Energy of knots and the infinitesimal cross ratio

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Abstract

This is a survey article on two topics. The Energy E of knots can be obtained by generalizing an electrostatic energy of charged knots in order to produce optimal knots. It turns out to be invariant under Möbius transformations. We show that it can be expressed in terms of the infinitesimal cross ratio, which is a conformal invariant of a pair of 1-jets, and give two kinds of interpretations of the real part of the infinitesimal cross ratio.

Key words and phrases. energy, knot, conformal geometry, cross ratio
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1 Introduction

This is a survey article based on the author's talk at the COE Conference "Groups, Homotopy and Configuration Spaces" in honor of Fred Cohen, the University of Tokyo, Japan, from July 5th to 11th in 2005[1].

In the first part of this paper we give an introduction to the theory of energy of knots. Energy of knots is a funcitional on the space of knots which blows up as a knot degenerates to a singular knot with double points. It was introduced to produce optimal knots. The first example, the energy E, was obtained by generalizing an electrostatic energy of charged knots ([O1]). Later on, it was proved to be invariant under Möbius transformations ([FHW]).

The second part of this paper is a survey and an announcement of a part of the joint work with Rémi Langevin ([LO1, LO2]). We give a new interpretation from a viewpoint of conformal geometry. The infinitesimal cross ratio is the cross ratio of $x, x+dx, y$, and $y+dy$, where these four points are considered complex numbers by identifying a sphere through them with the Riemann sphere $\mathbb{C} \cup \{\infty\}$. It can be considered a complex valued 2-form on $K \times K \setminus \Delta$. It is the unique conformal invariant of a pair of 1-jets of a given curve up to

[1]This article is to appear in the Proceedings [GHC].

multiplication by a constant. We show that the energy E can be expressed as the integration of the difference of the absolute value and the real part of the infinitesimal cross ratio. We then show that the real part of the infinitesimal cross ration can be interpreted in two ways: as the canonical symplectic form of the cotangent bundle of S^3 and as a signed area element with respect to the pseudo-Riemannian structure of the set of oriented 0-spheres in S^3.

2 Energy of knots

2.1 Motivation

Just like a minimal surface is modeled on the "optimal surface" of a soap film with a given boundary curve, one can ask whether we can define an "*optimal knot*", a beautiful knot which represents its knot type. The notion of energy of knots was introduced for this purpose. The basic philosophy is as follows.

Suppose there is a non-conductive knotted string which is charged uniformly in a non-conductive viscous fluid. Then it might evolve itself to decrease its electrostatic energy without intersecting itself because of Coulomb's repulsive force until it comes to a critical point of the energy. Then we might be able to define an "optimal knot" by an embedding that attains the minimum energy within its isotopy class. Thus our motivational problem, which was proposed by Fukuhara and Sakuma independently, can be stated as:

Problem 2.1 ([Fu], [Sak]) Give a functional e (which we will call an energy) on the space of knots $\mathcal{K}$ which satisfies the following conditions:

1. Let $[K]$ denote an isotopy class which contains a knot K. Define the energy of an isotopy class by $e([K]) = \inf_{K' \in [K]} e(K')$.
2. If a knot K_0 attains the minimum value of the functional e within its isotopy class, i.e. if $e(K_0) = e([K_0])$, we call K_0 an *e-minimizer* of the isotopy class $[K_0]$.
3. There is an e-minimizer in each isotopy class.

Our strategy can be illustrated conceptually in Figures 1-3.

Let $\mathcal{I}$ be the set of immersions from a circle into $\mathbb{R}^3$ (or S^3) and $\mathcal{D}$ the set of immersions that are not embeddings. Sometimes this set $\mathcal{D}$ is called the *discriminant* set, and an element of $\mathcal{D}$ is called a *singular knot*. Let $\mathcal{K}$ be the complement of $\mathcal{D}$ in $\mathcal{I}$, i.e. the space of knots. We will always assume that $\mathcal{I}$ is endowed with C^2-topology. Two knots K and K' can be joined by a continuous path in the space of the knots $\mathcal{K}$ if and only if K and K' are isotopic. Therefore each "cell" (an arcwise connected component) of $\mathcal{K}$ corresponds to an isotopy class.

Given a knot K (Figure 1). Suppose it can be evolved along the negative gradient flow of e (Figure 2). Assume that it converges to an e-minimizer K_0 as time goes to infinity.

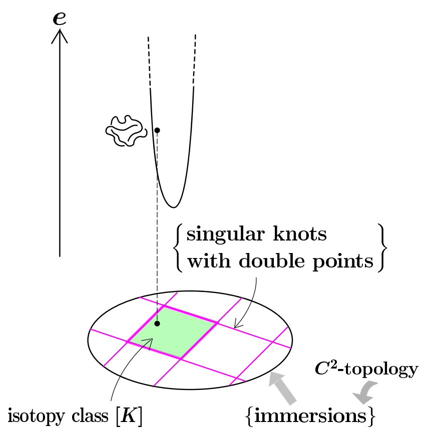

Figure 1:

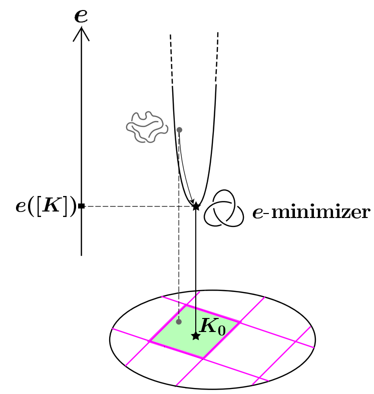

Figure 2:

If K_0 is isotopic to the original knot K then the problem is settled. As the isotopy class of a knot might be changed by a crossing change, it should be avoided while the knot is being evolved. Thus we are lead to the condition below.

Definition 2.2 We call a functional $e : \mathcal{K} \to \mathbb{R}$ a *self-repulsive energy* of knots, or simply, an *energy of knots*, if $e(K)$ blows up as K degenerates to a singular knot with double points (Figure 3).

If $e : \mathcal{K} \to \mathbb{R}$ is a self-repulsive energy of knots then each isotopy class is surrounded by infinitely high energy walls (Figure 1).

2.2 Renormalizations of electrostatic energy

The first example of such an energy, $E_\circ^{(2)}$, was defined as the renormalization of a "modified" electrostatic energy of uniformly charged knots. The electrostatic energy of a charged knot K is given by "E"$(K) = \displaystyle\iint_{K \times K} \frac{dxdy}{|x-y|}$ which turns out to be $+\infty$ for any knot, as it blows up at the diagonal $\Delta \subset K \times K$. We use a trick of subtacting a function which blows up in the same order at the diagonal to produce a finite valued functional $E^{(1)}$. But $E^{(1)}\big(\widehat{K}\big)$ does not blow up for a singular knot $\widehat{K}$ with double points, i.e. it is not a self-repulsive energy of knots. We obtain a self-repulsive energy if we make the power of $|x-y|$ in the integrand bigger than or equal to 2. Let us first study the case when it is equal to 2. It means that we consider the "modified" energy under the assumption that the magnitude of Coulomb's repulsive force between a pair of point charges of distance r is proportional to r^{-3}.

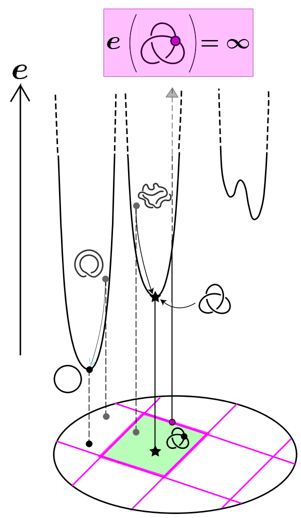

Figure 3: Our energy should blow up on the discriminant set.

