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Some quadratic equations in the free group of rank 2

DACIBERG L GONÇALVES
ELENA KUDRYAVTSEVA
HEINER ZIESCHANG

For a given quadratic equation with any number of unknowns in any free group F, with right-hand side an arbitrary element of F, an algorithm for solving the problem of the existence of a solution was given by Culler [8] using a surface method and generalizing a result of Wicks [46]. Based on different techniques, the problem has been studied by the authors [11, 12] for parametric families of quadratic equations arising from continuous maps between closed surfaces, with certain conjugation factors as the parameters running through the group F. In particular, for a one-parameter family of quadratic equations in the free group F_2 of rank 2, corresponding to maps of absolute degree 2 between closed surfaces of Euler characteristic 0, the problem of the existence of faithful solutions has been solved in terms of the value of the self-intersection index $\mu\colon F_2 \to \mathbb{Z}[F_2]$ on the conjugation parameter. The present paper investigates the existence of faithful, or non-faithful, solutions of similar families of quadratic equations corresponding to maps of absolute degree 0. The existence results are proved by constructing solutions. The non-existence results are based on studying two equations in $\mathbb{Z}[\pi]$ and in its quotient Q, respectively, which are derived from the original equation and are easier to work with, where π is the fundamental group of the target surface, and Q is the quotient of the abelian group $\mathbb{Z}[\pi \setminus \{1\}]$ by the system of relations $g \sim -g^{-1}$, $g \in \pi \setminus \{1\}$. Unknown variables of the first and second derived equations belong to π, $\mathbb{Z}[\pi]$, Q, while the parameters of these equations are the projections of the conjugation parameter to π and Q, respectively. In terms of these projections, sufficient conditions for the existence, or non-existence, of solutions of the quadratic equations in F_2 are obtained.

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1 Introduction

Equations in free groups have been extensively studied for many years: see Culler [8], Hmelevskiĭ [18, 19, 20], Lyndon [28, 29], Lyndon and Schupp [30, Sections 1.6 and 1.8], Makanin [32], Razborov [37], Steinberg [41] and Wicks [46]; see also Gonçalves

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and Zieschang [13], Grigorchuk and Kurchanov [14], Grigorchuk, Kurchanov and Zieschang [15], Ol'shanskiĭ [34], Osborne and Zieschang [36], and Zieschang [47, 48].

For a given quadratic equation $Q(z_1, \dots, z_q) = W$ with any number of unknowns $z_1, \dots, z_q$ in any free group F with an arbitrary right-hand side $W \in F$, the problem of the existence of a solution can be studied using Wicks forms (see Wicks [46], Culler [8] and Vdovina [42, 43, 44]) which are due to the geometric approach of Culler [8]. Special case of quadratic equations has been studied by the authors [11, 12, 26] for parametric families of quadratic equations which correpond to maps between closed surfaces, see (11). Also the notions of faithful and non-faithful solutions of such equations were there introduced, which correspond to the orientation-true maps and the maps which are not orientation-true, respectively (see Definitions 2.1(C), 3.2(a)). In particular, the problem of the existence of faithful solutions has been solved in [11, 12] in terms of the self-intersection index $\mu\colon F_2 \to \mathbb{Z}[F_2]$, for families of quadratic equations with two unknowns in the free group F_2 of rank 2, which correspond to maps of non-vanishing *absolute degree* (Definition 3.4) between closed surfaces of Euler characteristic 0. In this work, we study the existence of faithful, or non-faithful, solutions of the latter quadratic equations, which correspond to maps of absolute degree 0.

Specifically, let $F_2 = \langle a, b \mid \rangle$ be the free group of rank 2, v an element of F_2, and $\vartheta \in \{1, -1\}$. We consider the following equations in F_2 with the unknowns $z_1, z_2 \in F_2$:

(1) $$[z_1, z_2] = v[a, b]^{\vartheta} v^{-1} \cdot [a, b],$$

(2) $$[z_1, z_2] = v(a^2b^2)^{\vartheta} v^{-1} \cdot a^2b^2,$$

(3) $$z_1^2 z_2^2 = v[a, b]^{\vartheta} v^{-1} \cdot [a, b],$$

(4) $$z_1^2 z_2^2 = v(a^2b^2)^{\vartheta} v^{-1} \cdot a^2b^2.$$

Here $[a, b] = aba^{-1}b^{-1}$, and the conjugation factor $v \in F_2$ is called the *conjugation parameter* of the equation. The elements v, $R_\varepsilon(a, b) \in F_2$, where $R_\varepsilon(a, b)$ is defined below, can be regarded as the *coefficients* of the equation, see Lyndon and Schupp [30, Section 1.6]. The equations (1)–(4) have the form

(5) $$Q_\delta(z_1, z_2) = v\,(R_\varepsilon(a, b))^{\vartheta}\, v^{-1} \cdot R_\varepsilon(a, b)$$

where

$$Q_\delta(z_1, z_2) = \begin{cases} [z_1, z_2], & \delta = +\,, \\ z_1^2 z_2^2, & \delta = -\,, \end{cases} \qquad R_\varepsilon(a, b) = \begin{cases} [a, b], & \varepsilon = +\,, \\ a^2b^2, & \varepsilon = -\,. \end{cases}$$

As in [11], we denote by $w_\varepsilon\colon F_2 \to \{1, -1\}$ the homomorphism with $w_\varepsilon(a) = w_\varepsilon(b) = \varepsilon$, called the *orientation character*, see Definition 2.1 and Remark 3.3. Recall [12] that a solution (z_1, z_2) of (5) is called *faithful* if $w_\varepsilon(z_1) = w_\varepsilon(z_2) = \delta$, and otherwise the solution is called *non-faithful*, compare Definition 2.1(C). Of course, every solution of (1) is faithful, since $\varepsilon = \delta = +1$; every solution of (3) is non-faithful, since $\varepsilon = +1$ and $\delta = -1$.

We use the following geometric interpretations of the equations (1)–(4). These quadratic equations have two unknowns in the free group F_2 of rank 2. Such equations correspond to mappings from a compact surface of Euler characteristic -1 having one boundary component to the bouquet of two circles (see Culler [8]). The right-hand sides of (1)–(4) have special form which arises from maps between two closed surfaces of Euler characteristic 0, see Section 3. A solution is faithful if and only if the corresponding map is orientation-true, see [11] or Lemma 3.1.

Some faithful solutions of the equation (4), whose corresponding maps are self-maps of the Klein bottle, were listed in [12], see also Remark 2.2. The problem of the existence of faithful solutions of (5) with $w_\varepsilon(v) = \vartheta$ was solved by the authors in [11] in terms of the self-intersection index $\mu(v) \in \mathbb{Z}[F_2]$ of the conjugation parameter v, see Remark 2.2. These results are illustrated in Table 1 for special values of v.

The goal of the present paper is to investigate the existence of faithful, or non-faithful, solutions of equation (4) in the remaining cases formulated in detail as follows:

$$\text{(6)} \qquad \begin{array}{l} \text{the solution is faithful and } w_\varepsilon(v) = -\vartheta, \text{ or} \\ \text{the solution is non-faithful.} \end{array}$$

Such solutions actually correspond to mappings of absolute degree 0 (see Definition 3.4 and Corollary 3.11). Our main results are given in Tables 2 and 4, for faithful solutions, and in Tables 3 and 5, for non-faithful solutions, of an equation (8) which is equivalent to (5). The results are formulated in terms of the projection $\bar{v} \in \pi$ of the conjugation parameter $v \in F_2$ to the fundamental group π of the corresponding target surface via

$$p_\pi\colon\ F_2 \to \pi = F_2/N, \quad N = \langle\!\langle R_\varepsilon(a,b)\rangle\!\rangle,$$

as well as in terms of $p_Q(V) \in Q$, which is the image of $v_0^{-1}v \in N$ under the composition

$$\text{(7)} \qquad N \xrightarrow{q_N} \mathbb{Z}[\pi] \xrightarrow{p_Q} Q = (\mathbb{Z}[\pi \setminus \{1\}])/\langle g + g^{-1} \,|\, g \in \pi \setminus \{1\}\rangle,$$

where $v_0 \in F_2$ is a suitable representative of $\bar{v} \in \pi$ in F_2, see (39) and (40), while $V := q_N(v_0^{-1}v) \in \mathbb{Z}[\pi] \approx N/[N,N]$, see (25) and (26). Here $\langle\!\langle u_1, u_2, \ldots\rangle\!\rangle \subset G$ and $\langle u_1, u_2, \ldots\rangle \subset G$ denote the minimal normal subgroup and the minimal subgroup, respectively, containing the elements $u_1, u_2, \ldots \in G$ of a group G.

To establish the non-existence results given in Theorem 3.14 and Tables 2 and 3, we apply the Nielsen root theory for maps between closed surfaces (see Section 3), geometric results of Kneser [24] about maps of absolute degree 0 (see also Epstein [9]), and algebraic results (see Zieschang [47, 48], Zieschang, Vogt and Coldewey [50] and Ol'shanskiĭ [34]; see also Kudryavtseva, Weidmann and Zieschang [26, Corollary 2.4]) on epimorphisms of surface groups to free groups (Lemma 3.1, Propositions 3.6 and 3.8). These results allow us to reduce the problem of the existence of (faithful, or non-faithful, resp.) solutions of the equation (5) in F_2 satisfying the condition (6) to the problem of the existence of a (faithful, or non-faithful, resp.) solution of the following equation in the subgroup $N = \langle\langle \alpha\beta\alpha^{-\varepsilon}\beta^{-1}\rangle\rangle$ of $F_2 = \langle \alpha, \beta \mid \rangle$:

$$xyx^{-\delta}y^{-1} = v\big(\alpha\beta\alpha^{-\varepsilon}\beta^{-1}\big)^{\vartheta}v^{-1} \cdot \alpha\beta\alpha^{-\varepsilon}\beta^{-1}, \tag{8}$$

with the unknowns $x \in N$, $y \in F_2$, see ($1'$)–($4'$) in Section 3.3, and Corollaries 3.11 and 3.9(A) (see also Theorem 3.14). Here the free generators a, b of $F_2 = \langle a, b \mid \rangle$ and the unknowns z_1, z_2 are replaced by the new generators and unknowns via

$$\begin{array}{llllll} \alpha = a & \beta = b & \text{for } \varepsilon = 1, & \alpha = ab, & \beta = b^{-1} & \text{for } \varepsilon = -1, \\ x = z_1, & y = z_2 & \text{for } \delta = 1, & x = z_1z_2, & y = z_2^{-1} & \text{for } \delta = -1, \end{array} \tag{9}$$

thus $w_\varepsilon(\alpha) = 1$, $w_\varepsilon(\beta) = \varepsilon$. A solution (x, y) of (8) in N is called *faithful* if $w_\varepsilon(y) = \delta$. We also prove (Remark 3.12) that any solution of (8) in N satisfies

$$\bar{v} = \bar{y}^k \quad \text{and} \quad \vartheta\delta^k = -1 \qquad \text{for some} \quad k \in \mathbb{Z}.$$

To establish further non-existence results (Tables 4 and 5), we apply the algebraic approach developed in this paper (see Section 4) to the remaining cases of the equation (8), namely to those cases where the problem was not solved by the preceding methods (the so called "mixed" cases in Tables 2 and 3, see Remark 3.16 and Definition 3.15). From the equation (8) in N, two equations are derived using our algebraic approach, which have solutions corresponding to solutions of (8) if the latter exist. The *first derived equation* (Theorem 5.1) is

$$(1 - \delta\bar{y})\tilde{x} = 1 + \vartheta\bar{v}, \tag{10}$$

in the group ring $\mathbb{Z}[\pi] \approx N^{ab} = N/[N,N]$ of π (see Proposition 4.1), with two unknowns $\tilde{x} \in \mathbb{Z}[\pi]$, $\bar{y} \in \pi$, and the parameter $\bar{v} \in \pi$, see Theorem 5.1. A solution $(\tilde{x}, \bar{y})$ of (10) is called *faithful* if $w_\varepsilon(\bar{y}) = \delta$, and it is called *non-faithful* otherwise. For each solution of the equation (10) in the "mixed" cases (see above), we assign an equation in the quotient Q of N, see (7), namely the equations (2_2), (3_2), (4_2^{nf}) and (4_2^{f}), respectively, in Section 5.3. We use the fact that the quotient Q is isomorphic to $[N,N]/[F_2,[N,N]]$, see Proposition 4.5. The obtained in this way *second derived equation* (Theorem 5.10) has unknowns $X \in Q$, $Y \in \mathbb{Z}[\pi]$, a parameter $p_Q(V) \in Q$ determined by the conjugation parameter v, see (7), and some unknown integers which are parameters of the solutions of (10). We find all values of the parameter $p_Q(V)$ for which the second derived equation admits a solution (Theorems 6.4 and 6.8), and we use the obvious fact that the non-existence of a (faithful or non-faithful) solution of any of the derived equations implies the non-existence of a (faithful or non-faithful, resp.) solution of the corresponding quadratic equation (8).

The paper is organized as follows. In Section 2, we consider more general quadratic equations in free groups and briefly formulate some recent results of the authors about faithful solutions of such equations, including the equation (5) with $w_\varepsilon(v) = \vartheta$, which correspond to maps of absolute degree 2. In Section 3, we recall results of [11, 12], and Kudryavtseva, Weidmann and Zieschang [26] on the relationship between the quadratic equations and the Nielsen root theory, and derive some properties of solutions of (5) satisfying (6) from geometric results of Kneser [24] about maps having absolute degree 0 and algebraic results of Zieschang [47, 48], Zieschang, Vogt and Coldewey [50], and Ol'shanskiĭ [34] on homomorphisms of the surface groups to free groups. As a result, we obtain Tables 2 and 3, and reduce our problem to study the single equation (8) in N. In Section 4, we study some quotients of the subgroup $N = \langle\langle \alpha\beta\alpha^{-\epsilon}\beta^{-1}\rangle\rangle$ of the free group $F_2 = \langle \alpha, \beta \mid \rangle$ of rank 2. In particular, we prove that the quotient $[N,N]/[F_2,[N,N]]$ is isomorphic to the quotient Q in (7), see Proposition 4.5, and we obtain a presentation for the quotient $N/[F_2,[N,N]]$. In Section 5, we describe and derive two equations, namely the first and the second derived equations, see above, which are easier to work with than the original equation (8). The second derived equation is constructed when the first derived equation admits a solution, while the original quadratic equation does not necessarily admit a solution, see Example 6.11. In Section 6, we investigate the existence of a solution of the second derived equation in the "mixed" cases of Tables 2 and 3. The results of Sections 5 and 6 are summarized in Tables 4 and 5 of Section 7.

It is not clear whether our results can be obtained using Wicks forms. The results obtained here are entirely different from the type of results of Wicks [46] and Vdovina [42, 43, 44] using the Wicks forms, since we are able to consider certain families of equations at once, in contrast with methods which consider only one equation at the time.

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2 Recent results on quadratic equations

In the free group $F_r = \langle a_1, \dots, a_r \mid \rangle$ we consider quadratic equations of the form

$$Q(z_1, \dots, z_q) = (v_1 R v_1^{-1})^{c_1} \cdots (v_\ell R v_\ell^{-1})^{c_\ell} \quad \text{with} \quad R = R(a_1, \dots, a_r),$$

where Q and R are some "quadratic words" in variables $z_1, \dots, z_q$ and $a_1, \dots, a_r$, respectively, $q \geq 1$, $r \geq 1$ and all $c_j \neq 0$ are integers, $v_j \in F_r$. Here $z_1, \dots, z_q$ are considered as "unknowns", while $\ell, c_1, \dots, c_\ell$ and $v_1, \dots, v_\ell$ are "given parameters". Without loss of generality, one takes Q and R to be products of squares z_i^2 or commutators $[z_{2i-1}, z_{2i}] = z_{2i-1} z_{2i} z_{2i-1}^{-1} z_{2i}^{-1}$.

The following notation reflects the topological origin of the groups considered, namely fundamental groups of surfaces with boundary, see also Lemma 3.1.

Definition 2.1 Let r, q be integers ≥ 1 and $\varepsilon, \delta \in \{+1, -1\}$; often we will use ε, δ only as signs $+, -$.

(A) Let $F_{r,\varepsilon}$ denote the free group $F_r = \langle a_1, \dots, a_r \mid \rangle$ of rank r together with a homomorphism $w_\varepsilon : F_r \to \mathbb{Z}^* = \{1, -1\}$ called the *orientation character* where

$$w_+ : a_j \mapsto 1, \quad w_- : a_j \mapsto -1 \quad \text{for } 1 \leq j \leq r.$$

We call $F_{r,\varepsilon}$ a *free group with orientation character*. Define

$$Q_\delta(z_1, \dots, z_q) = \begin{cases} \prod_{i=1}^{q/2} [z_{2i-1}, z_{2i}], & \delta = +, \\ \prod_{i=1}^{q} z_i^2, & \delta = -, \end{cases}$$

$$\text{and} \qquad R_\varepsilon(a_1, \dots, a_r) = \begin{cases} \prod_{i=1}^{r/2} [a_{2i-1}, a_{2i}], & \varepsilon = +, \\ \prod_{i=1}^{r} a_i^2, & \varepsilon = -. \end{cases}$$

(B) In the group $F_{r,\varepsilon}$ we consider quadratic equations of the form

$$Q_\delta(z_1,\ldots,z_q) = \prod_{j=1}^{\ell} v_j \cdot (R_\varepsilon(a_1,\ldots,a_r))^{c_j} \cdot v_j^{-1}. \tag{11}$$

Here $c_j \neq 0$ are integers and $v_j \in F_r$; of course, when $\delta = +$ or $\varepsilon = +$ then q or r, resp., is even. Now $z_1,\ldots,z_q$ are considered as "unknowns", while $\ell, c_1,\ldots,c_\ell$ and $v_1,\ldots,v_\ell$ are "given parameters".

(C) If $w_\varepsilon(z_j) = \delta$, $1 \leq j \leq q$, then the solution $(z_1,\ldots,z_q)$ is called *faithful*, and otherwise it is called *non-faithful*. This gives the following restrictions for faithful solutions: if $\varepsilon = +$ then δ must be $+$, if $\varepsilon = -$ and $\delta = +$ then the length of each z_j must be even, if $\varepsilon = \delta = -$ then all lengths must be odd. Hence, one should only consider $(\delta,\varepsilon) \in \{(+,+),(-,-),(+,-)\}$ in the case of faithful solutions, and, similarly, $(\delta,\varepsilon) \in \{(+,-),(-,+),(-,-)\}$ in the case of non-faithful solutions.

Case	δ	ε	ϑ	conditions on v	faithful solution (z_1,z_2)
(1) a	$+$	$+$	$+$	$v=a$	$(a^2,\ b)$
b				$v=a^{-1}$	$(ba^{-1}b^{-1}a^{-1}b^{-1},\ ba^2b^{-1})$
c				$v=a^n,\ \lvert n\rvert \neq 1$	$\emptyset$
(2) a	$+$	$-$	$-$	$v=a^n$, n odd	(a^nb, b^{-2})
b			$+$	$v=a^n$, n even	$\emptyset$
c				$v=(ab)^n$	$\emptyset$
(3)	$-$	$+$		arbitrary v	$\emptyset$
(4) a	$-$	$-$	$+$	$v=ab$	$(aba,\ b)$
b				$v=(ab)^{-1}$	$(b^{-1}ab^3,\ b^{-2}ab^2)$
c				$v=(ab)^n,\ \lvert n\rvert \neq 1$	$\emptyset$
d				$v=a^n$, n even	$\emptyset$
e				$v=a^nb$, n odd	$(a^nba^{2-n},\ b)$
f			$-$	$v=a^n$, n odd	$(a^nb^{-1}a^{-n},\ b)$

Table 1: Faithful solutions of $Q_\delta(z_1,z_2) = vR_\varepsilon(a,b)^\vartheta v^{-1}R_\varepsilon(a,b)$ for some values of v with $w_\varepsilon(v) = \vartheta$

Many other values of v for which the equation has a faithful solution or not can be obtained from the solutions listed in Table 1, by applying an automorphism to v of the free group F_2 which sends $B := R_\varepsilon(a,b)$ to $B^{\pm 1}$, as given in [12, Corollary 7.2] and [11, Corollary 5.22].

Remark 2.2 In [11], the authors studied faithful solutions of the quadratic equation (11) in the case that all numbers $w_\varepsilon(v_j)c_j$, $1 \leq j \leq \ell$, have the same sign and

$A\cdot(r-1) = q-2+\ell$ where $A = |c_1|+\ldots+|c_\ell|$ (that is, the value q is "minimal", see Propositions 3.6 and 3.7(A)). We gave an algebraic criterion (see [11, Theorem 5.12], or [26, Theorem 5.17]) for the existence of a faithful solution of the quadratic equation (11) in terms of the self-intersection indices $\mu(v_1^{-1}v_j) \in \mathbb{Z}[F_{r,\varepsilon} \setminus \{1\}]$, $2 \le j \le \ell$, and the intersection indices $\lambda(v_1^{-1}v_i, v_1^{-1}v_j) \in \mathbb{Z}[F_{r,\varepsilon}]$, $2 \le j < i \le \ell$. As an application, we investigated the existence of faithful solutions of the quadratic equations (1)–(4) for some values of the conjugation parameter v with $\vartheta = w_\varepsilon(v)$, see [12, Corollary 7.2, Lemma 7.3], [11, Proposition 5.15, Corollary 5.22], or [26, Proposition 5.21]. The latter condition is equivalent to the fact that the corresponding maps have absolute degree 2, see Definition 3.4 and Corollary 3.11. These results are summarized in Table 1 above. Some faithful solutions from [12, Corollary 7.2] corresponding to maps of absolute degree 0 are given in Table 2, case (4a), and Table 4, case (4c).

3 Quadratic equations and Nielsen root theory

In Sections 3.1 and 3.2, we recall the notion of absolute degree of a map (Definition 3.4) and some results of [12] and [11] (see also [26]) about solutions of the quadratic equation (11). Then, in Section 3.3, we apply some of these results (Lemma 3.1, Propositions 3.6 and 3.8(A), (C), and Corollary 3.9(A)) to the equations (1)–(4) and summarize the obtained results in Theorem 3.14 and Tables 2 and 3. Other results of Sections 3.1 and 3.2 (Propositions 3.7 and 3.8(B), and Corollary 3.9(B), (C)) will not be used in our applications and can be skipped in the first reading (see also Remark 3.10).

Every solution of the equation (11) provides a continuous map $\bar f\colon\ \overline{M_1} \to \overline{M_2}$ between two closed surfaces (see below) with exactly ℓ roots having the multiplicities $c_1,\ldots,c_\ell$, see Section 3.2, [10], [11, 5.8, 5.21], or [12, Lemma 5.5(b)]. Here the closed surfaces $\overline{M_1}$ and $\overline{M_2}$ correspond to the quadratic words $Q_\delta(z_1,\ldots,z_q)$ and $R_\varepsilon(a_1,\ldots,a_r)$, respectively, and are defined as follows, see [11]. If $\varepsilon = 1$, we denote $\overline{M_2} := S_{r/2}$, a closed orientable surface of genus $r/2$; if $\varepsilon = -1$ then $\overline{M_2} := N_r$, a closed non-orientable surface of genus r (that is, the sphere with r crosscuts), thus N_{r+1} admits S_r as an orientable two-fold covering. Similarly, we denote $\overline{M_1} := S_{q/2}$ if $\delta = 1$, and $\overline{M_1} := N_q$ if $\delta = -1$.

In particular, the special quadratic equations (1)–(4) that we are going to study correspond to maps between closed surfaces of Euler characteristic 0. We investigate the existence of non-faithful solutions of these equations, and the existence of faithful solutions of the equations with $w_\varepsilon(v) = -\vartheta$, see (6), which actually correspond to mappings of absolute degree 0, see Corollary 3.11.

Consider two compact surfaces M_1 and M_2 having, respectively, ℓ and one boundary components, where M_1 (respectively, M_2) is obtained from $\overline{M_1}$ (respectively, $\overline{M_2}$) by removing the interiors of ℓ disjoint closed disks $D_1, \ldots, D_\ell \subset \overline{M_1}$ (respectively, the interior of a closed disk $D \subset \overline{M_2}$). Choose basepoints $P_1 \in \partial D_1$, $P_2 \in \partial D$. The fundamental groups of the surfaces admit the following *canonical presentations*:

$$\begin{aligned} \pi_1(M_1, P_1) &= \langle b_1, \ldots, b_q, d_1, \ldots, d_\ell \mid Q_\delta(b_1, \ldots, b_q) d_\ell^{-1} \ldots d_1^{-1} \rangle \\ &\approx F_{q+\ell-1}, \\ \pi_1(M_2, P_2) &= \langle a_1, \ldots, a_r, d \mid R_\varepsilon(a_1, \ldots, a_r) d^{-1} \rangle \\ &\approx F_r, \end{aligned} \tag{12}$$

$$\begin{aligned} \pi_1(\overline{M_1}, P_1) &= \langle b_1, \ldots, b_q \mid Q_\delta(b_1, \ldots, b_q) \rangle = F_q / \langle\langle Q_\delta(b_1, \ldots, b_q) \rangle\rangle, \\ \pi_1(\overline{M_2}, P_2) &= \langle a_1, \ldots, a_r \mid R_\varepsilon(a_1, \ldots, a_r) \rangle = F_r / \langle\langle R_\varepsilon(a_1, \ldots, a_r) \rangle\rangle, \end{aligned} \tag{13}$$

which correspond to some “canonical systems of cuts” on surfaces, see [11] or [26].

A continuous map $f\colon M_1 \to M_2$ is called *proper* if $\partial M_1 = f^{-1}(\partial M_2)$, that is, the boundary of the source is the preimage of the boundary of the target.

Lemma 3.1 ([12, Lemma 5.5], [26, Lemma 5.9]) *The existence of a solution $(z_1, \ldots, z_q)$ of the equation (11) is equivalent to the existence of a proper map $f\colon M_1 \to M_2$ such that $f(P_1) = P_2$ and the induced homomorphism $f_\#\colon \pi_1(M_1, P_1) \to \pi_1(M_2, P_2)$ sends*

$$f_\#(d_j) = v_1^{-1} v_j \cdot d^{c_j} \cdot v_j^{-1} v_1, \quad 1 \le j \le \ell.$$

Under this correspondence, the elements $z_1, \ldots, z_q$ of a solution are considered as the conjugates (with the conjugating factor v_1) of the images under $f_\#$ of the elements $b_1, \ldots, b_q$ of the canonical system of generators (12), that is $v_1^{-1} z_i v_1 = f_\#(b_i)$, $1 \le i \le q$.

The solution $(z_1, \ldots, z_q)$ is faithful if and only if the map f is orientation-true.

3.1 Absolute degree of a continuous map

The next two definitions are excerpted from [26, Definitions 4.5, 4.6] and introduce useful tools for studying continuous maps between manifolds of the same dimension.

Definition 3.2 (a) In a non-orientable manifold, the local orientation is either preserved or changed to the inverse when moved along a closed curve γ; according to this property γ is called *orientation-preserving* or *orientation-reversing*, respectively. Homotopic (even homologic) curves are the same with respect to orientation. On a surface, a simple loop γ is orientation-preserving if and only if γ is *two-sided*; otherwise the curve is *one-sided*. Following P Olum [35], a map $f\colon M_1 \to M_2$ is called *orientation-true* if orientation-preserving loops are sent to orientation-preserving ones and orientation-reversing loops to orientation-reversing ones.

(b) Following Hopf [22], Olum [35] and Skora [39], we distinguish three types of maps. A *map f is of Type I* if it is orientation-true. If f is not orientation-true and does not map orientation-reversing loops to null-homotopic ones then f is of *Type II*. The remaining maps are said to be of *Type III*; they are not orientation-true and map at least one orientation-reversing loop to a null-homotopic one. Of course, the type of a map can be determined by studying its effect on the fundamental group.

Remark 3.3 The orientation character $w_\varepsilon\colon F_{r,\varepsilon} \to \mathbb{Z}^* = \{1,-1\}$ defined in Definition 2.1(A) has the following geometric meaning. Consider the induced character $\pi_1(\bar{M}_2, P_2) \to \{1,-1\}$, see (13), which will be again denoted by w_ε. For any closed curve γ on $\bar{M}_2$ based at P_2, we have $w_\varepsilon([\gamma]) = 1$ if γ is orientation-preserving, and $w_\varepsilon([\gamma]) = -1$ if γ is orientation-reversing. Here $[\gamma] \in \pi_1(\bar{M}_2, P_2)$ denotes the homotopy class of γ.

For mappings between oriented closed manifolds, the notion $\deg(f)$, the degree of a map f, is well known, and there is a variety of ways to compute it. It is easily generalized to compact oriented manifolds with boundary if one restricts oneself to proper maps (see Lemma 3.1). For non-orientable manifolds one can also define the notion of a degree, as done by H Hopf [22], H Kneser[24] and D B A Epstein [9]. We recall the definition for surfaces as given by R Skora [39]; see also Brown and Schirmer [6].

Definition 3.4 (Absolute degree) Let $f\colon M_1 \to M_2$ be a proper map between compact surfaces.

(a) The *absolute degree of f*, denoted by $A(f)$, is defined as follows. There are three cases according to the type of the mapping f.

(I) f is of type I, that is, orientation-true. Let $\hat{M}_i = M_i$ and $k_i = 1$ if M_i is orientable and $\hat{M}_i$ be the 2–fold orientable covering of M_i and $k_i = 2$ otherwise. In particular, $\hat{M}_i$ is an orientable k_i–fold covering of M_i. Since f is orientation-true, there exists a lift $\hat{f}\colon \hat{M}_1 \to \hat{M}_2$. After fixing orientations on $\hat{M}_1$ and $\hat{M}_2$, the degree of $\hat{f}$ is defined, and we put
$$A(f) = \tfrac{k_2}{k_1}|\deg(\hat{f})|.$$

(II) If f is of type II, we define $A(f) = 0$.

(III) For f of type III, put $\ell = [\pi_1(M_2) : f_\#(\pi_1(M_1))]$ and let $\bar{M}_2 \to M_2$ be the ℓ–fold (unbranched) covering corresponding to the subgroup $f_\#(\pi_1(M_1))$. Now f has a lift $\bar{f}\colon M_1 \to \bar{M}_2$ which induces an epimorphism on the fundamental groups. Then $A(f)$ is either ℓ or 0 depending on whether the map

$$\bar{f}_*\colon\ \mathbb{Z}_2 = H_2(M_1, \partial M_1; \mathbb{Z}_2) \to H_2(\bar{M}_2, \partial \bar{M}_2; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } \ell < \infty, \\ 0 & \text{if } \ell = \infty \end{cases}$$

is bijective or not, respectively.

In particular, if $\ell = \infty$, then $A(f) = 0$. Further, if $A(f) \neq 0$ then $\ell \mid A(f)$.

(b) The *geometric degree of f* is the least non-negative integer d such that, for some disk $D \subset \mathring{M}_2$ and map g properly homotopic to f, the restriction of g to $g^{-1}(D)$ is a d–fold covering. *The geometric degree is never smaller than the absolute degree.*

For branched or unbranched coverings, the definition of the absolute degree does not give much new and the situation is much simpler.

Proposition 3.5 (a) *Every covering, branched or unbranched, is orientation-true.*

(b) *The geometric and the absolute degree of a (branched or unbranched) covering coincide and are equal to the order of the covering, that is, the number of leaves.*

(c) *The geometric and the absolute degree of any continuous map between closed surfaces coincide.*

Proof See Kneser [24]. □

3.2 Relation with the Nielsen root theory of maps

The geometric interpretation of solutions of (11) by means of proper maps $f\colon\ M_1 \to M_2$ between the compact surfaces M_1, M_2 with non-empty boundary (see Lemma 3.1) can be reformulated in terms of maps $\bar{f}\colon\ \overline{M_1} \to \overline{M_2}$ between the closed surfaces $\overline{M_1}$, $\overline{M_2}$ obtained from M_1, M_2 by attaching disks to the boundary components and radially extending the map f to the disks, see [11, 5.21]. Now, the centers of the disks in $\overline{M_1}$ form the preimage of the center c of the disk in $\overline{M_2}$.

The *root problem* for a map $\bar{f}\colon\ \overline{M_1} \to \overline{M_2}$ and a point $c \in \overline{M_2}$ is to find a map $\bar{g}$ homotopic to $\bar{f}$ which has the minimal number

$$MR[\bar{f}] := \min_{\bar{g} \simeq \bar{f}} |\bar{g}^{\ -1}(c)|$$

of roots $\bar{g}^{-1}(c)$ among all mappings $\bar{g}$ homotopic to $\bar{f}$. The roots of $\bar{f}$ split into *Nielsen equivalence classes* similar to the cases of the coincidence problem and intersection problem, see [11, 2.16] and [3, Definition 3.1]. It follows from Brooks [4], Epstein [9] and Kneser [24] that the number $NR[\bar{f}] = NC[\bar{f}, c]$ of *essential* Nielsen classes of roots (see Nielsen [33], or [3, Definition 3.6]) equals

$$NR[\bar{f}] = \begin{cases} [\pi_1(\overline{M_2}) : \bar{f}_\#(\pi_1(\overline{M_1}))] & \text{if } A(\bar{f}) > 0, \\ 0 & \text{if } A(\bar{f}) = 0, \end{cases} \tag{14}$$

where $A(\bar{f})$ denotes the absolute degree of $\bar{f}$. The map $\bar{f}$ has the *Wecken property for the root problem* if the general inequality

$$NR[\bar{f}] \leq MR[\bar{f}] \tag{15}$$

is an equality. The root problem for closed surfaces was completely solved in [1, 2, 12], including the study of the Wecken property.

Based on the Kneser congruence and the Kneser inequality, see [24] or [26, Theorem 4.20], and the geometric meaning of the equation (11), see Lemma 3.1, one obtains the following propositions.

Proposition 3.6 ([12, Proposition 5.8] or [26, Proposition 5.12]) *Suppose that equation (11) admits a solution $(z_1, \dots, z_q)$, and let $\bar{f}\colon \overline{M_1} \to \overline{M_2}$ be the corresponding map between closed surfaces admitting ℓ roots of multiplicities $w_\varepsilon(v_1)c_1, \dots, w_\varepsilon(v_\ell)c_\ell$. Let $A := w_\varepsilon(v_1)c_1 + \dots + w_\varepsilon(v_\ell)c_\ell$. If $A(\bar{f}) > 0$ then $A(\bar{f}) \cdot r \equiv q \mod 2$. If the solution is faithful then $A(\bar{f}) = |A|$.* □

Let, for an element $u \in \pi_1(M_2) = F_r = \langle a_1, \dots, a_r \,|\, \rangle$, the element

$$\bar{u} \in \pi_1(\overline{M_2}) = F_r / \langle\langle R_\varepsilon(a_1, \dots, a_r) \rangle\rangle = \langle a_1, \dots, a_r \,|\, R_\varepsilon(a_1, \dots, a_r) \rangle$$

denote its image under the natural projection $\pi_1(M_2) \to \pi_1(\overline{M_2})$. Denote by $H \subset \pi_1(\overline{M_2})$ the subgroup generated by the elements $\bar{z}_1, \dots, \bar{z}_q$. Denote by rank H the the minimal cardinality of a set of generators for H [27, Section II.2]. If H is a free group, or a free abelian group, this agrees with the usual definition of rank.

