1 \leq i_1 < i_2 < \ldots < i_k \leq p$ such that
 - $F_{i_j} = f_j$, $1 \leq j \leq k$, and
 - $i \notin \{i_1, i_2, \ldots, i_k\}$ implies F_i is a trivial loop.

This notion allows us to compare the digital homotopy properties of loops whose domains may have differing cardinality, since if $m_1 \leq m_2$, we can obtain a trivial extension of a loop $f : [0, m_1]_{\mathbb{Z}} \to X$ to $f' : [0, m_2]_{\mathbb{Z}} \to X$ via

$$f'(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq m_1; \\ f(m_1) & \text{if } m_1 \leq t \leq m_2. \end{cases}$$

We use the following notions to define the class of a pointed loop.

Definition 2.12. Let $f, g : [0, m]_{\mathbb{Z}} \to (X, x_0)$ be digital loops with basepoint x_0. If $H : [0, m]_{\mathbb{Z}} \times [0, M]_{\mathbb{Z}} \to X$ is a digital homotopy between f and g such that for all $t \in [0, M]_{\mathbb{Z}}$ we have

$$H(0, t) \;=\; H(m, t),$$

we say H is loop-preserving. If, further, for all $t \in [0, M]_{\mathbb{Z}}$ we have

$$H(0, t) \;=\; H(m, t) \;=\; x_0,$$

we say H holds the endpoints fixed.

Digital $\kappa-$loops f and g in X with the same basepoint p *belong to the same $\kappa-$loop class in X* if there are trivial extensions f' and g' of f and g, respectively, whose domains have the same cardinality, and a homotopy between f' and g' that holds the endpoints fixed [2].

Membership in the same loop class in (X, x_0) is an equivalence relation among digital $\kappa-$loops [2].

We denote by $[f]$ the loop class of a loop f in X. We have the following.

Proposition 2.13 ([2, 17])**.** *Suppose f_1, f_2, g_1, g_2 are digital loops in a pointed digital image (X, x_0), with $f_2 \in [f_1]$ and $g_2 \in [g_1]$. Then $f_2 \cdot g_2 \in [f_1 \cdot g_1]$.*

2.4. **Digital fundamental group.** Inspired by the fundamental group of a topological space, several researchers [22, 18, 2, 9] have developed versions of a fundamental group for digital images. These are not all equivalent; however, it is shown in [9] that the version of the fundamental group developed in that paper is equivalent to the version in [2]. In this paper, we use the version of the fundamental group developed in [2].

Material appearing in this section is largely quoted or paraphrased from other papers in digital topology. See, e.g., [2, 4].

Let (X, p, κ) be a pointed digital image. Consider the set $\Pi_1^\kappa(X, p)$ of κ-loop classes $[f]$ in X with basepoint p. By Proposition 2.13, the *product* operation

$$[f] * [g] \;=\; [f \cdot g]$$

is well-defined on $\Pi_1^\kappa(X, p)$. The operation $*$ is associative on $\Pi_1^\kappa(X, p)$ [17].

Lemma 2.14 ([2])**.** *Let (X, p) be a pointed digital image. Let $\overline{p} : [0, m]_{\mathbb{Z}} \to X$ be the constant function $\overline{p}(t) = p$. Then $[\overline{p}]$ is an identity element for $\Pi_1^\kappa(X, p)$.*

Lemma 2.15 ([2]). *If $f : [0, m]_{\mathbb{Z}} \to X$ represents an element of $\Pi_1(X, p)$, then the function $g : [0, m]_{\mathbb{Z}} \to X$ defined by*

$$g(t) = f(m - t) \text{ for } t \in [0, m]_{\mathbb{Z}}$$

is an element of $[f]^{-1}$ in $\Pi_1^{\kappa}(X, p)$.

Theorem 2.16 ([2]). *$\Pi_1^{\kappa}(X, p)$ is a group under the $*$ product operation, the* κ-fundamental group of (X, p).

It follows from the next result that in a connected digital image X, the digital fundamental group is independent of the choice of basepoint.

Theorem 2.17 ([2]). *Let X be a digital image with adjacency relation κ. If p and q belong to the same $\kappa-$component of X, then $\Pi_1^{\kappa}(X, p)$ and $\Pi_1^{\kappa}(X, q)$ are isomorphic groups.*

Despite the existence of images that are homotopy equivalent but not pointed homotopy equivalent ([13, 9], Example 2.9), notice that we do not require the homotopy equivalence in the following theorem to be a pointed homotopy equivalence.

Theorem 2.18 ([9]). *Let $f : (X, \kappa) \to (Y, \lambda)$ be a (κ, λ)-homotopy equivalence of connected digital images. Then $\Pi_1^{\kappa}(X, x_0)$ and $\Pi_1^{\lambda}(Y, f(x_0))$ are isomorphic groups.*

Similarly, in the following we do not require pointed contractibility.