Definition 2.3 ([O1]) Let $d_K(x,y)$ denote the (shorter) arc-length between x and y (Figure 4).

$$E_\circ^{(2)}(K) = \lim_{\varepsilon\to 0}\left\{\iint_{\{d_K(x,y)\geq\varepsilon\}\subset K\times K}\frac{dxdy}{|x-y|^2}-\frac{2}{\varepsilon}\right\} \tag{1}$$

$$= -4+\iint_{K\times K}\left(\frac{1}{|x-y|^2}-\frac{1}{d_K(x,y)^2}\right)dxdy\,, \tag{2}$$

where we assumed that the length of the knot K is equal to 1 in (1).

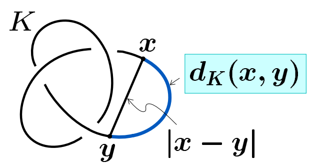

Figure 4: The arc-length and the chord length

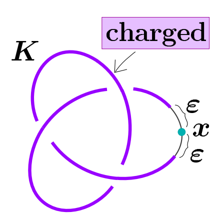

Figure 5: The subarc $\{y \in K|d_K(x,y)\geq\varepsilon\}$

The term $\displaystyle\int_{\{d_K(x,y)\geq\varepsilon\}}\frac{dy}{|x-y|^2}$ in (1) expresses the "voltage" at point x when the subarc $\{y\in K|d_K(x,y)\geq\varepsilon\}$ is charged (Figure 5). The renormalization in

(2) can be interpreted as taking the difference of the "extrinsic energy" based on the distance in the ambient space (chord length) and the "intrinsic energy" based on the distance in the knot (arc-length). If $\Gamma_\circ$ denotes a round circle then $E_\circ^{(2)}(\Gamma_\circ)=0$. It gives the smallest value of $E_\circ^{(2)}$ among all knots.

If $\widetilde{K}$ is an open long knot, i.e. the embedded line in $\mathbb{R}^3$ which tends asymptotically to a straight line at the both ends, its energy can be defined by dropping off the constant -4 ([FHW]):

$$E_\circ^{(2)}(\widetilde{K})=\iint_{\widetilde{K}\times\widetilde{K}}\left(\frac{1}{|x-y|^2}-\frac{1}{d_{\widetilde{K}}(x,y)^2}\right)dxdy.$$

If $\widetilde{\Gamma_\circ}$ is a straight line then $E_\circ^{(2)}(\widetilde{\Gamma_\circ})=0$.

2.3 Conformal invariance of $E_\circ^{(2)}$ and $E_\circ^{(2)}$-minimizers

The value of $E_\circ^{(2)}$ is invariant under rescaling or reparametrization. A *Möbius transformation* is a transformation of $\mathbb{R}^3\cup\{\infty\}$ which can be obtained as the composition of inversions in spheres (Figure 6).

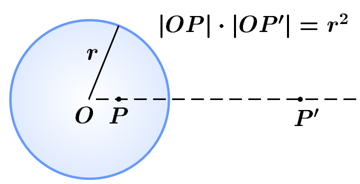

Figure 6: An inversion in a sphere

Theorem 2.4 ([FHW]) *Let K be a knot in $\mathbb{R}^3$ and T a Möbius transformation of $\mathbb{R}^3\cup\{\infty\}$. Then $E_\circ^{(2)}(T(K))=E_\circ^{(2)}(K)$. It holds even when $T(K)$ is an open long knot.*

We remark that the 2-form $\dfrac{dxdy}{|x-y|^2}$ on $K\times K\setminus\Delta$, which is an essential part of the integrand of (2) of the definition of $E^{(2)}(K)$, is invariant under Möbius transformations, in other words, if $K=f(S^1)$ then

$$f^*\left(T^*\left(\frac{dxdy}{|x-y|^2}\right)\right)=f^*\left(\frac{dxdy}{|x-y|^2}\right).$$

Using the conformal invariance Freedman, He, and Wang gave a partial affirmative answer to the motivational problem:

Theorem 2.5 ([FHW]) *There exists an $E_\circ^{(2)}$-minimizer for any isotopy class of a prime knot.*

Conjecture 2.6 ([KS]) On the other hand, Kusner and Sullivan conjectured through numerical experiments that there would be no $E_\circ^{(2)}$-minimizers in any isotopy class of a composite knot $[K_1 \sharp K_2]$, because both tangles representing $[K_1]$ and $[K_2]$ would "*pull tight*" to points if the knot evolves itself to decrease its energy (Figure 7).

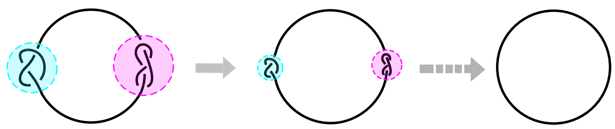

Figure 7: "Pull-tight"

They also conjecture that

$$E_\circ^{(2)}([K_1 \sharp K_2]) = E_\circ^{(2)}([K_1]) + E_\circ^{(2)}([K_2]).$$

This can be explained as follows. Consider an open long knot by an inversion in a sphere with center on the knot. If a knot pulls tight then the two tangles in the open long knot corresponding to $[K_1]$ and $[K_2]$ move in the opposite ways so that they become more and more disntant from each other (Figure 8). As the

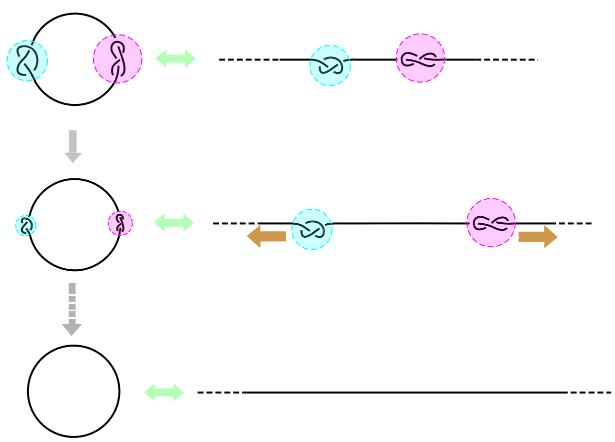

Figure 8: "Pull-tight" in open long knots

distance between the two tangles tends to $+\infty$ the interaction between them in the integral of $E_\circ^{(2)}$ tends to 0.

In the case of prime knots, the pull-tight can be avoided as follows. Suppose $\{K_n\}$ is a sequence of knots in an isotopy class of a prime knot $[K]$ with $\lim_{n\to\infty} E_\circ^{(2)}(K_n) = E_\circ^{(2)}([K])$. Applying Möbius transformations if necessary, we can obtain a new sequence of "*relaxed*" knots $\{K_n'\} \subset [K]$ so that the pull-tight

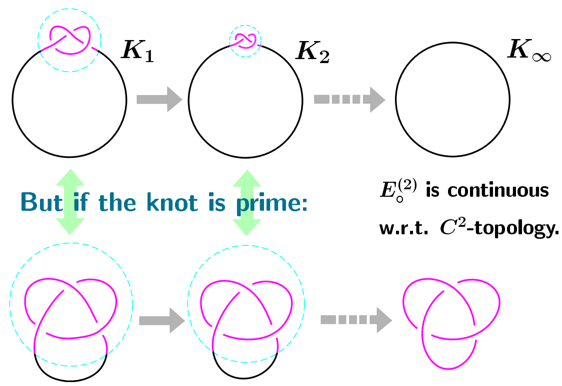

Figure 9: In the case of an isotopy class of a prime knot

does not occur and hence $\lim_{n\to\infty} K'_n$ belongs to the same isotopy class $[K]$. As $E_\circ^{(2)}(K'_n) = E_\circ^{(2)}(K_n)$ the limit $\lim_{n\to\infty} K'_n$ is an $E_\circ^{(2)}$-minimizer of $[K]$ (Figure 9).