Proposition 3.7 *Let, under the hypothesis of Proposition 3.6, $A(\bar{f}) > 0$. Then:*

(A) *$A(\bar{f}) \cdot (r-2) \leq q-2$ and $A(\bar{f}) \cdot (r-1) \leq q-2+MR[\bar{f}] \leq q-2+\ell$. In particular, if $MR[\bar{f}] = \ell = |\bar{f}^{-1}(c)|$, then $\bar{f}$ is a solution of the root problem for $\bar{f}$.*

(B) *If $A(\bar f)\cdot(r-2)=q-2$ then the solution is faithful, $\bar f$ is homotopic to an $|A|$–fold covering and $MR[\bar f]=NR[\bar f]=A(\bar f)=|A|$, thus $\bar f$ has the Wecken property for the root problem.*

(C) *$NR[\bar f]=[\pi_1(\overline{M_2}):H]\le\min\{\ell,A(\bar f)\}$. Furthermore, consider the subdivision of $\{1,\dots,\ell\}$ into $\ell_H=[\pi_1(\overline{M_2}):H]$ subsets where i,j belong to the same subset iff $\bar v_i\bar v_j^{-1}\in H$ (that is, $\bar v_i$ and $\bar v_j$ belong to the same Reidemeister root class); then each of these subsets is non-empty. If the solution is faithful then $w_\delta(\ker\bar f_\#)=\{1\}$, and the sum of $w_\varepsilon(v_j)c_j$ over all j belonging to the same subset equals $\frac{A}{\ell_H}$. If the solution is non-faithful then each of these sums is odd, $w_\delta(\ker\bar f_\#)=\{1,-1\}$, and $A(\bar f)=\ell_H=NR[\bar f]=MR[\bar f]$, thus $\bar f$ has the Wecken property for the root problem.*

Proof (A) Since $A(\bar f)>0$, it follows from the Kneser inequality [24] that $\chi(\overline{M_1})\le A(\bar f)\cdot\chi(\overline{M_2})$. Since $\chi(\overline{M_1})=2-q$, $\chi(\overline{M_2})=2-r$, this gives the first inequality. Since the map $\bar f$ has ℓ roots, we have $MR[\bar f]\le\ell$. Applying the Kneser inequality to a suitable proper map $g\colon M_1'\to M_2$ corresponding to a map $\bar g\colon\overline{M_1}\to\overline{M_2}$, which is homotopic to $\bar f$ and has $MR[\bar f]$ roots, one gets the inequality

$$\chi(\overline{M_1})-MR[\bar f]\le G(g)\cdot(\chi(\overline{M_2})-1),$$

where $G(g)$ denotes the geometric degree of g, see Definition 3.4(b) and [39, Theorem 4.1] (see also [12, Theorem 2.5(A)], in the case when $\bar f$ is orientation-true). On the other hand, $G(g)\ge G(\bar g)=A(\bar g)=A(\bar f)$, due to Proposition 3.5(c). This proves (A).

(B) Since $A(\bar f)>0$, and the Kneser inequality [24]

$$\chi(\overline{M_1})\le A(\bar f)\cdot\chi(\overline{M_2})$$

becomes an equality, it follows from [24] that the map $\bar f$ is homotopic to an $A(\bar f)$–fold covering (this also follows from the classification of maps of positive absolute degree, see [39, Theorem 1.1]). Therefore $\bar f$ is orientation-true, and

$$MR[\bar f]\le[\pi_1(\overline{M_2}):\bar f_\#(\pi_1(\overline{M_1}))]=A(\bar f).$$

By Lemma 3.1, the solution is faithful. Hence, by Proposition 3.6, $A(\bar f)=|A|$. Together with (14), (15), this proves the assertion.

(C) In the case of faithful solutions, this assertion follows from [12, Lemma 5.7] or [26, Lemma 5.18(b)]. If the solution is non-faithful then the map $\bar f$ is not orientation-true with $A(\bar f)>0$. Therefore $\bar f$ has Type III (see Definition 3.2(b)) or, equivalently, $w_\delta(\ker\bar f_\#)=\{1,-1\}$. It follows from [26, Proposition 4.19] that every sum under consideration is odd. Since the map $\bar f$ is not orientation-true, it has the Wecken property for the root problem, due to Kneser [23, 24] and (14), see also [9] or [12]. Indeed, Kneser [23, 24] proved that such $\bar{f}$ can be deformed to a map having 0 or ℓ_H roots depending on whether $A(\bar{f}) = 0$ or $A(\bar{f}) > 0$, and by (14) the latter number coincides with $NR[\bar{f}]$. Therefore $A(\bar{f}) = \ell_H = NR[\bar{f}] = MR[\bar{f}]$. □

By applying a suitable automorphism of the free group F_q, one obtains the following presentation of the fundamental group of the closed surface $\overline{M_1}$, in addition to (13), see Lyndon and Schupp [30, Chapter I, Proposition 7.6]:

$$\pi_1(\overline{M_1}, P_1) = \langle \xi_1, \dots, \xi_{[\frac{q+1}{2}]}, \eta_1, \dots, \eta_{[\frac{q}{2}]} \mid \mathcal{Q}_\delta(\xi_1, \dots, \xi_{[\frac{q+1}{2}]}, \eta_1, \dots, \eta_{[\frac{q}{2}]}) \rangle,$$

where

$$\mathcal{Q}_\delta\big(\xi_1, \dots, \xi_{[\frac{q+1}{2}]}, \eta_1, \dots, \eta_{[\frac{q}{2}]}\big) = \begin{cases} \prod_{i=1}^{\frac{q}{2}} [\xi_i, \eta_i], & \delta = 1, \\ \left(\prod_{i=1}^{\frac{q}{2}-1} [\xi_i, \eta_i]\right) \cdot [\xi_{\frac{q}{2}}, \eta_{\frac{q}{2}}]_-, & \delta = -1, q \text{ even}, \\ \left(\prod_{i=1}^{\frac{q-1}{2}} [\xi_i, \eta_i]\right) \cdot \xi^2_{\frac{q+1}{2}}, & \delta = -1, q \text{ odd}. \end{cases}$$

Here we use the notation

$$[x, y] \ = \ xyx^{-1}y^{-1}, \qquad [x, y]_- \ = \ xyxy^{-1}.$$

By applying the corresponding change of the unknowns, the equation (11) in $F_r = \langle a_1, \dots, a_r \mid \rangle$ rewrites in the following equivalent form:

$$\mathcal{Q}_\delta(x_1, \dots, x_{[\frac{q+1}{2}]}, y_1, \dots, y_{[\frac{q}{2}]}) = \prod_{j=1}^{\ell} v_j \cdot (R_\varepsilon(a_1, \dots, a_r))^{c_j} \cdot v_j^{-1}, \tag{16}$$

with the new unknowns $x_1, \dots, x_{[\frac{q+1}{2}]}, y_1, \dots, y_{[\frac{q}{2}]} \in F_r$. Similarly to Definition 2.1(C), a solution of the equation (16) is called *faithful* if

$$w_\varepsilon(x_i) = \begin{cases} 1, & 1 \le i \le [\frac{q}{2}], \\ -1, & i = \frac{q+1}{2},\ \delta = -1,\ q \text{ odd}, \end{cases} \qquad w_\varepsilon(y_i) = \begin{cases} 1, & 1 \le i \le [\frac{q-1}{2}], \\ \delta, & i = \frac{q}{2},\ q \text{ even}. \end{cases}$$

Otherwise the solution is called *non-faithful*. Actually, a solution of (11) is faithful if and only if the corresponding solution of (16) is faithful.

Suppose that, for a solution of (16), all $x_i \in N = \langle\langle R_\varepsilon(a_1, \dots, a_r) \rangle\rangle$, $1 \le i \le [\frac{q+1}{2}]$. (One easily shows that, in this case, both sides of the equation belong to N.) If one restricts oneself only to such solutions of (16), the obtained equation will be refered to as the *equation* (16) *in the subgroup* N of F_r. One checks that, for odd q, all solutions of (16) in N are non-faithful, while, for even q, a solution is faithful if and only if $w_\varepsilon(y_i) = 1$, $1 \le i \le \frac{q}{2} - 1$, and $w_\varepsilon\big(y_{\frac{q}{2}}\big) = \delta$.

For every solution of (16) in F_r, consider the corresponding homomorphism

$$h\colon\ F_q = \langle \xi_1, \dots, \xi_{[\frac{q+1}{2}]}, \eta_1, \dots, \eta_{[\frac{q}{2}]} \mid \ \rangle \to F_r = \langle a_1, \dots, a_r \mid \ \rangle$$

sending $\xi_i \mapsto x_i$, $\eta_i \mapsto y_i$. In particular, the subgroup H is the image of the composition

$$F_q \xrightarrow{h} F_r \xrightarrow{p_{r,\varepsilon}} F_r/\langle\langle R_\varepsilon(a_1, \dots, a_r)\rangle\rangle,$$

where $p_{r,\varepsilon}$ is the projection, see (13). It follows from Lemma 3.1 that $h = j_{v_1} f_\#$, where j_u is the conjugation by the element u in F_r, that is $j_u(v) = uvu^{-1}$, $u, v \in F_r$.

Proposition 3.8 *Suppose that, under the hypothesis of Proposition 3.6, $A(\bar f) = 0$. Denote by $(x_1, \dots, x_{[\frac{q+1}{2}]}, y_1, \dots, y_{[\frac{q}{2}]})$ the corresponding solution of the equation (16) in F_r, let $h\colon\ F_q \to F_r$ be the corresponding homomorphism, and $\rho := \operatorname{rank} H$. Then:*

(A) *$\rho \le [\frac{q}{2}]$, moreover there exists an automorphism φ of the free group F_q such that the word $\mathcal{Q}_\delta(\xi_1, \dots, \xi_{[\frac{q+1}{2}]}, \eta_1, \dots, \eta_{[\frac{q}{2}]}) \in F_q$ is preserved under φ, and*

$$h\varphi(\xi_i) \in N = \langle\langle R_\varepsilon(a_1, \dots, a_r)\rangle\rangle$$

for all $1 \le i \le [\frac{q+1}{2}]$. In other words, for the solution $\big(x'_1, \dots, x'_{[\frac{q+1}{2}]}, y'_1, \dots, y'_{[\frac{q}{2}]}\big)$ of the equation (16), which corresponds to the homomorphism $h' = h\varphi\colon\ F_q \to F_r$, one has $x'_i \in N$ for all $1 \le i \le [\frac{q+1}{2}]$. The solutions $\big(x_1, \dots, x_{[\frac{q+1}{2}]}, y_1, \dots, y_{[\frac{q}{2}]}\big)$ and $\big(x'_1, \dots, x'_{[\frac{q+1}{2}]}, y'_1, \dots, y'_{[\frac{q}{2}]}\big)$ of (16) are both faithful or both non-faithful.

(B) *$MR[\bar f] = NR[\bar f] = 0$; in particular, $\bar f$ has the Wecken property for the root problem.*

(C) *Consider the subdivision of $\{1, \dots, \ell\}$ into subsets where i,j belong to the same subset iff $\bar v_i \bar v_j^{-1} \in H$ (that is, $\bar v_i$ and $\bar v_j$ belong to the same Reidemeister root class). If $w_\delta(\ker \bar f_\#) = \{+1\}$ then, for each i with $1 \le i \le \ell$, the sum of $w_\delta(\bar f_\#^{-1}(\bar v_i \bar v_j^{-1}))c_j$ over all j belonging to the subset containing i vanishes. Otherwise (that is, if $w_\delta(\ker \bar f_\#) = \{1, -1\}$) each of these sums is even.*

Proof (A), (B) Since $A(\bar f) = 0$, it follows from Kneser [24] that the map $\bar f$ is homotopic to a map which is not surjective (see also Epstein [9]), thus $MR[\bar f] = NR[\bar f] = 0$. We also obtain that $\bar f$ is homotopic to a map whose image lies in the 1–skeleton of the target $\overline{M_2}$, and therefore $\bar f_\#$ admits a composition $\pi_1(\overline{M_1}, P_1) \to F \to \pi_1(\overline{M_2}, P_2)$ where the first homomorphism $g\colon\ \pi_1(\overline{M_1}, P_1) \to F$ is an epimorphism to a free group F. It follows that $\operatorname{rank} F \le [\frac{q}{2}]$, see Zieschang [47, 48], and Zieschang, Vogt and Coldewey[50] in the case of orientable M_1, and from Ol'shanskiĭ [34] in the general case (see also Lyndon and Schupp [30, Proposition 7.13], or [26, Corollary 2.4]). Therefore $\rho = \operatorname{rank} H \leq \operatorname{rank} F \leq \left[\frac{q}{2}\right]$.

In the case of orientable M_1, it has been proved in [48] using the Nielsen method (see also [50] or Grigorchuk, Kurchanov and Zieschang [15, Proposition 1.2]) that there exists a sequence of "elementary moves" of the system of generators $\xi_1, \dots, \xi_{q/2}, \eta_1, \dots, \eta_{q/2}$ of F_q, and a corresponding sequence of "elementary moves" of the "system of cuts" on $\overline{M_1}$ (see above), such that the resulting system of generators $\xi'_1, \dots, \xi'_{q/2}, \eta'_1, \dots, \eta'_{q/2}$ is also canonical (this means, there exists an automorphism φ of F_q such that $\xi'_i = \varphi(\xi_i)$, $\eta'_i = \varphi(\eta_i)$, and

$$\mathcal{Q}_\delta\big(\xi'_1, \dots, \xi'_{q/2}, \eta'_1, \dots, \eta'_{q/2}\big) = \mathcal{Q}_\delta\big(\xi_1, \dots, \xi_{q/2}, \eta_1, \dots, \eta_{q/2}\big)$$

in F_q), and $g(\bar{\xi}'_i) = 1$ in F for all $1 \leq i \leq \frac{q}{2}$. Here $\bar{u} \in \pi_1(\overline{M_1}, P_1)$ denotes the image of $u \in F_q$ under the projection $F_q \to F_q/\langle\langle \mathcal{Q}_\delta(\xi_1, \dots, \xi_{\frac{q}{2}}, \eta_1, \dots, \eta_{\frac{q}{2}})\rangle\rangle = \pi_1(\overline{M_1}, P_1)$. In the general case (that is, when M_1 is not necessarily oreintable), the existence of an automorphism φ of F_q having the analogous properties was proved by Ol'shanskiĭ [34, Theorem 1].

Since $g(\bar{\xi}'_i) = 1$ in F, it follows $\bar{f}_\#(\bar{\xi}'_i) = 1$ in $\pi_1(\overline{M_2}, P_2)$. Hence $f_\#(\xi'_i) \in N$. This gives $h\varphi(\xi_i) = h(\xi'_i) = j_{v_1} f_\#(\xi'_i) \in N$.

Let us prove the latter assertion of (A). Since the automorphism φ preserves the quadratic word $\mathcal{Q}_\delta\big(\xi_1, \dots, \xi_{[\frac{q+1}{2}]}, \eta_1, \dots, \eta_{[\frac{q}{2}]}\big)$, it also "preserves" the orientation character $w_\delta \colon F_q \to \{1, -1\}$, that is, $w_\delta = w_\delta \varphi$, see Definition 2.1(A), Remark 3.3, and Lyndon and Schupp [30, Chapter I, Proposition 7.6]. Now observe that the solution $\big(x_1, \dots, x_{[\frac{q+1}{2}]}, y_1, \dots, y_{[\frac{q}{2}]}\big)$ is faithful if and only if $w_\delta = w_\varepsilon h$. Similarly, $\big(x'_1, \dots, x'_{[\frac{q+1}{2}]}, y'_1, \dots, y'_{[\frac{q}{2}]}\big)$ is faithful if and only if $w_\delta = w_\varepsilon h'$. By the above, the latter equality is equivalent to $w_\delta \varphi = w_\varepsilon h \varphi$, and since φ is an automorphism, it is equivalent to $w_\delta = w_\varepsilon h$.

(C) In the case of faithful solutions, the assertion follows from [12, Lemma 5.7] or [26, Lemma 5.18(b)]. If the solution is non-faithful then the map $\bar{f}$ has Type II if $w_\delta(\ker \bar{f}_\#) = \{+1\}$, and it has Type III if $w_\delta(\ker \bar{f}_\#) = \{1, -1\}$. Since $A(\bar{f}) = 0$, it follows from [26, Proposition 4.19] that each sum under consideration vanishes if $\bar{f}$ has Type II, and it is even if $\bar{f}$ has Type III. □

Corollary 3.9 *Under the hypothesis of Proposition 3.6, the following properties hold:*

(A) *Suppose that $r = q = 2$. If $A(\bar{f}) > 0$ then $\operatorname{rank} H = 2$ and the solution is faithful. If $A(\bar{f}) = 0$ then $\operatorname{rank} H \leq 1$, moreover either $\delta = -1$ and $x \in N = \langle\langle R_\varepsilon(a_1, a_2)\rangle\rangle$, or*

$\delta = 1$ and $x' \in N$, for some solution (x', y') which is faithful (resp. non-faithful) if (x, y) is faithful (resp. non-faithful). If $x \in N$ then $A(\bar{f}) = 0$ and the following implications hold:

$$(17) \qquad (x, y) \text{ is faithful} \iff w_\varepsilon(y) = \delta,$$

$$(18) \qquad \ell = 2, c_2 \text{ odd} \implies \exists k \in \mathbb{Z}, \qquad c_1 = \begin{cases} -c_2\delta^k & \text{if } \bar{y} \neq 1 \text{ or } \delta = 1, \\ \text{odd} & \text{otherwise,} \end{cases}$$
$$\bar{v}_1 \bar{v}_2^{-1} = \bar{y}^k.$$

Here $(x, y) = (x_1, y_1)$ is the solution of (16) with $r = q = 2$ corresponding to the solution (z_1, z_2) of (5) via the standard transformation of unknowns, see (9).

(B) *Suppose that either the solution $(z_1, \dots, z_q)$ is non-faithful, or $A = 0$, or $|A| \cdot (r - 2) = q - 2$ (in particular, $r = q = 2$). Then the map $\bar{f}\colon \overline{M_1} \to \overline{M_2}$ has the Wecken property for the root problem: $MR[\bar{f}] = NR[\bar{f}] = |A(\bar{f})|$.*

(C) *Suppose the solution is faithful and $A \neq 0$. Then $|A| \cdot (r - 1) \leq q - 2 + \ell$, furthermore:*

If $|A| \cdot (r - 1) = q - 2 + \ell$, $\ell \geq 2$ and $w_\varepsilon(v_i)c_i \neq w_\varepsilon(v_j)c_j$ for some pair of indices $1 \leq i, j \leq \ell$, then $NR[\bar{f}] < MR[\bar{f}] = \ell$ and, thus, $\bar{f}$ does not have the Wecken property for the root problem. If $\ell' < |A| \cdot (r-1) - q + 2$ then $NR[\bar{f}] \leq \ell' < |A| \cdot (r-1) - q + 2 \leq MR[\bar{f}]$, where ℓ' is the maximal number of disjoint subsets of $\{1, \dots, \ell\}$ such that the union of the subsets is $\{1, \dots, \ell\}$ and the sum of $w_\varepsilon(v_j)c_j$ over all j belonging to the same subset does not depend on the subset and, hence, equals A/ℓ'.

Proof (A) Let $r = q = 2$. Suppose that $A(\bar{f}) > 0$. By Proposition 3.7(B), the solution is faithful and $\bar{f}$ is homotopic to a covering. Therefore $\ell_H = A(\bar{f})$ and $\bar{f}_\#\colon \pi_1(\overline{M_1}) \to \pi_1(\overline{M_2})$ is a monomorphism, hence $\operatorname{rank} H = \operatorname{rank} \pi_1(\overline{M_1}) = 2$.

Suppose that $A(\bar{f}) = 0$. By Proposition 3.8(A), $\rho = \operatorname{rank} H \leq [\frac{q}{2}] = 1$ and there exists an automorphism $\varphi \in \operatorname{Aut}(F_2)$ such that the relator $\xi\eta\xi^{-\delta}\eta^{-1} \in F_2$ is preserved by φ, and $x' := h\varphi(\xi) \in N$; moreover the corresponding solutions (x, y) and (x', y') are both faithful or both non-faithful. Thus φ is the desired automorphism when $\delta = 1$. In the case $\delta = -1$, it is well known that the cyclic subgroup $\langle \bar{\xi} \rangle$ of $\langle \xi, \eta \mid \xi\eta\xi\eta^{-1} \rangle$ (the fundamental group of the Klein bottle) generated by $\bar{\xi}$ is characteristic, hence $\varphi(\xi) = \xi^{\pm 1}\xi_1$ for some $\xi_1 \in \langle\langle \xi\eta\xi\eta^{-1} \rangle\rangle$. Since $h\varphi(\xi) \in N$ and $h(\xi_1) \in N$ (since $h(\xi\eta\xi\eta^{-1}) = xyxy^{-1}$ equals the right-hand side of the equation, thus it belongs to N), it follows that $x = h(\xi) \in N$.

Suppose that $x \in N$. Then $\bar{x} = 1$, thus $H = \langle \bar{y} \rangle$ and $\operatorname{rank} H \le 1$. It follows from the above that $A(\bar{f}) = 0$. The property (17) follows by observing that the solution $(x, y) = (h(\xi), h(\eta))$ is faithful if and only if $w_\varepsilon(h(\zeta)) = w_\delta(\zeta)$ for any $\zeta \in F_2 = \langle \xi, \eta \mid \rangle$ or, equivalently, for any $\zeta \in \{\xi, \eta\}$. For $\zeta = \xi$, this equality holds, since $h(\xi) = x \in N$ and $w_\delta(\xi) = 1$. For $\zeta = \eta$, the equality is equivalent to $w_\varepsilon(y) = \delta$.

Let us prove (18). Since $\ell = 2$ and c_2 is odd, it follows from Proposition 3.8(C) that $\bar{v}_1 \bar{v}_2^{-1}$ belongs to the subgroup $H = \langle \bar{y} \rangle$ and that $c_1 + c_2$ is even. Hence $\bar{v}_1 \bar{v}_2^{-1} = \bar{y}^k$, for some $k \in \mathbb{Z}$, and (18) is proved when $\bar{y} = 1$, $\delta = -1$. Let us assume that $\bar{y} \ne 1$ or $\delta = 1$. If $\bar{y} \ne 1$ then the kernel of the induced homomorphism

$$\bar{f}_\# \colon \pi_1(\overline{M_1}) = \langle \xi, \eta \mid \xi \eta \xi^{-\delta} \eta^{-1} \rangle \to \pi = \pi_1(\overline{M_2}) = \langle \alpha, \beta \mid B \rangle, \quad \bar{\xi} \mapsto \bar{x},\ \bar{\eta} \mapsto \bar{y},$$

is generated by $\bar{\xi}$. Since $w_\delta(\bar{\xi}) = 1$, we have $w_\delta(\ker \bar{f}_\#) = \{+1\}$. If $\delta = 1$, the equality $w_\delta(\ker \bar{f}_\#) = \{+1\}$ is obvious. Since $\bar{v}_1 \bar{v}_2^{-1} \in \langle \bar{y} \rangle = H$, it follows from Proposition 3.8(C) that $w_\delta(\bar{f}_\#^{-1}(\bar{v}_1 \bar{v}_2^{-1}))c_1 + c_2 = 0$. On the other hand, we have $\bar{v}_1 \bar{v}_2^{-1} = \bar{y}^k$, thus

$$w_\delta(\bar{f}_\#^{-1}(\bar{v}_1 \bar{v}_2^{-1})) = w_\delta(\bar{f}_\#^{-1}(\bar{y}^k)) = (w_\delta(\bar{\eta}))^k = \delta^k.$$

This proves the equality $c_1 \delta^k + c_2 = 0$, and thereby completes the proof of (18).

(B) If the solution is non-faithful then $\bar{f}$ has the Wecken property for the root problem, due to Propositions 3.7(C) and 3.8(B). Suppose that the solution is faithful. Then, by Proposition 3.6, $A(\bar{f}) = |A|$. If $A = 0$ or $|A| \cdot (r - 2) = q - 2$ then $\bar{f}$ has the Wecken property for the root problem, due to Propositions 3.8(B) and 3.7(B).

(C) As above, $A(\bar{f}) = |A|$. It follows from Proposition 3.7(A), (C) that $|A| \cdot (r - 1) \le q - 2 + MR[\bar{f}] \le q - 2 + \ell$ and $NR[\bar{f}] = \ell_H \le \ell'$. Hence, $NR[\bar{f}] \le \ell' < \ell = MR[\bar{f}]$ in the first case, and $NR[\bar{f}] \le \ell' < |A| \cdot (r-1) - q + 2 \le MR[\bar{f}]$ in the second case. □

Remark 3.10 Another way of proving the property (18) is given below in Theorem 5.1, using the corresponding first derived equation (which is similar to (36)), rather than Proposition 3.8(C). Both geometric and algebraic ways of proving Proposition 3.8(C) are given in [26, Proposition 4.19].

3.3 Applications to the quadratic equations (1)–(4)

Here we apply the results of Section 3.2 to study the existence of faithful, or non-faithful, solutions (z_1, z_2) of (5) satisfying the condition (6). For some values of $\bar{v} = p_\pi(v) \in \pi = F_2 / \langle\langle R_\varepsilon(a_1, a_2) \rangle\rangle$, we give some explicit faithful and non-faithful solutions in Tables 2 and 3, respectively, in terms of the new variables, as given in (9). The non-existence results stated in Tables 2 and 3 will be based on the results of Section 3.2.

Corollary 3.11 *A solution (z_1, z_2) of (5) satisfies the condition (6) if and only if the absolute degree $A(\bar{f})$ of the corresponding map $\bar{f}\colon \overline{M_1} \to \overline{M_2}$ (see Section 3.2) vanishes.*

Proof Suppose that the solution (z_1, z_2) does not satisfy the condition (6). Then the solution is faithful and $\vartheta \neq -w_\varepsilon(v)$, thus $A = w_\varepsilon(v)\vartheta + 1 \neq 0$. By Proposition 3.6, this gives $A(\bar{f}) = |A| > 0$.

Suppose that $A(\bar{f}) > 0$. By Corollary 3.9(A), the solution is faithful. By Proposition 3.6, this implies $A(\bar{f}) = |A| = |w_\varepsilon(v)\vartheta + 1|$. Since the latter expression is positive, we must have $\vartheta \neq -w_\varepsilon(v)$. Therefore the solution does not satisfy the condition (6). □

As in (8) and (9), let us rewrite the equations (1)–(4) in terms of the new generators α, β and the unknowns x, y, as given in (9). Thus $R_+(a,b) = [a,b] = [\alpha,\beta]$, $R_-(a,b) = a^2b^2 = \alpha\beta\alpha\beta^{-1}$, and we obtain the equation (8), which is written in detail as follows:

$$(1') \qquad [x,y] = v[\alpha,\beta]^{\vartheta}v^{-1}\cdot[\alpha,\beta],$$

$$(2') \qquad [x,y] = v(\alpha\beta\alpha\beta^{-1})^{\vartheta}v^{-1}\cdot\alpha\beta\alpha\beta^{-1},$$

$$(3') \qquad xyxy^{-1} = v[\alpha,\beta]^{\vartheta}v^{-1}\cdot[\alpha,\beta],$$

$$(4') \qquad xyxy^{-1} = v(\alpha\beta\alpha\beta^{-1})^{\vartheta}v^{-1}\cdot\alpha\beta\alpha\beta^{-1}.$$

In the new generators, the fundamental group $\pi = \pi_\varepsilon = \pi_1(\bar{M}_2)$ and the projection of $F_2 = \langle \alpha, \beta \mid \rangle$ to it have the form

$$p_\pi\colon F_2 \to \pi = F_2/N, \qquad \text{where} \quad N = \langle\langle B \rangle\rangle,\ B = \alpha\beta\alpha^{-\varepsilon}\beta^{-1}.$$

As above, denote

$$\bar{u} := p_\pi(u), \qquad u \in F_2.$$

Every element $\bar{u} \in \pi$ can be written in a unique way in the following *canonical form*:

$$(19) \qquad \bar{u} = \bar{\alpha}^r\bar{\beta}^s, \quad r, s \in \mathbb{Z}.$$

Remark 3.12 Let us apply Corollary 3.9(A) to study the existence of (faithful, or non-faithful) solutions of the equations (1′)–(4′) satisfying $A(\bar{f}) = 0$. Suppose that (x,y) is such a solution. In the case of the equations (3′) and (4′), we have $\delta = -1$;

hence $x \in N$. In the case of the equations ($1'$) and ($2'$), we have $\delta = 1$; hence there exists a solution (x', y') with $x' \in N$, where the solutions (x, y) and (x', y') are both faithful or both non-faithful. Thus we can restrict ourselves to study the existence of solutions (x, y) of ($1'$)–($4'$) satisfying $x \in N$. Such solutions have the properties (17) and (18), where one substitutes $v_1 = v$, $v_2 = 1$, $c_1 = \vartheta$, $c_2 = 1$. Thus the property (18) has the following form for the equation (8), or ($1'$)–($4'$):

$$\bar{v} = \bar{y}^k, \quad \vartheta\delta^k = -1, \qquad \text{for some} \quad k \in \mathbb{Z}. \tag{20}$$

Notation 3.13 If $F_2 = \langle t_1, t_2 \mid \rangle$ is a free group on two generators t_1, t_2, let $|u|_{t_i}$ denote the sum of the exponents of t_i which appear in a word $u \in F_2$. In the case of $\pi = \pi_-$, denote by $p_K^\alpha(u)$ and $p_K^\beta(u)$, the exponents of $\bar{\alpha}$, $\bar{\beta}$, respectively, which appear in the canonical form (19) of the element $\bar{u}$, thus $p_K^\alpha(u) := r$ and $p_K^\beta(u) := s$, see (19). We also denote the projection $p_\pi\colon F_2 \to \pi$ by p_T, or p_K, in the cases when π is the fundamental group of the 2–torus T ($\varepsilon = 1$), or the Klein bottle K ($\varepsilon = -1$), respectively. We will say that an element $\bar{u}$ of an abelian group is *divisible by* 2 if there exists an element $\bar{u}_1$ of the group such that $2\bar{u}_1 = \bar{u}$. (Here the additive notation for the group operation is used.)

The following Theorem 3.14 summarizes the above results about the existence of faithful, or non-faithful, solutions satisfying (6) of the quadratic equation (8). It can be regarded as the "first classification" of values of the conjugation parameter v with respect to the property that the corresponding quadratic equation admits a (faithful, or non-faithful) solution. These results are also summarized in Tables 2 and 3, and in the explicit solutions given in Tables 4 and 5. The cases which are not completely solved by Theorem 3.14 are marked as "mixed" cases in Tables 2 and 3.

Theorem 3.14 *Let $v \in F_2 = \langle \alpha, \beta \mid \rangle$, $\delta, \varepsilon, \vartheta \in \{1, -1\}$. For the quadratic equation (8), the existence of a faithful (resp. non-faithful) solution satisfying (6) is equivalent to the existence of a faithful (resp. non-faithful) solution satisfying $x \in N = \langle\langle \alpha\beta\alpha^{-\varepsilon}\beta^{-1} \rangle\rangle$. The following results on the existence of such faithful and non-faithful solutions hold, see Tables 2 and 3, respectively:*

1. *The equation ($1'$) has a faithful solution for any $v \in F_2$ and $\vartheta = -1$, see Table 2(1) for a solution, while it has no non-faithful solution for any $v \in F_2$ and $\vartheta \in \{1, -1\}$. So, in this case, the problem of the existence of solutions satisfying (6) is completely solved.*
2. *The equation ($2'$) with $w_-(v) = -\vartheta$ admits a faithful solution if and only if $\vartheta = -1$, see Table 2(2a) for a solution. For non-faithful solutions of ($2'$), we have:*

(a) *If $\vartheta = 1$, there is no solution.*

(b) *If $\vartheta = -1$ and $w_-(v) = -1$ then there is a solution, see Table 3(2b) for a solution.*

(c) *If $\vartheta = -1$, $w_-(v) = 1$ (thus $p_K^\beta(v)$ is even), and $p_K^\alpha(v) \neq 0$ then there is no solution.*

(d) *If $\vartheta = -1$, $w_-(v) = 1$ (thus $p_K^\beta(v)$ is even), and $p_K^\alpha(v) = 0$ then there is an element $v_1 \in p_K^{-1}(p_K(v))$, for which the equation admits a solution, see Table 5(2c,2d) for a solution.*

(3) *The equation ($3'$) has no faithful solution, while the following properties hold for its non-faithful solutions:*

(a) *If $\vartheta = 1$, there is a solution, see Table 3(3a) for a solution.*

(b) *If $\vartheta = -1$ and $p_T(v)$ is not divisible by 2, then there is no solution.*

(c) *If $\vartheta = -1$ and $p_T(v)$ is divisible by 2, then there is an element $v_1 \in p_T^{-1}(p_T(v))$, for which the equation admits a solution, see Table 5(3c) for a solution.*

(4) *The following properties hold for faithful solutions of the equation ($4'$) with $w_-(v) = -\vartheta$:*

(a) *If $w_-(v) = -1$ (thus $p_K^\beta(v)$ is odd) then there is a solution, see Table 2(4a) for a solution.*

(b) *If $w_-(v) = 1$ (thus $p_K^\beta(v)$ is even) and $p_K^\alpha(v) \neq 0$, then there is no solution.*

(c) *If $w_-(v) = 1$ (thus $p_K^\beta(v)$ is even) and $p_K^\alpha(v) = 0$, then there is an element $v_1 \in p_K^{-1}(p_K(v))$, for which the equation admits a solution, see Table 4(4d) for a solution.*

For non-faithful solutions of ($4'$), the following properties hold:

(d) *If $w_-(v) = -1$ (thus $p_K^\beta(v)$ is odd) then there is no solution.*

(e) *If $w_-(v) = 1$ (thus $p_K^\beta(v)$ is even) and $\vartheta = 1$, then there is a solution, see Table 3(4b) for a solution.*

(f) *If $w_-(v) = 1$ (thus $p_K^\beta(v)$ is even), $\vartheta = -1$, and moreover $p_K^\beta(v)$ is not divisible by 4 or $p_K^\alpha(v)$ is odd, then there is no solution.*

(g) *If $w_-(v) = 1$ (thus $p_K^\beta(v)$ is even), $\vartheta = -1$, $p_K^\beta(v)$ is divisible by 4, and $p_K^\alpha(v)$ is even, then there is an element $v_1 \in p_K^{-1}(p_K(v))$, for which the equation admits a solution, see Table 5(4d) for a solution.*

(5) *In each of the "mixed" cases 2d, 3c, 4c, 4g above, any solution with $x \in N$ satisfies (17) and (20), which imply $\bar{v} \in \langle \bar{y}^2 \rangle$.*

Proof By Corollary 3.11 and Remark 3.12, the existence of a solution satisfying (6) is equivalent to the existence of a solution satisfying $x \in N$, where the solutions are both faithful or both non-faithful. This proves the first desired assertion.