Corollary 2.19. *If (X, κ) is a contractible digital image, then $\Pi_1^{\kappa}(X, x_0)$ is a trivial group.*

Proof. Since X is contractible, X is homotopy equivalent to a one-point image, which has a trivial fundamental group. The assertion follows from Theorem 2.18. □

3. Fundamental groups of 2-spheres

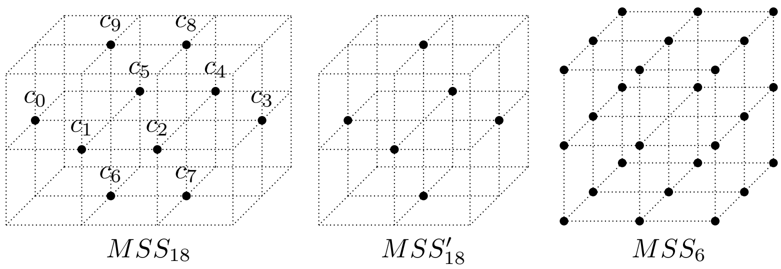

Figure 2. Three different digital images that model the 2-sphere

The following digital images are considered in [15]. Each in some sense models the 2-dimensional sphere S^2 in Euclidean 3-space (see Figure 2).

- $MSS_{18} = \{c_i\}_{i=0}^{9}$, where

$$c_0 = (0,0,0), c_1 = (1,1,0), c_2 = (1,2,0), c_3 = (0,3,0), c_4 = (-1,2,0),$$

$$c_5 = (-1,1,0), c_6 = (0,1,-1), c_7 = (0,2,-1), c_8 = (0,2,1), c_9 = (0,1,1).$$

- $MSS'_{18} = MSS'_{26} = \{(0,0,0),(1,1,0),(0,2,0),(-1,1,0),(0,1,-1),(0,1,1)\}$
- $MSS_6 = [0,2]_{\mathbb{Z}}^3 \setminus \{(1,1,1)\}$.

In this section we will show that fundamental groups Π_1^k for $k \in \{6,18,26\}$ for each of these images are trivial groups. These computations are attempted in [15, Lemma 3.3], but in each case the argument is incorrect or incomplete.

We begin with MSS_{18}, which is simplest for $k=6$.

Proposition 3.1. *Let $x \in MSS_{18}$. Then $\Pi_1^6(MSS_{18},x)$ is trivial.*

Proof. The 6-component of x in MSS_{18} is $\{x\}$. Thus, every 6-loop in (MSS_{18},x) is a trivial loop, and the assertion follows. □

For 18-adjacency we will use the following lemma:

Lemma 3.2. *Let $f : [0,m]_{\mathbb{Z}} \to MSS_{18}$ be a c_0-based 18-loop. Then there is a c_0-based 18-loop $f' : [0,m]_{\mathbb{Z}} \to MSS_{18} \setminus \{c_3\}$ with $[f] = [f']$ in $\Pi_1^{18}(MSS_{18},c_0)$.*

Proof. We may assume that f is not a trivial extension of another loop. Let t be the minimal number with $f(t) = c_3$. Since f is a loop based at c_0, and c_0 is 3 distant from c_3 in MSS_{18}, we must have $3 \le t \le m-3$. Since f is not a trivial extension of another loop, we must have $f(t-1) \leftrightarrow c_3 \leftrightarrow f(t+1)$, where $\leftrightarrow$ means 18-adjacent (and not equal).

Examining the structure of MSS_{18} we see that there will always be some point $c \in \{c_2,c_4,c_8,c_7\}$ with $f(t-1) \Leftrightarrow c \Leftrightarrow f(t+1)$, where $\Leftrightarrow$ means "equal or 18-adjacent." Now define $f_1 : [0,m]_{\mathbb{Z}} \to MSS_{18}$ by:

$$f_1(x) = \begin{cases} f(x) & \text{if } x \neq t, \\ c & \text{if } x = t. \end{cases}$$

By our choice of c, this f_1 will be continuous, and is 18-homotopic to f in one time step. Because $3 \le t \le m-3$, this homotopy holds the endpoints fixed. Thus $[f] = [f_1]$ in $\Pi_1^{18}(MSS_{18})$.

Note that f_1 meets the point c_3 one time fewer than f does. By applying the construction above, again, to f_1, we obtain a loop f_2 with $[f_2] = [f]$ which meets the point c_3 two fewer times than f does. Iterating this construction eventually gives a loop f' with $[f'] = [f]$ which never meets c_3, as desired. □

The following statement corresponds to [15, Lemma 3.3 (1)]. The argument given for proof in [15] merely demonstrates a specific 18-loop in MSS_{18} and shows that it is contractible, rather than showing that all such loops are contractible holding the endpoints fixed.

Proposition 3.3. *Let $x \in MSS_{18}$. Then $\Pi_1^{18}(MSS_{18},x)$ is a trivial group.*

Proof. It suffices to consider the case where $x = c_0$. Let $f : [0, M]_{\mathbb{Z}} \to MSS_{18}$ be a loop based at c_0. We will show that $[f] = [\bar{c}_0]$, where $\bar{c}_0$ is the constant path at c_0. By Lemma 3.2 we have a loop f' with $[f] = [f']$ and $c_3 \notin f'([0, M]_{\mathbb{Z}})$.

For $t \in \mathbb{Z}$, let $Q_t : \mathbb{Z}^3 \to \mathbb{Z}^3$ be defined by $Q_t(a, b, c) = (a, \min(b, t), c)$. Since f' avoids c_3, we have $Q_2 \circ f' = f'$. It is also clear that $f' \simeq Q_1 \circ f'$ by a one-step homotopy, and that $Q_1 \circ f'$ is a path meeting only points that are adjacent or equal to c_0. Thus $Q_1 \circ f' \simeq \bar{c}_0$ by a one-step homotopy, where $\bar{c}_0$ is the constant path at c_0. All of these homotopies fix the endpoints, and so we have $[f] = [f'] = [\bar{c}_0]$ as desired. □

Our final case for MSS_{18} uses 26-adjacency. Informally, since we have already shown that all loops in MSS_{18} are 18-contractible, it should follow that they are 26-contractible, since any 18-contraction is also automatically a 26-contraction. Further, we can easily see that every 26-loop in MSS_{18} is 26-homotopic in 1 step to an 18-loop in MSS_{18} with the homotopy holding the endpoints fixed. This intuition leads to the following lemma which is quite general. Below, $\leftrightarrows_\kappa$ means "κ-adjacent or equal".