Here are some remarks:

1. He showed that $E_\circ^{(2)}$-minimizers are smooth ([He]).

2. Suppose K is an $E_\circ^{(2)}$-minimizer of an isotopy class $[K]$. Then, for any Möbius transformation T, at least one of $T(K)$ and its mirror image $T(K)^*$ belongs to $[K]$. Since $E_\circ^{(2)}(T(K)) = E_\circ^{(2)}(T(K)^*) = E_\circ^{(2)}(K)$ it follows that $T(K)$ or $T(K)^*$ is an $E_\circ^{(2)}$-minimizer of $[K]$. Therefore, there are uncountably many $E_\circ^{(2)}$-minimizers for each isotopy class of a non-trivial prime knot.

3. The (cardinal) number of $E_\circ^{(2)}$-minimizers of an isotopy class of a prime knot modulo the action of the Möbius group is not known.

4. It is not known whether there exists an $E_\circ^{(2)}$-critical unknot which is not a round circle. If not, it implies Hatcher's results ([Ha]) that the set of unknots in S^3 deformation retracts onto the set of great circles.

   Numerical experiments show that $E_\circ^{(2)}$ can untie Ochiai's unknot ([KHG]) and "Freedman's unknot" ([KS]).

5. Using numerical experiments, Kusner and Sullivan conjecture that there exist unstable critical points in the isotopy class of a (p,q) torus knot if both p and q are greater than 2.

6. There are no known minimum values of $E_\circ^{(2)}$ of an isotopy class of a non-trivial knot which are obtained theoretically, like $6\pi^2$.

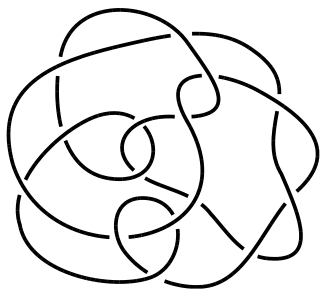

Figure 10: Ochiai's unknot

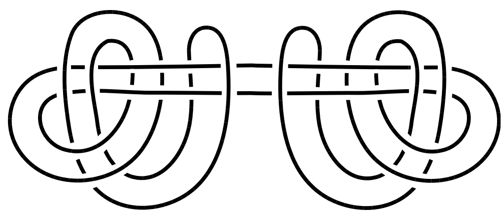

Figure 11: Freedman's unknot

7. It is an open problem whether $E_\circ^{(2)}$-minimizers are isolated in $\mathrm{Emb}(S^1, \mathbb{R}^3)/\sim$, where $\sim$ is generated by Möbius transformations and reparametrizations.

Various kinds of generalization of $E_\circ^{(2)}$ have been studied.

1. $E_\circ^{(2)}$ can be defined for a link $L = K_1 \cup \cdots \cup K_n$ ([FHW]). We do not need renormalization for the cross term $E(K_i, K_j) = \displaystyle\int_{x \in K_i} \int_{y \in K_j} \frac{dxdy}{|x-y|^2}$ $(i \neq j)$.
2. Similar energies are studied in [BO] and [BS]. The integrands are the products of the 2-form $\frac{dxdy}{|x-y|^2}$ and functions which kill the explosion of the integral at the diagonal.
3. A conformally invariant energy for surfaces was studied in [AS].
4. A conformally invariant energy for hypersurfaces using conformally defined angles was studied in [KS].
5. Fixing a knot K, $\displaystyle\iint_{K \times K} |x-y|^s dxdy$ can be considered a complex valued function of a complex variable s ([Bry]).

2.4 Generalization to produce energy minimizers

Conjecture 2.6 implies that $E_\circ^{(2)}$ does not give a completely affirmative solution to our motivational Problem 2.1. We have two ways to generalize $E_\circ^{(2)}$ so that all the isotopy classes have energy minimizers.

One is to make the power of $|x-y|$ in the integrand bigger than 2, and the other is to change the metric of the ambient space. In each case, our energies are no longer conformally invariant.

Definition 2.7 Let K be a knot with total length 1. Put

$$E^{(\alpha)}(K) = \iint_{K \times K} \left(\frac{1}{|x-y|^\alpha} - \frac{1}{d_K(x,y)^\alpha} \right) dxdy.$$

Theorem 2.8 ([O2], [O3]) *$E^{(\alpha)}$ is well-definede if $\alpha < 3$, and is self-repulsive if $\alpha \geq 2$. There exists an $E^{(\alpha)}$-minimizer for any isotopy class if $\alpha > 2$.*

Let M be a Riemannian manifold. Define

$$d_M(x,y) = \inf\{\text{Length of path joining } x \text{ and } y\},$$
$$E_M^{(\alpha)}(K) = \iint_{K\times K} \left(\frac{1}{d_M(x,y)^\alpha} - \frac{1}{d_K(x,y)^\alpha} \right) dxdy.$$

Theorem 2.9 ([O4]) *Let M be a compact manifold. Then there exists an $E_M^{(\alpha)}$-minimizer for any isotopy class if $\alpha > 2$.*

We conjecture that the Theorem above also holds for $\alpha = 2$ if $M = S^2$.

2.5 Related topics

Energy of knots gave rise to *geometric knot theory*, in which we study functionals to measure how complicated a knot is embedded and look for "optimal knots" with respect to those functionals.

One of the functionals which are intensively studied recently is the *rope length* (Cantarella, Kusner, Sullivan, Stasiak, et al.), which measures how long a rope of unit diameter is needed to make a given knot, or its equivalents, thickness (Buck, Rawdon, Simon, et al.) and global radius of curvature (Gonzalez, Maddocks, Smutny).

3 A viewpoint from conformal geometry

Joint work with Rémi Langevin.

We can give a new interpretation of $E_\circ^{(2)}$ using what is invariant under Möbius transformations, such as circles, spheres, and angles.

3.1 Minkowski space

The *Minkowski space* $\mathbb{R}^5_1$ is $\mathbb{R}^5$ with the non-degenerate indefinite quadratic form with index 1:

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = -{x_0}^2 + {x_1}^2 + \cdots + {x_4}^2.$$

The set of linear isomorphisms which preserve the Lorentz metric is called the *Lorentz group*:

$$O(4,1) = \left\{ A \in GL(5,\mathbb{R}) \,\middle|\, {}^tAJ_1^5A = J_1^5 \right\}, \text{ where } J_1^5 = \begin{pmatrix} -1 & & & O \\ & 1 & & \\ & & \ddots & \\ O & & & 1 \end{pmatrix}.$$

A vector $\boldsymbol{v}$ in $\mathbb{R}^5_1$ is called *spacelike* if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0$, *lightlike* if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ and $\boldsymbol{v} \neq \boldsymbol{0}$, and *timelike* if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle < 0$. The set of lightlike vectors and the origin $V = \left\{ \boldsymbol{v} \in \mathbb{R}^5_1 \,\middle|\, \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0 \right\}$ is called the *light cone*. The hyperquadric $\Lambda = \{ \boldsymbol{v} \in \mathbb{R}^5_1 | \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1 \}$ is called the *de Sitter space*. The 3-sphere S^3 can be realized in

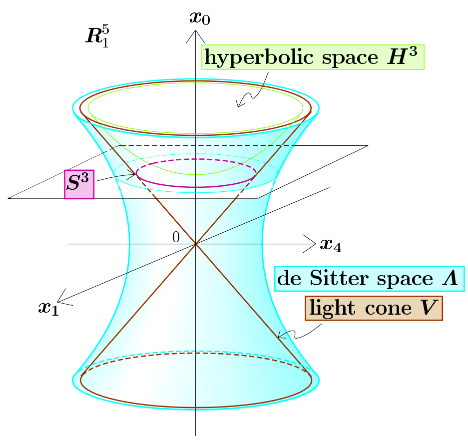

Figure 12:

$\mathbb{R}^5_1$ as the set of lines through the origin in the light cone $V = \{ \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0 \}$. We will denote it by $S^3(\infty)$. It can also be identified with the intersection of the light cone and the hyperplane $\{ \boldsymbol{x} \in \mathbb{R}^5_1 \,|\, x_0 = 1 \}$:

$$S^3(1) = \left\{ (1, x_1, x_2, x_3, x_4) \,\middle|\, {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = 1 \right\}.$$

The Lorentz group $O(4,1)$ acts on V and Λ. It also acts transitively on S^3 as the action on the set of lines in the light cone. This action is called a *Möbius transformation*.