By direct calculations in the free group F_2, or in the abelianised F_2, one readily obtains the following cases of Tables 2 and 3:

Table 2, cases (1), (2a), (2b), (3), (4a), and

Table 3, cases (1), (2a), (2b), (3a), (4b).

The corresponding arguments for each of these cases are given in the footnotes to these cases in Tables 2 and 3.

The following cases of Tables 2 and 3 are marked as "mixed" cases:

(21) Table 2, case (4c) and Table 3, cases (2d), (3c), (4e).

In each of these cases, an explicit value of the conjugation parameter $v_1 \in p_\pi^{-1}(v)$, together with an explicit solution of the corresponding quadratic equation, are given in Table 4, case (4d), and Table 5, cases (2c,2d), (3c), (4d), respectively. In the first of these cases, the solution was given in [12, Corollary 7.2]. Other three cases are justified by direct calculations in F_2 (actually in N). In the latter case, one also uses the following relation which is a simple consequence of the relation $\bar\alpha\bar\beta\bar\alpha\bar\beta^{-1} = 1$, in the fundamental group $\pi = \pi_-$ of the Klein bottle:

$$(\bar\alpha^r\bar\beta^{2s})^2 = \bar\alpha^{2r}\bar\beta^{4s}, \quad r, s \in \mathbb{Z}. \tag{22}$$

Let us prove (5) and the non-existence results stated in the remaining cases, namely:

Table 2, case (4b) and Table 3, cases (2c), (3b), (4a), (4c), (4d).

By Corollary 3.11 and Remark 3.12, we may assume that $x \in N = \langle\langle B \rangle\rangle$, and (17), (20) hold. In particular, $\bar x = 1$, $\bar v = \bar y^k$, for some $k \in \mathbb{Z}$.

Consider the cases (4b,c) of Table 2 and the cases (2c,d) of Table 3. Since the solution is faithful (resp. non-faithful), we have $w_-(\bar y) = -1$, see (17). We conclude that $\bar y^2$ is a power of $\bar\beta$, by applying the canonical form (19) of elements in $\pi = \pi_-$:

$$(\bar\alpha^r\bar\beta^{2s+1})^2 = \bar\beta^{4s+2}, \quad r, s \in \mathbb{Z}. \tag{23}$$

Since $\bar v = \bar y^k$, $w_-(\bar y) = -1$, $w_-(\bar v) = 1$, the integer k must be even. Therefore, $\bar v$ is a power of $\bar y^2$, thus also a power of $\bar\beta$.

In the cases (4a), (4c) of Table 3, we have $w_-(\bar v) = -1$ and $w_-(\bar y) = 1$, since the solution is non-faithful, see (17). This contradicts to $\bar v = \bar y^k$.

In the cases (3b,c), (4d,e) of Table 3, we have $\delta = \vartheta = -1$. It follows from the second part of (20) that $(-1)^k = 1$, thus k is even. This proves (5) and finishes the proof in the case (3b) of Table 3. In the case (4d) of Table 3, we have $w_-(\bar{y}) = 1$, since the solution is non-faithful, see (17). Together with the relation (22), this shows that the canonical form of $\bar{v} = \bar{y}^k$ is $\bar{\alpha}^{2m}\bar{\beta}^{4n}$, for some $m, n \in \mathbb{Z}$. □

Theorem 3.14 gives many cases for the values of $\bar{v} \in \pi$ such that all elements $v_1 \in p_\pi^{-1}(\bar{v})$ simultaneously have (or simultaneously do not have, respectively) the following property: the corresponding equation (8) has a solution satisfying (6), where the cases of faithful and non-faithful solutions are considered separately, see Tables 2 and 3, respectively. The remaining cases listed in (21) are marked in Tables 2 and 3 as "mixed" cases because of the following.

Case	δ	ε	ϑ	conditions on v		faithful solution (x, y)
				$w_\varepsilon(v)$		
(1)	+	+	−	+ [i]		$(vB^{-1}v^{-1},\ v^{-1})$ [iv]
(2) a	+	−	−	+		$(vB^{-1}v^{-1},\ v^{-1})$ [iv]
b			+	−		$\emptyset$ [ii]
(3)	−	+		+ [i]		$\emptyset$ [iii]
(4) a	−	−	+	−		$(B,\ B^{-1}v)$ [iv]
b			−	+	$p_K^\alpha(v) \neq 0$	$\emptyset$ [v]
c					$p_K^\alpha(v) = 0$	"mixed" case, see Table 4

Table 2: Faithful solutions of $xyx^{-\delta}y^{-1} = vB^{\vartheta}v^{-1}B$ with $B = \alpha\beta\alpha^{-\varepsilon}\beta^{-1}$, $w_\varepsilon(v) = -\vartheta$

Definition 3.15 A family of quadratic equations (8), with the conjugation parameter v running through the set $p_\pi^{-1}(\bar{v}_0)$, is called *mixed (with respect to the property of the existence of a faithful, respectively non-faithful, solution)* if there exist two parameter values $v_1, v_2 \in p_\pi^{-1}(\bar{v}_0)$ such that the equation with $v = v_1$ has a faithful (respectively, non-faithful) solution, while the equation with $v = v_2$ has no faithful (respectively, non-faithful) solution.

3.4 Comments to Tables 2 and 3

As above, we rewrite the equation (5) in the equivalent form (8), in terms of the new generators α, β of F_2, and the new unknowns x, y, using the transformation of variables (9). Thus the equations (1)–(4) are transformed to the equations (1′)–(4′), see

Case	δ	ε	ϑ	conditions on v		non-faithful solution (x, “y)
				$w_\varepsilon(v)$		
(1)	+	+		$+$[(i)]		$\emptyset$[(i)]
(2) a	$+$	$-$	$+$	$\pm$		$\emptyset$[(ii)]
b			$-$	$-$		$(vB^{-1}v^{-1}, v^{-1})$[(iv)]
c				+	$p_K^\alpha(v) \neq 0$	$\emptyset$[(v)]
d					$p_K^\alpha(v) = 0$	mixed case, see Table 5(2)
(3) a	$-$	+	$+$	$+$[(i)]		$([\alpha, \beta], [\beta, \alpha]v)$[(iv)]
b			$-$	$+$[(i)]	$2\nmid p_T(v)$	$\emptyset$[(v)]
c					$2 \mid p_T(v)$	mixed case, see Table 5(3)
(4) a	$-$	$-$	+	$-$		$\emptyset$[(v)]
b				+		$(B, B^{-1}v)$[(iv)]
c			$-$	$-$		$\emptyset$[(v)]
d				+	$4\nmid p_K^\beta(v)$ or $2\nmid p_K^\alpha(v)$	$\emptyset$[(v)]
e					$4 \mid p_K^\beta(v)$ and $2 \mid p_K^\alpha(v)$	mixed case, see Table 5(4)

Table 3: Non-faithful solutions of $xyx^{-\delta}y^{-1} = vB^\vartheta v^{-1}B$ with $B = \alpha\beta\alpha^{-\varepsilon}\beta^{-1}$

(i) Automatically for $\varepsilon = +1$.
(ii) The right-hand side of the equation (5) is not in $[F_2, F_2]$.
(iii) Automatically for $\varepsilon = +1$, $\delta = -1$.
(iv) Direct calculation.
(v) Using (17), and either (20) or the first derived equation (36), see Remark 3.10 and Theorem 5.1.

Section 3.3. In Tables 2 and 3 above, we summarize the results of Theorem 3.14 on the existence of faithful and non-faithful solutions of the latter equations, respectively.

Remark 3.16 The primary objective, for the remainder of this paper, is the study of the four cases (21) of Tables 2 and 3 (the "mixed" cases), which are not completely solved by Theorem 3.14. These cases are described in detail in Section 7. In Tables 4 and 5 below, we will show that the cases (21) are indeed the "mixed" cases with respect to the property of the existence of a solution, see Definition 3.15. A complete description of all words $v_1, v_2 \in p_\pi^{-1}(\bar{v})$ as in Definition 3.15, in a mixed case, does not seem to be an easy task.

4 Some quotients of the normal closure of an element of a free group

In this section, we denote by F a free group of finite rank ≥ 2, $B \in F$, $N = \langle\langle B\rangle\rangle$, and $\pi = F/N$. Thus, N is the normal closure of the element B, that is, the minimal normal subgroup of F containing B, while π is a one-relator group. We will assume that the word B is not a proper power of any element of F (although, in some of the assertions, the hypothesis above can be made weaker). In particular, all assertions of this section are valid if B is a strictly quadratic word in a set of free generators of F, see Lyndon and Schupp [30, Section I.7]. For $F = F_r = \langle a_1, \dots, a_r \mid \rangle$, such words are automorphic images of the words $R_+(a_1, \dots, a_r)$, $R_-(a_1, \dots, a_r)$, $r \geq 2$, see Definition 2.1(A) and [30, Chapter I, Proposition 7.6].

Consider the following normal subgroups of the group N:

$$N \supset [F,N] \supset N_1 = [N,N],\ [F,[F,N]] \supset [F,[N,N]] \supset [N,[N,N]].$$

We will construct presentations of the quotients N/N_1, $N_1/[N,N_1]$ and $N/[N,N_1]$ (see Section 4.1), $N_1/[F,N_1]$ and $N/[F,N_1]$ (see Section 4.2), and $N/[F,N]$ and $[F,N]/[F,[F,N]]$ (see Section 4.3). It will follow that the first, second, fourth, and sixth quotients are free abelian groups, the third and fifth quotients are the middle groups of extensions of free abelian groups, while the seventh one is isomorphic to $\pi^{ab} = \pi/[\pi,\pi]$, the abelianised group π. If N is the commutator subgroup $[F,F]$ then the latter quotient comes from the lower central series of the free group F, see also Remark 4.6.

As in Section 3, we will denote by $\bar{u} \in F$ the class of an element $u \in F$ in π.

4.1 The groups N/N_1, $N_1/[N,N_1]$, and $N/[N,N_1]$

Let us consider the short exact sequence

$$1 \to N_1 \to N \to N/N_1 \to 1.$$

Here, as above, $N = \langle\langle B\rangle\rangle$, $B \in F$, and B is not a proper power of any element of F. By the Nielsen–Schreier subgroup theorem [38], N is a free group, since it is a subgroup of a free group. Furthermore, it follows from Lyndon [27, Section 7] that $N^{ab} = N/N_1$, the abelianised group N, is isomorphic to the free abelian group which has a basis in a bijective correspondence with $\pi = F/N$, see [27, Introduction]. These results are formulated in more detail as follows.

Proposition 4.1 (Lyndon [27]) *Suppose that the relator $B \in F$ is not a proper power of any element of a free group F. Consider the short exact sequence*

$$1 \to N \to F \xrightarrow{p_\pi} \pi \to 1, \tag{24}$$

where $N = \langle\langle B \rangle\rangle$, the minimal normal subgroup which contains the relator B, while $\pi = F/N$, a group with a single defining relation. Then the group N is free and admits a free basis (for example, a Schreier basis) of the form B_u, $u \in W$, where $W = s(\pi) \subset F$, and $s\colon\ \pi \to F$ is a map with $p_\pi s = \mathrm{id}_\pi$. Furthermore, N^{ab}, the abelianised group N, is isomorphic to the abelian group $(\mathbb{Z}[\pi], +)$ of the group ring $\mathbb{Z}[\pi]$. Moreover, there exists a short exact sequence

$$1 \to [N,N] \xrightarrow{i_N} N \xrightarrow{q_N} (\mathbb{Z}[\pi], +) \to 0, \tag{25}$$

where i_N is the canonical inclusion, while q_N is an epimorphism sending

$$q_N\colon\ N \to (\mathbb{Z}[\pi], +), \qquad \prod_{i=1}^{r} B_{u_i}^{n_i} \longmapsto \sum_{i=1}^{r} n_i \bar{u}_i \in \mathbb{Z}[\pi], \tag{26}$$

for any $u_i \in F$, $n_i \in \mathbb{Z}$, where $B_u = uBu^{-1}$, $\bar{u} = p_\pi(u)$, $u \in F$. □

A similar assertion, for any element $B \in \pi$, was proved by Cohen and Lyndon [7]. In the case when the relator B is a strictly quadratic word in the free generators $a_1, \dots, a_r$ of the group $F = \langle a_1, \dots, a_r \mid \rangle$, $r \geq 2$, for example $B = R_\varepsilon(a_1, \dots, a_r)$, see Definition 2.1(A), an alternative proof of Proposition 4.1 can be obtained as follows. The subgroup N is a free group, as explained above. In the case when B is a strictly quadratic word, a free Schreier basis of N was explicitely constructed by Zieschang [49], Zieschang, Vogt and Coldewey [50]; see also Kudryavtseva, Weidmann and Zieschang [26, Proposition 4.9]. This immediately implies Proposition 4.1, see [26, Corollary 4.12].

Proposition 4.2 *Under the hypothesis of Proposition 4.1, consider the central short exact sequence*

$$1 \to N_1/[N,N_1] \to N/[N,N_1] \to N/N_1 \to 1.$$

Then $N_1/[N,N_1] \approx H_2(N/N_1) \approx \mathbb{Z}[J]$, a free abelian group with basis denoted by e_θ where θ runs over the set $J = (\pi \times \pi \setminus \Delta)/\Sigma_2$, and Σ_2 is the symmetric group in two symbols, which acts on $\pi \times \pi \setminus \Delta$ by permutations of the coordinates. A presentation of the group $N_1/[N,N_1]$ is obtained as follows: for each $\theta \in (\pi \times \pi \setminus \Delta)/\Sigma_2$ choose a pair $(\xi, \eta) \in \theta$, denote $\epsilon_{(\xi,\eta)} := e_\theta$, $\epsilon_{(\eta,\xi)} := -e_\theta$, and denote by J_1 the set of such pairs (ξ, η), thus $J_1 \subset \pi \times \pi \setminus \Delta$. Then there exists a short exact sequence

$$1 \to [N,N_1] \xrightarrow{i_{N_1}} N_1 \xrightarrow{q_{N_1}} \mathbb{Z}[J] \to 0, \tag{27}$$

where i_{N_1} is the canonical inclusion, while q_{N_1} is an epimorphism sending

$$(28)\qquad q_{N_1}\colon\ N_1 \to \mathbb{Z}[J], \qquad \left[\prod_{i=1}^{r} B_{u_i}^{n_i}, \prod_{j=1}^{s} B_{v_j}^{m_j}\right] \longmapsto \sum_{i=1}^{r}\sum_{j=1}^{s} n_i m_i \epsilon_{(\bar{u}_i,\bar{v}_j)} \in \mathbb{Z}[J],$$

where $u_i, v_j \in F$, $1 \le i \le r$, $1 \le j \le s$, and $B_u = uBu^{-1}$, $u \in F$. Furthermore, the group $N/[N,N_1]$ admits the following presentations:

$$(29)\quad N/[N,N_1] \approx \langle x_\xi,\ \xi \in \pi \mid [x_\xi,[x_\eta,x_\zeta]],\ \xi,\eta,\zeta \in \pi\rangle$$

$$\approx \left\langle \begin{array}{l} e_\theta,\ \theta \in (\pi\times\pi\setminus\Delta)/\Sigma_2, \\ x_\xi,\ \xi\in\pi \end{array} \ \middle|\ \begin{array}{l} [e_\theta,e_{\theta'}],\ [e_\theta,x_\xi],\ \theta,\theta' \in (\pi\times\pi\setminus\Delta)/\Sigma_2,\ \xi\in\pi, \\ [x_\xi,x_\eta]e^{-1}_{\{\xi,\eta\}},\ (\xi,\eta)\in J_1 \end{array} \right\rangle,$$

where $\{\xi,\eta\} \in (\pi\times\pi\setminus\Delta)/\Sigma_2$ denotes the class of $(\xi,\eta) \in J_1$ in $(\pi\times\pi\setminus\Delta)/\Sigma_2$.

Proof Recall that if $1 \to H \to G \to Q \to 1$ is a short exact sequence then we have a 5–term exact sequence

$$(30)\qquad H_2(G) \to H_2(Q) \to H/[G,H] \to H_1(G) \to H_1(Q) \to 0,$$

due to Stallings [40, Theorem 2.1]. Applying this to the short exact sequence

$$1 \to N_1 \to N \to N/N_1 \to 1$$

we obtain that $G = N$, $H = N_1$, $Q = N/N_1$, thus the first and third homomorphisms in the 5–term sequence are trivial. It follows that the second homomorphism $H_2(N/N_1) \to N_1/[N,N_1]$ is an isomorphism. Since $Q \approx (\mathbb{Z}[\pi],+)$ is a free abelian group, it follows from Brown [5, Theorem V.6.4] that $H_2(Q;\mathbb{Z}) \approx \Lambda^2(Q)$, the subgroup of grade 2 of the graded ring $\Lambda(Q)$ (the exterior graded ring of the group Q), where Q is at grade 0. This proves the desired presentation for the group $N_1/[N,N_1]$.

To prove that (28) defines a homomorphism, let us first show that there exists a unique homomorphism $q_{N_1}\colon\ N_1 \to \mathbb{Z}[J]$ satisfying (28). Denote by $p_{N_1}\colon\ N_1 \to N_1/[N,N_1]$ the canonical projection. Consider the free basis B_u, $u \in W$, of N given by Proposition 4.1. It follows from Magnus, Karrass and Solitar [31, Theorem 5.12] that the group $N_1/[N,N_1]$ is a free abelian group, where the elements $p_{N_1}([B_{s(u)},B_{s(v)}]) \in N_1/[N,N_1]$, $(u,v) \in J_1$, form a free abelian basis. Therefore the map sending $p_{N_1}([B_{s(u)},B_{s(v)}]) \mapsto e_{\{u,v\}}$, $(u,v) \in J_1$, uniquely extends to a homomorphism $\varphi_{N_1}\colon\ N_1/[N,N_1] \to \mathbb{Z}[J]$. Since φ_{N_1} sends the above basis of $N_1/[N,N_1]$ to a basis of $\mathbb{Z}[J]$, it is an isomorphism. The property (28) of the obtained projection $q_{N_1} := \varphi_{N_1}p_{N_1}$ follows from commutator calculus, see [31, Theorem 5.3].

Now the presentation (29) follows by observing that the natural epimorphism of $N/[N,N_1]$ to the group in the right-hand side of (29) sending $B_u[N,N_1] \mapsto x_{\bar{u}}$, $u \in W$, is well-defined. It has a trivial kernel, because one can easily construct its inverse. □

4.2 The groups $N_1/[F,N_1]$ and $N/[F,N_1]$

Here we will obtain the main results of this section, which will be applied in Section 5 to study the existence of solutions of the equations (1′)–(4′) in the mixed cases, see Remark 3.16.

Let us first recall some other facts from Lyndon [27] on the homology of one-relator groups.

Lemma 4.3 (Lyndon [27, Theorem 2.1]) *For any group π, the homology group $H_i(\pi, \mathbb{Z}[\pi])$ is trivial for $i \geq 1$, and isomorphic to $\mathbb{Z}$ for $i = 0$. Here the local coefficients $\mathbb{Z}[\pi]$ is the $\mathbb{Z}[\pi]$–module corresponding to the action of π on $\mathbb{Z}[\pi]$, which is given by the right multiplication.* □

The result above is also true if π acts on $\mathbb{Z}[\pi]$ by the left multiplication.

Lemma 4.4 *Under the hypothesis of Proposition 4.1, $H_i(\pi) = 0$, $i \geq 3$, while $H_2(\pi)$ is either $\mathbb{Z}$ or 0. The latter group is 0 if and only if $B \notin [F,F]$.*

Proof From [27, Corollaries 4.2 and 11.2] it follows that $H^i(\pi) = 0$ for $i \geq 3$, while $H^2(\pi)$ is a cyclic group, which is finite if and only if $B \notin [F,F]$. Using the universal coefficient theorem, we ontain the desired assertion, see also Brown [5, Example II.4.3]. □

Remark that we will not use in this paper that $H_i(\pi) = 0$ for $i \geq 4$.

Now denote $N_F = [F,N_1]$ where $N_1 = [N,N]$. Similarly to (7), denote by Q the quotient of the abelian group $\mathbb{Z}[\pi \setminus \{1\}]$ by the system of relations $g \sim -g^{-1}$, $g \in \pi \setminus \{1\}$:

$$Q = (\mathbb{Z}[\pi \setminus \{1\}])/\langle g + g^{-1} \mid g \in \pi \setminus \{1\}\rangle. \tag{31}$$

Proposition 4.5 *Under the hypothesis of Proposition 4.1, consider the central short exact sequence*

$$1 \to N_1/N_F \to N/N_F \to N^{ab} \to 1.$$

Then $N_1/N_F \approx H_2(F/N_1) \approx Q \approx \mathbb{Z}[I]$, for $I = (\pi \backslash \{1\})/\sim$, where the relation $\sim$ is given by identifying g with g^{-1}, for $g \in \pi \setminus \{1\}$. Moreover, there exists a short exact sequence

$$1 \to [F,N_1] \xrightarrow{i_{N_F}} N_1 \xrightarrow{q_{N_F}} Q \to 0, \tag{32}$$

where i_{N_F} is the canonical inclusion, while q_{N_F} is an epimorphism sending

$$(33)\qquad q_{N_F}\colon\ N_1 \to Q,\quad \Big[\prod_{i=1}^{r} B_{u_i}^{n_i}, \prod_{j=1}^{s} B_{v_j}^{m_j}\Big] \longmapsto p_Q\Big(\sum_{i=1}^{r}\sum_{j=1}^{s} n_i m_i \bar{u}_i^{-1}\bar{v}_j\Big) \in Q,$$

where $u_i, v_i \in F$, $p_Q\colon\ \mathbb{Z}[\pi] \to Q$ is the projection. Furthermore, the group N/N_F admits the following presentations:

(34)

$$\begin{aligned} N/N_F &\approx \langle x_\xi,\ \xi\in\pi \mid [x_\xi, x_\eta][x_{\zeta\xi}, x_{\zeta\eta}]^{-1},\ [x_\xi,[x_\eta,x_\zeta]],\ \xi,\eta,\zeta\in\pi\rangle \\ &\approx \left\langle \begin{array}{l} e_\theta,\ \theta\in\pi\setminus\{1\}, \\ x_\xi,\ \xi\in\pi \end{array} \,\middle|\, \begin{array}{l} [e_\theta, e_{\theta'}],\ [e_\theta, x_\xi],\ \theta,\theta'\in\pi\setminus\{1\},\ \xi\in\pi, \\ [x_\xi, x_\eta] e_{\xi^{-1}\eta}^{-1},\ (\xi,\eta)\in\pi\times\pi\setminus\Delta \end{array} \right\rangle. \end{aligned}$$

Proof To establish the isomorphism $N_1/N_F \approx H_2(F/N_1)$, consider the 5–term exact sequence

$$H_2(F) \to H_2(F/N_1) \to N_1/[F,N_1] \to F^{ab} \to (F/N_1)^{ab} \to 0$$

obtained from the short exact sequence $1 \to N_1 \to F \to F/N_1 \to 1$ by means of (30). Since $H_2(F) = 0$ and $F^{ab} \to (F/N_1)^{ab}$ is an isomorphism, it follows that $H_2(F/N_1) \to N_1/[F,N_1] = N_1/N_F$ is an isomorphism.

To establish the isomorphism $H_2(F/N_1) \approx Q$, consider the Hochschild–Serre spectral sequence [21] (also called the Lyndon–Hochschild–Serre spectral sequence) related to the short exact sequence $1 \to N/N_1 \to F/N_1 \to \pi \to 1$. Recall that this spectral sequence has the form

$$(35)\qquad E^2_{pq} = H_p(\pi, H_q(N/N_1)) \implies H_{p+q}(F/N_1),$$

where the local coefficients $H_q(N/N_1)$ is the $\mathbb{Z}[\pi]$–module corresponding to the action $\mathrm{Ad}^q_\pi\colon\ \pi \to \mathrm{Aut}(H_q(N/N_1))$, which is induced by the action $\mathrm{Ad}_\pi|_{N/N_1}\colon\ \pi \to \mathrm{Aut}(N/N_1)$ given by conjugation: $(gN)\cdot(xN_1) = gxg^{-1}N_1$, $g \in F$, $x \in N$. By Proposition 4.1, $H_1(N/N_1) \approx (\mathbb{Z}[\pi], +)$ and the action Ad^1_π is given by the left multiplication. Thus, from Lemma 4.3, we have that $E^2_{p1} = H_p(\pi, \mathbb{Z}[\pi]) = 0$ for $p \geq 1$. By Lemma 4.4, we have $E^2_{30} = H_3(\pi) = 0$, which implies $E^2_{02} = E^\infty_{02}$.

Let us show that $H_2(F/N_1) \approx E^2_{02}$. If $B \notin [F,F]$ then, by Lemma 4.4, $E^2_{20} = E^\infty_{20} = 0$, and thus we get $H_2(F/N_1) = E^2_{02}$. Consider the remaining case, $B \in [F,F]$. Observe that both groups $E^2_{20} = H_2(\pi)$ and $E^2_{01} = H_0(\pi, \mathbb{Z}[\pi])$ are isomorphic to $\mathbb{Z}$, due to Lemmas 4.4 and 4.3, respectively. On the other hand, the isomorphism $H_1(F/N_1) \approx F^{ab} \approx \pi^{ab} \approx H_1(\pi) = E^2_{10} = E^\infty_{10}$ and (35) for $p+q = 1$ imply $E^\infty_{01} = 0$. Thus the differential $d^2_{20}\colon\ E^2_{20} \to E^2_{01}$ is an isomorphism, and it follows that $H_2(F/N_1) \approx E^2_{02} = H_0(\pi, H_2(N^{ab}))$.

Now, due to Proposition 4.2, $H_2(N^{ab})$ is isomorphic to $\mathbb{Z}[J]$, where $J = (\pi \times \pi \setminus \Delta)/\Sigma_2$, and the corresponding action $\mathrm{Ad}^2_\pi\colon \pi \to \mathrm{Aut}(H_2(N^{ab})) \approx \mathrm{Aut}(\mathbb{Z}[J])$ is given by $\zeta \cdot e_{\{\xi,\eta\}} = e_{\{\zeta\xi,\zeta\eta\}}$, for each pair $(\xi,\eta) \in J_1$, and $\zeta \in \pi$, where $J_1 \subset \pi \times \pi \setminus \Delta$ is chosen to be invariant under Ad^2_π, see Proposition 4.2. Therefore $E^2_{02} = H_0(\pi, H_2(N^{ab})) \approx H_0(\pi, \mathbb{Z}[J])$ is isomorphic to the quotient of $\mathbb{Z}[J]$ by the system of relations $e_{\{\xi,\eta\}} \sim e_{\{\zeta\xi,\zeta\eta\}}$, $(\xi,\eta) \in J_1$, $\zeta \in \pi$. Hence it is isomorphic to $\mathbb{Z}[I] \approx Q$.

To prove that (33) defines a homomorphism q_{N_F}, observe that the canonical projection $N_1/[N,N_1] \to N_1/[F,N_1]$ factors through the canonical projection of the group $N_1/[N,N_1]$ onto the quotient of $N_1/[N,N_1]$ by the system of relations $p_{N_1}(n) \sim p_{N_1}(gng^{-1})$, $n \in N_1$, $g \in F$, where $p_{N_1}\colon N_1 \to N_1/[N,N_1]$ is the canonical projection, see also Proposition 4.2. Due to the isomorphism $N_1/[N,N_1] \to \mathbb{Z}[J]$ from Proposition 4.2, we obtain the system of relations $e_{\{\xi,\eta\}} \sim e_{\{\zeta\xi,\zeta\eta\}}$, $(\xi,\eta) \in J_1$, $\zeta \in \pi$, on $\mathbb{Z}[J]$. This system of relations determines the obvious equivalence relation $\sim$ on the basis $e_{\{\xi,\eta\}}$, $(\xi,\eta) \in J_1$, of $\mathbb{Z}[J]$. Thus the desired quotient of $\mathbb{Z}[J]$ is the free abelian group $\mathbb{Z}[J/\sim]$, where the equivalence classes of $\sim$ form a basis. This gives the desired isomorphism $N_1/[F,N_1] \approx \mathbb{Z}[J/\sim] = \mathbb{Z}[I]$. Now (33) follows by observing that the equivalence class of $e_{\{\bar u_i,\bar v_j\}} = q_{N_1}([B_{u_i}, B_{v_j}])$ in J corresponds to $p_Q(\bar u_i^{-1}\bar v_j) = q_{N_F}([B_{u_i}, B_{v_j}])$ under the isomorphism $\mathbb{Z}[I] \approx Q$.

Now the presentation (34) follows by observing that the natural epimorphism of $N/[F,N_1]$ to the group in the right-hand side of (34) sending $B_u[F,N_1] \mapsto x_{\bar u}$, $u \in W$, is well-defined. It has a trivial kernel, because one can easily construct its inverse. □

Remark 4.6 Our first derived equation is the "projection" of the equation (8) in N to the quotient $N^{ab} = N/[N,N]$, see Theorem 5.1. Our second derived equation is the "projection" of the equation (8) to the quotient $[N,N]/[F,[N,N]]$, via choosing suitable representatives of the solutions of the first derived equation, see Theorem 5.10. Observe that the subgroup $[N,N]$ is the second term $\Gamma^2(N)$ of the lower central series

$$\Gamma^1(N) = N, \quad \Gamma^{i+1}(N) = [N, \Gamma^i(N)], \quad i \geq 1,$$

of the group N, while the subgroup $[F,[N,N]]$ is the third term $\Gamma^3_F(N)$ of the lower central series

$$\Gamma^1_F(N) = N, \quad \Gamma^{i+1}_F(N) = [F, \Gamma^i(N)], \quad i \geq 1,$$

with respect to the action of the group F on N by conjugation, that is $g \cdot x = gxg^{-1}$, $g \in F$, $x \in N$. We recall (see Hilton [16], or Hilton, Mislin and Roitberg [17, Section II.2]) that if a group G acts on a group H then the *lower central series with respect to the action of G on H* is defined as

$$\Gamma^1_G(H) = H, \quad \Gamma^{i+1}_G(H) = \mathrm{gr}\{(g\cdot x)yx^{-1}y^{-1} \mid g\in G,\ x\in\Gamma^i(H),\ y\in H\},\ i\geq 1.$$

Here $g\cdot x$ means the action of the automorphism defined by g on the element x, $\Gamma^i(H)$ is the usual lower central series of the group H, and $\mathrm{gr}\,S$ denotes the minimal subgroup of H containing a subset $S\subset H$.

4.3 The groups $N/[F,N]$ and $[F,N]/[F,[F,N]]$

Here we study the quotients corresponding to the subgroups $N\supset[F,N]\supset[F,[F,N]]$. One can apply them to study existence of solutions of equations in free groups. However, the results of this subsection are not used in our applications, and can be skipped in the first reading.

Proposition 4.7 *Under the hypothesis of Proposition 4.1, the group $N/[F,N]$ is isomorphic to $\mathbb{Z}$. Moreover, there exists a short exact sequence*

$$1\to[F,N]\xrightarrow{i}N\xrightarrow{\varepsilon q_N}\mathbb{Z}\to 0,$$

where i is the canonical inclusion, $q_N\colon N\to(\mathbb{Z}[\pi],+)$ is the epimorphism given by (26), while $\varepsilon\colon\mathbb{Z}[\pi]\to\mathbb{Z}$ is the augmentation.

Proof The first assertion follows in a straightforward way from the 5–term exact sequence obtained from the short exact sequence (24), namely $1\to N\to F\to\pi\to 1$, by means of (30). In detail, if $B\in[F,F]$ then $H_2(F)=0$, $F^{ab}\to\pi^{ab}$ is an isomorphism, and $H_2(\pi)\approx\mathbb{Z}$, due to Lemma 4.4. Therefore $H_2(\pi)\to N/[F,N]$ is an isomorphism, hence $N/[F,N]\approx\mathbb{Z}$. Suppose that $B\notin[F,F]$. Then $\ker(F^{ab}\to\pi^{ab})\approx\mathbb{Z}$, and $H_2(\pi)=0$, due to Lemma 4.4. Therefore $N/[F,N]\approx\ker(F^{ab}\to\pi^{ab})\approx\mathbb{Z}$.

To prove the second assertion, observe that the composition $\varepsilon q_N\colon N\to\mathbb{Z}$ sends $B\mapsto 1$, hence it is an epimorphism. Next we show that the kernel of εq_N equals $[F,N]$. The inclusion $\ker(\varepsilon q_N)\supset[F,N]$ follows from the fact that $[F,N]$ is generated by the elements $[u,B_v]\in[F,N]$, $u,v\in F$, due to commutator calculus, while $[u,B_v]=B_{uv}B_v^{-1}$ is mapped to $1-1=0$ under εq_N, thus $[u,B_v]\in\ker(\varepsilon q_N)$. The converse inclusion follows by observing that any element $u\in\ker(\varepsilon q_N)$ has the form $u=B_{u_1}^{c_1}\dots B_{u_r}^{c_r}$, for some $r,c_1,\dots,c_r\in\mathbb{Z}$, $r\geq 0$, $u_1,\dots,u_r\in F$, where $c_1+\dots+c_r=0$. Clearly, the projection of u to the quotient $N/[F,N]$ equals the projection of the element $B^{c_1}\dots B^{c_r}=B^0=1$ to $N/[F,N]$, thus $u\in[F,N]$. $\square$

Proposition 4.8 *Under the hypothesis of Proposition 4.1,*

$$[F,N]/[F,[F,N]] \approx H_2(F/[F,N]) \approx H_1(\pi) \approx \pi^{ab}.$$

Moreover, there exists a short exact sequence

$$1 \to [F,[F,N]] \xrightarrow{i_F} [F,N] \xrightarrow{q_F} \pi^{ab} \to 0,$$

where i_F is the canonical inclusion, q_F is an epimorphism sending $q_F\colon [n,g] \mapsto \varepsilon q_N(n) \cdot p_{ab}(\bar{g}) \in \pi^{ab}$, $n \in N$, $g \in F$. Here one uses an additive notation for the group operation in π^{ab}, $p_{ab}\colon \pi \to \pi^{ab}$ denotes the canonical projection, $q_N\colon N \to (\mathbb{Z}[\pi],+)$ is the epimorphism defined by (26), $\varepsilon\colon \mathbb{Z}[\pi] \to \mathbb{Z}$ is the augmentation.

Proof Let us prove the first assertion. Consider the 5–term exact sequence

$$H_2(F) \to H_2(F/[F,N]) \to [F,N]/[F,[F,N]] \to F^{ab} \to (F/[F,N])^{ab} \to 0$$

obtained from the short exact sequence $1 \to [F,N] \to F \to F/[F,N] \to 1$ by means of (30). Since $H_2(F)=0$ and $F^{ab} \to (F/[F,N])^{ab}$ is an isomorphism, it follows that $H_2(F/[F,N]) \to [F,N]/[F,[F,N]]$ is an isomorphism.