Lemma 3.4. *Let X be a digital image with two adjacency relations κ and λ. Assume that if $x \leftrightarrows_\kappa y$ then $x \leftrightarrows_\lambda y$, and that if $x \leftrightarrows_\lambda y$ then there is a κ-path in X of length 2 from x to y. If $\Pi_1^\kappa(X, x_0)$ is trivial for some $x_0 \in X$, then $\Pi_1^\lambda(X, x_0)$ is trivial.*

Proof. Let $h : [0, m]_{\mathbb{Z}} \to X$ be a λ-loop based at x_0. Let $\bar{h} : [0, 2m]_{\mathbb{Z}} \to X$ be the trivial extension of h defined by

$$\bar{h}(s) = \begin{cases} h(s/2) & \text{if } s \text{ is even;} \\ \bar{h}(s-1) & \text{if } s \text{ is odd.} \end{cases}$$

Since $\bar{h}(2u) \leftrightarrows_\lambda \bar{h}(2(u+1))$, there is a κ-path, which is therefore a λ-path, of length 2 from $\bar{h}(2u)$ to $\bar{h}(2(u+1))$ through some $x_u \in X$. Therefore, the function $H : [0, 2m]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}} \to X$ defined by

$$H(s,t) = \begin{cases} \bar{h}(s) & \text{if } s \text{ is even;} \\ \bar{h}(s) & \text{if } s \text{ is odd and } t = 0; \\ x_u & \text{if } s = 2u+1 \text{ and } t = 1, \end{cases}$$

is a λ-homotopy from $\bar{h}$ to a κ-loop h' that keeps the endpoints fixed. Therefore, we have

$$[h]_\lambda = [\bar{h}]_\lambda = [h']_\lambda, \tag{3.1}$$

where the subscript λ indicates that we are considering the loop class in $\Pi_1^\lambda(X, x_0)$.

Since $\Pi_1^\kappa(X, x_0)$ is trivial, there is a trivial extension h'' of h' that is κ-homotopic, hence λ-homotopic, to a trivial loop $\overline{x_0}$ keeping the endpoints fixed. Therefore, $[h']_\lambda = [h'']_\lambda = [\overline{x_0}]_\lambda$. With equation (3.1), this implies $[h]_\lambda = [\overline{x_0}]_\lambda$. The assertion follows. □

The hypothesis above concerning paths of length 2 could possibly be weakened, but some form of this restriction is necessary. As a counterexample to a more general statement, consider the example in Figure 3. Here $X \subset \mathbb{Z}^2$, and 4-adjacency implies 8-adjacency, but $\Pi_1^4(X)$ is trivial and $\Pi_1^8(X)$ is infinite cyclic.

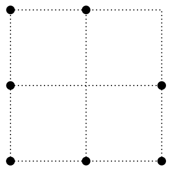

FIGURE 3. 4-adjacency implies 8-adjacency, but $\Pi_1^4(X)$ is trivial while $\Pi_1^8(X)$ is not.

The lemma immediately leads to the following (which does not appear in [15]):

Proposition 3.5. *Let $x \in MSS_{18}$. Then $\Pi_1^{26}(MSS_{18}, x)$ is a trivial group.*

Proof. We will apply Lemma 3.4 with $\kappa = 18$ and $\lambda = 26$. Observe that any two 26-adjacent points of MSS_{18} can be connected by a 18-path of length 2. Then since $\Pi_1^{18}(MSS_{18})$ is trivial, Lemma 3.4 shows that $\Pi_1^{26}(MSS_{18})$ is trivial. □

Since $MSS'_{18} = MSS'_{26}$, Proposition 3.6 below encompasses [15, Lemma 3.3 (2), (4)], and agrees with the assertions given in that paper. The proofs given in [15] are incomplete. The argument for Lemma 3.3 (2) claims only that MSS'_{18} is contractible, implying the use of a result like Corollary 2.19 to complete the proof. No result like Corollary 2.19 appears in [15]; in fact our proof of Corollary 2.19 depends on a nontrivial recent result in [9]. The argument for Lemma 3.3 (4) in [15] merely asserts without proof that every 6-loop in MSS'_k is 6-nullhomotopic; further, the argument neglects to require such nullhomotopies to fix the endpoints.

Proposition 3.6. *Let $x \in MSS'_{18}$. Then $\Pi_1^k(MSS'_{18}, x)$ is a trivial group, $k \in \{6, 18, 26\}$.*

Proof. For $k = 6$, the 6-component of x in MSS'_{18} is $\{x\}$. Thus, every 6-loop in (MSS'_{18}, x) is a trivial loop, and the assertion follows.