3.2 de Sitter space as the set of spheres

Put $\mathcal{S}(2,3) = \{ \Sigma \,|\, \text{an oriented 2-sphere in } S^3 \}$. Then there is a bijection between $\mathcal{S}(2,3)$ and the de Sitter space Λ. Let Σ be an oriented 2-sphere in S^3. In the Minkowski space $\mathbb{R}^5_1$, Σ can be realized as the intersecton of S^3 and an oriented 4-dimensional subsapce Π through the origin (Figure 13). Let $\sigma \in \Lambda$ be the endpoint of the positive unit normal vector to Π. Then the map $\varphi : \mathcal{S}(2,3) \ni \Sigma \mapsto \sigma \in \Lambda$ is the bijection we want. Moreover, since this bijection is defined only by means of the pseudoorthogonality, it is preserved under the action of the Lorentz group $O(4,1)$, i.e. $\varphi(A \cdot \Sigma) = A\varphi(\Sigma)$ for $A \in O(4,1)$.

3.3 Willmore Conjecture

In his attempt to solve the Willmore Conjecture, Langevin has been interested in defining conformally invariant functionals on the space of surfaces and knots

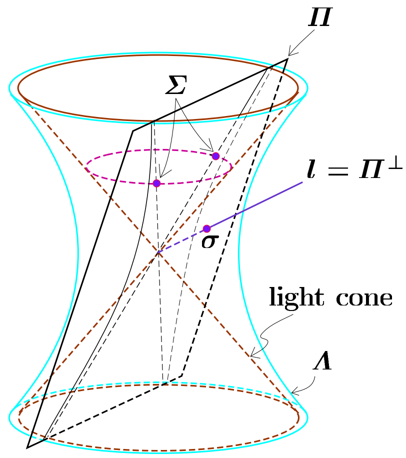

Figure 13: The bijection between $\mathcal{S}(2,3)$ and Λ

by means of integral geometry in the Minkowski space. One of his functionals turned out to be a self-repulsive energy of knots ([LO1]). That was the beginning of our joint work. Let us make a short comment on the Willmore Conjecture.

Let $\iota : T^2 \to \mathbb{R}^3$ be a smooth embedding. Let κ_1, κ_2 be the principal curvatures. The Willmore functional W is defined by

$$W(\iota)=\int_{T^2}\left(\frac{\kappa_1+\kappa_2}{2}\right)^2 dv=\int_{T^2}\left(\frac{\kappa_1-\kappa_2}{2}\right)^2 dv.$$

The second equality comes from the Gauss-Bonnet theorem. The integrand of the right hand side is known to be invariant under Möbius transformations, so is W. The Willmore Conjecture asserts:

Conjecture 3.1 We have $W(\iota) \geq 2\pi^2$. The equality holds if and only if $\iota(T^2)$ is a torus of revolution $T_{\sqrt{2},1}$ modulo Möbius transformations, where $T_{\sqrt{2},1}$ can be obtained by rotating around the z-axis a circle with radius 1 in the xz-plane whose center is distant form the z-axis by $\sqrt{2}$.

One of the important contributions is the following theorem due to Bryant. Fix a stereographic projection $\pi : S^3 \to \mathbb{R}^3 \cup \{\infty\}$.

Theorem 3.2 ([Br]) *Let $\Sigma_{\frac{2}{\kappa_1(p)+\kappa_2(p)}}(p)$ be a sphere which is tangent to $\iota(T^2)$ at p with curvature $\frac{\kappa_1(p)+\kappa_2(p)}{2}$. Define $\psi_\iota : T^2 \to \Lambda$ by*

$$\psi_\iota(p) = \varphi \circ \pi^{-1}\left(\Sigma_{\frac{2}{\kappa_1(p)+\kappa_2(p)}}(p)\right),$$

where $\varphi : \mathcal{S}(2,3) \to \Lambda$ is the bijection given in Subsection 3.2. *Then $W(\iota)$ is equal to the area of $\psi_\iota(T^2)$.*

The area is given with respect to the indefinite metric of $\Lambda \subset \mathbb{R}^5_1$, which is $SO(4,1)$-invariant. It does not depend on the stereographic projection π.

3.4 Infinitesimal cross ratio

Let us introduce the *infinitesimal cross ratio* Ω, which plays an important role in our study. It is a complex valued 2-form on $K \times K \setminus \Delta$. (The imaginary part might not be smooth.) It is conformally invariant.

We explain that its real part can be interpreted in two ways. They correspond to two kinds of interpretations of $S^3 \times S^3 \setminus \Delta$ which contains $K \times K \setminus \Delta$. One is as the total space of the cotangent bundle T^*S^3, which enables us to consider $\mathfrak{Re}\,\Omega_K$ as the pull-back of the canonical symplectic form of the cotangent bundle T^*S^3. The other is as the set of oriented 0-spheres in S^3 which has a natural pseudo-Riemannian structure coming from that of the Minkowski space, which enables us to consider $\mathfrak{Re}\,\Omega_K$ as a signed area form.

Let us begin with a geometric definition of the infinitesimal cross ratio.

Let $\varSigma = \varSigma_K(x, y)$ denote the 2-sphere which is tangent to the knot K at both x and y. We call it a *bitangent sphere*. It can be considered the 2-sphere $\varSigma(x, x + dx, y, y + dy)$ that passes through four points $x, x + dx, y$, and $y + dy$ (Figure 14). It is generically determined uniquely unless these four points are concircular, which is a condimension 2 phenomenon. Identify $\Sigma_K(x, y)$ with

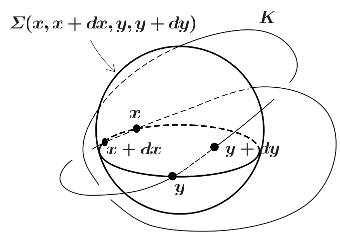

Figure 14:

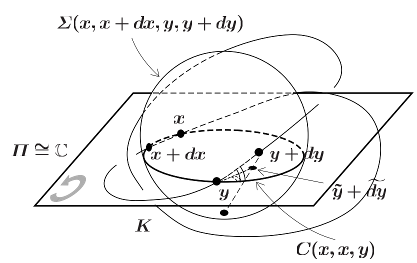

Figure 15: A stereographic projection

the Riemann sphere $\mathbb{C} \cup \{\infty\}$ through a stereographic projection p (Figure 15). Then the four points $x, x + dx, y$, and $y + dy$ can be considered a quadruplet of complex numbers $\tilde{x} = p(x)$, $\tilde{x} + \widetilde{dx} = p(x+dx)$, $\tilde{y} = p(y)$, and $\tilde{y} + \widetilde{dy} = p(y+dy)$. Let $\Omega_K(x, y)$ be the cross ratio $(\tilde{x} + \widetilde{dx}, \tilde{y}; \tilde{x}, \tilde{y} + \widetilde{dy})$:

$$\Omega_K(x, y) = \frac{(\tilde{x} + \widetilde{dx}) - \tilde{x}}{(\tilde{x} + \widetilde{dx}) - (\tilde{y} + \widetilde{dy})} : \frac{\tilde{y} - \tilde{x}}{\tilde{y} - (\tilde{y} + \widetilde{dy})} \sim \frac{\widetilde{dx}\widetilde{dy}}{(\tilde{x} - \tilde{y})^2}.$$

To be precise, we need an orientation of $\varSigma$ to avoid the ambiguity of complex conjugacy of the infinitesimal cross ratio (see the Remark below).