Similarly to the proof of Proposition 4.5, consider the Hochschild–Serre spectral sequence related to the short exact sequence $1 \to N/[F,N] \to F/[F,N] \to \pi \to 1$. Since $N/[F,N] \approx \mathbb{Z}$ by Proposition 4.7, the spectral sequence has the form

$$E^2_{pq} = H_p(\pi, H_q(N/[F,N])) \approx H_p(\pi, H_q(\mathbb{Z})) \implies H_{p+q}(F/[F,N]),$$

similarly to (35), where the local coefficients $H_q(\mathbb{Z})$ is the trivial $\mathbb{Z}[\pi]$–module. Since $H_0(\mathbb{Z}) \approx H_1(\mathbb{Z}) \approx \mathbb{Z}$ and $H_q(\mathbb{Z})=0$ for $q \geq 2$, while the homology of π vanishes in dimension 3 (due to Lemma 4.4), the only possible non-vanishing E^2 terms are those with $q=0,1$ and $p \neq 3$. In particular, $E^2_{02}=0$, $E^2_{30}=H_3(\pi)=0$, and therefore $E^\infty_{11}=E^2_{11}=H_1(\pi)$.

Let us show that $E^\infty_{20}=E^3_{20}=0$. If $B \notin [F,F]$ then, by Lemma 4.4, $E^2_{20}=H_2(\pi)=0$. Suppose that $B \in [F,F]$. Then $E^2_{20}=H_2(\pi) \approx \mathbb{Z}$ by Lemma 4.4, moreover the projection $(F/[F,N])^{ab} \to H_1(\pi)=E^2_{10}$ is an isomorphism. Therefore $d^2_{20}\colon E^2_{20} \to E^2_{01}$ is an isomorphism, hence $E^\infty_{20}=E^3_{20}=0$.

Since $E^2_{02}=0$ and $E^3_{20}=0$, we have the desired isomorphism $H_2(F/[F,N]) \approx E^\infty_{11} = E^2_{11} = H_1(\pi)$.

Let us prove the second assertion. We shall represent elements of the quotient $[F,N]/[F,[F,N]]$ by elements of $[F,N]$, identified under the congruence relation $g_1 \equiv g_2$ *modulo* $[F,[F,N]]$, and shall write $g_1 \equiv g_2$ whenever $g_1 g_2^{-1} \in [F,[F,N]]$.

Observe that every element $w \in [F,N]$ can be written in the form $w \equiv [B,u]$, for some $u \in F$, due to the following congruences: $[B_u, v] \equiv [B, u^{-1}vu]$ and $[B,uv] = [B,u][u,[B,v]][B,v] \equiv [B,u][B,v]$ for any $u,v \in F$.

Let us show that there exists an epimorphism $p\colon\ \pi^{ab} \to [F,N]/[F,[F,N]]$ sending $p_{ab}(\bar{u}) \mapsto p_F([B,u])$, $u \in F$, where $p_F\colon\ [F,N] \to [F,N]/[F,[F,N]]$ is the canonical projection. To show that such a map p is well-defined, we use commutator calculus and the following observations. Using $[B,B_v] = [B,[v,B]] \in [N,[F,N]] \subset [F,[F,N]]$, $v \in F$, one shows that $[N,N] \subset [F,[F,N]]$, which implies $[B,un] \equiv [B,u][B,n] \equiv [B,u]$ for any $u \in F$ and $n \in N$. Furthermore, using one of the Witt–Hall identities (see Magnus, Karrass and Solitar [31, Theorem 5.1, (11)]) one can show that $[N,[F,F]] \subset [F,[F,N]]$, which implies $[B,uf'] \equiv [B,u][B,f'] \equiv [B,u]$ for any $u \in F$ and $f' \in [F,F]$. The map p is a homomorphism, since $[B,uv] \equiv [B,u][B,v]$ for any $u,v \in F$, see above. Therefore the map p is an epimorphism.

Since $\pi^{ab} \approx [F,N]/[F,[F,N]]$, and π^{ab} is a finitely-generated abelian group, it follows that any epimorphism $\pi^{ab} \to [F,N]/[F,[F,N]]$ is an isomorphism. Therefore the epimorphism p is an isomorphism. It follows that the composition $p^{-1}p_F\colon\ [F,N] \to \pi^{ab}$ is an epimorphism and satisfies the desired properties. □

5 Derived equations in $\mathbb{Z}[\pi]$ and $\mathbb{Z}[\pi \setminus \{1\}]/\sim$

The quadratic equations under consideration are the equations (1′)–(4′) of Section 3.3 with two unknowns $x \in N$, $y \in F_2$ in the free group $F_2 = \langle \alpha, \beta \mid \rangle$ of rank 2, see Theorem 3.14. Actually these equations are in the subgroup $N = \langle\langle B \rangle\rangle$ where $B = \alpha\beta\alpha^{-\varepsilon}\beta^{-1}$. To prove some further non-existence results, we will apply the algebraic approach developed in Section 4. For each of the equations (2′), (3′) and (4′) in N, we will construct two *derived equations*, which are in fact "projections" of the equation to the abelian quotients N/N_1 and $N_1/[F_2,N_1]$, respectively, described in Section 4, see Propositions 4.1 and 4.5, where $N_1 = [N,N]$. The first derived equation is an equation in the group ring $\mathbb{Z}[\pi]$ of the fundamental group $\pi = \pi_\varepsilon = F_2/N$ of the corresponding target surface (this group ring, as an abelian group, is isomorphic to the abelianised group N, see Proposition 4.1). The second derived equation is an equation in the quotient Q of $\mathbb{Z}[\pi]$, see (7), and it is obtained by "projecting" the equation to this quotient (actually, to $N_1/[F_2,N_1] \approx Q$, see Proposition 4.5), via choosing suitable representatives of the solutions (if there exists any) of the first derived equation.

5.1 The first derived equation

Here we will construct the *first derived equation* for each of the equations (2′), (3′) and (4′) of Section 3.3. Due to Theorem 3.14, or Corollaries 3.9(A) and 3.11, we can assume, without loss of generality, that $x \in N$, for a solution (x, y) of (8). So, the left-hand side of the equation (8) is the product of x and $yx^{-\delta}y^{-1}$ where both elements belong to N. The right-hand side is also the product of two elements of N, whose projections to $N^{ab} = N/[N,N] \approx (\mathbb{Z}[\pi], +)$ are $\vartheta\bar{v}$ and 1, respectively, where $\bar{v} = p_\pi(v)$ and $p_\pi \colon F_2 \to \pi$ is the projection, see (25), (26). So, we can project both sides of the equation to $N^{ab} = N/[N,N]$, and we get:

Theorem 5.1 *Suppose that (x, y) is a solution of the equation (8) with $x \in N = \langle\langle \alpha\beta\alpha^{-\varepsilon}\beta^{-1}\rangle\rangle$, $y \in F_2 = \langle \alpha, \beta \mid \rangle$. Let $\tilde{x} = q_N(x) \in \mathbb{Z}[\pi]$, $\bar{y} = p_\pi(y) \in \pi = F_2/N$ be the images of x, y under the projections $q_N \colon N \to (\mathbb{Z}[\pi], +) \approx N^{ab} = N/[N,N]$ and $p_\pi \colon F_2 \to \pi$, respectively. Here the natural identification of N^{ab}, the abelianised group N, with the group $(\mathbb{Z}[\pi], +)$ is given by (25), (26). Then the pair $(\tilde{x}, \bar{y})$ satisfies the following equation called the first derived equation:*

$$(1 - \delta\bar{y})\tilde{x} = 1 + \vartheta\bar{v} \tag{36}$$

in the group ring $\mathbb{Z}[\pi]$, with the "unknowns" $\bar{y} \in \pi$ and $\tilde{x} \in \mathbb{Z}[\pi]$. Moreover, the properties (17), (20) are valid. Furthermore, any solution $(\tilde{x}, \bar{y})$ of (36) satisfies (20).

Proof The group N^{ab} is isomorphic to the abelian group $(\mathbb{Z}[\pi], +)$, see Proposition 4.1. Under this isomorphism, the element $B = \alpha\beta\alpha^{-\varepsilon}\beta^{-1} \in N$ is identified with $1 \in \pi \subset \mathbb{Z}[\pi]$, and B_u with $\bar{u} \in \pi \subset \mathbb{Z}[\pi]$, thus the right-hand side of the equation is identified with $1 + \vartheta\bar{v}$. Moreover, the conjugation of $B_u = uBu^{-1}$, $u \in F_2$, by an element $z \in F_2$ equals B_{zu}, which is identified with $\bar{z}\bar{u} \in \pi \subset \mathbb{Z}[\pi]$. It follows that the projection of the left-hand side to N^{ab} equals $\tilde{x} - \delta\bar{y}\tilde{x} = (1 - \delta\bar{y})\tilde{x}$, which gives (36). The properties (17), (20) are due to Remark 3.12.

Let us derive the property (20) from (36). Suppose that $(\tilde{x}, \bar{y}) \in \mathbb{Z}[\pi] \times \pi$ is a solution of (36). Consider the left action of the infinite cyclic group $\langle t\rangle \approx \mathbb{Z}$ on π via $t \cdot g = \bar{y}g$, $g \in \pi$. Consider the orbits $\mathcal{O}_g = \mathbb{Z} \cdot g$, $g \in \pi$. For any $\tilde{x} \in \mathbb{Z}[\pi]$, denote by $\tilde{x}_g \in \mathbb{Z}[\mathcal{O}_g]$ the image of $\tilde{x}$ under the projection $\mathbb{Z}[\pi] \to \mathbb{Z}[\mathcal{O}_g]$, $g \in \pi$. It follows from (36) that $(1 - \delta\bar{y})\tilde{x}_1 = 1 + \vartheta\bar{v}$ if $\bar{v} \in \mathcal{O}_1$, and $(1 - \delta\bar{y})\tilde{x}_1 = 1$ if $\bar{v} \notin \mathcal{O}_1$. Since the augmentation of the left-hand side is even, this implies $\bar{v} \in \mathcal{O}_1 = \langle\bar{y}\rangle$, thus $\bar{v} = \bar{y}^k$ for some $k \in \mathbb{Z}$. If $\bar{y} = 1$ and $\delta = -1$, the property (20) is now obvious, since it is equivalent to $1 = 1^k$ and $(-1)^k\vartheta = -1$, for some $k \in \mathbb{Z}$. In the remaining case ($\bar{y} \neq 1$ or $\delta = 1$), consider the homomorphism $\chi\colon \langle\bar{y}\rangle \to \mathbb{Z}^* = \{1,-1\} \subset \mathbb{Z}$ sending $\bar{y} \mapsto \delta$ (it is well-defined, since π is a torsion free group). By extending χ linearly to the group ring $\mathbb{Z}[\langle\bar{y}\rangle]$, one obtains the *$\chi$–twisted augmentation* $\varepsilon_\chi\colon \mathbb{Z}[\langle\bar{y}\rangle] \to \mathbb{Z}$. From above, we have $(1-\delta\bar{y})\tilde{x}_1 = 1+\vartheta\bar{y}^k$, where $\tilde{x}_1 \in \mathbb{Z}[\langle\bar{y}\rangle]$, a Laurent polynomial in $\bar{y}$. Since χ–twisted augmentation of the left-hand side vanishes, we have $0 = \varepsilon_\chi(1+\vartheta\bar{y}^k) = 1+\vartheta\delta^k$. This completes the derivation of (20) from (36). □

5.2 Solutions of the first derived equation in the "mixed" cases

Here we study separately the solutions of the first derived equation (36) in the *mixed cases* described in Remark 3.16 and Definition 3.15, see also Tables 2 and 3. Recall that, for any solution $(\tilde{x},\bar{y}) \in (\mathbb{Z}[\pi]) \times \pi$ of the first derived equation (36) in a mixed case, $\bar{v}$ belongs to the cyclic subgroup of $\pi = \pi_\varepsilon$ generated by $\bar{y}^2$, see Theorem 3.14(5). We will call a solution $(\tilde{x},\bar{y})$ of (36) *faithful* if $w_\varepsilon(\bar{y}) = \delta$, see (17).

Case of the equation (2′)

Here $\delta = 1$, $\varepsilon = -1$. We will only consider non-faithful solutions of (36) for $\vartheta = -1$, $v \in F_2$ such that $\bar{v} = p_K(v) = \bar{\beta}^{2n}$, $n \in \mathbb{Z}$, see Remark 3.16. Denote $c_L = \beta\alpha^{-L}$, $L \in \mathbb{Z}$.

Lemma 5.2 *For the equation (2′) with $\vartheta = -1$, the non-faithful solutions of the first derived equation (36) are described by*

$$(2_1) \qquad (1-\bar{y})\tilde{x} = 1-\bar{v}, \quad \text{where} \quad w_-(\bar{y}) = -1,$$

in $\mathbb{Z}[\pi]$, with the unknowns $\tilde{x} \in \mathbb{Z}[\pi]$, $\bar{y} \in \pi$, where $\pi = \pi_-$. For v satisfying $\bar{v} = p_K(v) = \bar{\beta}^{2n}$, $n \in \mathbb{Z}$, the solutions of the equation (2_1) are given by

$$\bar{y} = \bar{c}_L^{\ell} = (\bar{\alpha}^L\bar{\beta})^{\ell}, \quad \tilde{x} = \frac{1-\bar{c}_L^{2n}}{1-\bar{c}_L^{\ell}} = \begin{cases} 1+\bar{c}_L^{\ell}+\bar{c}_L^{2\ell}+\ldots+\bar{c}_L^{2n-\ell}, & n/\ell > 0, \\ 0, & n = 0, \\ -\bar{c}_L^{-\ell}-\bar{c}_L^{-2\ell}-\ldots-\bar{c}_L^{2n}, & n/\ell < 0, \end{cases}$$

where $L \in \mathbb{Z}$ is arbitrary, and ℓ runs over the set of all odd divisors of n. For $n = 0$ we assume that ℓ is any odd number.

Proof The equation (2_1) follows from Theorem 5.1. Suppose $\bar{y} = \bar{\alpha}^L\bar{\beta}^\ell$. Because the solution is non-faithful, it follows that ℓ is odd. Since $\bar{v} = \bar{\beta}^{2n}$ belongs to the subgroup generated by $\bar{y}$, it follows that $\bar{v} = \bar{y}^k = (\bar{\alpha}^L \bar{\beta}^\ell)^k$, for some $k \in \mathbb{Z}$. This implies that ℓ is a divisor of $2n$. Thus $\bar{y}$ has the form given by the second part of the Lemma. It follows by a straightforward calculation that all values of $(\tilde{x}, \bar{y})$ given by Lemma are solutions. That they are the only solutions follows from the fact that the group ring $\mathbb{Z}[\pi_-]$ has no zero divisors, since π_- is a solvable torsion free group, see Kropholler, Linnell and Moody [25, Theorem 1.4]. □

Remark 5.3 Later, the following representatives $(x_{L,\ell}, y_{L,\ell}) \in N \times F_2$ of the solutions $(\tilde{x}, \bar{y})$ of (2_1) from Lemma 5.2 will be used:

$$y_{L,\ell} = c_L^\ell = (\beta\alpha^{-L})^\ell,$$

$$x_{L,\ell} = \begin{cases} B_{c_L^{2n-\ell}} B_{c_L^{2n-2\ell}} B_{c_L^{2n-3\ell}} \dots B_{c_L^\ell} B, & n/\ell > 0, \\ 1, & n = 0, \\ B^{-1}_{c_L^{2n}} B^{-1}_{c_L^{2n+\ell}} B^{-1}_{c_L^{2n+2\ell}} \dots B^{-1}_{c_L^{-2\ell}} B^{-1}_{c_L^{-\ell}}, & n/\ell < 0, \end{cases}$$

where $B = \alpha\beta\alpha\beta^{-1}$, $\ell \neq 0$ is any odd number if $n = 0$, or any odd divisor of n if $n \neq 0$, thus the number of factors in the expression for $x_{L,\ell}$ is even and equal to $2|n/\ell|$.

Case of the equation $(3')$

Here $\delta = -1$, $\varepsilon = 1$, and all solutions are non-faithful. We will consider only the case where $\vartheta = -1$ and $\bar{v} = p_T(v) = \bar{\alpha}^{2m}\bar{\beta}^{2n}$, see Remark 3.16.

If $|m| + |n| > 0$, let us denote $d = \gcd(m, n)$ and $c = \alpha^{m/d}\beta^{n/d}$.

Lemma 5.4 *For the equation $(3')$ with $\vartheta = -1$, the solutions of the first derived equation (36) are described by*

$$(3_1) \qquad (1 + \bar{y})\tilde{x} = 1 - \bar{v}$$

in $\mathbb{Z}[\pi]$, with the unknowns $\tilde{x} \in \mathbb{Z}[\pi]$, $\bar{y} \in \pi$, where $\pi = \pi_+$. For v satisfying $\bar{v} = p_T(v) = \bar{\alpha}^{2m}\bar{\beta}^{2n}$, $m, n \in \mathbb{Z}$, $|m| + |n| > 0$, all solutions of this equation are given by

$$\bar{y} = \bar{c}^\ell,$$

$$\tilde{x} = \frac{1 - \bar{c}^{2d}}{1 + \bar{c}^\ell} = \begin{cases} 1 - \bar{c}^\ell + \bar{c}^{2\ell} - \ldots + \bar{c}^{2d-2\ell} - \bar{c}^{2d-\ell}, & \ell > 0, \\ \bar{c}^{\,-\ell} - \bar{c}^{\,-2\ell} + \ldots - \bar{c}^{2d+2\ell} + \bar{c}^{2d+\ell} - \bar{c}^{2d}, & \ell < 0, \end{cases}$$

where $\ell \neq 0$ is any divisor of $d = \gcd(m, n)$, $\bar{c} = \bar{\alpha}^{m/d}\bar{\beta}^{n/d}$. If v satisfies $\bar{v} = p_T(v) = 1$ then all solutions are given by $\tilde{x} = 0$ and $\bar{y} \in \pi$ is any element.

Proof The equation (3_1) follows from Theorem 5.1. Let $\bar{v} = \bar{\alpha}^{2m}\bar{\beta}^{2n}$ and $\bar{y} = \bar{\alpha}^r\bar{\beta}^s$, $m, n, r, s \in \mathbb{Z}$. Suppose $|m| + |n| > 0$. Since $\bar{v}$ belongs to the subgroup generated by $\bar{y}^2$, it follows that $\bar{v} = \bar{y}^{2k} = \bar{\alpha}^{2kr}\bar{\beta}^{2ks}$, for some $k \in \mathbb{Z}$. This implies $kr = m$, $ks = n$, thus k is a divisor of d, and $\bar{y} = \bar{\alpha}^{m/k}\bar{\beta}^{n/k} = \bar{c}^{\ell}$ where $\ell = d/k$. Thus $\bar{y}$ has the form given by the second part of the Lemma. It follows by a straightforward calculation that all values of $(\tilde{x}, \bar{y})$ given by Lemma are solutions. That they are the only solutions follows from the fact that the group ring $\mathbb{Z}[\pi_+]$ has no zero divisors.

Suppose $m = n = 0$, thus $\bar{v} = 1$, and the right-hand side of the equation (3_1) vanishes. Since $1 + \bar{y} \neq 0$ in $\mathbb{Z}[\pi]$ for any $\bar{y} \in \pi$, it follows that $\tilde{x} = 0$, since the group ring $\mathbb{Z}[\pi_+]$ has no zero divisors, since it is a polynomial ring. □

Remark 5.5 Suppose that $\bar{v} \neq 1$, thus $\bar{v} = \bar{\alpha}^{2m}\bar{\beta}^{2n}$ with $|m| + |n| > 0$. Denote $d = \gcd(m, n)$, $c = \alpha^{m/d}\beta^{n/d}$, $B = \alpha\beta\alpha^{-1}\beta^{-1}$. Later, the following representatives $(x_\ell, y_\ell) \in N \times F_2$ of the solutions $(\tilde{x}, \bar{y})$ of (3_1) from Lemma 5.4 will be used:

$$y_\ell = c^\ell,$$
$$x_\ell = \begin{cases} B_{c^{2d-2\ell}}B_{c^{2d-4\ell}}\dots B_{c^{2\ell}}BB_{c^\ell}^{-1}B_{c^{3\ell}}^{-1}\dots B_{c^{2d-3\ell}}^{-1}B_{c^{2d-\ell}}^{-1}, & \ell > 0,\\ B_{c^{2d}}^{-1}B_{c^{2d+2\ell}}^{-1}\dots B_{c^{-4\ell}}^{-1}B_{c^{-2\ell}}^{-1}B_{c^{-\ell}}B_{c^{-3\ell}}\dots B_{c^{2d+3\ell}}B_{c^{2d+\ell}}, & \ell < 0,\end{cases}$$

where $\ell \neq 0$ is any divisor of d, thus the number of factors in the expression for x_ℓ is even and equal to $2d/|\ell|$.

For $\bar{v} = 1$, we will use the representatives $x_{L,\ell} = 1$ and $y_{L,\ell} = \alpha^L\beta^\ell$, where $L, \ell \in \mathbb{Z}$. Actually L, ℓ coincide with the exponents in the canonical form of $\bar{y} \in \pi_+$, see (19).

Case of the equation ($4'$)

Here $\delta = \varepsilon = -1$. First we consider the case of faithful solutions where $\vartheta = -1$ and $\bar{v} = p_K(v) = \bar{\beta}^{2n}$, see Remark 3.16.

As above, we denote $c_L = \beta\alpha^{-L}$.

Lemma 5.6 *For the equation ($4'$) with $\vartheta = -1$, the faithful solutions of the first derived equation (36) are described by*

$$(4_1^{\mathrm{f}}) \qquad (1 + \bar{y})\tilde{x} = 1 - \bar{v}, \quad \text{where} \quad w_-(\bar{y}) = -1,$$

in $\mathbb{Z}[\pi]$, with the unknowns $\tilde{x} \in \mathbb{Z}[\pi]$, $\bar{y} \in \pi$, where $\pi = \pi_-$. For v satisfying $\bar{v} = p_K(v) = \bar{\beta}^{2n}$, $n \in \mathbb{Z}$, the solutions of this equation are given by

$$\bar{y} = \bar{c}_L^{\ell} = (\bar{\alpha}^L \bar{\beta})^{\ell},$$

$$\tilde{x} = \frac{1 - \bar{c}_L^{2n}}{1 + \bar{c}_L^{\ell}} = \begin{cases} 1 - \bar{c}_L^{\ell} + \bar{c}_L^{2\ell} - \ldots + \bar{c}_L^{2n-2\ell} - \bar{c}_L^{2n-\ell}, & n/\ell > 0, \\ 0, & n = 0, \\ \bar{c}_L^{-\ell} - \bar{c}_L^{-2\ell} + \ldots + \bar{c}_L^{2n+\ell} - \bar{c}_L^{2n}, & n/\ell < 0, \end{cases}$$

where $L \in \mathbb{Z}$ is arbitrary, and ℓ runs over the set of all odd divisors of n. For $n = 0$ we assume that ℓ is any odd number. Compare Lemma 5.2.

Proof Similar to that of Lemma 5.2. □

Remark 5.7 Later, the following representatives $(x_{L,\ell}, y_{L,\ell}) \in N \times F_2$ of the solutions $(\tilde{x}, \bar{y})$ of (4_1^{f}) from Lemma 5.6 will be used:

$$y_{L,\ell} = c_L^{\ell},$$

$$x_{L,\ell} = \begin{cases} B_{c_L^{2d-2\ell}} B_{c_L^{2d-4\ell}} \ldots B_{c_L^{2\ell}} B B_{c_L^{\ell}}^{-1} B_{c_L^{3\ell}}^{-1} \ldots B_{c_L^{2d-3\ell}}^{-1} B_{c_L^{2d-\ell}}^{-1}, & n/\ell > 0, \\ 1, & n = 0, \\ B_{c_L^{2d}}^{-1} B_{c_L^{2d+2\ell}}^{-1} \ldots B_{c_L^{-4\ell}}^{-1} B_{c_L^{-2\ell}}^{-1} B_{c_L^{-\ell}} B_{c_L^{-3\ell}} \ldots B_{c_L^{2d+3\ell}} B_{c_L^{2d+\ell}}, & n/\ell < 0, \end{cases}$$

where $c_L = \beta\alpha^{-L}$, $B = \alpha\beta\alpha\beta^{-1}$, $\ell \neq 0$ is any odd number if $n = 0$, or any odd divisor of n if $n \neq 0$, thus the number of factors in the expression for $x_{L,\ell}$ is even and equal to $2|n/\ell|$. Compare Remarks 5.3 and 5.5.

Now consider the case of non-faithful solutions where $\vartheta = -1$ and $\bar{v} = p_K(v) = \bar{\alpha}^{2m}\bar{\beta}^{4n}$, see Remark 3.16. If $|m| + |n| > 0$, we denote $d = \gcd(m, n)$, $c = \alpha^{m/d}\beta^{2n/d}$.

Lemma 5.8 *For the equation ($4'$) with $\vartheta = -1$, the non-faithful solutions of the first derived equation (36) are described by*

(4_1^{nf}) $$(1 + \bar{y})\tilde{x} = 1 - \bar{v}, \quad \text{where} \quad w_-(\bar{y}) = 1,$$

in $\mathbb{Z}[\pi]$, with the unknowns $\tilde{x} \in \mathbb{Z}[\pi]$, $\bar{y} \in \pi$, where $\pi = \pi_-$. For v satisfying $\bar{v} = p_K(v) = \bar{\alpha}^{2m}\bar{\beta}^{4n}$, $m, n \in \mathbb{Z}$, $|m| + |n| > 0$, all non-faithful solutions of this equation are given by the same formulae as in Lemma 5.4:

$$\bar{y} = \bar{c}^{\ell},$$

$$\tilde{x} = \frac{1 - \bar{c}^{2d}}{1 + \bar{c}^{\ell}} = \begin{cases} 1 - \bar{c}^{\ell} + \bar{c}^{2\ell} - \ldots + \bar{c}^{2d-2\ell} - \bar{c}^{2d-\ell}, & \ell > 0, \\ \bar{c}^{-\ell} - \bar{c}^{-2\ell} + \ldots - \bar{c}^{2d+2\ell} + \bar{c}^{2d+\ell} - \bar{c}^{2d}, & \ell < 0, \end{cases}$$

where $\ell \neq 0$ is any divisor of $d = \gcd(m,n)$, $\bar{c} = \bar{\alpha}^{m/d}\bar{\beta}^{2n/d}$. If v satisfies $\bar{v} = p_K(v) = 1$ then all non-faithful solutions are given by: $\tilde{x} = 0$ and $\bar{y} \in \pi$ is any element satisfying $w_-(\bar{y}) = 1$.

Proof Similar to that of Lemma 5.4 (see also the end of the proof of Lemma 5.2). □

Remark 5.9 Suppose that $\bar{v} \neq 1$, thus $\bar{v} = \bar{\alpha}^{2m}\bar{\beta}^{4n}$ with $|m| + |n| > 0$. Denote $d = \gcd(m,n)$, $c = \alpha^{m/d}\beta^{2n/d}$, $B = \alpha\beta\alpha\beta^{-1}$. We will later use the following representatives $(x_\ell, y_\ell) \in N \times F_2$ of the solutions $(\tilde{x}, \bar{y})$ of (4_1^{nf}) from Lemma 5.8. We define these representatives by the same formulae as in Remark 5.5.

For $\bar{v} = 1$, we will use the following representatives: $x_{L,\ell} = 1$, $y_{L,\ell} = \alpha^L\beta^{2\ell}$, where $L, \ell \in \mathbb{Z}$. Actually $L, 2\ell$ coincide with the exponents in the canonical form of $\bar{y} \in \pi_-$, see (19).

5.3 The second derived equation in the “mixed” cases

In order to find further properties of the solutions of the equations ($2'$), ($3'$) and ($4'$) in the “mixed” cases (see Section 3.3, Tables 2 and 3, Remark 3.16, and Definition 3.15), we will construct the *second derived equation* (2_2) (resp. (3_2) or (4_2^{f}), (4_2^{nf})) for the equation ($2'$) (resp. ($3'$), or ($4'$)). More specifically, for every solution of one of the first derived equations (2_1), (3_1), (4_1^{f}), and (4_1^{nf}) (see Lemmas 5.2, 5.4, 5.6 and 5.8) we will construct an equation in the free abelian group Q, see (7), which is the quotient

$$(37) \qquad Q = Q_\varepsilon = (\mathbb{Z}[\pi \setminus \{1\}])/\langle g + g^{-1} \mid g \in \pi \setminus \{1\}\rangle, \text{ with } \pi = \pi_\varepsilon = F_2/N,$$

of the free abelian group $\mathbb{Z}[\pi \setminus \{1\}]$ by the system of relations $g \sim -g^{-1}$, $g \in \pi \setminus \{1\}$, where $N := \langle\langle \alpha\beta\alpha^{-\varepsilon}\beta^{-1} \rangle\rangle$. (This quotient is isomorphic to $[N,N]/[F_2,[N,N]]$, see Proposition 4.5.) Consequently, we will obtain (in Theorem 5.10) an equation, which we will call the *second derived equation*, in two unknown “polynomials” $X \in Q$, $Y \in \mathbb{Z}[\pi]$, and some integer unknowns which enumerate the solutions of the first derived equation.

As an application of the second derived equation, we will obtain the non-existence results stated in Tables 4 and 5, see Section 7. For this, we will use the following property of the derived equations, which follows from Theorems 5.1 and 5.10: the non-existence of a solution of either the first or the second derived equation implies the non-existence of a (faithful or non-faithful) solution of the corresponding quadratic equation (8) in N.

Case of the equation ($2'$)

Here $\delta = 1$, $\varepsilon = -1$, and we may assume that $\vartheta = -1$ and $\bar v = \bar\beta^{2n} \in \pi$, $n \in \mathbb{Z}$, $\pi = \pi_-$, see Table 3 and Remark 3.16. We consider the following pair of derived equations (corresponding to non-faithful solutions). The first derived equation is (2_1) in $\mathbb{Z}[\pi]$, $\pi = \pi_-$, with the unknowns $\bar y \in \pi$ and $\tilde x \in \mathbb{Z}[\pi]$, see Lemma 5.2. By this Lemma, the solutions have the form $\bar y = \bar y_{L,\ell} = \bar\alpha^L \bar\beta^\ell$, $\tilde x = \tilde x_{L,\ell} = \frac{1-\bar\beta^{2n}}{1-\bar\alpha^L\bar\beta^\ell}$ for all $L, \ell \in \mathbb{Z}$ such that

$$\ell \mid n \quad \text{if} \quad n \neq 0, \quad \text{and} \quad \ell \text{ is odd} \tag{38}$$

(the latter condition corresponds to the fact that a solution to be found is non-faithful). Our *second derived equation* will be the following equation in the quotient $Q = Q_-$, see (37):

$$p_Q\left(\frac{1-\bar\beta^{-2n}}{1-\bar\beta^{-\ell}}\cdot\varphi^L(Y)\right) = p_Q\left(\varphi^L(V) - \frac{1-\bar\beta^{-2n}}{1-\bar\beta^2}\bar\beta\frac{1-\bar\alpha^L}{1-\bar\alpha} + \frac{1-\bar\beta^{-2n}}{1-\bar\beta^\ell}\right), \tag{2_2}$$

where $p_Q\colon \mathbb{Z}[\pi] \to \mathbb{Z}[\pi \setminus \{1\}] \to Q$ is the projection, $\varphi \in \mathrm{Aut}(F_2)$ denotes the automorphism sending $\alpha \mapsto \alpha$, $\beta \mapsto \beta\alpha$, and $B = B_- = \alpha\beta\alpha\beta^{-1} \mapsto B$, as well as the induced automorphism of Q. The parameter $V \in \mathbb{Z}[\pi]$ of the equation (2_2) is defined via

$$v = v_0 \prod B_{v_i}^{n_i}, \quad V = \sum n_i \bar v_i, \quad v_0 = \beta^{2n}, \tag{39}$$

see (26), while the unknowns are (L, ℓ, X, Y) with $L, \ell \in \mathbb{Z}$ as in (38), and $X \in Q$, $Y \in \mathbb{Z}[\pi]$. Remark that the unknown X does not contribute to the equation (2_2), thus X can be arbitrary.

In the special case $n = 0$, the second derived equation (2_2) has the form $0 = p_Q(V)$ with the unknowns $L, \ell \in \mathbb{Z}$, ℓ odd, and $X \in Q$, $Y \in \mathbb{Z}[\pi]$. Since no unknown contributes to this equation, a solution exists if and only if $p_Q(V) = 0$, moreover if $p_Q(V) = 0$ then arbitrary values of the unknowns determine a solution.

Case of the equation ($3'$)

Here $\delta = -1$, $\varepsilon = 1$, and we may assume that $\vartheta = -1$ and $\bar v = \bar\alpha^{2m}\bar\beta^{2n} \in \pi$, $m, n \in \mathbb{Z}$, $\pi = \pi_+$, see Table 3 and Remark 3.16. For the equation ($3'$) (it has only non-faithful solutions) we consider the following pair of derived equations. The first derived equation is (3_1) in $\mathbb{Z}[\pi]$, $\pi = \pi_+$, with the unknowns $\bar y \in \pi$ and $\tilde x \in \mathbb{Z}[\pi]$, see Lemma 5.4. By this Lemma, for $|m| + |n| > 0$ the solutions have the form $\bar y = \bar y_\ell = \bar c^\ell$, $\tilde x = \tilde x_\ell = \frac{1-\bar c^{2d}}{1+\bar c^\ell}$

where $d = \gcd(m,n)$, $\ell \in \mathbb{Z}$ such that $\ell \mid d$, and $c = \alpha^{m/d}\beta^{n/d} \in \pi$, while for $m = n = 0$ the solutions have the form $\bar{y} = \bar{y}_{L,\ell} = \bar{\alpha}^L\bar{\beta}^\ell$, $\tilde{x} = \tilde{x}_{L,\ell} = 0$ where $L, \ell \in \mathbb{Z}$. Our *second derived equation* will be the following equation in the quotient $Q = Q_+$, see (37):

$$(3_2) \qquad \begin{cases} 2X - p_Q\left(\dfrac{1-\bar{c}^{\,-2d}}{1+\bar{c}^{\,-\ell}}\cdot Y\right) = p_Q\left(V + \dfrac{1-\bar{c}^{\,-2d}}{1-\bar{c}^{2\ell}}\right) & \text{if } |m|+|n|>0, \\ 2X = p_Q(V) & \text{if } m=n=0, \end{cases}$$

where $p_Q\colon \mathbb{Z}[\pi] \to \mathbb{Z}[\pi \setminus \{1\}] \to Q$ is the projection. The parameter $V \in \mathbb{Z}[\pi]$ of the equation (3_2) is defined similarly to above, with $\pi = \pi_+$, $B = B_+ = \alpha\beta\alpha^{-1}\beta^{-1}$, via

$$(40) \qquad v = v_0 \prod B_{v_i}^{n_i}, \quad V = \sum n_i \bar{v}_i, \quad v_0 = \begin{cases} c^{2d}, & |m|+|n|>0, \\ 1, & m=n=0, \end{cases}$$

while the unknowns are either (ℓ, X, Y) with $\ell \in \mathbb{Z}$, $\ell \mid d$, $X \in Q$, $Y \in \mathbb{Z}[\pi]$ if $|m|+|n|>0$, or (L, ℓ, X, Y) with $L, \ell \in \mathbb{Z}$, $X \in Q$, $Y \in \mathbb{Z}[\pi]$ if $m = n = 0$.