For $k \in \{18, 26\}$, we observe that Proposition 4.1 of [4] shows that MSS'_{18} is 26-contractible; indeed, its proof shows that MSS'_{18} is pointed 26-contractible; and the same argument shows that MSS'_{18} is pointed 18-contractible. The assertion follows from Corollary 2.19. □

[15] states incorrectly in Lemma 3.3 (3) that $\Pi_1^6(MSS_6)$ is a free group with two generators. Specifically it is claimed that the following equatorial loop represents a nontrivial generator of $\Pi_1^6(MSS_6)$:

$$D = ((0,0,1),(1,0,1),(2,0,1),(2,1,1),(2,2,1),(1,2,1),(0,2,1),(0,1,1),(0,0,1))$$

But D is in fact trivial in $\Pi_1^6(MSS_6)$. Consider the following sequence of loops, starting with a trivial extension of D:

$$((0,0,1),(0,0,1),(1,0,1),(2,0,1),(2,1,1),(2,2,1),(1,2,1),(0,2,1),(0,1,1),(0,0,1),(0,0,1))$$
$$((0,0,1),(0,0,2),(1,0,2),(2,0,2),(2,1,2),(2,2,2),(1,2,2),(0,2,2),(0,1,2),(0,0,2),(0,0,1))$$
$$((0,0,1),(0,0,2),(1,0,2),(2,0,2),(2,1,2),(2,1,2),(1,1,2),(0,1,2),(0,1,2),(0,0,2),(0,0,1))$$
$$((0,0,1),(0,0,2),(1,0,2),(2,0,2),(2,0,2),(2,0,2),(1,0,2),(0,0,2),(0,0,2),(0,0,2),(0,0,1))$$
$$((0,0,1),(0,0,2),(1,0,2),(1,0,2),(1,0,2),(1,0,2),(1,0,2),(0,0,2),(0,0,2),(0,0,2),(0,0,1))$$
$$((0,0,1),(0,0,2),(0,0,2),(0,0,2),(0,0,2),(0,0,2),(0,0,2),(0,0,2),(0,0,2),(0,0,2),(0,0,1))$$
$$((0,0,1),(0,0,1),(0,0,1),(0,0,1),(0,0,1),(0,0,1),(0,0,1),(0,0,1),(0,0,1),(0,0,1),(0,0,1))$$

This sequence of loops gives a homotopy that holds the endpoints fixed, from a trivial extension of D to a trivial loop, and thus D represents a trivial fundamental group element. In fact, arguments similar to those used in Lemma 3.2 and Proposition 3.3 can be repeated for MSS_6: any 6-loop can be first moved to avoid the point $(1,2,1)$, and then composed with Q_t to contract fully. We obtain the following, which also follows as a case of Theorem 3.1 of the paper [4]:

Proposition 3.7. *Let $x \in MSS_6$. Then $\Pi_1^6(MSS_6, x)$ is a trivial group.*

Immediately we can also compute the other fundamental groups of MSS_6 (this result does not appear in [15]):

Proposition 3.8. *Let $x \in MSS_6$. Then $\Pi_1^k(MSS_6, x)$ is a trivial group for $k \in \{18, 26\}$.*

Proof. For $k = 18$, apply Lemma 3.4 using $\kappa = 6$ and $\lambda = 18$.
For $k = 26$ we apply Lemma 3.4 using $\kappa = 18$ and $\lambda = 26$. □

4. Fundamental groups for connected sums of digital 2-spheres

Han [14] defines the *connected sum* of two digital surfaces X and Y, denoted $X\sharp Y$. Roughly, the idea behind this operation is that one removes the interior of a minimal (depending on the adjacency used) simple closed curve from each of X and Y such that there is an isomorphism F between these simple closed curves, and sews together the remainders of X and Y along these simple close curves by identifying points that are matched by F. This minimal simple closed curve, together with its interior, is denoted A_k, and called a *digital disk*. [15] uses three different digital disks, shown in Figure 4. See [14] for details of the definition of the $\sharp$ operation.

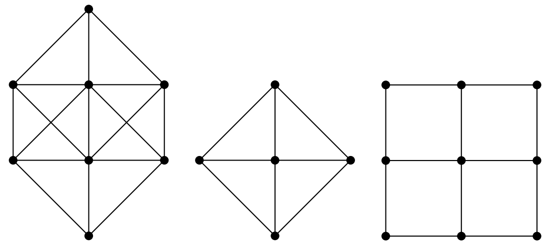

FIGURE 4. Various "digital disks" that are used to form connected sums

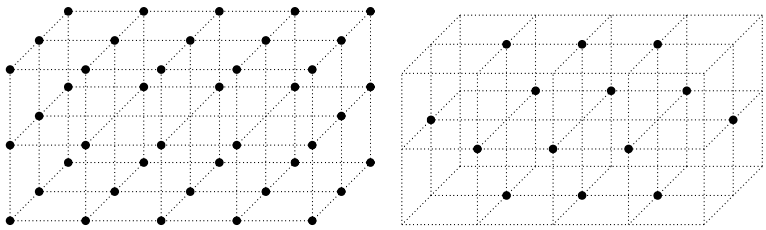

FIGURE 5. Connected sums of some images from Figure 2.

Combining the spheres from Figure 2 by the $\sharp$ operation gives new digital images, shown in Figure 5. In this section we show that both of these images have trivial fundamental groups.

Theorem 3.4(1) of [15] asserts that $\Pi_1^6(MSS_6 \sharp MSS_6)$ is a group with two generators, but this is a propagation of errors from the computation of $\Pi_1^6(MSS_6)$ in that paper. Theorem 3.4(2) of [15] says $\Pi_1^{18}(MSS_{18} \sharp MSS_{18})$ is trivial but justifies it only by showing that a single 18-loop is contractible. In fact, following again the arguments used for Lemma 3.2 and Proposition 3.3 we obtain a correct proof for the following, corresponding to Theorem 3.4(1) and Theorem 3.4(2) of [15].