Then $\Omega_K(x, y)$ is independent of the choice of the stereographic projection p. Suppose we use another stereographic projection. Then we get another quadruplet of complex numbers, which can be obtained from the former by a linear fractional transformation. Since a linear fractional transformation does not change the cross ratio, we get the same value.

We call $\Omega_K(x,y)$ the *infinitesimal cross ratio* of the knot K. The real part of it has a pole of order 2 at the diagonal $\Delta \subset K \times K$. It satisfies

$$(T\times T)^*\left(\Omega_{T(K)}(Tx,Ty)\right)=\Omega_K(x,y)$$

for any Möbius transformation T, where $T\times T$ is the diagonal action

$$T\times T : K\times K\setminus\Delta \ni (x,y)\mapsto (Tx,Ty)\in T(K)\times T(K)\setminus\Delta.$$

Remark: Let $\mathcal{S}$ be the set of quadruplets of ordered four points in S^3 which are not concircular. We can define a continuous map from $\mathcal{S}$ to the set of oriented spheres Λ. (It is given by a similar formula to (6) which shall be given later.) The composite with the cross ratio map gives a continuous map from $\mathcal{S}$ to $\mathbb{C}\setminus\mathbb{R}$. Since $\mathcal{S}$ is connected its image is contained in one of the two half-planes. Our convention implies that the imaginary part of the cross ratio of any quadruplet of ordered four points in S^3 is non-negative (the reader is referred to [O5] for details).

3.5 Conformal angles and cosine formula

Definition 3.3 (Doyle and Schramm) Let $C(x,x,y)$ be an oriented circle tangent to K at x which passes through y whose orientation coincides with that of K at x. Let θ be the angle from $C(x,x,y)$ to $C(y,y,x)$ at point y. We call it the *conformal angle* between x and y and denote it by $\theta=\theta_K(x,y)$.

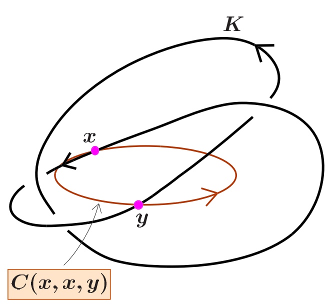

Figure 16:

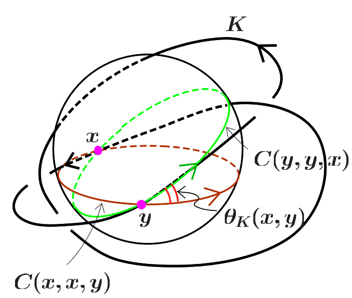

Figure 17: The conformal angle

Generically $C(x,x,y)$ and $C(y,y,x)$ are different. The bitangnet sphere $\Sigma_K(x,y)$ is then the unique sphere that contains both $C(x,x,y)$ and $C(y,y,x)$. We assume that the sign of $\theta_K(x,y)$ is given with respect to the orientation of $\Sigma_K(x,y)$. Our convention of the orientation of bitangent spheres ([O5]) implies that the conformal angle always satisfies $0\le\theta_K(x,y)\le\pi$.

Proposition 3.4 *The absolute value of the infinitesimal cross ration $\Omega_K(x,y)$ is equal to $\dfrac{dxdy}{|x-y|^2}$ and the argument is equal to $\theta_K(x,y)$. Therefore we have*

$$\Omega_K(x,y) = e^{i\theta_K(x,y)}\frac{dxdy}{|x-y|^2}.$$

Remark: (1) The conformal angle is of the order of $|x-y|^2$ near the diagonal.

(2) The conformal angle behaves like an absolute value of a smooth function. Therefore, the imaginary part of the infinitesimal cross ratio may have singularity at $\{(x,y) \in K \times K \setminus \Delta \,|\, \theta_K(x,y) = 0\}$.

Doyle and Schramm gave a *cosine formula* of $E_\circ^{(2)}(K)$ ([AS], [KS]):

$$E_\circ^{(2)}(K) = \iint_{K\times K\setminus\Delta} \frac{(1-\cos\theta_K(x,y))}{|x-y|^2}\,dxdy.$$

This is another proof of the conformal invariance of $E_\circ^{(2)}$.

Proposition 3.4 and the cosine formula imply

Proposition 3.5 ([LO1]) *The energy $E_\circ^{(2)}$ can be expressed in terms of the infinitesimal cross ratio Ω_K as*

$$E_\circ^{(2)}(K) = \iint_{K\times K\setminus\Delta} \left(|\Omega_K| - \mathfrak{Re}\,\Omega_K\right).$$

3.6 $\mathfrak{Re}\,\Omega_K$ and the canonical symplectic form of T^*S^3

We give the first interpretation of the real part of the infinitesimal cross ratio as the pull-back of the canonical symplectic form of the cotangent bundle T^*S^3.

Definition 3.6 Let T^*M be a cotangent bundle of an m-dimensional manifold M. Let $(q_1,\cdots,q_m,p_1,\cdots,p_m)$ be local coordinates of T^*M, where $(q_1,\cdots,q_m)$ are local coordinates of M and $(p_1,\cdots,p_m)$ are local coordinates of fibers associated with the basis $\{dq_1,\cdots,dq_m\}$. The *canonical symplectic form ω_M of the cotangent bundle* T^*M is a globally defined non-vanishing 2-form which can locally be expressed by

$$\omega_M = \sum dq_i \wedge dp_i.$$

It is an exact form. In fact, there is a 1-form θ of T^*M which can locally be expressed by $\theta = \sum p_i dq_i$ that satisfies $\omega_M = -d\theta$. This θ is called the tautological form of T^*M. It can be defined globally as follows. Let $T(T^*M)$ be a tangent bundle of T^*M. Then θ is given by

$$(\theta(x,v))(w) = v(d\pi(w)) \in \mathbb{R}, \qquad (x,v) \in T_x^*M, \quad w \in T_{(x,v)}T^*M,$$

where $d\pi : T_{(x,v)}T^*M \to T_xM$ is induced by the projection $\pi : T^*M \to M$.

The space $S^n \times S^n \setminus \Delta$ can be identified with the total space of the cotangent bundle T^*S^n as follows. Assume $S^n \subset \mathbb{R}^{n+1}$. Let $\boldsymbol{x} \in S^n$. Let $\Pi_{\boldsymbol{x}} = (\mathrm{Span}\langle \boldsymbol{x} \rangle)^\perp$ be the n-plane in $\mathbb{R}^{n+1}$ through the origin which is orthogonal to $\boldsymbol{x}$, and $p_{\boldsymbol{x}} : S^n \setminus \{\boldsymbol{x}\} \to \Pi_{\boldsymbol{x}}$ be a stereographic projection. We identify $T_{\boldsymbol{x}}S^n$ with $\Pi_{\boldsymbol{x}} \cong \mathbb{R}^n$, and $T_{\boldsymbol{x}}S^n$ with $T^*_{\boldsymbol{x}}S^n$ by

$$T_{\boldsymbol{x}}S^n \ni \boldsymbol{u} \mapsto (T_{\boldsymbol{x}}S^n \ni \boldsymbol{v} \mapsto (\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}) \in T^*_{\boldsymbol{x}}S^n.$$

Then the composition of identifications

$$\varphi_{\boldsymbol{x}} : S^n \setminus \{\boldsymbol{x}\} \xrightarrow[p_{\boldsymbol{x}}]{\cong} \Pi_{\boldsymbol{x}} \xrightarrow{\cong} T_{\boldsymbol{x}}S^n \xrightarrow{\cong} T^*_{\boldsymbol{x}}S^n$$

induces a canonical bijection φ:

$$S^n \times S^n \setminus \Delta = \bigcup_{\boldsymbol{x} \in S^n} \{\boldsymbol{x}\} \times (S^n \setminus \{\boldsymbol{x}\}) \ni (\boldsymbol{x}, \boldsymbol{y}) \overset{\varphi}{\mapsto} (\boldsymbol{x}, \varphi_{\boldsymbol{x}}(\boldsymbol{y})) \in \bigcup_{\boldsymbol{x} \in S^n} T^*_{\boldsymbol{x}}S^n = T^*S^n.$$

Let us write the pull-back $\varphi^*\omega_{S^n}$ of the canonical symplectic form ω_{S^n} of T^*S^n by the same letter ω_{S^n}.