In the special case $m = n = 0$, the second derived equation (3_2) has the form $2X = p_Q(V)$ with the unknowns $L, \ell \in \mathbb{Z}$, $X \in Q$, $Y \in \mathbb{Z}[\pi]$. It admits a solution if and only if $2 \mid p_Q(V)$, moreover for $2 \mid p_Q(V)$ the value of X is uniquily determined, while the unknowns (L, ℓ, Y) take arbitrary values, in order to determine a solution.

Case of the equation ($4'$), non-faithful solutions

Here $\delta = \varepsilon = -1$, and we may assume that $\vartheta = -1$ and $\bar{v} = \bar{\alpha}^{2m}\bar{\beta}^{4n}$, $m, n \in \mathbb{Z}$, $\pi = \pi_-$, see Table 3 and Remark 3.16. We consider the following pair of derived equations (corresponding to non-faithful solutions). The first derived equation is (4_1^{nf}) in $\mathbb{Z}[\pi]$, $\pi = \pi_-$, with the unknowns $\bar{y} \in \pi$ and $\tilde{x} \in \mathbb{Z}[\pi]$ such that $w_-(\bar{y}) = 1$, see Lemma 5.8. By this Lemma, for $|m|+|n| > 0$ the solutions have the form $\bar{y} = \bar{y}_\ell = \bar{c}^\ell$, $\tilde{x} = \tilde{x}_\ell = \frac{1-\bar{c}^{2d}}{1+\bar{c}^\ell}$ where $d = \gcd(m,n)$, $\ell \in \mathbb{Z}$ such that $\ell \mid d$, and $c = \alpha^{m/d}\beta^{2n/d} \in \pi$, while for $m = n = 0$ the solutions have the form $\bar{y} = \bar{y}_{L,\ell} = \bar{\alpha}^L\bar{\beta}^{2\ell}$, $\tilde{x} = \tilde{x}_{L,\ell} = 0$ where $L, \ell \in \mathbb{Z}$. Our *second derived equation* will be the following equation in the quotient $Q = Q_-$, see (37):

$$(4_2^{\mathrm{nf}}) \qquad \begin{cases} 2X - p_Q\left(\dfrac{1-\bar{c}^{\,-2d}}{1+\bar{c}^{\,-\ell}}\cdot Y\right) = p_Q\left(V + \dfrac{1-\bar{c}^{\,-2d}}{1-\bar{c}^{2\ell}}\right) & \text{if } |m|+|n|>0, \\ 2X = p_Q(V) & \text{if } m=n=0. \end{cases}$$

Here the projection p_Q and the polynomial $V \in \mathbb{Z}[\pi]$ are defined as in (40) with $\pi = \pi_-$, $B = B_- = \alpha\beta\alpha\beta^{-1}$, $c = \alpha^{m/d}\beta^{2n/d}$, while the unknowns are either (ℓ, X, Y)

with $\ell \in \mathbb{Z}$, $\ell \mid d$, $X \in Q$, $Y \in \mathbb{Z}[\pi]$ if $|m|+|n|>0$, or (L,ℓ,X,Y) with $L,\ell \in \mathbb{Z}$, $X \in Q$, $Y \in \mathbb{Z}[\pi]$ if $m=n=0$.

In the special case $m=n=0$, the second derived equation (4_2^{nf}) has the form $2X = p_Q(V)$ with the unknowns $L,\ell \in \mathbb{Z}$, $X \in Q$, $Y \in \mathbb{Z}[\pi]$. As above, it admits a solution if and only if $2 \mid p_Q(V)$, moreover for $2 \mid p_Q(V)$ the value of X is uniquily determined, while the unknowns (L,ℓ,Y) take arbitrary values, in order to determine a solution.

Case of the equation ($4'$), faithful solutions

Here $\delta = \varepsilon = -1$, and we may assume that $\vartheta = -1$ and $\bar{v} = \bar{\beta}^{2n}$, $n \in \mathbb{Z}$, $\pi = \pi_-$, see Table 2 and Remark 3.16. We consider the following pair of derived equations (corresponding to faithful solutions). The first derived equation is (4_1^{f}) in $\mathbb{Z}[\pi]$, $\pi = \pi_-$, with the unknowns $\bar{y} \in \pi$ and $\tilde{x} \in \mathbb{Z}[\pi]$ such that $w_-(\bar{y}) = -1$, see Lemma 5.6. By this Lemma, the solutions have the form $\bar{y} = \bar{y}_{L,\ell} = \bar{\alpha}^L \bar{\beta}^\ell$, $\tilde{x} = \tilde{x}_{L,\ell} = \frac{1-\bar{\beta}^{2n}}{1+\bar{\alpha}^L\bar{\beta}^\ell}$ where $L,\ell \in \mathbb{Z}$ satisfy (38) (the latter condition in (38) corresponds to the fact that a solution to be found is faithful). Our *second derived equation* will be the following equation in the quotient $Q = Q_-$, see (37):

$$(4_2^{\mathrm{f}})\quad 2\varphi^L(X) - p_Q\left(\frac{1-\bar{\beta}^{-2n}}{1-\bar{\beta}^{-\ell}} \cdot \varphi^L(Y)\right) = p_Q\left(\varphi^L(V) - \frac{1-\bar{\beta}^{-2n}}{1-\bar{\beta}^2}\bar{\beta}\frac{1-\bar{\alpha}^L}{1-\bar{\alpha}} + \frac{1-\bar{\beta}^{-2n}}{1-\bar{\beta}^{2\ell}}\right).$$

Here the projection p_Q, the automorphism φ of Q, and the polynomial $V \in \mathbb{Z}[\pi]$ are defined as in (39), while the unknowns are (L,ℓ,X,Y) with $L,\ell \in \mathbb{Z}$ as in (38), and $X \in Q$, $Y \in \mathbb{Z}[\pi]$.

In the special case $n=0$, the second derived equation (4_2^{f}) has the form $2X = p_Q(V)$ with the unknowns $L,\ell \in \mathbb{Z}$, ℓ odd, and $X \in Q$, $Y \in \mathbb{Z}[\pi]$. As above, it admits a solution if and only if $2 \mid p_Q(V)$. Moreover, if $2 \mid p_Q(V)$ then the value of X is uniquily determined, while the unknowns (L,ℓ,Y) take arbitrary values, in order to determine a solution.

Theorem 5.10 *Under the hypothesis of Theorem 5.1, suppose that $v_0 \in F_2$ is the representative of $\bar{v} \in \pi$, as in (39) or (40), and (x,y) is a solution of one of the equations ($2'$), ($3'$) or ($4'$) from Section 3.3, in a "mixed" case, see Remark 3.16 and Tables 2 and 3. Let $(x_{L,\ell}, y_{L,\ell})$ be the corresponding representative, given by*

Remarks 5.3, 5.5, 5.7 and 5.9, of the solution $(\tilde{x},\bar{y})\in(\mathbb{Z}[\pi])\times\pi$ *of the corresponding first derived equation* (2_1), (3_1), (4_1^{f}) *or* (4_1^{nf}) *(see Lemmas 5.2, 5.4, 5.6 and 5.8) where the subscript* L *is not necessarily present. Let* $X\in Q$, $Y,V\in\mathbb{Z}[\pi]$ *be the images of the elements* $x_{L,\ell}^{-1}x\in[N,N]$, $y_{L,\ell}^{-1}y$, $v_0^{-1}v\in N$ *under the projections* $q_{N_F}\colon [N,N]\to Q\approx[N,N]/[F_2,[N,N]]$ *and* $q_N\colon N\to(\mathbb{Z}[\pi],+)\approx N^{ab}$*, respectively:*

$$X=q_{N_F}(x_{L,\ell}^{-1}x)\in Q,\quad Y=q_N(y_{L,\ell}^{-1}y)\in\mathbb{Z}[\pi],\quad V=q_N(v_0^{-1}v)\in\mathbb{Z}[\pi],$$

where the natural identifications $N^{ab}\approx(\mathbb{Z}[\pi],+)$ *and* $[N,N]/[F_2,[N,N]]\approx Q$ *are given by (25), (26), and (32), (33). Then the quadruple* (L,ℓ,X,Y) *(or the triple* (ℓ,X,Y)*, respectively) satisfies the corresponding equation* (2_2), (3_2), (4_2^{f}) *or* (4_2^{nf})*, described above, called the* second derived equation*.*

5.4 Derivation of the second derived equation

Here we give a proof of Theorem 5.10, that is we derive the equations (2_2) and (3_2) from the equations $(2')$ and $(3')$, respectively, and the equations (4_2^{f}) and (4_2^{nf}) from the equation $(4')$, in the "mixed" cases, see Section 3.3 and Remark 3.16.

The following three technical Lemmas will be useful for deriving the second derived equations (2_2) and (4_2^{f}) from the equations $(2')$ and $(4')$, respectively.

Lemma 5.11 *In the free group* $F_2=\langle\alpha,\beta\mid\rangle$*, put* $B=\alpha\beta\alpha\beta^{-1}$ *and denote* $B_u=uBu^{-1}$, $u\in F_2$*. Then, for any* $L\in\mathbb{Z}$*,*

$$\alpha^L\beta\alpha^L\beta^{-1}=\begin{cases}B_{\alpha^{L-1}}B_{\alpha^{L-2}}\dots B_\alpha B, & L\ge 0,\\ B_{\alpha^L}^{-1}B_{\alpha^{L+1}}^{-1}\dots B_{\alpha^{-1}}^{-1}, & L<0.\end{cases}$$

If $N=\langle\langle B\rangle\rangle$ *and* $\pi=\pi_-=F_2/N$ *then, under the projection* $q_N\colon N\to(\mathbb{Z}[\pi],+)\approx N^{ab}=N/[N,N]$*, see (25), (26), the element* $\alpha^L\beta\alpha^L\beta^{-1}$ *is mapped to* $q_N(\alpha^L\beta\alpha^L\beta^{-1})=\frac{1-\bar{\alpha}^L}{1-\bar{\alpha}}$*.*

Proof Let us calculate $\alpha^L\beta\alpha^L\beta^{-1}$. For $L\ge 0$ we prove the formula by induction. For $L=0,1$ the formula is obviously true. From the formula for $L\ge 1$ we get the formula for $L+1$ as follows:

$$\begin{aligned}\alpha^{L+1}\beta\alpha^{L+1}\beta^{-1}&=\alpha(\alpha^L\beta\alpha^L\beta^{-1})\alpha^{-1}\alpha\beta\alpha\beta^{-1}=\alpha(B_{\alpha^{L-1}}B_{\alpha^{L-2}}\dots B_\alpha B)\alpha^{-1}\cdot B\\&=(B_{\alpha^L}B_{\alpha^{L-1}}\dots B_{\alpha^2}B_\alpha)B=B_{\alpha^L}B_{\alpha^{L-1}}\dots B_\alpha B.\end{aligned}$$

Using the above formula, we get the formula for $L<0$:

$$\alpha^L\beta\alpha^L\beta^{-1}=\alpha^L(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})^{-1}\alpha^{-L}=\alpha^L(B_{\alpha^{-L-1}}B_{\alpha^{-L-2}}\dots B_\alpha B)^{-1}\alpha^{-L}$$

$$= \alpha^L(B^{-1}B_\alpha^{-1}\dots B^{-1}_{\alpha^{-L-2}}B^{-1}_{\alpha^{-L-1}})\alpha^{-L} = B^{-1}_{\alpha^L}B^{-1}_{\alpha^{L+1}}\dots B^{-1}_{\alpha^{-2}}B^{-1}_{\alpha^{-1}}.$$

In the abelianised group N, which is identified with $(\mathbb{Z}[\pi_-],+)$, see Proposition 4.1, we have

$$q_N(\alpha^L\beta\alpha^L\beta^{-1}) = q_N(B_{\alpha^{L-1}}B_{\alpha^{L-2}}\dots B_\alpha B) = \bar\alpha^{L-1}+\bar\alpha^{L-2}+\dots+\bar\alpha+1 = \frac{1-\bar\alpha^L}{1-\bar\alpha}$$

if $L \geq 0$, and

$$q_N(\alpha^L\beta\alpha^L\beta^{-1}) = q_N(B^{-1}_{\alpha^L}B^{-1}_{\alpha^{L+1}}\dots B^{-1}_{\alpha^{-1}}) = -\bar\alpha^L-\bar\alpha^{L+1}-\dots-\bar\alpha^{\,-1} = \frac{1-\bar\alpha^L}{1-\bar\alpha}$$

if $L < 0$. □

Remark 5.12 Under the assumptions of Lemma 5.11, one can prove the following generalization of the formulae from this Lemma, for arbitrary $L,\ell \in \mathbb{Z}$ where ℓ is odd:

$$\alpha^L\beta^\ell\alpha^L\beta^{-\ell} = \begin{cases} \prod_{k=1}^L\left[\left(\prod_{i=1}^{(\ell-1)/2}B^{-1}_{\alpha^{L+1-k}\beta^{\ell-2i}}\right)\prod_{j=0}^{(\ell-1)/2}B_{\alpha^{L-k}\beta^{2j}}\right], & \ell>0,\\ \prod_{k=1}^L\left[\left(\prod_{i=0}^{(-\ell-1)/2}B_{\alpha^{L+1-k}\beta^{\ell+2i}}\right)\prod_{j=1}^{(-\ell-1)/2}B^{-1}_{\alpha^{L-k}\beta^{-2j}}\right], & \ell<0\end{cases}$$

if $L \geq 0$, and

$$\alpha^L\beta^\ell\alpha^L\beta^{-\ell} = \begin{cases} \prod_{k=L}^{-1}\left[\left(\prod_{j=(1-\ell)/2}^{0}B^{-1}_{\alpha^k\beta^{-2j}}\right)\prod_{i=(1-\ell)/2}^{-1}B_{\alpha^{k+1}\beta^{\ell+2i}}\right], & \ell>0,\\ \prod_{k=L}^{-1}\left[\left(\prod_{j=(1+\ell)/2}^{-1}B_{\alpha^k\beta^{2j}}\right)\prod_{i=(1+\ell)/2}^{0}B^{-1}_{\alpha^{k+1}\beta^{\ell-2i}}\right], & \ell<0\end{cases}$$

if $L<0$. Observe that the formulae for $L<0$ can be easily obtained from the formulae for $L>0$ via the identity $\alpha^L\beta^\ell\alpha^L\beta^{-\ell} = \alpha^L(\alpha^{-L}\beta^\ell\alpha^{-L}\beta^{-\ell})^{-1}\alpha^{-L}$. The above formulae (for $L>0$) can be proved either by straightforward calculations, or geometrically, by identifying the subgroup $N = \langle\langle B\rangle\rangle$ with the fundamental group of a suitable covering of the punctured Klein bottle, and interpreting elements of N as based loops on this covering, considered up to the based homotopy. In more detail, we consider a punctured Klein bottle $K^* = K\setminus \mathring{D}$ with base point $P\in\partial D$, where K is the Klein bottle, and $D\subset K$ a closed disk. We interprete K as the quotient of the Euclidean plane $\tilde K$ by the free action of the group $\pi = F_2/N$ on $\tilde K$ by isometries of the plane, in a usual way. We can also identify $\pi_1(K,P)=\pi$, $\pi_1(K^*,P)=F_2$, and the element $B=\alpha\beta\alpha\beta^{-1}\in F_2$ with the homotopy class of the (suitably oriented) boundary circle ∂K^*. Consider the covering $\tilde K^*$ of K^* corresponding to the subgroup $N=\langle\langle B\rangle\rangle$. It is a punctured plane with infinitely many punctures, moreover the inclusion $K^*\hookrightarrow K$ lifts to an inclusion $\tilde K^*\hookrightarrow\tilde K$. Let us consider a based loop γ on K^*, whose homotopy class equals $[\gamma]=\alpha^L\beta^\ell\alpha^L\beta^{-\ell}\in F_2$. Since $[\gamma]\in N$, this loop lifts to the covering $\tilde K^*$. The obtained based loop $\tilde\gamma$ on $\tilde K^*$ can be considered as a rectangle of "width" L and "height" ℓ on the plane $\tilde{K}$. Representing the elements B_u, $u \in F_2$, by suitable based loops on $\tilde{K}^*$, one can decompose the element $[\tilde{\gamma}] \in N$ into the product of B_u, $u \in F_2$, in many different ways. One can check that the above formulae give one of the ways for such a decomposition.

Lemma 5.13 *Suppose $n, L \in \mathbb{Z}$, $n \neq 0$, $c_L = \beta\alpha^{-L}$, $B = \alpha\beta\alpha\beta^{-1}$. Then*

$$\beta^{-2n} c_L^{2n} = \begin{cases} \prod_{j=0}^{n-1} \beta^{1-2n+2j}(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})\beta^{2n-2j-1}, & n > 0, \\ \prod_{j=1}^{-n} \beta^{1-2n-2j}(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})^{-1}\beta^{2n+2j-1}, & n < 0, \end{cases}$$

thus the element $\beta^{-2n} c_L^{2n} \in F_2$ belongs to the subgroup $N = \langle\langle B \rangle\rangle$. If $\pi = \pi_- = F_2/N$ then, under the projection $q_N \colon N \to (\mathbb{Z}[\pi], +) \approx N^{ab} = N/[N,N]$, see (25), (26), the element $\beta^{-2n} c_L^{2n}$ is mapped to $q_N(\beta^{-2n} c_L^{2n}) = -\bar{\beta}\frac{1-\bar{\beta}^{-2n}}{1-\bar{\beta}^2} \cdot \frac{1-\bar{\alpha}^{-L}}{1-\bar{\alpha}}$.

Proof Suppose $n > 0$. Then

$$\begin{aligned} \beta^{-2n} c_L^{2n} &= \beta^{1-2n} \cdot (\alpha^{-L}\beta\alpha^{-L}\beta^{-1}) \cdot \beta^2(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})\beta^{-2} \cdot \beta^4(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})\beta^{-4} \\ &\quad \cdots \beta^{2n-4}(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})\beta^{4-2n} \cdot \beta^{2n-2}(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})\beta^{2-2n} \cdot \beta^{2n-1} \\ &= \beta^{1-2n} \cdot \left(\prod_{j=0}^{n-1} \beta^{2j}(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})\beta^{-2j} \right) \cdot \beta^{2n-1} \\ &= \prod_{j=0}^{n-1} \beta^{1-2n+2j}(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})\beta^{2n-2j-1} \in N \end{aligned}$$

by Lemma 5.11. In the abelianised group N, which is identified with $(\mathbb{Z}[\pi_-], +)$, see Proposition 4.1, we obtain

$$q_N(\beta^{-2n} c_L^{2n}) = \bar{\beta}^{1-2n} \cdot \sum_{j=0}^{n-1} \bar{\beta}^{2j} q_N(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})$$

which, by Lemma 5.11, equals

$$\bar{\beta}^{1-2n} \cdot \sum_{j=0}^{n-1} \bar{\beta}^{2j} \frac{1-\bar{\alpha}^{-L}}{1-\bar{\alpha}} = \bar{\beta}^{1-2n} \cdot \frac{1-\bar{\beta}^{2n}}{1-\bar{\beta}^2} \cdot \frac{1-\bar{\alpha}^{-L}}{1-\bar{\alpha}} = -\bar{\beta}\frac{1-\bar{\beta}^{-2n}}{1-\bar{\beta}^2} \cdot \frac{1-\bar{\alpha}^{-L}}{1-\bar{\alpha}}.$$

Suppose $n<0$. Then we have

$$\begin{aligned}\beta^{-2n}c_L^{2n} &= \beta^{-2n}(\beta\alpha^{-L})^{2n} = \beta^{-2n}(\alpha^L\beta^{-1})^{-2n}\\ &= \beta^{1-2n}\cdot\beta^{-2}(\beta\alpha^L\beta^{-1}\alpha^L)\beta^2\cdot\beta^{-4}(\beta\alpha^L\beta^{-1}\alpha^L)\beta^4\cdot\beta^{-6}(\beta\alpha^L\beta^{-1}\alpha^L)\beta^6\\ &\quad\cdots\beta^{2n+2}(\beta\alpha^L\beta^{-1}\alpha^L)\beta^{-2n-2}\cdot\beta^{2n}(\beta\alpha^L\beta^{-1}\alpha^L)\beta^{-2n}\cdot\beta^{2n-1}\\ &= \beta^{1-2n}\cdot\left(\prod_{j=1}^{-n}\beta^{-2j}(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})^{-1}\beta^{2j}\right)\cdot\beta^{2n-1}\\ &= \prod_{j=1}^{-n}\beta^{1-2n-2j}(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})^{-1}\beta^{2n+2j-1}\in N\end{aligned}$$

by Lemma 5.11. In the abelianised group N, which is identified with $(\mathbb{Z}[\pi_-],+)$, see Proposition 4.1, we obtain

$$q_N(\beta^{-2n}c_L^{2n}) = -\bar\beta^{1-2n}\cdot\sum_{j=1}^{-n}\bar\beta^{-2j}q_N(\alpha^{-L}\beta\alpha^{-L}\beta^{-1})$$

which, by Lemma 5.11, equals

$$-\bar\beta^{1-2n}\cdot\sum_{j=1}^{-n}\bar\beta^{-2j}\frac{1-\bar\alpha^{-L}}{1-\bar\alpha} = \bar\beta^{1-2n}\cdot\frac{1-\bar\beta^{2n}}{1-\bar\beta^2}\cdot\frac{1-\bar\alpha^{-L}}{1-\bar\alpha} = -\bar\beta\frac{1-\bar\beta^{-2n}}{1-\bar\beta^2}\cdot\frac{1-\bar\alpha^{-L}}{1-\bar\alpha}.$$

□

Derivation of (3_2)

Here $B=\alpha\beta\alpha^{-1}\beta^{-1}$, $N=\langle\langle B\rangle\rangle$. As in Lemma 5.4, we assume that $\bar v=\bar\alpha^{2m}\bar\beta^{2n}$, $m,n\in\mathbb{Z}$.

Suppose $|m|+|n|>0$, thus $v=c^{2d}P_v$ where $c=\alpha^{m/d}\beta^{n/d}$, $d=\gcd(m,n)$, $P_v\in N$, thus $P_v=\prod B_{v_i}^{n_i}=\prod_{i=1}^r B_{v_i}^{n_i}$, see (40). It follows from Lemma 5.4 that any solution (x,y) of $(3')$ has the form $x=x_\ell\xi$, $y=y_\ell\eta$, for some $\ell\in\mathbb{Z}$ with $\ell\mid d$, $\xi\in[N,N]$, and $\eta\in N$, where x_ℓ,y_ℓ are given by Remark 5.5. The equation $(3')$ has the form

$$xyxy^{-1}=c^{2d}P_v\,B^{-1}\,P_v^{-1}c^{-2d}\,B.$$

Thus, the equation has the following form in the new unknowns ℓ,ξ,η:

$$x_\ell\xi\;y_\ell\eta\;x_\ell\xi\;\eta^{-1}y_\ell^{-1}=c^{2d}P_v\,B^{-1}\,P_v^{-1}c^{-2d}\,B. \tag{41}$$

We will start by analyzing both sides of this equality modulo $[F_2,[N,N]]$, and we will complete by using the presentation (33) of $[N,N]/[F_2,[N,N]]$. We shall represent elements of $N/[F_2,[N,N]]$ by elements of N, identified under the congruence relation $g_1 \equiv g_2$ *modulo* $[F_2,[N,N]]$, and shall write $g_1 \equiv g_2$ whenever $g_1 g_2^{-1} \in [F_2,[N,N]]$.

The right-hand side of (41) modulo $[F_2,[N,N]]$ equals

$$\begin{aligned} c^{2d} P_v\, B^{-1}\, P_v^{-1} c^{-2d}\, B &= c^{2d}[P_v, B^{-1}]B^{-1}c^{-2d}B \\ &\equiv c^{2d}B^{-1}c^{-2d}B[P_v,B^{-1}] = B_{c^{2d}}^{-1}B[P_v,B^{-1}]. \end{aligned}$$

The left-hand side of (41) modulo $[F_2,[N,N]]$ equals

$$\xi^2 x_\ell y_\ell x_\ell y_\ell^{-1} \cdot y_\ell [x_\ell^{-1},\eta] y_\ell^{-1} \equiv \xi^2 x_\ell y_\ell x_\ell y_\ell^{-1}[x_\ell^{-1},\eta], \tag{42}$$

since the elements ξ, $[x_\ell^{-1},\eta]$ belong to $[N,N]$ and, hence, they commute with any element of F_2 in the quotient $F_2/[F_2,[N,N]]$. Let us calculate $x_\ell y_\ell x_\ell y_\ell^{-1}$ in the quotient $F_2/[F_2,[N,N]]$. We have, by Remark 5.5,

$$\begin{aligned} x_\ell y_\ell x_\ell y_\ell^{-1} &= \big(B_{c^{2d-2\ell}}B_{c^{2d-4\ell}}\dots B_{c^{2\ell}}B\cdot B_{c^{\ell}}^{-1}B_{c^{3\ell}}^{-1}\dots B_{c^{2d-3\ell}}^{-1}B_{c^{2d-\ell}}^{-1}\big)\cdot c^{\ell} \\ &\quad \cdot \big(B_{c^{2d-2\ell}}B_{c^{2d-4\ell}}\dots B_{c^{2\ell}}B\cdot B_{c^{\ell}}^{-1}B_{c^{3\ell}}^{-1}\dots B_{c^{2d-3\ell}}^{-1}B_{c^{2d-\ell}}^{-1}\big)\cdot c^{-\ell} \\ &= B_{c^{2d-2\ell}}B_{c^{2d-4\ell}}\dots B_{c^{2\ell}}B\cdot B_{c^{2\ell}}^{-1}B_{c^{4\ell}}^{-1}\dots B_{c^{2d-2\ell}}^{-1}B_{c^{2d}}^{-1} \end{aligned}$$

$$\equiv B_{c^{2d}}^{-1}B\cdot \prod_{j=1}^{\frac{d}{\ell}}[B,B_{c^{2j\ell}}^{-1}] \qquad \text{if} \quad \ell>0; \tag{43}$$

$$\begin{aligned} x_\ell y_\ell x_\ell y_\ell^{-1} &= \big(B_{c^{2d}}^{-1}B_{c^{2d+2\ell}}^{-1}\dots B_{c^{-4\ell}}^{-1}B_{c^{-2\ell}}^{-1}\cdot B_{c^{-\ell}}B_{c^{-3\ell}}\dots B_{c^{2d+3\ell}}B_{c^{2d+\ell}}\big)\cdot c^{\ell} \\ &\quad \cdot \big(B_{c^{2d}}^{-1}B_{c^{2d+2\ell}}^{-1}\dots B_{c^{-4\ell}}^{-1}B_{c^{-2\ell}}^{-1}\cdot B_{c^{-\ell}}B_{c^{-3\ell}}\dots B_{c^{2d+3\ell}}B_{c^{2d+\ell}}\big)\cdot c^{-\ell} \\ &= B_{c^{2d}}^{-1}B_{c^{2d+2\ell}}^{-1}\dots B_{c^{-4\ell}}^{-1}B_{c^{-2\ell}}^{-1}\cdot BB_{c^{-2\ell}}\dots B_{c^{2d+4\ell}}B_{c^{2d+2\ell}} \end{aligned}$$

$$\equiv B_{c^{2d}}^{-1}B\cdot \prod_{j=1}^{-\frac{d}{\ell}-1}[B_{c^{-2j\ell}}^{-1},B] \qquad \text{if} \quad \ell<0. \tag{44}$$

Therefore, after cancelling the common factor $B_{c^{2d}}^{-1}B$ from the both sides, the equation has the following form in the quotient $F_2/[F_2,[N,N]]$:

$$\xi^2\cdot\left(\prod_{j=1}^{\frac{d}{\ell}}[B,B_{c^{2j\ell}}^{-1}]\right)\cdot[x_\ell^{-1},\eta]\equiv[P_v,B^{-1}] \qquad \text{for} \quad \ell>0,$$

$$\xi^2\cdot\left(\prod_{j=1}^{-\frac{d}{\ell}-1}[B_{c^{-2j\ell}}^{-1},B]\right)\cdot[x_\ell^{-1},\eta]\equiv[P_v,B^{-1}] \qquad \text{for} \quad \ell<0.$$

Both sides of the latter equation belong to $N_1=[N,N]$. After identification of $N_1/[F_2,N_1]$ with $Q=(\mathbb{Z}[\pi\setminus\{1\}])/\sim$, see Proposition 4.5, and denoting $X=$

$q_{N_F}(\xi) \in Q \approx N_1/[F_2,N_1]$, $Y = q_N(\eta) \in \mathbb{Z}[\pi]$, $V = q_N(P_v) \in \mathbb{Z}[\pi]$, we get, using (33) and Lemma 5.4, the equation

$$(45) \qquad \begin{cases} 2X + p_Q\left(\sum_{j=1}^{\frac{d}{\ell}} \bar{c}^{\,-2j\ell} - \frac{1-\bar{c}^{\,-2d}}{1+\bar{c}^{\,-\ell}} \cdot Y\right) = p_Q(V), & \ell > 0, \\ 2X + p_Q\left(-\sum_{j=0}^{-\frac{d}{\ell}-1} \bar{c}^{2j\ell} - \frac{1-\bar{c}^{\,-2d}}{1+\bar{c}^{\,-\ell}} \cdot Y\right) = p_Q(V), & \ell < 0, \end{cases}$$

which coincides with the desired equation (3_2) for $|m|+|n|>0$.

Consider the case $m = n = 0$, that is $\bar{v} = 1$. We have $v = P_v \in N$ where $P_v = \prod B_{v_i}^{n_i} = \prod_{i=1}^r B_{v_i}^{n_i}$. It follows from Lemma 5.4 that any solution (x,y) of ($3'$) has the form $x = x_{L,\ell}\xi$, $y = y_{L,\ell}\eta$ with $x_{L,\ell} = 1$, $y_{L,\ell} = \alpha^L\beta^\ell$, for some $L,\ell \in \mathbb{Z}$, $\xi \in [N,N]$, and $\eta \in N$, see Remark 5.5. Similarly to above, we obtain the equation (41) where x_ℓ, y_ℓ, c^{2d} are replaced by $x_{L,\ell} = 1$, $y_{L,\ell}$, 1, respectively. It follows from $x_{L,\ell} = 1$ that $x_{L,\ell}y_{L,\ell}x_{L,\ell}y_{L,\ell}^{-1} = 1$ and $[x_{L,\ell}^{-1},\eta] = 1$, hence the left-hand side modulo $[F_2,[N,N]]$ equals ξ^2. As above, the right-hand side modulo $[F_2,[N,N]]$ equals $[P_v,B^{-1}]$. After identification of $N_1/[F_2,N_1]$ with $Q = \mathbb{Z}[\pi\setminus\{1\}]/\sim$, and denoting $X = q_{N_F}(\xi) \in Q \approx N_1/[F_2,N_1]$, $Y = q_N(\eta) \in \mathbb{Z}[\pi]$, $V = q_N(P_v) \in \mathbb{Z}[\pi]$, we get, using (33), the desired equation

$$2X = p_Q(V).$$

This finishes the derivation of (3_2) from ($3'$).

Derivation of (4_2^{nf})

Here $B = \alpha\beta\alpha\beta^{-1}$, $N = \langle\langle B\rangle\rangle$. As in Lemma 5.8, we assume that $\bar{v} = \bar{\alpha}^{2m}\bar{\beta}^{4n}$, $m,n \in \mathbb{Z}$.

Suppose $|m|+|n|>0$, thus $v = c^{2d}P_v$ where $c = \alpha^{m/d}\beta^{2n/d}$, $d = \gcd(m,n)$, $P_v \in N$, thus $P_v = \prod B_{v_i}^{n_i} = \prod_{i=1}^r B_{v_i}^{n_i}$. It follows from Lemma 5.8 that any non-faithful solution (x,y) of ($4'$) has the form $x = x_\ell\xi$, $y = y_\ell\eta$ for some $\ell \in \mathbb{Z}$ with $\ell \mid d$, $\xi \in [N,N]$, and $\eta \in N$, where x_ℓ, y_ℓ are given by Remark 5.9.

The rest of the derivation is similar to that of (3_2).

Derivation of (2$_2$)

Here $B=\alpha\beta\alpha\beta^{-1}$, $N=\langle\langle B\rangle\rangle$. As in Lemma 5.2, we assume that $v=\beta^{2n}P_v$, where $P_v=\prod B_{v_i}^{n_i}$, $n,n_i\in\mathbb{Z}$, $v_i\in F_2$. By this Lemma, any non-faithful solution (x,y) of ($2'$) has the form $x=x_{L,\ell}\xi$, $y=y_{L,\ell}\eta$ for $L,\ell\in\mathbb{Z}$ as in (38), $\xi\in[N,N]$, and $\eta\in N$, where $x_{L,\ell},y_{L,\ell}$ are given by Remark 5.3. The equation ($2'$) has the form

$$xyx^{-1}y^{-1}=\beta^{2n}P_v\,B^{-1}\,P_v^{-1}\beta^{-2n}\,B.$$

Thus, the equation has the following form in the new unknowns L,ℓ,ξ,η:

$$x_{L,\ell}\xi\;y_{L,\ell}\eta\;\xi^{-1}x_{L,\ell}^{-1}\;\eta^{-1}y_{L,\ell}^{-1}=\beta^{2n}P_v\,B^{-1}\,P_v^{-1}\beta^{-2n}\,B.$$

As above, we will analyze both sides of this equality modulo $[F_2,[N,N]]$, and will write $g_1\equiv g_2$ whenever $g_1g_2^{-1}\in[F_2,[N,N]]$.