Theorem 4.1. *$\Pi_1^6(MSS_6 \sharp MSS_6)$ and $\Pi_1^{18}(MSS_{18} \sharp MSS_{18})$ are trivial groups.*

Theorem 3.4(3) and Theorem 3.4(4) of [15] both make assertions about $\Pi_1^{18}(SS_{18})$. However, SS_{18} is not defined in [15].

If "SS_{18}" is interpreted as "MSS_{18}", then Theorem 3.4(3) of [15] would say that $\Pi_1^{18}(MSS_{18} \sharp MSS_{18})$ is isomorphic to $\Pi_1^{18}(MSS_{18})$. This assertion would be redundant in light of Lemma 3.3(1) of [15], which (as corrected above at Proposition 3.1) says $\Pi_1^{18}(MSS_{18})$ is trivial, and Theorem 3.4(2) of [15] (as corrected above at Theorem 4.1), which says that $\Pi_1^{18}(MSS_{18} \sharp MSS_{18})$ is a trivial group.

Again interpreting "SS_{18}" as "MSS_{18}", then Theorem 3.4(4) of [15] would say that $\Pi_1^{18}(MSS'_{18}\sharp MSS_{18})$ is isomorphic to $\Pi_1^{18}(MSS_{18})$. This follows since each is trivial. That $\Pi_1^{18}(MSS_{18})$ is trivial is shown above in Proposition 3.3. That $\Pi_1^{18}(MSS'_{18}\sharp MSS_{18})$ is trivial follows from an argument similar to that used to prove Theorem 4.1, or by observing that $MSS'_{18}\sharp MSS_{18}$ is (18,18)-isomorphic to MSS_{18}.

Theorem 3.4(5) of [15] says that $\Pi_1^{k}(MSS'_{k}\sharp MSS'_{k})$ is a trivial group for $k \in \{18, 26\}$. The assertion is correct, and the argument given in proof is basically correct; its only flaw is in its dependence on the incorrectly proven Lemma 3.3(2) of [15]. Since our Proposition 3.6 gives a correct proof of Lemma 3.3(2) of [15], we can accept the assertion of Theorem 3.4(5) of [15].

Theorem 3.4(6) of [15] makes an assertion about $\Pi_1^{18}(SS_{26})$. However, SS_{26} isn't defined in [15]. If we interpret "SS_{26}" as "MSS'_{26}" then Han's Theorem 3.4(6) would say that $\Pi_1^{26}(MSS'_{26}\sharp MSS'_{26})$ is isomorphic to $\Pi_1^{26}(MSS'_{26})$. This assertion can be correctly proven by observing that $(MSS'_{26}\sharp MSS_{26})$ is $(26, 26)$-isomorphic to MSS'_{26}.

5. Fundamental groups for images without holes

In [15], attempts are made to derive fundamental groups for certain digital surfaces without holes. Errors in these efforts are discussed in this section. We also obtain some related original results.

Definition 5.1 ([15])**.** A digital image (X, κ) *has no κ-hole* if every κ-path in X is κ-nullhomotopic in X.

In the definition above, we must understand "path" in the sense of Definition 2.1, as any path in the sense of Definition 2.4 is nullhomotopic. Recall from Definition 2.10 that a path in the sense of Definition 2.1 is nullhomotopic when its inclusion map is nullhomotopic. We show below that, in the case where each component of X is finite, the no hole condition is equivalent to contractibility of each component.

Using Definition 5.1, it is claimed as Theorem 3.5 of [15] that a closed k-surface $X \subset \mathbb{Z}^3$ with no k-holes has trivial fundamental group for $k \in \{18, 26\}$. However, the argument given fails to require homotopies between loops to hold the endpoints fixed.

By Definition 2.5, a condition that is necessary for a connected image X to be contractible or to have no holes is that X must have a finite upper bound for lengths of shortest paths between distinct points, since there are finitely many "time steps" in a homotopy. We will use the following.

Proposition 5.2. *Let (X, κ) be a digital image. Then X is finite and connected if and only if X is a κ-path.*

Proof. First assume that X is finite and connected. Let $X = \{x_i\}_{i=0}^m$. Since X is connected, there is a path P_i in X from x_{i-1} to x_i, $i \in \{1, 2, \ldots, m\}$. By traversing P_1 followed by P_2 followed by $\ldots$ followed by P_m, we see that X is the path $\bigcup_{i=1}^m P_i$.

The converse is clear from the definition of path and connectivity - any path must be finite and connected. □

Proposition 5.3. *Let (X, κ) be a digital image such that each component is finite. Then X has no κ-hole if and only if every component of X is κ-contractible.*

Proof. Suppose X has no κ-hole. Let A be a κ-component of X. Then A is finite, and by Proposition 5.2, A is a κ-path. Since X has no κ-hole, the inclusion $i : A \to X$ is nullhomotopic in A, and thus A is contractible.

Conversely, suppose every component of X is κ-contractible. Since every path $P \subset X$ is a connected set, we must have P contained in some component A of X. By restricting a contraction of A to P, we have a nullhomotopy of P in X. Thus, X has no κ-hole. □

The importance of the finiteness restriction in Proposition 5.3 is demonstrated in the following example.

Example 5.4. $\mathbb{Z}$ has no 2-hole, but is connected and not 2-contractible.