Theorem 3.7 ([LO1])

1. *The* 2*-form* ω_{S^n} *on* $S^n \times S^n \setminus \Delta$ *is invariant under the diagonal action of a Möbius transformation:* $(T \times T)^*\omega_{S^n} = \omega_{S^n}$.

2. (folklore) *Let* $\omega_{\mathrm{cr}} = \dfrac{dw \wedge dz}{(w-z)^2}$ *be a complex* 2*-form on* $\mathbb{C} \times \mathbb{C} \setminus \Delta$. *It can be considered the cross ratio of* $w, w + dw, z, z + dz$*:*

$$\frac{(w + dw) - w}{(w + dw) - (z + dz)} : \frac{z - w}{z - (z + dz)} = \frac{dwdz}{(w - z)^2}\,.$$

Then $\mathfrak{Re}\,\omega_{\mathrm{cr}} = -\frac{1}{2}\,\omega_{S^2}$ *through the identification of* S^2 *and* $\mathbb{C} \cup \{\infty\}$ *by a stereographic projection.*

3. *The real part of the infinitesimal cross ratio can be expressed as the pull-back of the canonical symplectic form of the cotangent bundle* T^*S^3 *by the inclusion* $\iota : K \times K \setminus \Delta \hookrightarrow S^3 \times S^3 \setminus \Delta$*:*

$$\mathfrak{Re}\,\Omega(x, y) = -\frac{1}{2}\iota^*\omega_{S^3}.$$

As a corollary, $\mathfrak{Re}\,\Omega(x, y)$ *is an exact form.*

3.7 Pseudo-Riemannian structure of the set of spheres

We introduce some of the results in [LO2] in what follows.

We give the second interpretation of the real part of the infinitesimal cross ratio using the pseudo-Riemannian structure of the set of oriented 0-spheres in S^3.

Let $\mathcal{S}(q,n)$ denote the set of oriented q-spheres in S^n. As we saw in Subsection 3.2, when $n=3$ and $q=2=3-1$, $\mathcal{S}(2,3)$ can be identified with the de Sitter space Λ in $\mathbb{R}^5_1$. The restriction of the indefinite metric of $\mathbb{R}^5_1$ to each tangent space of Λ induces an indefinite non-degenerate quadratic form of index 1. Let us consider the generalization to the cases with bigger codimensions. We assume $n-q\geq 2$ in this subsection.

Theorem 3.8 ([LO2]) *The dimension of* $\mathcal{S}(q,n)$ *is given by* $(q+2)(n-q)$. *There is a natural pseudo-Riemannian structure on* $\mathcal{S}(q,n)$ *of index* $n-q$. *Namely, each tangent space* $T_p\mathcal{S}(q,n)$ *admits an indefinite non-degenerate quadratic form* g *such that* $T_p\mathcal{S}(q,n)$ *can be decomposed as the direct sum* $T_p\mathcal{S}(q,n)\cong V_+\oplus V_-$ *such that* $\dim V_+=(q+1)(n-q)$, $\dim V_-=n-q$, *and that the restriction of* g *to* V_+ (*or, to* V_-) *is positive definite* (*or respectively, negative definite*).

This indefinite non-degenerate quadratic form g induces an indefinte pseudo-inner product.

Just like in the case of $n=3$, S^n can be realized in the Minkowski space $\mathbb{R}^{n+2}_1$ with the metric

$$\langle \boldsymbol{x},\boldsymbol{x}\rangle = -{x_0}^2+{x_1}^2+\cdots+{x_{n+1}}^2$$

as the set of lines through the origin in the light cone $V=\{\langle \boldsymbol{v},\boldsymbol{v}\rangle=0\}$, and an oriented q-sphere Σ in S^n can be considered the intersection of S^n and an oriented $(q+2)$-plane Π_Σ through the origin. Therefore, $\mathcal{S}(q,n)$ can be identified with the Grassmann manifold

$$\widetilde{Gr}_-(q+2;\mathbb{R}^{n+2}_1)=\left\{\Pi\subset\mathbb{R}^{n+2}_1\;\middle|\;\begin{array}{l}\text{oriented }(q+2)\text{-plane through }\mathbf{0}\\ \Pi\text{ intersects the light cone transversally}^*\end{array}\right\}.$$

($*$ The second condition above is equivalent to say that Π is again a Minkowski space, i.e. the restriction of $\langle\,,\,\rangle$ to Π is a non-degenerate indefinite quadratic form of index 1.) It is a homogeneous space

$$\widetilde{Gr}_-(q+2;\mathbb{R}^{n+2}_1)\cong SO(n+1,1)/SO(n-q)\times SO(q+1,1),$$

and Theorem follows from Propspsition 3.2.6 of [KY].

Let us introduce more constructive explanation which is useful in the study of conformal geometry.

Let Π be an oriented $(q+2)$-dimensional vector subspace in $\mathbb{R}^{n+2}_1$, and let $\{\boldsymbol{x}_1,\cdots,\boldsymbol{x}_{q+2}\}$ be an ordered basis of Π which gives the orientation of Π. Let M be a $(q+2)\times(n+2)$-matrix given by

$$M=\begin{pmatrix}\boldsymbol{x}_1\\ \vdots\\ \boldsymbol{x}_{q+2}\end{pmatrix}=\begin{pmatrix}x_{1\,0} & x_{1\,1} & \cdots & x_{1\,n+1}\\ \vdots & \vdots & \ddots & \vdots\\ x_{q+2\,0} & x_{q+2\,1} & \cdots & x_{q+2\,n+1}\end{pmatrix}.$$

Let $I = (i_1, \cdots, i_{q+2})$ be a multi-index $(0 \leq i_k \leq n+1)$. Define $p_I = p_{i_1 \cdots i_{q+2}}$ by

$$p_{i_1 \cdots i_{q+2}} = \begin{vmatrix} x_{1\, i_1} & x_{1\, i_2} & \cdots & x_{1\, i_{q+2}} \\ x_{2\, i_1} & x_{2\, i_2} & \cdots & x_{2\, i_{q+2}} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(q+2)\, i_1} & x_{(q+2)\, i_2} & \cdots & x_{(q+2)\, i_{q+2}} \end{vmatrix}. \tag{3}$$

Then $p_{i_1 \cdots i_{q+2}}$ is alternating in the suffixes i_k. The exterior product of $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{q+2}$ is given by

$$\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_{q+2} = \sum_{0 \leq i_1 < \cdots < i_{q+2} \leq n+1} p_{i_1 \cdots i_{q+2}}\, \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_{q+2}} \in \bigwedge^{q+2} \mathbb{R}^{n+2}_1.$$

Let $N = \binom{n+2}{q+2}$. We identify $\overset{q+2}{\wedge}\ \mathbb{R}^{n+2}_1$ with $\mathbb{R}^N$ by expressing $\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_{q+2} \in \overset{q+2}{\wedge}\ \mathbb{R}^{n+2}_1$ by $(\cdots, p_{i_1 \cdots i_{q+2}}, \cdots) \in \mathbb{R}^N$.