The right-hand side modulo $[F_2,[N,N]]$ equals

$$\begin{aligned}&\beta^{2n}P_v\,B^{-1}\,P_v^{-1}\beta^{-2n}\,B=\beta^{2n}[P_v,B^{-1}]B^{-1}\beta^{-2n}B\\ &\equiv\beta^{2n}B^{-1}\beta^{-2n}B[P_v,B^{-1}]=B_{\beta^{2n}}^{-1}B[P_v,B^{-1}].\end{aligned}\tag{46}$$

The left-hand side modulo $[F_2,[N,N]]$ equals

$$[x_{L,\ell},y_{L,\ell}]\cdot y_{L,\ell}[x_{L,\ell},\eta]y_{L,\ell}^{-1}\equiv[x_{L,\ell},y_{L,\ell}]\cdot[x_{L,\ell},\eta],$$

since the elements ξ, $[x_{L,\ell},\eta]$ belong to $[N,N]$ and, hence, they commute with any element of F_2 in the quotient $F_2/[F_2,[N,N]]$. Let us calculate $[x_{L,\ell},y_{L,\ell}]$ in $F_2/[[N,N],F_2]$. Denote $c_L=\beta\alpha^{-L}$, thus $\tilde{x}_{L,\ell}=\frac{1-\bar{c}_L^{2n}}{1-\bar{c}_L^{\ell}}$ and $\bar{y}_{L,\ell}=\bar{c}_L^{\ell}$. For $n/\ell\geq 0$ we have, by Remark 5.3,

$$\begin{aligned}[x_{L,\ell},y_{L,\ell}]&=\left(B_{c_L^{2n-\ell}}B_{c_L^{2n-2\ell}}\dots B_{c_L^{\ell}}B\right)\cdot c_L^{\ell}\cdot\left(B^{-1}B_{c_L^{\ell}}^{-1}\dots B_{c_L^{2n-2\ell}}^{-1}B_{c_L^{2n-\ell}}^{-1}\right)\cdot c_L^{-\ell}\\ &=B_{c_L^{2n-\ell}}B_{c_L^{2n-2\ell}}\dots B_{c_L^{\ell}}B\cdot B_{c_L^{\ell}}^{-1}B_{c_L^{2\ell}}^{-1}\dots B_{c_L^{2n-\ell}}^{-1}B_{c_L^{2n}}^{-1}\\ &\equiv B_{c_L^{2n}}^{-1}B\cdot\prod_{j=1}^{2\frac{n}{\ell}}[B,B_{c_L^{j\ell}}^{-1}],\end{aligned}$$

while for $n/\ell<0$ we have, by Remark 5.3,

$$\begin{aligned}[x_{L,\ell},y_{L,\ell}]&=\left(B_{c_L^{2n}}^{-1}B_{c_L^{2n+\ell}}^{-1}\dots B_{c_L^{-2\ell}}^{-1}B_{c_L^{-\ell}}^{-1}\right)\cdot c_L^{\ell}\cdot\left(B_{c_L^{-\ell}}B_{c_L^{-2\ell}}\dots B_{c_L^{2n+\ell}}B_{c_L^{2n}}\right)\cdot c_L^{-\ell}\\ &=B_{c_L^{2n}}^{-1}B_{c_L^{2n+\ell}}^{-1}\dots B_{c_L^{-2\ell}}^{-1}B_{c_L^{-\ell}}^{-1}\cdot BB_{c_L^{-\ell}}\dots B_{c_L^{2n+2\ell}}B_{c_L^{2n+\ell}}\\ &\equiv B_{c_L^{2n}}^{-1}B\cdot\prod_{j=1}^{-2\frac{n}{\ell}-1}[B_{c_L^{-j\ell}}^{-1},B].\end{aligned}$$

Therefore, after multiplying the both sides by $B^{-1}B_{c_L^{2n}}$, the equation has the following form in the quotient $F_2/[F_2,[N,N]]$:

$$\left(\prod_{j=1}^{2\frac{n}{\ell}}[B,B^{-1}_{c_L^{j\ell}}]\right)\cdot[x_{L,\ell},\eta]\equiv B^{-1}_{\beta^{2n}}B_{c_L^{2n}}\cdot[P_v,B^{-1}] \qquad \text{for} \quad n/\ell\ge 0,$$

$$\left(\prod_{j=1}^{-2\frac{n}{\ell}-1}[B^{-1}_{c_L^{-j\ell}},B]\right)\cdot[x_{L,\ell},\eta]\equiv B^{-1}_{\beta^{2n}}B_{c_L^{2n}}\cdot[P_v,B^{-1}] \qquad \text{for} \quad n/\ell< 0.$$

Observe that

$$B^{-1}_{\beta^{2n}}B_{c_L^{2n}}=\beta^{2n}[B^{-1},\beta^{-2n}c_L^{2n}]\beta^{-2n}\equiv[B^{-1},\beta^{-2n}c_L^{2n}]\in[N,N], \tag{47}$$

due to Lemma 5.13. In particular, both sides of the obtained equation belong to $N_1=[N,N]$. After identification of $N_1/[F_2,N_1]$ with $Q=\mathbb{Z}[\pi\backslash\{1\}]/\sim$, see (32), (33), and denoting $X=q_{N_F}(\xi)\in Q\approx N_1/[F_2,N_1]$, $Y=q_N(\eta)\in\mathbb{Z}[\pi]$, $V=q_N(P_v)\in\mathbb{Z}[\pi]$, we get, using Lemma 5.13 and (33), the equation

$$p_Q\left(-\frac{1-\bar c_L^{-2n}}{1-\bar c_L^{\ell}}+\frac{1-\bar c_L^{-2n}}{1-\bar c_L^{-\ell}}\cdot Y\right)=p_Q\left(V+\bar\beta\frac{1-\bar\beta^{-2n}}{1-\bar\beta^{2}}\cdot\frac{1-\bar\alpha^{-L}}{1-\bar\alpha}\right). \tag{48}$$

Since the automorphism φ^L sends $\bar c_L=\bar\alpha^L\bar\beta\mapsto\bar\beta$, $\bar\beta\mapsto\bar\beta\bar\alpha^L$, and leaves fixed $\bar\alpha$ and $\bar\beta^2$, we obtain, after applying the automorphism φ^L to both sides of the latter equation, the desired equation (2_2).

Derivation of (4_2^{f})

Here $B=\alpha\beta\alpha\beta^{-1}$, $N=\langle\langle B\rangle\rangle$. As in Lemma 5.6, we assume that $v=\beta^{2n}P_v$, where $P_v=\prod B_{v_i}^{n_i}$, $n\in\mathbb{Z}$. By this Lemma, any faithful solution (x,y) of ($4'$) has the form $x=x_{L,\ell}\xi$, $y=y_{L,\ell}\eta$ for $L,\ell\in\mathbb{Z}$ as in (38), $\xi\in[N,N]$, and $\eta\in N$, where $x_{L,\ell},y_{L,\ell}$ are given by Remark 5.7. The equation ($4'$) has the left-hand side similar to that of ($3'$) and the right-hand side as in ($2'$):

$$xyxy^{-1}=\beta^{2n}P_v\,B^{-1}\,P_v^{-1}\beta^{-2n}\,B.$$

Thus, the equation has the following form in the new unknowns L,ℓ,ξ,η:

$$x_{L,\ell}\xi\;y_{L,\ell}\eta\;x_{L,\ell}\xi\;\eta^{-1}y_{L,\ell}^{-1}=\beta^{2n}P_v\,B^{-1}\,P_v^{-1}\beta^{-2n}\,B.$$

As above, we will analyze both sides of this equality modulo $[F_2,[N,N]]$, and will write $g_1\equiv g_2$ whenever $g_1g_2^{-1}\in[F_2,[N,N]]$.

As in (46), the right-hand side modulo $[F_2,[N,N]]$ equals $B_{\beta^{2n}}^{-1}B[P_v,B^{-1}]$. Similarly to (42), one shows that the left-hand side modulo $[F_2,[N,N]]$ is equal to $\xi^2 x_{L,\ell}y_{L,\ell}x_{L,\ell}y_{L,\ell}^{-1}[x_{L,\ell}^{-1},\eta]$. Moreover, using Remark 5.7, we have, similarly to (43) and (44),

$$x_{L,\ell}y_{L,\ell}x_{L,\ell}y_{L,\ell}^{-1} \equiv \begin{cases} B_{c_L^{2n}}^{-1}B\cdot \prod_{j=1}^{\frac{n}{\ell}}[B,B_{c_L^{2j\ell}}^{-1}], & n/\ell \geq 0,\\ B_{c_L^{2n}}^{-1}B\cdot \prod\limits_{j=1}^{-\frac{n}{\ell}-1}[B_{c_L^{-2j\ell}}^{-1},B], & n/\ell<0.\end{cases}$$

Therefore, after multiplying the both sides by $B^{-1}B_{c_L^{2n}}$, the equation has the following form in the quotient $F_2/[F_2,[N,N]]$:

$$\xi^2\cdot\left(\prod_{j=1}^{\frac{n}{\ell}}[B,B_{c_L^{2j\ell}}^{-1}]\right)\cdot[x_{L,\ell}^{-1},\eta]\equiv B_{\beta^{2n}}^{-1}B_{c_L^{2n}}\cdot[P_v,B^{-1}] \qquad \text{for} \quad n/\ell\geq 0,$$

$$\xi^2\cdot\left(\prod_{j=1}^{-\frac{n}{\ell}-1}[B_{c_L^{-2j\ell}}^{-1},B]\right)\cdot[x_{L,\ell}^{-1},\eta]\equiv B_{\beta^{2n}}^{-1}B_{c_L^{2n}}\cdot[P_v,B^{-1}] \qquad \text{for} \quad n/\ell< 0.$$

By (47), both sides of the obtained equation belong to $N_1=[N,N]$. After identification of $N_1/[F_2,N_1]$ with $Q=(\mathbb{Z}[\pi\setminus\{1\}])/\sim$, see Proposition 4.5, and denoting $X=q_{N_F}(\xi)\in Q\approx N_1/[F_2,N_1]$, $Y=q_N(\eta)\in\mathbb{Z}[\pi]$, $V=q_N(P_v)\in\mathbb{Z}[\pi]$, we get, similarly to (45) for the left-hand side, and to (48) for the right-hand side, the equation

$$2X-p_Q\left(\frac{1-\bar{c}_L^{\,-2n}}{1-\bar{c}_L^{2\ell}}+\frac{1-\bar{c}_L^{\,-2n}}{1+\bar{c}_L^{\,-\ell}}\cdot Y\right)=p_Q\left(V+\bar{\beta}\frac{1-\bar{\beta}^{-2n}}{1-\bar{\beta}^2}\cdot\frac{1-\bar{\alpha}^{\,-L}}{1-\bar{\alpha}}\right).$$

Since the automorphism φ^L sends $\bar{c}_L=\bar{\alpha}^L\bar{\beta}\mapsto\bar{\beta}$, $\bar{\beta}\mapsto\bar{\beta}\bar{\alpha}^L$, and leaves fixed $\bar{\alpha}$ and $\bar{\beta}^2$, we obtain, after applying the automorphism φ^L to both sides of the latter equation, the desired equation (4_2^{f}).

This finishes the proof of Theorem 5.10.

6 Solutions of the second derived equations

In this section we give a necessary and sufficient condition for each of the second derived equations (2_2), (3_2), (4_2^{f}) and (4_2^{nf}), see Section 5.3, to have a solution. As a consequence, we will describe, in each of the mixed cases, many infinite families of v's for which the equation (8) has no solution, see Remark 3.16 and Tables 4 and 5. Unfortunately it is not true that if the second derived equation has a solution then the original equation also has a solution, see Example 6.11. As we noticed in Remark 3.16, for a given $\bar{v} \in \pi$ which corresponds to a mixed case, it is not an easy task to classify all the elements in $p^{-1}(\bar{v})$ with respect to the property that the corresponding equation (8) has a solution or has no solution. In fact we do not know $\bar{v}$ for which the answer is completely known.

In the following three assertions, we list some identities in the quotient $Q = \mathbb{Z}[\pi \setminus \{1\}]/\sim$, see (37) and (31), which will be used for solving the second derived equations (2_2), (3_2), (4_2^{f}) and (4_2^{nf}).

As above, $\pi = \pi_\pm$ denotes the group $\pi_\varepsilon = \langle \alpha, \beta \mid \alpha\beta\alpha^{-\varepsilon}\beta^{-1}\rangle$, $\varepsilon \in \{1, -1\}$, and $\bar{u} \in \pi$ denotes the class of an element $u \in F_2 = \langle \alpha, \beta \mid \rangle$ in π. Consider the natural projection $p_Q \colon \mathbb{Z}[\pi] \to Q$, see (7). It has the kernel

$$K = \ker p_Q = \mathbb{Z}[\{1\}] \oplus \langle \{g + g^{-1} \mid g \in \pi \setminus \{1\}\}\rangle, \tag{49}$$

where $\langle S\rangle$ denotes the minimal abelian subgroup of $(\mathbb{Z}[\pi], +)$ containing a subset $S \subset \mathbb{Z}[\pi]$. We will represent elements of Q by elements of $\mathbb{Z}[\pi]$, identified under the congruence relation $X_1 \equiv X_2$ *modulo* K, and shall write $X_1 \equiv X_2$ whenever $X_1 - X_2 \in K$.

Lemma 6.1 *For any $x \in \pi$, $k \in \mathbb{Z}$, the following congruences in $\mathbb{Z}[\pi]$ hold modulo K:*

$$\text{(a)}\ \frac{1-x^{2k}}{1-x}x^{1-k} \equiv x^k, \qquad \text{(b)}\ \frac{1-x^{2k}}{1-x^2}x^{1-k} \equiv 0, \qquad \text{(c)}\ \frac{1-x^{2k}}{1-x^2}x^{-k} \equiv x^{-k}.$$

Proof (a) The difference of the left-hand side and the right-hand side equals

$$\frac{x^{1-k}-x^k}{1-x} = \frac{x^{1-k}-x}{1-x} + \frac{x-1}{1-x} + \frac{1-x^k}{1-x} = \frac{x^{-k}-1}{x^{-1}-1} - 1 + \frac{1-x^k}{1-x} \equiv 0;$$

$$\text{(b)} \qquad \frac{1-x^{2k}}{1-x^2}x^{1-k} = \begin{cases} x^{1-k} + x^{3-k} + \ldots + x^{k-3} + x^{k-1} \equiv 0, & k > 0, \\ 0, & k = 0, \\ -x^{-1-k} - x^{-3-k} - \ldots - x^{k+3} - x^{k+1} \equiv 0, & k < 0; \end{cases}$$

$$\text{(c)} \qquad \frac{1-x^{2k}}{1-x^2}x^{-k} = \begin{cases} x^{-k} + x^{2-k} + \ldots + x^{k-4} + x^{k-2} \equiv x^{-k}, & k > 0, \\ 0, & k = 0, \\ -x^{-2-k} - x^{-4-k} - \ldots - x^{k+2} - x^k \equiv x^{-k}, & k < 0. \end{cases}$$

This completes the proof. $\square$

Corollary 6.2 *For any $x \in \pi = \pi_-$ and $n, L, \ell, k, m \in \mathbb{Z}$ with $\ell \mid n$ and ℓ odd, the following congruences in $\mathbb{Z}[\pi_-]$ hold modulo K:*

(a) $$\bar\beta^n \equiv \frac{1-\bar\beta^{2n}}{1-\bar\alpha^L\bar\beta^\ell}\bar\alpha^L\bar\beta^{\ell-n} \quad \text{if } n \text{ is even,}$$

(b) $$\frac{1-x^{2k}}{1-x^2}x^{2m} \equiv \frac{1-x^{2k}}{1+x}\cdot\frac{x^{2m}-x^{1-k}}{1-x} \equiv \frac{1-x^{2k}}{1-x}\cdot\frac{x^{2m}+(-1)^k x^{1-k}}{1+x},$$

(c) $$\frac{1-\bar\beta^{2n}}{1-\bar\beta^{2\ell}}\bar\beta^{2k\ell}\bar\alpha^m \equiv \frac{1-\bar\beta^{2n}}{1-\bar\alpha^L\bar\beta^\ell}\cdot\frac{\bar\beta^{2k\ell}+\bar\alpha^L\bar\beta^{\ell-n}}{1+\bar\alpha^L\bar\beta^\ell}\bar\alpha^m \quad \text{if } n \text{ is even.}$$

Lemma 6.3 *For any $n, L, \ell \in \mathbb{Z}$ with $\ell \mid n$ and ℓ odd, there exists $Z_1 \in \mathbb{Z}[\pi_-]$ satisfying the following congruence in $\mathbb{Z}[\pi_-]$ modulo K, for any $m \in \mathbb{Z}$:*

$$\frac{1-\bar\beta^{-2n}}{1-\bar\beta^2}\bar\beta\bar\alpha^m \equiv (1-\bar\beta^{2n})\cdot Z_1\cdot\bar\alpha^m + \begin{cases} 0, & n \text{ even,} \\ \bar\beta^n\bar\alpha^m, & n \text{ odd.}\end{cases}$$

Proof If n is even, we put $Z_1 := -\frac{1-\bar\beta^n}{1-\bar\beta^2}\bar\beta^{1-2n}$; then

$$(1-\bar\beta^{2n})\cdot Z_1\cdot\bar\alpha^m = \frac{1-\bar\beta^{-2n}}{1-\bar\beta^2}(1-\bar\beta^n)\bar\beta\bar\alpha^m \equiv \frac{1-\bar\beta^{-2n}}{1-\bar\beta^2}\bar\beta\bar\alpha^m,$$

where the latter congruence is due to Lemma 6.1(b). If n is odd, we put $Z_1 := -\frac{1-\bar\beta^{n-1}}{1-\bar\beta^2}\bar\beta^{1-2n}$; then

$$(1-\bar\beta^{2n})\cdot Z_1\cdot\bar\alpha^m = \frac{1-\bar\beta^{-2n}}{1-\bar\beta^2}(1-\bar\beta^{n-1})\bar\beta\bar\alpha^m \equiv \frac{1-\bar\beta^{-2n}}{1-\bar\beta^2}\bar\beta\bar\alpha^m - \bar\beta^n\bar\alpha^m,$$

where the latter congruence is due to Lemma 6.1(c). □

Denote $Q' = Q \otimes \mathbb{Z}_2$, and consider the natural projection

(50) $$p_{Q'}\colon\ \mathbb{Z}_2[\pi] \to Q' \approx (\mathbb{Z}_2[\pi \setminus \{1\}])/\langle g+g^{-1} \mid g \in \pi\setminus\{1\}\rangle,$$

compare (7). In this section, we will only consider the unsolved case $\bar v \neq 1$.

Case of the equations (3_2) and (4_2^{nf})

Observe that these equations have similar form, where (3_2) is in Q_+, while (4_2^{nf}) is in Q_-, see Section 5.3.

More specifically, for the equation (3_2), we have $B = [\alpha, \beta]$, $\pi = \pi_+$, $Q = Q_+$, $\bar v = \bar\alpha^{2m}\bar\beta^{2n} = \bar c^{2d} \in \pi$, where $\bar c = \bar\alpha^{m/d}\bar\beta^{n/d}$, $m, n \in \mathbb{Z}$, $|m| + |n| > 0$, $d = \gcd(m, n)$.

For the equation (4_2^{nf}), we have $B=\alpha\beta\alpha\beta^{-1}$, $\pi=\pi_-$, $Q=Q_-$, $\bar{v}=\bar{\alpha}^{2m}\bar{\beta}^{4n}=\bar{c}^{2d}\in\pi$, where $\bar{c}=\bar{\alpha}^{m/d}\bar{\beta}^{2n/d}$, $m,n\in\mathbb{Z}$, $|m|+|n|>0$, $d=\gcd(m,n)$, thus $\bar{c}$ is orientation-preserving, and is not a proper power of an orientation-preserving element of $\pi=\pi_-$.

Observe that the existence of a solution (ℓ,X,Y) of the equation (3_2) in $Q=Q_+$ is equivalent to the existence of a solution (ℓ,Z') of the following equation in $Q'=Q\otimes\mathbb{Z}_2$, with the same $\ell\mid d$ and $V':=V \mod 2\in\mathbb{Z}_2[\pi_+]$, $Z':=Z \mod 2\in\mathbb{Z}_2[\pi_+]$ where $Z=\bar{c}^{\ell-2d}\cdot Y+\frac{\bar{c}^{\,-2d}-\bar{c}^{\ell-d}}{1-\bar{c}^{\ell}}$:

$$(\bar{3}_2)\qquad p_{Q'}\left(\frac{1-\bar{c}^{2d}}{1+\bar{c}^{\ell}}\cdot Z'\right)=p_{Q'}(V'),$$

due to Corollary 6.2(b).

Similarly, the existence of a solution (ℓ,X,Y) of the equation (4_2^{nf}) in $Q=Q_-$ is equivalent to the existence of a solution (ℓ,Z') of the following equation in $Q'=Q\otimes\mathbb{Z}_2$, with the same $\ell\mid d$ and $V':=V \mod 2\in\mathbb{Z}_2[\pi_-]$, $Z':=Z \mod 2\in\mathbb{Z}_2[\pi_-]$ where $Z=\bar{c}^{\ell-2d}\cdot Y+\frac{\bar{c}^{\,-2d}-\bar{c}^{\ell-d}}{1-\bar{c}^{\ell}}$:

$$(\bar{4}_2^{\mathrm{nf}})\qquad p_{Q'}\left(\frac{1-\bar{c}^{2d}}{1+\bar{c}^{\ell}}\cdot Z'\right)=p_{Q'}(V').$$

In the following Theorem 6.4 and Proposition 6.5, we will formulate necessary and sufficient conditions for each of the equations ($\bar{3}_2$) and ($\bar{4}_2^{\mathrm{nf}}$) to have a solution, when $\bar{v}\neq 1$.

Denote $\bar{u}=\bar{c}^{d}\in\pi=\pi_\varepsilon$, thus $\bar{v}=\bar{u}^2$. Consider the left actions on π of the free groups $G=\langle t,i\mid\rangle$, $\hat{G}=\langle\hat{t},\hat{\imath}\mid\rangle$ of rank 2, where the actions of the generators t,i and $\hat{t},\hat{\imath}$ are defined by

$$(51)\qquad t\cdot g=\bar{c}g,\qquad i\cdot g=g^{-1},\qquad g\in\pi,$$

$$(52)\qquad \hat{t}\cdot g=\bar{u}g=\bar{c}^{d}g,\qquad \hat{\imath}\cdot g=g^{-1},\qquad g\in\pi.$$

Clearly, $\hat{G}$ can be considered as a subgroup of G, with the inclusion map $\hat{G}\hookrightarrow G$, $\hat{t}\mapsto t^{d}$, $\hat{\imath}\mapsto i$. Denote $\mathcal{O}_g:=G\cdot g$ and $\hat{\mathcal{O}}_g:=\hat{G}\cdot g$, the orbits of an element $g\in\pi$ under the actions of G and $\hat{G}$, respectively. Clearly $\hat{\mathcal{O}}_h\subset\mathcal{O}_g$ for any $g\in\pi$, $h\in\mathcal{O}_g$. Define the *$\hat{G}$–augmentation*

$$(53)\qquad \hat{\varepsilon}_g\colon\ \mathbb{Z}_2[\hat{\mathcal{O}}_g]\to\mathbb{Z}_2,\quad \sum_{k=1}^{r}m_k\bar{u}_k\mapsto\sum_{k=1}^{r}m_k,\quad m_k\in\mathbb{Z}_2,\ \bar{u}_k\in\hat{\mathcal{O}}_g,\ g\in\pi,$$

the restriction of the usual augmentation $\mathbb{Z}_2[\pi]\to\mathbb{Z}_2$ to the subgroup $\mathbb{Z}_2[\hat{\mathcal{O}}_g]\subset\mathbb{Z}_2[\pi]$.

Theorem 6.4 *Suppose $\bar c \in \pi$, $d \in \mathbb{N}$, $V \in \mathbb{Z}[\pi]$ are defined by the element $v \in F_2$, $\bar v = \bar c^{2d} \neq 1$, as in (40). Consider the actions (51), (52) of the groups $G, \hat G$ on π. Each of the equations ($\bar 3_2$), ($\bar 4_2^{\mathrm{nf}}$) has the following properties:*

(A) *For every fixed $\ell \mid d$, the corresponding equation with the unknown $Z' \in \mathbb{Z}_2[\pi]$ splits into the system of independent equations in the subspaces $(\mathbb{Z}_2[\mathcal{O}_g \setminus \{1\}])/\sim$ with the unknowns $Z'_g \in \mathbb{Z}_2[\mathcal{O}_g]$, where $g \in \pi$.*

(B) *The following conditions are pairwise equivalent:*

- (i) *the equation admits a solution;*
- (ii) *the equation admits a solution with $\ell = d$;*
- (iii) *for every $h \in \pi \setminus \hat{\mathcal{O}}_1$, the projection $\hat V'_h$ of the element $V' := V \mod 2 \in \mathbb{Z}_2[\pi]$ to the subspace $\mathbb{Z}_2[\hat{\mathcal{O}}_h]$ has vanishing $\hat G$–augmentation: $\hat\varepsilon_h(\hat V'_h) = 0$.*

Proof (A) Clearly, the equivalence $g \sim g^{-1}$, $g \in \pi \setminus \{1\}$, on $\pi \setminus \{1\}$ induces an equivalence relation on $\mathcal{O}_g \setminus \{1\}$, for each orbit $\mathcal{O}_g$. Moreover, two elements of $\mathbb{Z}_2[\pi \setminus \{1\}]$ are equivalent if and only if their projections to each subspace $\mathbb{Z}_2[\mathcal{O}_g \setminus \{1\}]$ are equivalent. Since $\frac{1-\bar c^{2d}}{1+\bar c^{\ell}} \cdot Z'$ belongs to $\mathbb{Z}_2[\mathcal{O}_g]$ whenever $Z' \in \mathbb{Z}_2[\mathcal{O}_g]$, the induced equations in the quotients of $\mathbb{Z}_2[\mathcal{O}_g \setminus \{1\}]$ by $\sim$ are pairwise independent (for every fixed ℓ).

(B) Consider the natural projection $p_{Q'}\colon \mathbb{Z}_2[\pi] \to Q' = Q \otimes \mathbb{Z}_2$, see (50). It has the kernel

$$K' = \ker p_{Q'} = \mathbb{Z}_2[\{1\}] \oplus \langle \{g + g^{-1} \mid g \in \pi \setminus \{1\}\} \rangle,$$

where $\langle S \rangle$ denotes the minimal abelian subgroup of $(\mathbb{Z}_2[\pi], +)$ containing a subset $S \subset \mathbb{Z}_2[\pi]$, compare (49). Similarly to Lemma 6.1, Corollary 6.2 and Lemma 6.3, we will represent elements of Q' by elements of $\mathbb{Z}_2[\pi]$, identified under the congruence relation $X_1 \equiv X_2$ *modulo* K', and shall write $X_1 \equiv X_2$ whenever $X_1 - X_2 \in K'$.

(i) $\Longrightarrow$ (ii) Suppose that (ℓ, Z') is a solution. Then the left-hand side equals

$$\frac{1-\bar c^{2d}}{1+\bar c^{\ell}} \cdot Z' = \frac{1-\bar c^{2d}}{1+\bar c^{d}} \cdot \frac{1+\bar c^{d}}{1+\bar c^{\ell}} \cdot Z'.$$

Since the right-hand sides of ($\bar 3_2$) and ($\bar 4_2^{\mathrm{nf}}$) do not depend on ℓ, the pair $(d, \frac{1+\bar c^{d}}{1+\bar c^{\ell}} \cdot Z')$ is a solution.

(ii) $\Longrightarrow$ (iii) Suppose (d, Z') is a solution, thus

$$(1 - \bar c^{d}) \cdot Z' \equiv V'.$$

It follows that $V' = U' + W'$, where U' is a linear combination of the elements of the form $(1-\bar{c}^d)g_1$, $g_1 \in \pi$, while $W' \in K$ is a linear combination of the elements of the form $g_2 + g_2^{-1}$ and g_3, $g_2 \in \pi \setminus \{1\}$, $g_3 = 1 \in \pi$.

Take any $h \in \pi \setminus \hat{\mathcal{O}}_1$. It follows that $\hat{V}'_h$ is a linear combination of $(1-\bar{c}^d)g_1$, $g_2 + g_2^{-1}$, and g_3, where $g_1 \in \hat{\mathcal{O}}_h$, $g_2 \in \hat{\mathcal{O}}_h \setminus \{1\}$, $g_3 = 1 \in \pi \cap \hat{\mathcal{O}}_h$. Since $g_3 = 1 \notin \hat{\mathcal{O}}_h$, the coefficient at g_3 in this linear combination vanishes. Therefore the augmentation of this linear combination vanishes, thus $\hat{\varepsilon}_h(\hat{V}'_h) = 0$.

(iii) $\Longrightarrow$ (i) Suppose $\hat{\varepsilon}_h(\hat{V}'_h) = 0$ for any $h \in \pi \setminus \hat{\mathcal{O}}_1$. Since $\hat{V}'_h \in \mathbb{Z}_2[\hat{\mathcal{O}}_h]$, and $\hat{\mathcal{O}}_h$ is an orbit with respect to the action of the group $\hat{G}$ on π, it follows from $\hat{\varepsilon}_h(\hat{V}'_h) = 0$ that $\hat{V}'_h$ is a linear combination of the elements of the form $(1-\bar{c}^d)g_1$ and $g_2 + g_2^{-1}$, where $g_1 \in \hat{\mathcal{O}}_h$, $g_2 \in \hat{\mathcal{O}}_h \setminus \{1\}$. Similarly, since one of the elements $\hat{V}'_1, \hat{V}'_1 + 1 \in \mathbb{Z}_2[\hat{\mathcal{O}}_1]$ has vanishing $\hat{G}$–augmentation, it follows that $\hat{V}'_1$ is a linear combination of the elements of the form $(1-\bar{c}^d)g_1$, $g_2 + g_2^{-1}$, and g_3, where $g_1 \in \hat{\mathcal{O}}_1$, $g_2 \in \hat{\mathcal{O}}_1 \setminus \{1\}$, $g_3 = 1 \in \hat{\mathcal{O}}_1$.

This immediately gives $\hat{V}'_h \equiv (1-\bar{c}^d)\cdot \hat{Z}'_h$, for some $\hat{Z}'_h \in \mathbb{Z}_2[\hat{\mathcal{O}}_h]$, for every $h \in \pi$. Since V' equals the sum of $\hat{V}'_h \in \mathbb{Z}_2[\hat{\mathcal{O}}_h]$ over all $\hat{G}$–orbits $\hat{\mathcal{O}}_h \subset \pi$, we obtain the desired decomposition $V' \equiv (1-\bar{c}^d)\cdot Z'$, for some $Z' \in \mathbb{Z}_2[\pi]$. Hence (d, Z') is a solution. □

Proposition 6.5 *Suppose $\bar{u} \in \pi$, $w_\varepsilon(\bar{u}) = 1$, $\bar{v} = \bar{u}^2$, where $\pi = \pi_\varepsilon = \langle \alpha, \beta \mid \alpha\beta\alpha^{-\varepsilon}\beta^{-1}\rangle$. Consider the corresponding action (52) of the group $\hat{G}$ on π. Then the orbits $\hat{\mathcal{O}}_h$, $h \in \pi$, under this action have the following form:*

(A) *Suppose $\varepsilon = 1$ and $\bar{u} = \bar{\alpha}^m\bar{\beta}^n$, $m, n \in \mathbb{Z}$. Then, for $h = \bar{\alpha}^p\bar{\beta}^q$, $p, q \in \mathbb{Z}$, one has*

$$\hat{\mathcal{O}}_h = \{\bar{u}^k h^{\pm 1} \mid k \in \mathbb{Z}\} = \{\bar{\alpha}^{p+km}\bar{\beta}^{q+kn} \mid k \in \mathbb{Z}\} \cup \{\bar{\alpha}^{-p+km}\bar{\beta}^{-q+kn} \mid k \in \mathbb{Z}\}.$$

(B) *Suppose $\varepsilon = -1$, thus $\bar{u} = \bar{\alpha}^m\bar{\beta}^{2n}$, $m, n \in \mathbb{Z}$. If $w_-(h) = 1$ then $h = \bar{\alpha}^p\bar{\beta}^{2q}$, for some $p, q \in \mathbb{Z}$, and*

$$\hat{\mathcal{O}}_h = \{\bar{u}^k h^{\pm 1} \mid k \in \mathbb{Z}\} = \{\bar{\alpha}^{p+km}\bar{\beta}^{2q+2kn} \mid k \in \mathbb{Z}\} \cup \{\bar{\alpha}^{-p+km}\bar{\beta}^{-2q+2kn} \mid k \in \mathbb{Z}\}.$$

If $w_-(h) = -1$ then $h = \bar{\alpha}^p\bar{\beta}^{2q+1}$, for some $p, q \in \mathbb{Z}$, and

$$\begin{aligned}\hat{\mathcal{O}}_h &= \{\bar{\alpha}^{km}\bar{\beta}^{(2k+4r)n}h^{\pm 1} \mid k, r \in \mathbb{Z}\}\\ &= \{\bar{\alpha}^{p+km}\bar{\beta}^{2q+1+(2k+4r)n} \mid k, r \in \mathbb{Z}\} \cup \{\bar{\alpha}^{p+km}\bar{\beta}^{-(2q+1)+(2k+4r)n} \mid k, r \in \mathbb{Z}\},\end{aligned}$$

moreover, in the latter case, the set of all such orbits is in one-to-one correspondence with the set $\mathbb{Z}_{|m|} \oplus \mathbb{Z}_{|n|}$, where one denotes $\mathbb{Z}_0 = \mathbb{Z}$, $\mathbb{Z}_1 = \{0\}$; in particular, the number of such orbits is either $|mn|$ if $mn \neq 0$, or infinite if $mn = 0$.

Proof (A) Suppose $h = \bar\alpha^p\bar\beta^q$, and denote $\bar{\mathcal{O}}_h = \{\bar u^k h^{\pm 1} \mid k \in \mathbb{Z}\}$. Obviously, $h \in \bar{\mathcal{O}}_h$, and $\bar{\mathcal{O}}_h$ is invariant under the action of $\hat G$ (since $\pi = \pi_+$ is abelian), hence $\hat{\mathcal{O}}_h \subset \bar{\mathcal{O}}_h$. The converse inclusion follows from the fact that any element of $\bar{\mathcal{O}}_h$ is obtained from h or h^{-1} by the left multiplication by $\bar u^k$, for some $k \in \mathbb{Z}$. This proves $\hat{\mathcal{O}}_h = \bar{\mathcal{O}}_h$.

The equality of the two presentations for the set $\hat{\mathcal{O}}_h$ follows from the fact that the group $\pi = \pi_+$ is abelian.

(B) Suppose $w_-(h) = 1$, thus $h = \bar\alpha^p\bar\beta^{2q}$. Denote $\bar{\mathcal{O}}_h = \{\bar u^k h^{\pm 1} \mid k \in \mathbb{Z}\}$. Obviously, $h \in \bar{\mathcal{O}}_h$. Since $w_-(\bar u) = w_-(h) = 1$, the elements $\bar u$ and h commute, therefore $\bar{\mathcal{O}}_h$ is invariant under the action of $\hat G$, hence $\hat{\mathcal{O}}_h \subset \bar{\mathcal{O}}_h$. The converse inclusion follows from the fact that any element of $\bar{\mathcal{O}}_h$ is obtained from h or h^{-1} by the left multiplication by $\bar u^k$, for some $k \in \mathbb{Z}$. This proves $\hat{\mathcal{O}}_h = \bar{\mathcal{O}}_h$.

The equality of the two presentations for the set $\hat{\mathcal{O}}_h$ follows from the fact that the subgroup of $\pi = \pi_-$ generated by $\bar\alpha, \bar\beta^2$ is abelian.