Proof. Let $P = \{y_i\}_{i=0}^m$ be a 2-path in $\mathbb{Z}$, in the sense of Definition 2.1. Then P is a digital interval: $P = [a, b]_{\mathbb{Z}}$. Therefore, the function $H : P \times [0, b-a]_{\mathbb{Z}} \to \mathbb{Z}$ defined by

$$H(s, t) = p(\max\{0, s - t\})$$

is a nullhomotopy of P. It follows that $\mathbb{Z}$ has no 2-holes.

Clearly, $\mathbb{Z}$ is 2-connected. $\mathbb{Z}$ is not 2-contractible, as given $x \in \mathbb{Z}$, there is no finite bound on the length of 2-paths from $y \in \mathbb{Z}$ to x, and a homotopy has only finitely many steps in which the distance between points can be lessened by at most 2. □

The following is our modified and corrected version of Theorem 3.5 of [15]. The result in that paper is stated only for closed digital surfaces (which are automatically finite), but our theorem holds more generally for any digital image with finite components.

Theorem 5.5. *If (X, κ) has no κ-hole and each component of X is finite, then $\Pi_1^\kappa(X, x_0)$ is trivial for any x_0.*

Proof. Let $x_0 \in X$. By Proposition 5.3, the component A of x_0 in X is contractible. It follows from Corollary 2.19 that $\Pi_1^\kappa(X, x_0) = \Pi_1^\kappa(A, x_0)$ is trivial. □

A closely related version of the "no hole" condition can be formulated in terms of paths viewed as functions according to Definition 2.4.

Definition 5.6. A digital image (X, κ) *has no loophole* if every κ-loop in X is κ-nullhomotopic in X by a loop-preserving homotopy.

As with Han's "no hole" condition, we can show that a space with no loopholes has trivial fundamental group. The following is another version of Theorem 3.5 of [15].

Theorem 5.7. *If (X, κ) has no loopholes, then $\Pi_1^\kappa(X, x_0)$ is trivial for any x_0.*

Proof. Let $f : [0, k]_{\mathbb{Z}} \to X$ be a loop in X based at x_0. We must show that $[f] = [\bar{x}_0]$, where $\bar{x}_0$ is the constant loop at x_0. Since X has no loopholes, f is homotopic to $\bar{x}_0$ by a loop-preserving homotopy, say $H : [0, k]_{\mathbb{Z}} \times [0, m]_{\mathbb{Z}} \to X$. Since H is loop-preserving, we have $H(0, t) = H(k, t)$ for each t. Let $p(t) = H(0, t)$, so p is the path taken by the basepoints of the loops during the homotopy. Since both f and $\bar{x}_0$ have basepoint x_0, this path p is a loop at x_0.

For $t \in [0, m]_{\mathbb{Z}}$, let $p_t : [0, m]_{\mathbb{Z}} \to X$ be defined by $p_t(s) = p(\min\{s, t\})$. Then p_t is a path from $p(0) = x_0$ to $p(t)$ and let p_t^{-1} be the reverse path.

Let $\bar{x}_0$ be a constant path of length m. Let $\bar{H} : [0, k + 2m]_{\mathbb{Z}} \times [0, m]_{\mathbb{Z}} \to X$ be defined by

$$\bar{H}(s, t) = (p_t \cdot H_t \cdot p_t^{-1})(s),$$

where $H_t(s) = H(s, t)$. Then $\bar{H}$ is a homotopy from $p_0 \cdot f \cdot p_0^{-1} = \bar{x}_0 \cdot f \cdot \bar{x}_0$, a trivial extension of f, to $p \cdot \bar{x}_0 \cdot p^{-1}$, a trivial extension of $p \cdot p^{-1}$, and $\bar{H}$ holds the endpoints fixed. Therefore,

$$[f] = [p \cdot p^{-1}]. \tag{5.1}$$

Since the function $K : [0, 2m]_{\mathbb{Z}} \times [0, m]_{\mathbb{Z}} \to X$ defined by

$$K(s, t) = (p_t \cdot p_t^{-1})(s)$$

is a homotopy from $p \cdot p^{-1}$ to $\bar{x}_0 \cdot \bar{x}_0$ that holds the endpoints fixed, we have

$$[p \cdot p^{-1}] = [\bar{x}_0]. \tag{5.2}$$

From equations (5.1) and (5.2), we have $[f] = [\bar{x}_0]$, as desired. $\square$

In the proof of Theorem 5.7 we also proved the following.

Lemma 5.8. *Let f be a loop in X based at x_0. If f is homotopic to the constant loop $\bar{x}_0$ by a loop preserving homotopy, then $[f] = [\bar{x}_0]$ in $\Pi_1^\kappa(X, x_0)$.*

Note that Lemma 5.8 need not be true when the constant loop $\bar{x}_0$ is replaced by some other loop. See the discussion following Definition 2.8 of [5] for an example of two loops that are homotopic by a loop-preserving homotopy but are not equivalent in the fundamental group.

The converses of Theorem 5.7 and Lemma 5.8 are not true. The following example shows an image with a loophole and trivial fundamental group.

Example 5.9. Let $(X, 6) \subset (\mathbb{Z}^3, 6)$ be given by $X = \delta([0, 4]_{\mathbb{Z}}^3) \setminus \{(4, 2, 2)\}$. This image X is analogous to MSS_6, but larger, with the center of one "side" deleted. A schematic of this image is shown in Figure 6. Let f be the 8 point loop in X which circles the deleted point $(4, 2, 2)$.