Let $[\Pi]$ denote an unoriented $(q+2)$-space which is obtained from Π by forgetting its orientation. Then it can be identified by the homogeneous coordinates $[\cdots, p_{i_1 \cdots i_{q+2}}, \cdots] \in \mathbb{R}P^{N-1}$. They are called the *Plücker coordinates* or *Grassmann coordinates*. They do not depend on the choice of $(q+2)$ linearly independent vectors which span $[\Pi]$. Let $Gr\,(q+2, n+2)$ be the Grassmann manifold of the set of all $(q+2)$-dimensional vector subspaces in $\mathbb{R}^{n+2}$. The mapping

$$Gr\,(q+2, n+2) \ni [\Pi] \mapsto [\cdots, p_{i_1 \cdots i_{q+2}}, \cdots] \in \mathbb{R}P^{N-1}$$

is called the *Grassmann mapping*.

The Plücker coordinates $p_{i_1 \cdots i_{q+2}}$ are not independent. They satisfy the *Plücker relations*:

$$\sum_{k=1}^{q+3} (-1)^k p_{i_1 \cdots i_{q+1} j_k} p_{j_1 \cdots \widehat{j_k} \cdots j_{q+3}} = 0, \tag{4}$$

where $\widehat{j_k}$ indicates that the index j_k is being removed. (We remark that the digits $i_1, \cdots, i_{q+1}, j_k$ in the multi-index above are not necessarily ordered according to their sizes.) All the Plücker relations are not necessarily independent.

The pseudo-Riemannian structure of $\overset{q+2}{\wedge}\ \mathbb{R}^{n+2}_1$ is given by

$$\langle \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_{q+2}},\, \boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_{q+2}} \rangle = - \begin{vmatrix} \langle \boldsymbol{e}_{i_1}, \boldsymbol{e}_{j_1} \rangle & \cdots & \langle \boldsymbol{e}_{i_1}, \boldsymbol{e}_{j_{q+2}} \rangle \\ \vdots & \ddots & \vdots \\ \langle \boldsymbol{e}_{i_{q+2}}, \boldsymbol{e}_{j_1} \rangle & \cdots & \langle \boldsymbol{e}_{i_{q+2}}, \boldsymbol{e}_{j_{q+2}} \rangle \end{vmatrix},$$

which can be obtained by generalizing a formula in the case of codimension 1, $\bigwedge^4 \mathbb{R}^5_1 \cong \mathbb{R}^5_1$. Therefore,

$$\{\boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_{q+2}}\}_{0\le i_1<\cdots<i_{q+2}\le n+1}$$

can serve as a pseudoorthonormal basis of $\bigwedge^{q+2} \mathbb{R}^{n+2}_1$ which satisfies

$$\langle \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_{q+2}}, \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_{q+2}}\rangle = \begin{cases} -1 & \text{if} \quad i_1 \ge 1, \\ +1 & \text{if} \quad i_1 = 0. \end{cases}$$

It follows that if $\boldsymbol{v} = (\cdots, p_{i_1\cdots i_{q+2}}, \cdots) \in \bigwedge^{q+2} \mathbb{R}^{n+2}_1 \cong \mathbb{R}^N$ then

$$\langle \boldsymbol{v}, \boldsymbol{v}\rangle = -\sum_{1\le i_1<\cdots<i_{q+2}} {p_{i_1\cdots i_{q+2}}}^2 + \sum_{i_1=0<i_2<\cdots<i_{q+2}} {p_{0 i_2\cdots i_{q+2}}}^2. \tag{5}$$

Put $N_1 = \binom{n+1}{q+2}$ and $N_2 = \binom{n+1}{q+1}$. Then $\bigwedge^{q+2} \mathbb{R}^{n+2}_1 \cong \mathbb{R}^N$ can be decomposed to a direct sum $\mathbb{R}^{N_1}_- \oplus \mathbb{R}^{N_2}_+$, where the restriction of $\langle\, ,\rangle$ to $\mathbb{R}^{N_1}_-$ (or $\mathbb{R}^{N_2}_+$) is negative (or respectively, positive) definite. We denote $\mathbb{R}^N$ with the metric $\langle\, ,\rangle$ given by (5) by $\mathbb{R}^N_{N_1}$.

Theorem 3.9 ([LO2]) *Let $N = \binom{n+2}{q+2}$ and $N_1 = \binom{n+1}{q+2}$ as before.*

(1) *Let $\Pi = \mathrm{Span}\langle \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{q+2}\rangle$ be an oriented $(q+2)$-dimensional vector subspace in $\mathbb{R}^{n+2}_1$ spanned by $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{q+2}$. Put $\boldsymbol{p} = \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_{q+2} \in \mathbb{R}^N_{N_1}$. Then Π intersects the light cone V transversally if and only if $\langle \boldsymbol{p}, \boldsymbol{p}\rangle > 0$.*

(2) *Let $S^{N-1}_{N_1}$ be the unit pseudosphere:*

$$S^{N-1}_{N_1} = \left\{ \boldsymbol{v} = (\cdots, p_{i_1\cdots i_{q+2}}, \cdots) \in \mathbb{R}^N_{N_1} \,\middle|\, \langle \boldsymbol{v}, \boldsymbol{v}\rangle = 1 \right\}$$

and $\widetilde{Q}_P(q+2; \mathbb{R}^{n+2}_1)$ be the quadric satisfying the Plücker relations:

$$\widetilde{Q}_P(q+2; \mathbb{R}^{n+2}_1) = \left\{ (\cdots, p_{i_1\cdots i_{q+2}}, \cdots) \,\middle|\, \sum_{k=1}^{q+3} (-1)^k p_{i_1\cdots i_{q+1} j_k} p_{j_1\cdots \widehat{j_k}\cdots j_{q+3}} = 0 \right\}.$$

Then the set $\mathcal{S}(q,n)$ of oriented q-dimensional spheres in S^n can be identified with the intersection of $S^{N-1}_{N_1}$ and $\widetilde{Q}_P(q+2; \mathbb{R}^{n+2}_1)$:

$$\mathcal{S}(q,n) \cong S^{N-1}_{N_1} \cap \widetilde{Q}_P(q+2; \mathbb{R}^{n+2}_1) \subset \mathbb{R}^N_{N_1}.$$

Let us denote the right hand side by $\Theta(q,n)$.

(3) *Let $\Sigma(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{q+2})$ denote an oriented q-sphere Σ which is given as the intersection of S^n and an oriented vector subspace $\mathrm{Span}\langle \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{q+2}\rangle$. Then the bijection $\psi_G : \mathcal{S}(q,n) \to \Theta(q,n)$ is given by*

$$\psi_G(\Sigma(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{q+2})) = \frac{\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_{q+2}}{\sqrt{\langle \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_{q+2}, \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_{q+2}\rangle}}. \tag{6}$$

We show that a Möbius transformation of S^n induces a pseudoorthogonal transformation of $\Theta(q,n) \subset \bigwedge^{q+2} \mathbb{R}^{n+2}_1$. Let $O(N_2, N_1)$ denote the pseudoorthogonal group.