Suppose now $w_-(h) = -1$, thus $h = \bar\alpha^p\bar\beta^{2q+1}$. Denote

$$\bar{\mathcal{O}}_h = \{\bar\alpha^{p+km}\bar\beta^{2q+1+(2k+4r)n} \mid k, r \in \mathbb{Z}\} \cup \{\bar\alpha^{p+km}\bar\beta^{-(2q+1)+(2k+4r)n} \mid k, r \in \mathbb{Z}\}.$$

Obviously $h \in \bar{\mathcal{O}}_h$. Let us show that $\bar{\mathcal{O}}_h$ is invariant under the action of $\hat G$. Since $\bar\beta^2$ commutes with any element of $\pi = \pi_-$, and $\bar\alpha^p\bar\beta = \bar\beta\bar\alpha^{-p}$, we have, for any $s \in \mathbb{Z}$,

$$\begin{aligned}
\bar u^s \cdot \bar\alpha^{p+km}\bar\beta^{2q+1+2(k+2r)n} &= (\bar\alpha^m\bar\beta^{2n})^s \cdot \bar\alpha^{p+km}\bar\beta^{2q+1+2(k+2r)n} \\
&= \bar\alpha^{sm}\bar\beta^{2sn} \cdot \bar\alpha^{p+km}\bar\beta^{2q+1+2(k+2r)n} \\
&= \bar\alpha^{p+(s+k)m}\bar\beta^{2q+1+2(s+k+2r)n} \in \bar{\mathcal{O}}_h, \\
(\bar\alpha^{p+km}\bar\beta^{2q+1+(2k+4r)n})^{-1} &= \bar\alpha^{p+km}\bar\beta^{-(2q+1)-(2k+4r)n} \in \bar{\mathcal{O}}_h,
\end{aligned}$$

and similarly $\bar u^s \cdot \bar\alpha^{p+km}\bar\beta^{-(2q+1)+2(k+2r)n} \in \bar{\mathcal{O}}_h$, $(\bar\alpha^{p+km}\bar\beta^{-(2q+1)+(2k+4r)n})^{-1} \in \bar{\mathcal{O}}_h$. Therefore $\hat{\mathcal{O}}_h \subset \bar{\mathcal{O}}_h$. The converse inclusion follows by observing that

$$\bar\alpha^{p+km}\bar\beta^{2q+1+(2k+4r)n} = \hat t^k \cdot \bar\alpha^p\bar\beta^{2q+1+4rn} = \hat t^k \hat{\imath}\hat t^{-r}\hat{\imath}\hat t^r \cdot \bar\alpha^p\bar\beta^{2q+1} \in \hat{\mathcal{O}}_h,$$

therefore any element of $\bar{\mathcal{O}}_h$ belongs to the orbit $\hat{\mathcal{O}}_h$ of $h = \bar\alpha^p\bar\beta^{2q+1}$ under the action of $\hat G$. This proves $\hat{\mathcal{O}}_h \supset \bar{\mathcal{O}}_h$ and, hence, $\hat{\mathcal{O}}_h = \bar{\mathcal{O}}_h$.

The equality of the two presentations for the set $\hat{\mathcal{O}}_h$ follows from the identities

$$\begin{aligned}
\bar\alpha^{km}\bar\beta^{(2k+4r)n} \cdot h &= \bar\alpha^{km}\bar\beta^{(2k+4r)n} \cdot \bar\alpha^p\bar\beta^{2q+1} \\
&= \bar\alpha^{p+km}\bar\beta^{2q+1+(2k+4r)n}, \\
\text{and} \qquad \bar\alpha^{km}\bar\beta^{(2k+4r)n} \cdot h^{-1} &= \bar\alpha^{km}\bar\beta^{(2k+4r)n} \cdot \bar\beta^{-(2q+1)}\bar\alpha^{-p} \\
&= \bar\alpha^{p+km}\bar\beta^{-(2q+1)+(2k+4r)n}.
\end{aligned}$$

This completes the proof. □

Case of the equations (2_2) and (4_2^{f})

For each of the equations (2_2) and (4_2^{f}), see Section 5.3, we have $B=\alpha\beta\alpha\beta^{-1}$, $\pi=\pi_-$, $Q=Q_-$, $\bar v=\bar\beta^{2n}$, $n\in\mathbb{Z}$. Both equations have the unknowns (L,ℓ,X,Y) with $L,\ell\in\mathbb{Z}$ as in (38), $X\in Q$, $Y\in\mathbb{Z}[\pi]$.

Observe that the existence of a solution (L,ℓ,X,Y) of the equation (2_2) in $Q=Q_-$ is equivalent to the existence of a solution (L,ℓ,Z) of the following equation in Q, with the same $V\in\mathbb{Z}[\pi_-]$, $L,\ell\in\mathbb{Z}$ satisfying (38), and with $Z=\bar c_L^{\ell-2n}Y+\bar c_L^{-2n}+C$, $\bar c_L=\bar\beta\bar\alpha^{-L}=\varphi^{-L}(\bar\beta)$, $C\in\mathbb{Z}[\pi_-]$:

$$(\bar 2_2)\qquad p_Q\left(\frac{1-\bar\beta^{2n}}{1-\bar\beta^{\ell}}\cdot\varphi^L(Z)\right)=p_Q(\varphi^L(V))+\begin{cases}0, & n\text{ even},\\ -p_Q\left(\bar\beta^n\frac{1-\bar\alpha^L}{1-\bar\alpha}\right), & n\text{ odd},\end{cases}$$

where C is determined by Lemma 6.3. As in (2_2), this equation is equivalent to $0=p_Q(V)$ if $n=0$.

Similarly, the existence of a solution (L,ℓ,X,Y) of the equation (4_2^{f}) in $Q=Q_-$ is equivalent to the existence of a solution (L,ℓ,Z') of the following equation in $Q'=Q\otimes\mathbb{Z}_2$, with the same $L,\ell\in\mathbb{Z}$ satisfying (38), with $V':=V \mod 2\in\mathbb{Z}_2[\pi_-]$, $Z':=Z \mod 2\in\mathbb{Z}_2[\pi_-]$, and with $Z=\bar c_L^{\ell-2n}Y+\frac{\bar c_L^{-2n}+(-1)^n\bar c_L^{\ell-n}}{1+\bar c_L^{\ell}}+C$, $C\in\mathbb{Z}[\pi_-]$ from above:

$$(\bar 4_2^{\mathrm{f}})\qquad p_{Q'}\left(\frac{1-\bar\beta^{2n}}{1-\bar\beta^{\ell}}\cdot\varphi^L(Z')\right)=p_{Q'}(\varphi^L(V'))+\begin{cases}0, & n\text{ even},\\ -p_{Q'}\left(\bar\beta^n\frac{1-\bar\alpha^L}{1-\bar\alpha}\right), & n\text{ odd},\end{cases}$$

due to Corollary 6.2(b). This equation is equivalent to $0=p_{Q'}(V')$ if $n=0$.

Below (see Theorem 6.8 and Proposition 6.10), we will formulate necessary and sufficient conditions for each of the equations $(\bar 2_2)$, $(\bar 4_2^{\mathrm{f}})$ to have a solution, when $\bar v\neq 1$.

From now on, for the remainder of this section, let us fix an integer $n\neq 0$, and denote by s the exponent of 2 in the prime factorization of $|n|$; put $\mu=\frac{n}{|n|}2^s$, $\ell_{\max}=\frac{|n|}{2^s}=\frac{n}{\mu}$, the greatest odd divisor of n. Consider the left actions on $\pi=\pi_-$ of the groups

$$\begin{aligned}
G_L &:= \langle t_L, i \mid i^2,\ (t_Lit_L)^2,\ (it_L)^4\rangle,\\
\hat G_L &:= \langle \hat t_L,\hat\imath \mid \hat\imath^2,\ (\hat t_L\hat\imath\hat t_L)^2,\ (\hat\imath\hat t_L)^4\rangle,\\
\tilde G_L &:= \langle \tilde t,\tilde\imath,\tilde j_L \mid \tilde\imath^2,\ \tilde j_L^2,\ (\tilde\imath\tilde j_L)^2,\ (\tilde\imath\tilde t)^2,\ (\tilde j_L\tilde t)^2\rangle\approx\mathbb{Z}\rtimes_{\psi_2}(\mathbb{Z}_2\oplus\mathbb{Z}_2),\\
&\qquad \psi_2\colon\ \mathbb{Z}_2\oplus\mathbb{Z}_2\to\mathrm{Aut}(\mathbb{Z}),\quad \psi_2(\tilde\imath)(\tilde t):=\tilde t^{-1}=:\psi_2(\tilde j_L)(\tilde t),\\
\tilde G &:= \langle \tilde t,\tilde\imath \mid \tilde\imath^2,\ (\tilde\imath\tilde t)^2\rangle\approx\mathbb{Z}\rtimes_{\psi_1}\mathbb{Z}_2,\qquad \psi_1\colon\ \mathbb{Z}_2\to\mathrm{Aut}(\mathbb{Z}),\quad \psi_1(\tilde\imath)(\tilde t):=\tilde t^{-1},
\end{aligned}$$

where $L \in \mathbb{Z}$, and the actions of the generators $\tilde{t}, \tilde{\imath}, \tilde{j}_L$, $\hat{t}_L, \hat{\imath}$, and t_L, i are defined by

(54) $\quad t_L \cdot g = \bar{\alpha}^L \bar{\beta} g, \qquad i \cdot g = g^{-1}, \qquad g \in \pi,$

(55) $\quad \hat{t}_L \cdot g = \bar{\alpha}^L \bar{\beta}^{\ell_{\max}} g, \quad \hat{\imath} \cdot g = g^{-1}, \qquad g \in \pi,$

(56) $\quad \tilde{t} \cdot g = \bar{\beta}^{2n} g, \qquad \tilde{\imath} \cdot g = g^{-1}, \quad \tilde{j}_L \cdot g = \bar{\alpha}^L \bar{\beta}^{\ell_{\max}} (\bar{\alpha}^L \bar{\beta}^{\ell_{\max}} g)^{-1}, \quad g \in \pi.$

Clearly, we have the inclusions $\tilde{G} \subset \tilde{G}_L \hookrightarrow \hat{G}_L \hookrightarrow G_L$ with $\tilde{t} \mapsto \hat{t}_L^{2\mu}$, $\hat{t}_L \mapsto t_L^{\ell_{\max}}$, $\tilde{\imath} \mapsto \hat{\imath} \mapsto i$, $\tilde{j}_L \mapsto \hat{t}_L \hat{\imath} \hat{t}_L$, which respect the actions.

This provides the following alternative approach for defining the groups $\tilde{G}$, $\tilde{G}_L$, $\hat{G}_L$. We will henceforth identify these groups with the corresponding subgroups of the group G_L by denoting

$$\tilde{t} = \hat{t}_L^{2\mu} = t_L^{2n} =: t, \quad \hat{t}_L = t_L^{\ell_{\max}}, \quad \tilde{\imath} = \hat{\imath} = i, \quad \tilde{j}_L = \hat{t}_L i \hat{t}_L =: j_L.$$

Thus the subgroups $\tilde{G} \subset \tilde{G}_L \subset \hat{G}_L \subset G_L$ admit the following presentations by means of generators and defining relations:

$$\begin{aligned} \hat{G}_L :=& \langle \hat{t}_L, i \mid i^2,\ (\hat{t}_L i \hat{t}_L)^2,\ (i \hat{t}_L)^4 \rangle, \\ \tilde{G}_L :=& \langle t, i, j_L \mid i^2,\ j_L^2,\ (i j_L)^2,\ (it)^2,\ (j_L t)^2 \rangle \approx \mathbb{Z} \rtimes_{\psi_2} (\mathbb{Z}_2 \oplus \mathbb{Z}_2), \\ \tilde{G} =& \langle t, i \mid i^2,\ (it)^2 \rangle \approx \mathbb{Z} \rtimes_{\psi_1} \mathbb{Z}_2. \end{aligned}$$

Observe that the defined in this way group $\tilde{G}$ depends on L. However the above presentations of the groups by means of generators and defining relations provide an obvious group isomorphism $G_L \approx G_{L'}$ for $L, L' \in \mathbb{Z}$. Although this isomorphism does not respect the actions of $G_L, G_{L'}$ on π if $L \neq L'$ (since these actions determine different orbits), the induced isomorphism of the corresponding subgroups $\tilde{G} \subset G_L$ and $\tilde{G} \subset G_{L'}$ respects the actions. This gives the natural identification of different subgroups $\tilde{G} \subset G_L$, respecting their actions on π.

One easily checks that

(57)
$$\begin{aligned} i \cdot (\bar{\beta}^{2q} \bar{\alpha}^p) &= \bar{\beta}^{-2q} \bar{\alpha}^{-p}, & j_L \cdot (\bar{\beta}^{2q} \bar{\alpha}^p) &= \bar{\beta}^{-2q} \bar{\alpha}^p, \\ i \cdot (\bar{\beta}^{2q+1} \bar{\alpha}^p) &= \bar{\beta}^{-(2q+1)} \bar{\alpha}^p, & j_L \cdot (\bar{\beta}^{2q+1} \bar{\alpha}^p) &= \bar{\beta}^{-(2q+1)} \bar{\alpha}^{-p-2L}, \\ t \cdot (\bar{\beta}^{q} \bar{\alpha}^p) &= \bar{\beta}^{q+2n} \bar{\alpha}^p, & \hat{t}_L \cdot (\bar{\beta}^{2q} \bar{\alpha}^p) &= \bar{\beta}^{2q+\ell_{\max}} \bar{\alpha}^{p-L}, \\ t_L \cdot (\bar{\beta}^{2q} \bar{\alpha}^p) &= \bar{\beta}^{2q+1} \bar{\alpha}^{p-L}. \end{aligned}$$

Denote $\mathcal{O}_{g,L} := G_L \cdot g$, $\hat{\mathcal{O}}_{g,L} := \hat{G}_L \cdot g$, $\tilde{\mathcal{O}}_{g,L} := \tilde{G}_L \cdot g$, and $\tilde{\mathcal{O}}_g := \tilde{G} \cdot g$, the orbits of an element $g \in \pi$ under the actions of G_L, $\hat{G}_L$, $\tilde{G}_L$, and $\tilde{G}$, respectively. Clearly $\tilde{\mathcal{O}}_h \subset \tilde{\mathcal{O}}_{g,L} \subset \hat{\mathcal{O}}_{f,L} \subset \mathcal{O}_{e,L}$ for any $e \in \pi, f \in \mathcal{O}_{f,L}$, $g \in \hat{\mathcal{O}}_{f,L}$, $h \in \tilde{\mathcal{O}}_{g,L}$.

An element $g \in \pi$ (together with its orbit $\tilde{\mathcal{O}}_g$) is called $\tilde{G}$*–regular* if g has a trivial stabilizer with respect to the action of $\tilde{G}$ on π (thus $\mathrm{Stab}_{\tilde{G}}(g) = \{1\}$, so the natural map $\tilde{G} \to \tilde{G} \cdot g$ is bijective). Otherwise g (together with its orbit $\tilde{\mathcal{O}}_g$) is called $\tilde{G}$*–singular*.

Lemma 6.6 *An element $g \in \pi$ is $\tilde{G}$–singular if and only if either $w_-(g) = 1$ and $g = \bar{\beta}^{nk}$ (thus nk is even), or $w_-(g) = -1$ and $g = \bar{\beta}^{nk}\bar{\alpha}^m$ (thus nk is odd), for some $k, m \in \mathbb{Z}$. Moreover, the stabilizer $\mathrm{Stab}_{\tilde{G}}(g)$ of a $\tilde{G}$–singular element $g = \bar{\beta}^{nk}\bar{\alpha}^m$ under the action of $\tilde{G}$ is the cyclic subgroup of $\tilde{G}$ generated by the element $t^k i \in \tilde{G}$. Here the element $t^k i$ is conjugate in $\tilde{G}$ either to the element i if k is even, or to the element ti if k is odd.* □

In the case of the equation ($\bar{4}_2^{\mathrm{f}}$), we define the augmentations

$$\tilde{\varepsilon}_g\colon\ \mathbb{Z}_2[\tilde{\mathcal{O}}_g] \to \mathbb{Z}_2, \quad \tilde{\varepsilon}_{g,L}\colon\ \mathbb{Z}_2[\tilde{\mathcal{O}}_{g,L}] \to \mathbb{Z}_2, \quad \hat{\varepsilon}_{g,L}\colon\ \mathbb{Z}_2[\hat{\mathcal{O}}_{g,L}] \to \mathbb{Z}_2, \tag{58}$$

called the $\tilde{G}$*–augmentation*, $\tilde{G}_L$*–augmentation*, and $\hat{G}_L$*–augmentation*, respectively, as the restrictions of the usual augmentation $\mathbb{Z}_2[\pi] \to \mathbb{Z}_2$ to $\mathbb{Z}_2[\tilde{\mathcal{O}}_g]$, $\mathbb{Z}_2[\tilde{\mathcal{O}}_{g,L}]$, and $\mathbb{Z}_2[\hat{\mathcal{O}}_{g,L}]$, respectively, for every $g \in \pi$.

In order to define similar augmentations in the case of the equation ($\bar{2}_2$), the following constructions will be useful. Consider the character

$$\chi_L\colon\ G_L \to \mathbb{Z}^* = \{1, -1\}, \quad t_L \mapsto -1,\ i \mapsto -1,$$

thus $\hat{t}_L = t_L^{\ell_{\max}} \mapsto -1$, $t = t_L^{2n} \mapsto 1$. Denote $\chi := \chi_L|_{\tilde{G}}$. For every $\tilde{G}$–regular element $g \in \pi$, define the χ*–twisted* $\tilde{G}$*–augmentation*

$$\tilde{\varepsilon}_g\colon\ \mathbb{Z}[\tilde{\mathcal{O}}_g] \to \mathbb{Z}, \quad r \cdot g \mapsto \chi(r), \quad r \in \tilde{G}, \tag{59}$$

by the linear extension of the latter formula. The χ–twisted $\tilde{G}$–augmentation $\tilde{\varepsilon}_g$ is well-defined for any $\tilde{G}$–regular element $g \in \pi$, since the equality $r_1 \cdot g = r_2 \cdot g$ implies $r_1^{-1} r_2 \in \mathrm{Stab}_{\tilde{G}}(g) = \{1\}$, hence $r_1 = r_2$. We also have $\tilde{\varepsilon}_{r \cdot g} = \chi(r)\tilde{\varepsilon}_g$, for any $r \in \tilde{G}$, and for any $\tilde{G}$–regular element $g \in \pi$.

An element $g \in \pi$ (together with its $\tilde{G}_L$–orbit) is called $\tilde{G}_L$*–defective*, or simply *defective*, if there exists $r \in \mathrm{Stab}_{\tilde{G}_L}(g)$ with $\chi_L(r) = -1$. In other words, $\chi_L(\mathrm{Stab}_{\tilde{G}_L}(g)) = \{1, -1\}$ for defective g, and $\chi_L(\mathrm{Stab}_{\tilde{G}_L}(g)) = \{1\}$ for non-defective g. For every element $g \in \pi$, define the χ_L*–twisted* $\tilde{G}_L$*–augmentation*

$$\tilde{\varepsilon}_{g,L}\colon\ \mathbb{Z}[\tilde{\mathcal{O}}_{g,L}] \to \begin{cases} \mathbb{Z}, & g \text{ non-defective}, \\ \mathbb{Z}_2, & g \text{ defective}, \end{cases} \qquad r \cdot g \mapsto \begin{cases} \chi_L(r) \in \mathbb{Z}, \\ 1 \in \mathbb{Z}_2, \end{cases} \quad r \in \tilde{G}_L, \tag{60}$$

by the linear extension of the latter formula. If n is odd, we similarly define the χ_L*–twisted* $\hat{G}_L$*–augmentation*

$$\hat{\varepsilon}_{g,L}\colon\ \mathbb{Z}[\hat{\mathcal{O}}_{g,L}] \to \begin{cases} \mathbb{Z}, & g \text{ non-defective}, \\ \mathbb{Z}_2, & g \text{ defective}, \end{cases} \qquad r \cdot g \mapsto \begin{cases} \chi_L(r) \in \mathbb{Z}, \\ 1 \in \mathbb{Z}_2, \end{cases} \quad r \in \hat{G}_L, \tag{61}$$

by the linear extension of the latter formula. One easily checks that

$$\hat{\varepsilon}_{g,L}(\hat{V}_{g,L}) = \tilde{\varepsilon}_{g,L}(\tilde{V}_{g,L}) - \tilde{\varepsilon}_{h,L}(\tilde{V}_{h,L}) \quad \text{where} \quad h := \bar{\alpha}^L \bar{\beta}^n g,\ n \text{ odd},$$

$$\tilde{\varepsilon}_{g,L}(\tilde{V}_{g,L}) = \begin{cases} \tilde{\varepsilon}_g(\tilde{V}_g), & g \in \{(\bar{\alpha}^L\bar{\beta})^k \mid k \in \mathbb{Z}\}, \\ \tilde{\varepsilon}_g(\tilde{V}_g) \mod 2, & g \in \{\bar{\beta}^{2kn}\bar{\alpha}^m \mid k,m \in \mathbb{Z},\ m \neq 0\}, \\ \tilde{\varepsilon}_g(\tilde{V}_g) + \tilde{\varepsilon}_{ij_L \cdot g}(\tilde{V}_{ij_L \cdot g}), & \text{otherwise}. \end{cases} \tag{62}$$

Observe that, if an element $g \in \pi$ is defective, then all elements $h \in \tilde{\mathcal{O}}_{g,L}$ (as well as $h \in \hat{\mathcal{O}}_{g,L}$ if n is odd) are also defective (since $\chi_L(r) = \chi_L(srs^{-1})$ for any $r,s \in \tilde{G}_L$), furthermore $\tilde{\varepsilon}_{h,L} = \tilde{\varepsilon}_{g,L}$ is the usual augmentation on $\mathbb{Z}[\tilde{\mathcal{O}}_{g,L}]$ reduced modulo 2. For non-defective $g \in \pi$, the χ_L–twisted $\tilde{G}_L$–augmentation $\tilde{\varepsilon}_{g,L}$ is well-defined, since the equality $r_1 \cdot g = r_2 \cdot g$ implies $r_1^{-1}r_2 \in \mathrm{Stab}_{\tilde{G}_L}(g)$, hence $\chi_L(r_1^{-1}r_2) = 1$ and $\chi_L(r_1) = \chi_L(r_2)$. We have $\tilde{\varepsilon}_{r\cdot g,L} = \chi_L(r)\tilde{\varepsilon}_{g,L}$, for any $r \in \tilde{G}_L$, and for any non-defective $g \in \pi$.

Lemma 6.7 *An element $g \in \pi = \pi_-$ is defective if and only if $g = \bar{\beta}^{nk}\bar{\alpha}^m$ for some $k,m \in \mathbb{Z}$. In particular, all $\tilde{G}$–singular elements are defective.*

Proof The assertion easily follows from the following formulae for the stabilizer $\mathrm{Stab}_{\tilde{G}_L}(g)$ of an element $g \in \pi = \pi_-$. Suppose that g is not of the form $\bar{\beta}^{nk}\bar{\alpha}^m$, $k,m \in \mathbb{Z}$. If $g = (\bar{\alpha}^L\bar{\beta})^k$, $k \in \mathbb{Z}$, then $\mathrm{Stab}_{\tilde{G}_L}(g)$ is the cyclic subgroup of $\tilde{G}_L$ generated by ij_L; otherwise $\mathrm{Stab}_{\tilde{G}_L}(g) = \{1\}$. Therefore $\chi_L(\mathrm{Stab}_{\tilde{G}_L}(g)) = \{1\}$, hence g is non-defective. Suppose that $g = (\bar{\alpha}^L\bar{\beta})^{nk}\bar{\alpha}^m$, $k,m \in \mathbb{Z}$. If $m \neq 0$ then $\mathrm{Stab}_{\tilde{G}_L}(g)$ is the cyclic subgroup of $\tilde{G}_L$ generated by $t^k i$ (if nk is odd) or by $t^k j_L$ (if nk is even); otherwise $\mathrm{Stab}_{\tilde{G}_L}(g)$ is generated by two elements ij_L, $t^k i$. Therefore $\chi_L(\mathrm{Stab}_{\tilde{G}_L}(g)) = \{1,-1\}$, hence g is defective. □

Denote $\mathcal{Z} := \mathbb{Z}$, $V' := V \in \mathbb{Z}[\pi]$ for the equation ($\bar{2}_2$), and $\mathcal{Z} := \mathbb{Z}_2$, $V' := V \mod 2 \in \mathbb{Z}_2[\pi]$ for the equation ($\bar{4}_2^{\mathrm{f}}$), where $\pi = \pi_-$. For any $g \in \pi = \pi_-$, denote by $\tilde{V}'_g$, $\tilde{V}'_{g,L}$, $\hat{V}'_{g,L}$, and $V'_{g,L}$ the projections of the element $V' \in \mathcal{Z}[\pi]$ to $\mathcal{Z}[\tilde{\mathcal{O}}_g]$, $\mathcal{Z}[\tilde{\mathcal{O}}_{g,L}]$, $\mathcal{Z}[\hat{\mathcal{O}}_{g,L}]$, and $\mathcal{Z}[\mathcal{O}_{g,L}]$, respectively.

Theorem 6.8 *Suppose $n \in \mathbb{Z} \setminus \{0\}$ and $V \in \mathbb{Z}[\pi]$ are defined by an element $v \in F_2$, $\bar{v} = \bar{\beta}^{2n} \neq 1$, as in (39). For every $L \in \mathbb{Z}$, consider the left actions (54), (55), (56) of the groups $\tilde{G} \subset \tilde{G}_L \subset \hat{G}_L \subset G_L$ on π, and the corresponding (twisted) augmentations $\tilde{\varepsilon}_g$, $\tilde{\varepsilon}_{g,L}$, $\hat{\varepsilon}_{g,L}$, see (58), (59), (60), (61). Each of the equations ($\bar{2}_2$) and ($\bar{4}_2^{\mathrm{f}}$) has the following properties:*

(A) *For every fixed* $L, \ell \in \mathbb{Z}$ *as in* (38), *the corresponding equation with the unknown* $Z' \in \mathcal{Z}[\pi]$ *splits into the system of independent equations in the subspaces* $(\mathcal{Z}[\mathcal{O}_{g,L} \setminus \{1\}])/\sim$ *with the unknowns* $Z'_g \in \mathcal{Z}[\mathcal{O}_{g,L}]$*, where* $g \in \pi$.

(B) *The following conditions* (i), (ii) *and* (iii) *are pairwise equivalent:*

(i) *the equation admits a solution;*

(ii) *the equation admits a solution with* $\ell = \ell_{\max}$*;*

(iii) *the following conditions* (iii$_1$) *and* (iii$_2$) *hold for* $\ell := \ell_{\max}$ *(compare Lemmas 6.6 and 6.7):*

(iii$_1$) *If* n *is even then, for every pair of elements* $g, h \in \pi \setminus \{\bar{\beta}^{kn} \mid k \in \mathbb{Z}\}$ *(thus both* g, h *are* $\tilde{G}$*–regular) with* $h = \bar{\beta}^{2\ell r} g$, $r \in \mathbb{Z}$*, one has*

$$\tilde{\varepsilon}_g(\tilde{V}'_g) = \tilde{\varepsilon}_h(\tilde{V}'_h) \in \mathcal{Z};$$

(iii$_2$) *There exists* $L \in \mathbb{Z}$ *satisfying the following conditions. For every pair of elements* $g, h \in \pi$ *with* $g \notin \{\bar{\beta}^{2\ell k} \bar{\alpha}^m \mid k, m \in \mathbb{Z}\}$, $w_-(g) = 1$*, and* $h = \bar{\alpha}^L \bar{\beta}^\ell g$ *(thus both* g, h *are non-defective), one has*

$$\tilde{\varepsilon}_{g,L}(\tilde{V}'_{g,L}) = \tilde{\varepsilon}_{h,L}(\tilde{V}'_{h,L}) \in \mathcal{Z}.$$

Moreover, if n *is odd then, for every* $m \in \mathbb{N}$*, the pair of elements* $g = \bar{\alpha}^m$, $h = \bar{\alpha}^L \bar{\beta}^n g$ *(thus both* g, h *are defective) satisfies the following equality in* $\mathbb{Z}_2$*:*

$$\tilde{\varepsilon}_{g,L}(\tilde{V}'_{g,L}) + \tilde{\varepsilon}_{h,L}(\tilde{V}'_{h,L}) = \begin{cases} 1, & 0 < m \le P(L), \\ 0, & m > P(L) \end{cases} \qquad P(L) := \begin{cases} L-1, & L \ge 1, \\ -L, & L \le 0. \end{cases}$$

Remarks 6.9 (A) Condition (iii) is equivalent to the following condition:

(iv) the following conditions (iv$_e$) and (iv$_o$) hold (compare Lemmas 6.6 and 6.7):

(iv$_e$) Suppose that n is even, and put $\ell := \ell_{\max}$. Then, for every pair of elements $g, h \in \pi \setminus \{\bar{\beta}^{kn} \mid k \in \mathbb{Z}\}$ (thus both g, h are $\tilde{G}$–regular) with $h = \bar{\beta}^{2\ell r} g$, $r \in \mathbb{Z}$, one has

$$\tilde{\varepsilon}_g(\tilde{V}'_g) = \tilde{\varepsilon}_h(\tilde{V}'_h) \in \mathcal{Z}.$$

Moreover, there exists $L \in \mathbb{Z}$ such that, for every pair of elements $g, h \in \pi$ with $g \notin \{\bar{\beta}^{2\ell k} \bar{\alpha}^m \mid k, m \in \mathbb{Z}\}$, $w_-(g) = 1$, and $h = \bar{\alpha}^L \bar{\beta}^\ell g$ (thus both g, h are non-defective), one has

$$\tilde{\varepsilon}_{g,L}(\tilde{V}'_{g,L}) = \tilde{\varepsilon}_{h,L}(\tilde{V}'_{h,L}) \in \mathcal{Z}.$$

(iv$_\mathrm{o}$) Suppose that n is odd. Then there exists $L \in \mathbb{Z}$ satisfying the following conditions. For every element $g \in \pi \setminus \{\bar\beta^{2nk}\bar\alpha^m \mid k, m \in \mathbb{Z}\}$ with $w_-(g) = 1$ (thus both $g, \bar\alpha^L\bar\beta^n g$ are non-defective), one has

$$\hat\varepsilon_{g,L}(\hat V'_{g,L}) = 0 \in \mathcal{Z}.$$

Moreover, if $g = \bar\alpha^m$ with $m \in \mathbb{N}$ (thus both $g, \bar\alpha^L\bar\beta^n g$ are defective), then

$$\hat\varepsilon_{g,L}(\hat V'_{g,L}) = \begin{cases} 1, & 0 < m \le P(L), \\ 0, & m > P(L) \end{cases} \quad \text{in } \mathbb{Z}_2.$$

(B) Condition (iii$_1$) (respectively, the first part of (iv$_\mathrm{e}$)) is equivalent to the similar condition where g, h run through the sets $g \in \{\bar\beta^{2k}\bar\alpha^m \mid -\ell < 2k < \ell,\ m > 0\} \cup \{\bar\beta^{2k} \mid 0 < 2k < \ell\} \cup \{\bar\beta^{2k+1}\bar\alpha^m \mid 0 < 2k+1 \le \ell,\ m \in \mathbb{Z}\}$ (thus g is automatically $\tilde G$–regular), and $h = \bar\beta^{2\ell r} g$ is $\tilde G$–regular with $1 \le r < |n|/\ell$.

(C) The first part of the condition (iii$_2$) (respectively, the second part of (iv$_\mathrm{e}$) or the first part of (iv$_\mathrm{o}$)) is equivalent to the similar condition where g runs through the set $\{\bar\beta^{2k}\bar\alpha^m \mid 0 < 2k < \ell_{\max},\ m \ge 0\}$.

Proof (A) Similar to the proof of Theorem 6.4(A).

(B) (i) $\Longrightarrow$ (ii) Suppose that (L, ℓ, Z') is a solution. Then the left-hand side equals

$$\frac{1-\bar\beta^{2n}}{1-\bar\beta^{\ell}} \cdot \varphi^L(Z') = \frac{1-\bar\beta^{2n}}{1-\bar\beta^{\ell_{\max}}} \cdot \frac{1-\bar\beta^{\ell_{\max}}}{1-\bar\beta^{\ell}} \cdot \varphi^L(Z').$$

Since the right-hand sides of ($\bar 2_2$) and ($\bar 4^{\mathrm{f}}_2$) do not depend on ℓ, the triple

$$\left(L, \ell_{\max}, \varphi^{-L}\left(\tfrac{1-\bar\beta^{\ell_{\max}}}{1-\bar\beta^{\ell}}\right) \cdot Z'\right)$$

is a solution.

(ii) $\Longrightarrow$ (iii) Consider the case of the equation ($\bar 2_2$). Suppose $(L, \ell_{\max}, Z)$ is a solution, and denote $\ell := \ell_{\max}$. Observe that, under the assumption $\ell = \ell_{\max}$, the equation ($\bar 2_2$) is equivalent to the following congruence in $\mathbb{Z}[\pi_-]$ modulo K:

$$V \equiv \frac{1-\bar\beta^{2n}}{1-\bar\alpha^L\bar\beta^{\ell}} \cdot (Z + C_1) + \begin{cases} 0, & n \text{ even}, \\ -\frac{1-\bar\alpha^L}{1-\bar\alpha}, & n \text{ odd}, \end{cases} \tag{63}$$

where $C_1 := 0$ if n is even, $C_1 := \frac{1-\bar\alpha^L}{1-\bar\alpha}$ if n is odd and > 0, $C_1 := -\bar\alpha^L\bar\beta^{-n}\frac{1-\bar\alpha^L}{1-\bar\alpha}$ if n is odd and < 0. The first summand of the right-hand side of this congruence is a linear combination of the elements $\frac{1-\bar\beta^{2n}}{1-\bar\alpha^L\bar\beta^{\ell}} f \in \mathbb{Z}[\pi]$, $f \in \pi$, with integer coefficients, and thus a linear combination of the elements

$$U = (1 + \bar\beta^{2\ell} + \bar\beta^{4\ell} + \ldots + \bar\beta^{2|n|-2\ell})(1 + \bar\alpha^L\bar\beta^{\ell})f, \quad f \in \pi.$$

In particular, it is a linear combination of the elements $\frac{1-\bar\beta^{2n}}{1-\bar\beta^{2\ell}}f$, $f\in\pi$, and thus a linear combination of the elements

$$W=(1+\bar\beta^{2\ell}+\bar\beta^{4\ell}+\ldots+\bar\beta^{2|n|-2\ell})f,\quad f\in\pi.$$

In order to prove (iii$_1$), consider the polynomial $W\in\mathbb{Z}[\pi]$ from above and observe that, for any $\tilde G$–regular element $h_r:=\bar\beta^{2\ell r}f$, $r\in\mathbb{Z}$, the χ–twisted $\tilde G$–augmentation of $\tilde W_{h_r}\in\mathbb{Z}[\tilde{\mathcal{O}}_{h_r}]$ (based at h_r) equals

$$\tilde\varepsilon_{h_r}(\tilde W_{h_r})=-\tilde\varepsilon_{h_r^{-1}}(\tilde W_{h_r^{-1}})=\begin{cases}1, & q \text{ even and } p\neq 0, \text{ or } \ell\nmid q,\\ 0, & q \text{ odd or } p=0, \text{ and } \ell\mid q,\end{cases}\tag{64}$$

where $f=\bar\beta^q\bar\alpha^p$, the "canonical" form of $f\in\pi$, similar to (19). Obviously, for any element $h\in\pi\setminus\{h_r,h_r^{-1}\mid r\in\mathbb{Z}\}$, the χ–twisted $\tilde G$–augmentation of $\tilde W_h$ (based at h) vanishes. Observe also that the right-hand side of (64) does not depend on r. This shows that the element $W\in\mathbb{Z}[\pi]$ satisfies the condition (iii$_1$). For n even, this implies that V also satisfies (iii$_1$), since V is a linear combination of such elements W, together with the elements $f+f^{-1}$, $f\in\pi\setminus\{1\}$, and $1\in\pi$.