By Theorem 3.12 of [13], the only loops homotopic to f by loop-preserving homotopies are rotations of f. Thus f does not contract by a loop-preserving homotopy, and so X has a loophole. But simple modifications to the arguments used in Section 3 will show that $\Pi_1^6(X, x_0)$ is trivial for any $x_0 \in X$.

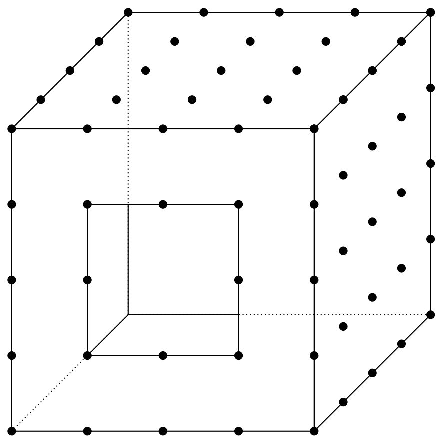

Figure 6. A schematic of the image used in Example 5.9. (Dots have been omitted for points on 3 sides of X.) The loop circling the "hole" on the front face is not contractible by a loop-preserving homotopy, but a trivial extension is pointed contractible.

The no loophole condition and the no hole condition are closely related, but not equivalent. Under a the same finiteness condition used above, "no loophole" is weaker than "no hole":

Proposition 5.10. *Let X be a digital image such that each component of X is finite. If X has no hole, then X has no loophole.*

Proof. Let $f : [0, m]_{\mathbb{Z}} \to X$ be a loop in X, and we will show that it is nullhomotopic by a loop-preserving homotopy. Let $A \subset X$ be the component of X containing the points of the path f. Since X has no hole and the components of X are finite, Proposition 5.3 shows that A is contractible. Let $G : A \times [0, k]_{\mathbb{Z}} \to X$ be a contraction of A, say $G(x, k) = a_0$ for all $x \in A$.

Then define $H : [0, m]_{\mathbb{Z}} \times [0, k]_{\mathbb{Z}} \to X$ as $H(t, s) = G(f(t), s)$. Being a composition of continuous functions, H has the necessary continuity properties to be a homotopy from f to $\bar{a}_0$, the constant path at a_0. Furthermore H is loop preserving since, for any s we have:

$$H(0, s) = G(f(0), s) = G(f(m), s) = H(m, s)$$

and thus each stage of H is a loop. Thus f is nullhomotopic by a loop-preserving homotopy as desired. $\square$

The converse to Proposition 5.10 is false, as shown by the following example.

Example 5.11. $(MSS_6, 6)$ has no loopholes, but has a hole.

Proof. The proof of Proposition 3.3 is easily modified to show that $(MSS_6, 6)$ has no loophole.

MSS_6 is finite, connected, and not contractible [1]. It follows from Proposition 5.3 that MSS_6 has a hole. □

It is claimed, as Theorem 3.6 of [15], that if X and Y are digital surfaces in $\mathbb{Z}^3$ with no k-holes, $k \in \{18, 26\}$, then $\Pi_1^k(X \sharp Y)$ is a trivial group. The argument given depends on Theorem 3.5 of [15], the flaws in which are discussed above. Although our Theorem 5.5 could be used to overcome this deficiency, the argument for Theorem 3.6 of [15] also claims without proof or citation that $X \sharp Y$ has no k-holes. We neither have a proof nor a counterexample for this assertion at the current writing. Thus, Theorem 3.6 of [15] must be regarded as unproven, and we state as open questions:

Open Question 5.12. If X and Y are digital surfaces in $\mathbb{Z}^3$ with no k-holes, is $\Pi_1^k(X \sharp Y)$ trivial?

Open Question 5.13. If X and Y are digital surfaces in $\mathbb{Z}^3$ with no k-holes, does $X \sharp Y$ have no k-holes?

Open Question 5.14. If X and Y are digital surfaces in $\mathbb{Z}^3$ with no k-loopholes, does $X \sharp Y$ have no k-loopholes?

6. Euler characteristic

In this section, we correct and extend several statements that appear in Section 5 of [15] concerning the Euler characteristic $\chi(X)$ of a digital image X. Some of the errors in [15] were previously noted in [7]; they are recalled here for completeness.

A digital image X can be considered to be a graph. When X is finite, let $V = V(X)$ be the number of vertices, i.e., the number of distinct points of X; let $E = E(X)$ be the number of distinct edges of X, where an edge is given by each adjacent pair of points; and let $F = F(X)$ be the number of distinct faces, where a face is an unordered triple of distinct vertices each pair of which is adjacent. More generally, a *k-simplex in X of dimension d* is a set of $d+1$ distinct members of X, each pair of which is k-adjacent.

The definition of the Euler characteristic in [14] is

$$\chi(X) = V - E + F.$$

This definition is satisfactory if X has no simplices of dimension greater than 2. However, the latter assumption is not always correct, even for digital surfaces; e.g., MSC_8^* has 3-simplices. Thus, a better definition of the Euler characteristic is that of [7]:

$$\chi(X) = \chi(X, k) = \sum_{q=0}^{m} (-1)^q \alpha_q,$$

where m is the largest integer d such that (X, k) has a simplex of dimension d and α_q is the number of distinct q-dimensional k-simplices in X.