Definition 3.10 Let $N = \binom{n+2}{q+2}$ as before. Define

$$\Psi_{q,n} : M_{n+2}(\mathbb{R}) \ni A = (a_{ij}) \mapsto \widetilde{A} = (\tilde{a}_{IJ}) \in M_N(\mathbb{R}),$$

where $I = (i_1 \cdots i_{q+2})$ and $J = (j_1 \cdots j_{q+2})$ are multi-indices, and $\tilde{a}_{IJ}$ is given by

$$\tilde{a}_{IJ} = \begin{vmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_{q+2}} \\ \vdots & \ddots & \vdots \\ a_{i_{q+2} j_1} & \cdots & a_{i_{q+2} j_{q+2}} \end{vmatrix}.$$

Proposition 3.11 ([LO2]) *Let $N = \binom{n+2}{q+2}$, $N_1 = \binom{n+1}{q+2}$ and $N_2 = \binom{n+1}{q+1}$ as before. Let $A = (a_{ij}) \in M_{n+2}(\mathbb{R})$.*

1. *A matrix $\widetilde{A} \in M_N(\mathbb{R})$ satisfies*

$$(A\boldsymbol{x}_1) \wedge \cdots \wedge (A\boldsymbol{x}_{q+2}) = \widetilde{A}\,(\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_{q+2}) \quad (\forall \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{q+2} \in \mathbb{R}^{n+2}_1)$$

   if and only if $\widetilde{A} = \Psi(A)$.

2. *If $A \in O(n+1,1)$ then $\Psi(A) \in O(N_2, N_1)$.*

3. *The restriction of $\Psi_{q,n}$ to $O(n+1,1)$, which shall also be denoted by $\Psi_{q,n}$,*

$$\Psi_{q,n} : O(n+1,1) \ni A \mapsto \widetilde{A} \in O(N_2, N_1)$$

   is a homomorphism.

4. *Let $\psi_G : \mathcal{S}(q,n) \to \Theta(q,n)$ be the bijection given in the above* Theorem. *Then,*

$$\psi_G(A \cdot \Sigma) = \Psi_{q,n}(A)\psi_G(\Sigma)$$

   for $\Sigma \in \mathcal{S}(q,n)$ and $A \in O(n+1,1)$.

3.8 $\mathfrak{Re}\,\Omega$ as a signed area form

We give the second interpretation of the real part of the infinitesimal cross ratio.

When $q = 0$ and $n = 3$ the set $\mathcal{S}(0,3)$ of oriented 0-spheres in S^3 is a subspace of $\mathbb{R}^{10}_6$ since $N = \binom{5}{2} = 10$ and $N_1 = \binom{4}{2} = 6$. At the same time, it is identical with $S^3 \times S^3 \setminus \Delta$. It admits the pseudo-Riemannian structure of index 3 (Theorem 3.8). The pseudoorthonormal basis can be given by mutually pseudoorthogonal *pencils*, as is illustrated in Figure 18 (which is a picture in $\mathbb{R}^3$ obtained through a stereographic projection).

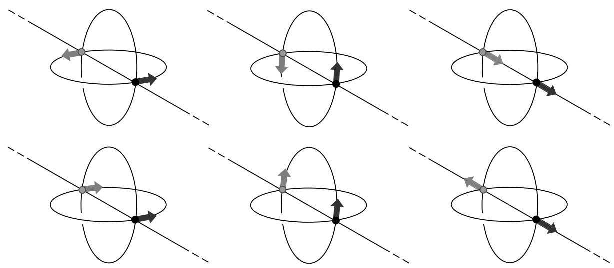

Figure 18: 3 spacelike pencils (above) and 3 timelike pencils (below)

Let (x, y) be a pair of distinct points of a knot K. Then it can be considered a point in $\mathcal{S}(0,3) \cong \Theta(0,3)$. Let it be denoted by $\boldsymbol{s}(x,y)$. Namely, $\boldsymbol{s}$ induces a map

$$\boldsymbol{s} : K \times K \setminus \Delta \hookrightarrow S^3 \times S^3 \setminus \Delta \xrightarrow{\cong} \mathcal{S}(0,3) \cong \Theta(0,3) \subset \mathbb{R}^{10}_6.$$

The image $\boldsymbol{s}(K \times K \setminus \Delta)$ is a surface in $\Theta(0,3)$. Its area element is given by

$$dv = \sqrt{\begin{vmatrix} \langle \boldsymbol{s}_x, \boldsymbol{s}_x \rangle & \langle \boldsymbol{s}_x, \boldsymbol{s}_y \rangle \\ \langle \boldsymbol{s}_y, \boldsymbol{s}_x \rangle & \langle \boldsymbol{s}_y, \boldsymbol{s}_y \rangle \end{vmatrix}}\, dxdy\,,$$

where $\boldsymbol{s}_x$ and $\boldsymbol{s}_y$ denote $\dfrac{\partial \boldsymbol{s}}{\partial x}(x,y)$ and $\dfrac{\partial \boldsymbol{s}}{\partial y}(x,y)$ in $T_{\boldsymbol{s}(x,y)} \times (0,3)$. It turns out that $\langle \boldsymbol{s}_x, \boldsymbol{s}_x \rangle = \langle \boldsymbol{s}_y, \boldsymbol{s}_y \rangle = 0$. Therefore

$$dv = \sqrt{-\langle \boldsymbol{s}_x, \boldsymbol{s}_y \rangle^2}\, dxdy.$$

Definition 3.12 Define a *signed area form* α of the surface $\boldsymbol{s}(K \times K \setminus \Delta)$ by

$$\alpha = \langle \boldsymbol{s}_x, \boldsymbol{s}_y \rangle \, dx \wedge dy.$$

Theorem 3.13 ([LO2]) *The real part of the infinitesimal cross ratio is equal to the half of the signed area form of the surface $\boldsymbol{s}(K \times K \setminus \triangle)$ with respect to the pseudo-Riemannian structure of $\mathcal{S}(0,3)$:*

$$\mathfrak{Re}\, \Omega_K(x,y) = \frac{1}{2} \, \langle \boldsymbol{s}_x, \boldsymbol{s}_y \rangle \, dx \wedge dy.$$

Let $\gamma_1 \cup \gamma_2$ be a 2-component link. The infinitesimal cross ratio $\Omega(x,y)$ ($x \in \gamma_1, y \in \gamma_2$) can be defined in the same way. The above Theorem and Theorem 3.7 imply that the signed area form $\alpha = 2\,\mathfrak{Re}\,\Omega$ of the surface $\boldsymbol{s}(\gamma_1 \times \gamma_2) \subset \Theta(0,3)$ is an exact form. Therefore, Stokes' theorem implies that the signed area of $\boldsymbol{s}(\gamma_1 \times \gamma_2)$ vanishes:

$$\int_{\gamma_1 \times \gamma_2} \alpha = \int_{x \in \gamma_1} \int_{y \in \gamma_2} \langle \boldsymbol{s}_x, \boldsymbol{s}_y \rangle \, dx \wedge dy = 0.$$

3.9 The imaginary part of the infinitesimal cross ratio

Unlike the real part, the imaginary part $\Im\mathfrak{m}\,\Omega$ of the infinitesimal cross ratio does not have a nice global interpretation. It cannot be expressed as a pull-back of a globally defined 2-form. (We cannot generalize the imaginary part of $\omega_{\rm cr}=\dfrac{dw\wedge dz}{(w-z)^2}$ to $S^n\times S^n\setminus\Delta$ for $n\geq 3$.) It might be singular at $(x,y)\in K\times K\setminus\triangle$ where the conformal angle $\theta_K(x,y)$ vanishes.

The imaginary part $\Im\mathfrak{m}\,\Omega$ of the infinitesimal cross ratio can be considered a local transversal area element of geodesics in $\mathbb{H}^4$ joining pairs of points on the knot K. To be precise, let $S^3\cong\partial\mathbb{H}^4$, $l(x,y)$ be a geodesic in $\mathbb{H}^4$ joining a pair of points x and y on K, Π_0 be any totally geodesic 3-space of $\mathbb{H}^4$ which is perpendicular to $l(x_0,y_0)$, and $S(x,y)=l(x,y)\cap\Pi_0$ be a surface in Π_0. Then $\Im\mathfrak{m}\,\Omega(x_0,y_0)$ is equal to the quater of the area element of $S(x,y)$ at (x_0,y_0) ([LO2]).

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