At statement (5.1) of [15], it is inferred that, using 18-adjacency in $\mathbb{Z}^3$,

$$V(MSS_{18}) = 10, \quad E(MSS_{18}) = 20, \quad F(MSS_{18}) = 12,$$

and therefore that $\chi(MSS_{18}) = 2$. In fact, one sees easily (see Figure 2) that $F(MSS_{18}) = 8$, namely, the faces are

$$\langle c_0, c_1, c_9\rangle, \langle c_0, c_1, c_6\rangle, \langle c_0, c_5, c_6\rangle, \langle c_0, c_5, c_9\rangle,$$
$$\langle c_2, c_3, c_7\rangle, \langle c_2, c_3, c_8\rangle, \langle c_3, c_4, c_7\rangle, \langle c_3, c_4, c_8\rangle,$$

and therefore, as noted in [7], we have the following.

Example 6.1. $\chi(MSS_{18}) = -2$.

Theorem 5.2 of [15] claims that for closed k-surfaces X and Y,

$$\chi(X\sharp Y) = \chi(X) + \chi(Y) - 2.$$

This formula is attractive because it matches the classical formula for the Euler characteristic of a connected sum of surfaces. Unfortunately we will see that the formula holds only in some cases. The argument given in [15] makes some counting errors, and fails to count 3-simplices. A correct formula must include the Euler characteristic of A_k.

Lemma 6.2. *For closed digital surfaces X and Y, we have*

$$\chi(X\sharp Y) = \chi(X) + \chi(Y) - 2\chi(A_k).$$

Proof. Recall that $\delta(A_k)$ denotes the boundary of A_k. The construction of $X\sharp Y$ can be thought of as deleting A_k from each of X and Y, and then reinserting only one copy of $\delta(A_k)$. Because X and Y are digital surfaces with A_k embedded inside, no simplex of X or of Y has vertices in both the interior and exterior of A_k. Thus when we delete A_k from each of X and Y, this deletes only simplices of A_k, and when we reinsert $\delta(A_k)$, this inserts only simplices of $\delta(A_k)$. Thus in each dimension q we have:

$$\alpha_q(X\sharp Y) = \alpha_q(X) + \alpha_q(Y) - 2\alpha_q(A_k) + \alpha_q(\delta(A_k)),$$

where α_q is the number of q-simplices. Taking the alternating sum above we obtain:

$$\chi(X\sharp Y) = \chi(X) + \chi(Y) - 2\chi(A_k) + \chi(\delta(A_k)).$$

It remains to show that $\chi(\delta(A_k)) = 0$. We can check easily in Figure 4 that in each possible case for A_k, the boundary $\delta(A_k)$ is a simple cycle of points. Thus $\chi(\delta(A_k)) = 0$ as desired. $\square$

The next result was obtained for digital surfaces in [15] and generalized in [7].

Proposition 6.3. *Isomorphic digital images have the same Euler characteristic.*

Example 6.4. We have the following.

- $\chi(MSC_8^*) = 1$.
- $\chi(MSC_8'^*) = 1$.
- $\chi(MSC_4^*) = -3$.

Proof. See Figure 4. For $(MSC_8^*, 8)$, we see there are 8 vertices, 17 edges, 12 faces, 2 3-simplices, and no simplices of dimension greater than 3, so

$$\chi(MSC_8^*) = 8 - 17 + 12 - 2 = 1.$$

For $(MSC_8'^*, 8)$, we see there are 5 vertices, 8 edges, 4 faces, and no simplices of dimension greater than 2, so

$$\chi(MSC_8'^*) = 5 - 8 + 4 = 1.$$

For $(MSC_4^*, 4)$, we see there are 9 vertices, 12 edges, and 0 simplices of dimension greater than 1, so

$$\chi(MSC_4^*) = 9 - 12 = -3.$$

□

The computations above immediately give a corrected version of Theorem 5.2 of [15]:

Theorem 6.5. *For closed digital surfaces X and Y,*

$$\chi(X\sharp Y) = \begin{cases} \chi(X) + \chi(Y) - 2 & \text{if } A_k \approx_{(k,8)} MSC_8^*; \\ \chi(X) + \chi(Y) - 2 & \text{if } A_k \approx_{(k,8)} MSC_8'^*; \\ \chi(X) + \chi(Y) + 6 & \text{if } A_k \approx_{(k,4)} MSC_4^*. \end{cases}$$

Proof. The assertion follows from Lemma 6.2 and Example 6.4. □

Example 5.3 of [15] claims incorrectly that

$$\chi(MSS_{18}\sharp MSS_{18}) = \chi(MSS_{18}) = 2$$

and that

$$\chi(MSS'_{18}\sharp MSS_{18}) = \chi(MSS'_{18}) = 2.$$

Examples 6.6 and 6.7 below correct these errors.

Example 6.6 ([7]).

- $\chi(MSS_{18}\sharp MSS_{18}) = -6$.
- $\chi(MSS_{18}) = -2$.
- $\chi(MSS'_{18}) = 2$.

Example 6.7. $\chi(MSS'_{18}\sharp MSS_{18}) = -2$.

Proof. Using $A_{18} \approx_{(18,8)} MSC_8'^*$, it is easily observed that $MSS'_{18}\sharp MSS_{18}$ and MSS_{18} are 18-isomorphic. From Proposition 6.3, $\chi(MSS'_{18}\sharp MSS_{18}) = \chi(MSS_{18})$. The assertion follows from Example 6.6. □

7. Further remarks

We have given corrections to many errors that appear in [15] concerning fundamental groups and Euler characteristics of 2-sphere-like digital images. We have also presented some original results related to these ideas, including an example that shows that contractibility does not imply pointed contractibility among digital images, and our results concerning "no loopholes."

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