In order to prove (iii$_2$), let us consider the integer L and the polynomial $U\in\mathbb{Z}[\pi]$ from above. Recall that U has the form $U=(1+\bar\beta^{2\ell}+\bar\beta^{4\ell}+\ldots+\bar\beta^{2|n|-2\ell})(1+\bar\alpha^L\bar\beta^\ell)\cdot f$, for some $f\in\pi$, $f=\bar\beta^q\bar\alpha^p$, $p,q\in\mathbb{Z}$. Take any $g\in\pi\setminus\{\bar\beta^{2\ell r}\bar\alpha^m\mid r,m\in\mathbb{Z}\}$ with $w_-(g)=1$; thus the elements g and $h:=\hat\imath_L\cdot g=\bar\alpha^L\bar\beta^\ell g$ are automatically non-defective, see Lemma 6.7. If g belongs to the set $S_{f,L}:=\{(\bar\alpha^L\bar\beta^\ell)^r f\mid r\in\mathbb{Z}\}$ then $\ell\nmid q$, thus the χ_L–twisted $\tilde G_L$–augmentation of $\tilde U_{g,L}\in\mathbb{Z}[\tilde{\mathcal{O}}_{g,L}]$ (based at g) equals

$$\begin{aligned}\tilde\varepsilon_{g,L}(\tilde U_{g,L})&=\tilde\varepsilon_{j_L\cdot g^{-1},L}(\tilde U_{j_L\cdot g^{-1},L})\\ &=-\tilde\varepsilon_{g^{-1},L}(\tilde U_{g^{-1},L})=-\tilde\varepsilon_{j_L\cdot g,L}(\tilde U_{j_L\cdot g,L})=1\in\mathbb{Z}.\end{aligned}$$

If $g,g^{-1},j_L\cdot g,j_L\cdot g^{-1}\notin S_{f,L}$ then the χ_L–twisted $\tilde G_L$–augmentation of $\tilde U_{g,L}$ (based at g) vanishes. Observe that $g\in S_{f,L}$ if and only if $h=\hat\imath_L\cdot g\in S_{f,L}$, for any $g\in\pi$ (without assumption $w_-(g)=1$). Hence $g^{-1}\in S_{f,L}$ if and only if $j_L\cdot h\in S_{f,L}$; $j_L\cdot g\in S_{f,L}$ if and only if $h^{-1}\in S_{f,L}$; $j_L\cdot g^{-1}\in S_{f,L}$ if and only if $j_L\cdot h^{-1}\in S_{f,L}$. Together with the above properties of the χ_L–twisted $\tilde G_L$–augmentation, this proves the desired equality $\tilde\varepsilon_{g,L}(\tilde U_{g,L})=\tilde\varepsilon_{h,L}(\tilde U_{h,L})\in\mathbb{Z}$, thereby proving the first part of (iii$_2$) for the element $U\in\mathbb{Z}[\pi]$. Therefore V also satisfies the first part of (iii$_2$), since V is a linear combination of such elements U, together with the elements $f+f^{-1}$, $\bar\alpha^r$, and $1\in\pi$, where $f\in\pi\setminus\{1\}$, $r\in\mathbb{Z}$.

Suppose that n is odd, and take any element $g=\bar\alpha^m$ with $m\in\mathbb{N}$. Denote, similarly to above, $h:=\bar\alpha^L\bar\beta^n g$ (thus both g,h are defective). It is obvious that $\tilde\varepsilon_{g,L}(\tilde U_{g,L})=0\in\mathbb{Z}_2$

if and only if $g, g^{-1} \notin S_{f,L}$, moreover $\tilde{\varepsilon}_{h,L}(\tilde{U}_{g,L}) = 0 \in \mathbb{Z}_2$ if and only if $h, j_L \cdot h \notin S_{f,L}$. Since $g, g^{-1} \notin S_{f,L}$ is equivalent to $h, j_L \cdot h \notin S_{f,L}$, we obtain $\tilde{\varepsilon}_{g,L}(\tilde{U}_{g,L}) + \tilde{\varepsilon}_{h,L}(\tilde{U}_{h,L}) = 0$. Therefore $\tilde{\varepsilon}_{g,L}(\tilde{V}_{g,L}) + \tilde{\varepsilon}_{h,L}(\tilde{V}_{h,L}) = \tilde{\varepsilon}_{g,L}(\tilde{D}_{g,L}) + \tilde{\varepsilon}_{h,L}(\tilde{D}_{h,L})$ where $D := -\frac{1-\bar{\alpha}^L}{1-\bar{\alpha}} \in \mathbb{Z}[\pi]$, since $V - D$ is a linear combination of such elements U, together with the elements $f + f^{-1}, f \in \pi \setminus \{1\}$, and $1 \in \pi$. One easily computes

$$\tilde{\varepsilon}_{h,L}(\tilde{D}_{h,L}) = 0, \quad \tilde{\varepsilon}_{g,L}(\tilde{D}_{g,L}) = \begin{cases} 1, & 0 < m \le P(L), \\ 0, & m > P(L) \end{cases} \qquad \text{in } \mathbb{Z}_2.$$

This completes the proof of (iii$_2$).

Consider the case of the equation ($\bar{4}_2^{\mathrm{f}}$). Suppose $(L, \ell_{\max}, Z')$ is a solution, and denote $\ell := \ell_{\max}$. It follows from ($\bar{4}_2^{\mathrm{f}}$) that the congruence (63) in $\mathbb{Z}_2[\pi_-]$ holds modulo K', where the coefficients are reduced modulo 2. It follows from the case of ($\bar{2}_2$) that V' satisfies the mod 2 analogue of the condition (iii).

(iii) $\Longrightarrow$ (i) Let us consider the case of the equation ($\bar{2}_2$). Suppose n is odd, put $\ell := \ell_{\max} = |n|$.

Step 1 For every $L \in \mathbb{Z}$ and for every polynomial $V \in \mathbb{Z}[\pi]$, there exists a (unique) presentation satisfying the following congruence modulo K:

$$\mathcal{V} := V + \frac{1 - \bar{\alpha}^L}{1 - \bar{\alpha}} \equiv U + W_1 + W_2 + W_3 + R, \tag{65}$$

where U is a linear combination of $(1 - \bar{\beta}^{2n})h$, $h \in \pi$, while $W_1, W_2, W_3, R \in \mathbb{Z}[\pi]$ have the form

$$\begin{aligned}
W_1 &= \sum_{m>0,\ 0<2k<\ell} (a^+_{g_{m,k}} g_{m,k} + a^-_{g_{m,k}} ij_L \cdot g_{m,k} + b^+_{g_{m,k}} h_{m,k} + b^-_{g_{m,k}} ij_L \cdot h_{m,k}), \\
W_2 &= \sum_{0<2k<\ell} (a_{g_{0,k}} g_{0,k} + b_{g_{0,k}} h_{0,k}), \\
W_3 &= \sum_{m>0} (a_{g_{m,0}} g_{m,0} + b^+_{g_{m,0}} h_{m,0} + b^-_{g_{m,0}} ij_L \cdot h_{m,0}), \\
R &= b_1 h_{0,0},
\end{aligned}$$

where $g_{m,k} := \bar{\beta}^{2k} \bar{\alpha}^m$, $h_{m,k} := \bar{\alpha}^L \bar{\beta}^\ell g_{m,k}$, and $a^\pm_g, b^\pm_g, a_g, b_g \in \mathbb{Z}$ with the additional condition that $b_1, b^+_{\bar{\alpha}^m} \in \{0, 1\}$, $b^-_{\bar{\alpha}^m} \in \{a_{\bar{\alpha}^m} - b^+_{\bar{\alpha}^m},\ a_{\bar{\alpha}^m} - b^+_{\bar{\alpha}^m} + 1\}$, $m > 0$ (these coefficients correspond to $\tilde{G}$–singular elements $h_{m,0}$). Here uniqueness follows from the equalities

$$\begin{aligned}
a^+_g &= \tilde{\varepsilon}_g(\tilde{\mathcal{V}}_g), & a^-_g &= \tilde{\varepsilon}_{ij_L \cdot g}(\tilde{\mathcal{V}}_{ij_L \cdot g}), & & \\
b^+_g &= \tilde{\varepsilon}_h(\tilde{\mathcal{V}}_h), & b^-_g &= \tilde{\varepsilon}_{ij_L \cdot h}(\tilde{\mathcal{V}}_{ij_L \cdot h}), & h &:= \bar{\alpha}^L \bar{\beta}^\ell g,
\end{aligned}$$

while $a_g = \tilde{\varepsilon}_g(\tilde{\mathcal{V}}_g)$ for W_2, W_3; $b_g = \tilde{\varepsilon}_h(\tilde{\mathcal{V}}_h)$ for W_2; furthermore $b_g^+ \bmod 2 = \tilde{\varepsilon}_h(\tilde{\mathcal{V}}_h)$ and $b_g^- \bmod 2 = \tilde{\varepsilon}_{ij_L \cdot h}(\tilde{\mathcal{V}}_{ij_L \cdot h}) \in \mathbb{Z}_2$ for $\tilde{G}$–singular $h = h_{m,0}$ in W_3, R, where $\tilde{\varepsilon}_h$ is defined similarly to the case of $\tilde{G}$–regular h, by reducing $\bmod 2$.

Step 2 Observe that every summand of the sums W_1, W_2, W_3 has the form

$$\begin{aligned} & a^+ g + a^- ij_L \cdot g + b^+ h + b^- ij_L \cdot h \\ & \qquad \equiv a^+ g - a^-(j_L \cdot g + h + h^{-1}) + b^+ h - b^-(j_L \cdot h + g + g^{-1}) \\ & \qquad = (a^+ - b^-)g + (b^+ - a^-)h - a^-(h^{-1} + j_L \cdot g) - b^-(g^{-1} + j_L \cdot h), \end{aligned} \tag{66}$$

where $w_-(g) = 1$ and $h := \bar{\alpha}^L \bar{\beta}^\ell g$. Here $a^\pm := a_g^\pm$, $b^\pm := b_g^\pm$ for W_1; $a^+ := a_g$, $a^- := 0$, $b^+ := b_g$, $b^- := 0$ for W_2; $a^+ := a_g$, $a^- := 0$, $b^\pm := b_g^\pm$ for W_3.

Let us show that $a^+ - b^- = b^+ - a^-$, provided that $L \in \mathbb{Z}$ is taken as in the condition (iii). Indeed, from the above formulae for $a_g^\pm$, $b_g^\pm$, we have that, for W_1 and W_2,

$$\begin{aligned} a^+ - b^- - (b^+ - a^-) &= a^+ + a^- - (b^+ + b^-) \\ &= a_g^+ + a_g^- - (b_g^+ + b_g^-) \\ &= \tilde{\varepsilon}_g(\tilde{\mathcal{V}}_g) + \tilde{\varepsilon}_{ij_L \cdot g}(\tilde{\mathcal{V}}_{ij_L \cdot g}) - (\tilde{\varepsilon}_h(\tilde{\mathcal{V}}_h) + \tilde{\varepsilon}_{ij_L \cdot h}(\tilde{\mathcal{V}}_{ij_L \cdot h})) \\ &= \tilde{\varepsilon}_{g,L}(\tilde{\mathcal{V}}_{g,L}) - \tilde{\varepsilon}_{h,L}(\tilde{\mathcal{V}}_{h,L}) \\ &= \hat{\varepsilon}_{g,L}(\hat{\mathcal{V}}_{g,L}), \end{aligned}$$

see (62). Now, if $L \in \mathbb{Z}$ is taken as in the condition (iii), then the latter expression vanishes, due to (iv_o) or (iii_2). Similarly, for $g = \bar{\alpha}^m$, $m > 0$, as in W_3, we obtain $(b_g^+ + b_g^- - a_g) \mod 2 = \hat{\varepsilon}_{g,L}(\hat{\mathcal{V}}_{g,L}) = 0 \in \mathbb{Z}_2$, due to (62) and the second part of (iii_2). Since $b_g^+ + b_g^- - a_g \in \{0, 1\}$, see above, we have $a_g = b_g^+ + b_g^-$.

Since $a^+ - b^- = b^+ - a^-$, the expression (66) equals

$$(a^+ - b^-)(1 + \bar{\alpha}^L \bar{\beta}^\ell) g - a^-(1 + \bar{\alpha}^L \bar{\beta}^\ell) h^{-1} - b^-(1 + \bar{\alpha}^L \bar{\beta}^\ell) j_L \cdot h = (1 + \bar{\alpha}^L \bar{\beta}^\ell) Z_g,$$

where $Z_g := (a^+ - b^-)g - a^- h^{-1} - b^- j_L \cdot h$.

Step 3 For the remainder term R, observe that $h_{0,0} = \bar{\alpha}^L \bar{\beta}^\ell \equiv 1 + \bar{\alpha}^L \bar{\beta}^\ell$. This shows that every summand in the right-hand side of (65) is divisible (modulo K) by $1 + \bar{\alpha}^L \bar{\beta}^\ell$. Hence it is also divisible by $\frac{1 - \bar{\beta}^{2n}}{1 - \bar{\alpha}^L \bar{\beta}^\ell}$, since $\ell = |n|$. This means that V has the form (63) and therefore ($\bar{2}_2$) admits a solution.

Suppose that n is even and that $V \in \mathbb{Z}[\pi]$ satisfies (iii_1), or the first part of (iv_e). Put $\ell := \ell_{\max}$.

Step 1 For every $L \in \mathbb{Z}$ and for every polynomial $V \in \mathbb{Z}[\pi]$ satisfying (iii_1), there exists a (unique) presentation satisfying the following congruence modulo K:

$$V \equiv U + W_1 + W_2 + W_3 + R, \tag{67}$$

where U is a linear combination of $(1-\bar\beta^{2n})h$, $h \in \pi$, while $W_1, W_2, W_3, R \in \mathbb{Z}[\pi]$ have the form

$$W_1 = \sum_{\substack{m>0,\\ 0<2k<\ell}} \frac{1-\bar\beta^{2n}}{1-\bar\beta^{2\ell}}(a^+_{g_{m,k}} g_{m,k} + a^-_{g_{m,k}} ij_L \cdot g_{m,k} + b^+_{g_{m,k}} h_{m,k} + b^-_{g_{m,k}} ij_L \cdot h_{m,k}),$$

$$W_2 = \sum_{0<2k<\ell} \frac{1-\bar\beta^{2n}}{1-\bar\beta^{2\ell}}(a_{g_{0,k}} g_{0,k} + b_{g_{0,k}} h_{0,k}),$$

$$W_3 = \sum_{m>0} \frac{1-\bar\beta^{2n}}{1-\bar\beta^{2\ell}} a_{g_{m,0}} g_{m,0},$$

$$R = a_{g_{0,n/2}} g_{0,n/2} = a_{\bar\beta^n} \bar\beta^n,$$

where $g_{m,k} := \bar\beta^{2k}\bar\alpha^m$, $h_{m,k} := \bar\alpha^L \bar\beta^\ell g_{m,k}$, $a^\pm_g, b^\pm_g, a_g, b_g \in \mathbb{Z}$ with the additional condition that $a_{\bar\beta^n} \in \{0,1\}$ (this coefficient corresponds to the $\tilde G$–singular elements $\bar\beta^{(2q+1)n}$, $q \in \mathbb{Z}$). Here uniqueness follows from the equalities

$$a^+_g = \tilde\varepsilon_g(\tilde V_g), \qquad a^-_g = \tilde\varepsilon_{ij_L \cdot g}(\tilde V_{ij_L \cdot g}),$$
$$b^+_g = \tilde\varepsilon_h(\tilde V_h), \qquad b^-_g = \tilde\varepsilon_{ij_L \cdot h}(\tilde V_{ij_L \cdot h}), \qquad h := \bar\alpha^L \bar\beta^\ell g,$$

moreover $a_g = \tilde\varepsilon_g(\tilde V_g)$ (as an equality modulo 2 if $g = \bar\beta^n$), $b_g = \tilde\varepsilon_h(\tilde V_h)$ (observe that, for any g, h as in W_1, W_2, W_3, and for any $q \in \mathbb{Z}$, the elements $\bar\beta^{2\ell q} g, \bar\beta^{2\ell q} h$ are $\tilde G$–regular). Here $g_{0,n/2} = \bar\beta^n$ appears in the remainder term R (corresponding to the case $k = m = 0$), since the condition (iii_1), or the first part of $(\mathrm{iv_e})$, poses no restriction to the coefficients of V at $\bar\beta^{(2q+1)n}$, $q \in \mathbb{Z}$. Actually, V admits similar presentations, with the additional terms

$$\frac{1-\bar\beta^{2n}}{1-\bar\beta^{2\ell}}(b^+_{g_{m,0}} h_{m,0} + b^-_{g_{m,0}} ij_L \cdot h_{m,0}) \equiv 0 \qquad \text{in } W_3,\ m > 0,$$

$$\text{and} \qquad \frac{1-\bar\beta^{2n}}{1-\bar\beta^{2\ell}}(a_1 + b_1 h_{0,0}) \equiv a_1 \bar\beta^n \qquad \text{in } R,$$

where the coefficients $b^\pm_{g_{m,0}}$, a_1, b_1 are arbitrary integers. In the presentation (67), these terms are omitted, in order to have uniqueness.

Step 2 Fix arbitrary $g \in \pi$, $q \in \mathbb{Z}$, denote $\tilde g := \bar\beta^{2\ell q} g$, $\tilde{\tilde g} := \bar\beta^{-2\ell q} g$, $h := \bar\alpha^L \bar\beta^\ell g$.

One easily observes

$$\widetilde{ij_L \cdot g}{=}ij_L \cdot \tilde{g}, \qquad \tilde{h}{=}\widetilde{\bar{\alpha}^L\bar{\beta}^\ell g}{=}\bar{\alpha}^L\bar{\beta}^\ell\tilde{g}, \qquad \widetilde{ij_L \cdot h}{=}ij_L \cdot \tilde{h}{=}ij_L \cdot \bar{\alpha}^L\bar{\beta}^\ell\tilde{g},$$
$$\widetilde{g^{-1}}{=}(\tilde{\tilde{g}})^{-1}, \quad \widetilde{h^{-1}}{=}(\tilde{\tilde{h}})^{-1}, \qquad \widetilde{j_L \cdot g}{=}j_L \cdot \tilde{\tilde{g}}, \qquad \widetilde{j_L \cdot h}{=}j_L \cdot \tilde{\tilde{h}}.$$

Hence, by applying to each summand of the sums W_1, W_2 the arguments of Step 2 of the case of n odd, it follows that the condition (iii$_2$), or the second part of (iv$_\mathrm{e}$), implies

$$\frac{1-\bar{\beta}^{2n}}{1-\bar{\beta}^{2\ell}}(a_g^+ g + a_g^- ij_L \cdot g + b_g^+ h + b_g^- ij_L \cdot h) \equiv \frac{1-\bar{\beta}^{2n}}{1-\bar{\beta}^{2\ell}}(1+\bar{\alpha}^L\bar{\beta}^\ell)E_g = \frac{1-\bar{\beta}^{2n}}{1-\bar{\alpha}^L\bar{\beta}^\ell}E_g,$$

for some $E_g \in \mathbb{Z}[\pi]$, where $g \in \pi$ as in W_1, W_2. Therefore $W_1 + W_2 = \frac{1-\bar{\beta}^{2n}}{1-\bar{\alpha}^L\bar{\beta}^\ell}E$, for $E := \sum_g E_g$.

Step 3 For the term W_3, we observe that

$$\frac{1-\bar{\beta}^{2n}}{1-\bar{\beta}^{2\ell}}\bar{\alpha}^m \equiv \frac{1-\bar{\beta}^{2n}}{1-\bar{\alpha}^L\bar{\beta}^\ell}F,$$

for some $F \in \mathbb{Z}[\pi]$, due to Corollary 6.2(c). For the remainder term R, we observe that

$$\bar{\beta}^n \equiv \frac{1-\bar{\beta}^{2n}}{1-\bar{\alpha}^L\bar{\beta}^\ell}\bar{\alpha}^L\bar{\beta}^{\ell-n},$$

due to Corollary 6.2(a). This shows that every summand of the right-hand side of (67) is divisible (modulo K) by $\frac{1-\bar{\beta}^{2n}}{1-\bar{\alpha}^L\bar{\beta}^\ell}$. This means that V has the form (63), therefore ($\bar{2}_2$) admits a solution.

For the equation ($\bar{4}_2^\mathrm{f}$), the implication (iii) $\Longrightarrow$ (i) immediately follows from the case of the equation ($\bar{2}_2$). □

Define the notions of a defective $\tilde{G}$–orbit and a defective $\hat{G}_L$–orbit, similarly to the definition of a defective $\tilde{G}_L$–orbit, see above (60). Below we consider a set S as a subset of the abelian group $\mathbb{Z}[S]$.

Proposition 6.10 *Suppose that $\bar{v} = \bar{u}^{2\mu} = \bar{\beta}^{2n}$ in the group*

$$\pi = \pi_- = \langle \alpha, \beta \mid \alpha\beta\alpha\beta^{-1} \rangle,$$

where $n \in \mathbb{Z} \setminus \{0\}$, $\bar{u} = \bar{\beta}^\ell$, $\mu = \frac{n}{|n|}2^s$, $s \geq 0$, $\ell = \ell_{\max} > 0$ odd, $n = \mu\ell$. Consider the corresponding actions (55), (56) of the groups $\tilde{G} \subset \tilde{G}_L \subset \hat{G}_L$ on π. Then the orbits $\tilde{\mathcal{O}}_h$, $\tilde{\mathcal{O}}_{h,L}$, $\hat{\mathcal{O}}_{h,L}$, $h \in \pi$, under these actions have the following form:

(A) *For* $h = \bar\beta^{2q}\bar\alpha^{p}$, $p, q \in \mathbb{Z}$, *one has*

$$\begin{aligned}
\tilde{\mathcal{O}}_h &= \{\bar v^k h^{\pm 1} \mid k\in\mathbb{Z}\} = \{\bar\beta^{2q+2kn}\bar\alpha^{p} \mid k\in\mathbb{Z}\}\cup\{\bar\beta^{-2q+2kn}\bar\alpha^{\,-p} \mid k\in\mathbb{Z}\},\\
\tilde{\mathcal{O}}_{h,L} &= \{\bar v^k h^{\pm 1} \mid k\in\mathbb{Z}\}\cup\{\bar v^k\bar\alpha^L\bar\beta^\ell h^{\pm1}\bar\alpha^L\bar\beta^{-\ell} \mid k\in\mathbb{Z}\}\\
&= \{\bar\beta^{2q+2kn}\bar\alpha^{p} \mid k\in\mathbb{Z}\}\cup\{\bar\beta^{-2q+2kn}\bar\alpha^{\,-p} \mid k\in\mathbb{Z}\}\\
&\qquad \cup\{\bar\beta^{2q+2kn}\bar\alpha^{\,-p} \mid k\in\mathbb{Z}\}\cup\{\bar\beta^{-2q+2kn}\bar\alpha^{p} \mid k\in\mathbb{Z}\},\\
\hat{\mathcal{O}}_{h,L} &= \{\bar u^{2k} h^{\pm 1} \mid k\in\mathbb{Z}\}\sqcup\{\bar u^{2k}(\bar\alpha^L\bar\beta^\ell h)^{\pm1} \mid k\in\mathbb{Z}\}\\
&= (\{\bar\beta^{2q+2k\ell}\bar\alpha^{p} \mid k\in\mathbb{Z}\}\cup\{\bar\beta^{-2q+2k\ell}\bar\alpha^{\,-p} \mid k\in\mathbb{Z}\})\\
&\qquad \sqcup(\{\bar\beta^{-2q-\ell+2k\ell}\bar\alpha^{p-L} \mid k\in\mathbb{Z}\}\cup\{\bar\beta^{2q+\ell+2k\ell}\bar\alpha^{p-L} \mid k\in\mathbb{Z}\}).
\end{aligned}$$

(B) *For* $h = \bar\beta^{2q+\ell}\bar\alpha^{p}$, $p, q \in \mathbb{Z}$, *one has*

$$\begin{aligned}
\tilde{\mathcal{O}}_h &= \{\bar v^k h^{\pm 1} \mid k\in\mathbb{Z}\} = \{\bar\beta^{2q+\ell+2kn}\bar\alpha^{p} \mid k\in\mathbb{Z}\}\cup\{\bar\beta^{-2q-\ell+2kn}\bar\alpha^{p} \mid k\in\mathbb{Z}\},\\
\tilde{\mathcal{O}}_{h,L} &= \{\bar v^k h^{\pm 1} \mid k\in\mathbb{Z}\}\cup\{\bar v^k\bar\alpha^L\bar\beta^\ell h^{\pm1}\bar\alpha^L\bar\beta^{-\ell} \mid k\in\mathbb{Z}\}\\
&= \{\bar\beta^{2q+\ell+2kn}\bar\alpha^{p} \mid k\in\mathbb{Z}\}\cup\{\bar\beta^{-2q-\ell+2kn}\bar\alpha^{p} \mid k\in\mathbb{Z}\}\\
&\qquad \cup\{\bar\beta^{2q+\ell+2kn}\bar\alpha^{\,-p-2L} \mid k\in\mathbb{Z}\}\cup\{\bar\beta^{-2q-\ell+2kn}\bar\alpha^{\,-p-2L} \mid k\in\mathbb{Z}\},\\
\hat{\mathcal{O}}_{h,L} &= \{\bar u^{2k} h^{\pm 1} \mid k\in\mathbb{Z}\}\sqcup\{\bar u^{2k}(\bar\alpha^L\bar\beta^{-\ell} h)^{\pm1} \mid k\in\mathbb{Z}\}\\
&= (\{\bar\beta^{2q+\ell+2k\ell}\bar\alpha^{p} \mid k\in\mathbb{Z}\}\cup\{\bar\beta^{-2q-\ell+2k\ell}\bar\alpha^{p} \mid k\in\mathbb{Z}\})\\
&\qquad \sqcup(\{\bar\beta^{2q+2k\ell}\bar\alpha^{p+L} \mid k\in\mathbb{Z}\}\cup\{\bar\beta^{-2q+2k\ell}\bar\alpha^{\,-p-L} \mid k\in\mathbb{Z}\}).
\end{aligned}$$

(C) *For any* $h \in F_2$, *let us enumerate consecutively the subsets appearing in the above decompositions of the orbits* $\tilde{\mathcal{O}}_h$, $\tilde{\mathcal{O}}_{h,L}$ *and* $\hat{\mathcal{O}}_{h,L}$, *thus the decompositions have the forms*

$$\begin{aligned}
\tilde{\mathcal{O}}_h &= S_{h,1}\cup S_{h,2},\\
\tilde{\mathcal{O}}_{h,L} &= S_{h,3}\cup S_{h,4}\cup S_{h,5}\cup S_{h,6},\\
\hat{\mathcal{O}}_{h,L} &= (S_{h,7}\cup S_{h,8})\sqcup(S_{h,9}\cup S_{h,10}).
\end{aligned}$$

Then, for any non-defective orbit, the restriction to this orbit of the corresponding twisted augmentation (based at h*) sends* $S_{h,2k+1}\to 1$, $S_{h,2k}\to -1$, *see* (59), (60) *and* (61).

Proof The above presentations of the orbits $\tilde{\mathcal{O}}_h$ follow from Proposition 6.5 with $\varepsilon=-1$, $\bar u = \bar\beta^{2n}$. In other cases, the proof is similar to the proof of Proposition 6.5, using (54)–(57). □

Example 6.11 Let us investigate existence of non-faithful solutions of the equation (2′) with $v = B_\alpha B_{\alpha^{-1}}$, $B = \alpha\beta\alpha\beta^{-1}$, $\vartheta = -1$, see Section 3.3. Observe that this is a "mixed" case (see Table 3) with $\bar{v} = 1 \in \pi$, $V = \alpha + \alpha^{-1} \in \mathbb{Z}[\pi]$, thus $p_Q(V) = 0 \in Q$. This means that the first and the second derived equations (2_1) and (2_2) admit solutions, see Section 5.3 and Lemma 5.2. However, we will use the method of Wicks [46] to show that the equation ($2'$) does not admit non-faithful solutions. In more detail, consider the cyclically reduced word W obtained from the right-hand side $vB^{-1}v^{-1}B = [v, B^{-1}]$ of the equation ($2'$). For each word W_i obtained from W by cyclic permutation, see below, we will find all presentations of this word in the form $abca^{-1}b^{-1}c^{-1}$ due to Wicks, see [46]. We will observe that the corresponding "canonical" solutions $(x, y) = (ab, cb)$ are faithful. This allows one to conclude that the equation ($2'$) with $v = B_\alpha B_{\alpha^{-1}}$, $B = \alpha\beta\alpha\beta^{-1}$, $\vartheta = -1$ has only faithful solutions.

Here $W_i = V_iU_i$ where U_i, V_i are the subwords of W which are defined by the properties $W = U_iV_i$, $|U_i| = i$, and $|W| = |U_i| + |V_i|$, where $|\cdot|$ means the length of the word. The cyclically reduced word W has the form

$$W = xxyxy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1}x^{-1}x^{-1}yx^{-1}y^{-1}xyx^{-1}y^{-1}x^{-1}yxy^{-1},$$

and it has length 26. By a straightforward calculation, the words W_i, W_{i+13} with $i = 0, 1, 6, 7, 8, 9, 10$, and only such, have the Wicks form $abca^{-1}b^{-1}c^{-1}$, or $ded^{-1}e^{-1}$, with non-empty subwords a, b, c, d, e:

$$\begin{aligned}
W = W_0 \quad & \text{with } (a,b,c) = (\alpha,\ \alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta\alpha\beta^{-1}\alpha,\ \beta\alpha^{-1}\beta^{-1});\\
W_1 \quad & \text{with } (a,b,c) = (\alpha\beta\alpha\beta^{-1}\alpha^{-1}\beta\alpha\beta^{-1}\alpha,\ \beta\alpha^{-1}\beta^{-1},\ \alpha^{-1});\\
W_6 \quad & \text{with } (d,e) = (\beta\alpha\beta^{-1},\ \alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}\alpha^{-1}\beta\alpha^{-1}\beta^{-1}\alpha);\\
W_7 \quad & \text{with } (d,e) = (\alpha,\ \beta^{-1}\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}\alpha^{-1}\beta\alpha^{-1}\beta^{-1}\alpha\beta);\\
W_8 \quad & \text{with } (d,e) = (\beta^{-1}\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}\alpha^{-1}\beta\alpha^{-1}\beta^{-1}\alpha\beta,\ \alpha^{-1});\\
W_9 \quad & \text{with } (d,e) = (\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}\alpha^{-1}\beta\alpha^{-1}\beta^{-1}\alpha,\ \beta\alpha^{-1}\beta^{-1});\\
W_{10} \quad & \text{with } (a,b,c) = (\beta\alpha^{-1}\beta^{-1},\ \alpha^{-1},\ \alpha^{-1}\beta\alpha^{-1}\beta^{-1}\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}).
\end{aligned}$$

Here the Wicks form for the word W_{i+13} is obtained from the Wicks form of W_i in the obvious way.

Example 6.12 Similarly to Example 6.11, one can investigate existence of solutions of the equations ($3'$) and ($4'$) with $v = B_\alpha B_{\alpha^{-1}}$, $\vartheta = -1$, where $B = \alpha\beta\alpha^{-1}\beta^{-1}$ for ($3'$), while $B = \alpha\beta\alpha\beta^{-1}$ for ($4'$), see Section 3.3. In this case, one considers the "non-orientable" forms $abcbac^{-1}$ and $a^2bc^2b^{-1}$, due to Wicks [45].

7 Tables for the "mixed" cases of Tables 2 and 3

In this section, we summarize the main results of Sections 5 and 6 in two tables below. Table 4 deals with faithful solutions in the so called "mixed" case (4c) of Table 2, while Table 5 deals with non-faithful solutions in the "mixed" cases (2d), (3c), (4e) of Table 3, see Remark 3.16. Observe that $\vartheta = -1$, $w_\varepsilon(v) = 1$ in all "mixed" cases.

Specifically, we denote $B = \alpha\beta\alpha^{-\varepsilon}\beta^{-1} \in F_2 = \langle \alpha, \beta \mid \rangle$ and study the equation

$$xyx^{-\delta}y^{-1} = vB^{-1}v^{-1}B$$

in the group $N = \langle\langle B \rangle\rangle$ with two unknowns $x \in N$, $y \in F_2$. A solution of this equation is called *faithful* if $w_\varepsilon(y) = \delta$. The parameters $\varepsilon, \delta \in \{1, -1\}$ and the conjugation parameter $v \in F_2$ of the equation are not arbitrary, but run through the following families, corresponding to the "mixed" cases, see Remark 3.16 and Definition 3.15:

"Mixed" case for faithful solutions (case (4c) of Table 2):

(4) $\delta = \varepsilon = -1$, $\bar v = \bar\beta^{2n}$, $n \in \mathbb{Z}$.

"Mixed" cases for non-faithful solutions (cases (2d), (3c), (4e) of Table 3):

(2) $\delta = 1$, $\varepsilon = -1$, $\bar v = \bar\beta^{2n}$, $n \in \mathbb{Z}$;

(3) $\delta = -1$, $\varepsilon = 1$, $\bar v = \bar\alpha^{2m}\bar\beta^{2n}$, $m, n \in \mathbb{Z}$;

(4) $\delta = \varepsilon = -1$, $\bar v = \bar\alpha^{2m}\bar\beta^{4n}$, $m, n \in \mathbb{Z}$.

As above, $\bar v \in \pi$ denotes the class of $v \in F_2$ in $\pi = F_2/N$. In each of these four "mixed" cases, let us write the element v in the following canonical form:

$$v = \beta^{2n} \prod B_{v_i}^{n_i};$$
$$v = \beta^{2n} \prod B_{v_i}^{n_i};$$
$$v = c^{2d} \prod B_{v_i}^{n_i}, \qquad c = \alpha^{m/d}\beta^{n/d} \text{ if } |m|+|n|>0, \qquad v = \prod B_{v_i}^{n_i} \text{ if } m=n=0;$$
$$v = c^{2d} \prod B_{v_i}^{n_i}, \qquad c = \alpha^{m/d}\beta^{2n/d} \text{ if } |m|+|n|>0, \qquad v = \prod B_{v_i}^{n_i} \text{ if } m=n=0;$$

respectively, where $\prod B_{v_i}^{n_i} = \prod_{i=1}^{r} B_{v_i}^{n_i}$, $v_i \in F_2$, $n_i \in \mathbb{Z}$, $B_{v_i} = v_i B v_i^{-1}$, $1 \le i \le r$, $d = \gcd(m,n)$ if $|m|+|n| > 0$.

Denote $\mathcal{Z} = \mathbb{Z}$ if $\delta = 1$, $\mathcal{Z} = \mathbb{Z}_2$ if $\delta = -1$. Denote by $\bar u \in \pi = F_2/N$ the image of $u \in F_2$ under the projection $F_2 \to \pi$, by $V \in \mathbb{Z}[\pi]$ the polynomial $V = \sum n_i \bar v_i \in \mathbb{Z}[\pi]$, and by $V' \in \mathcal{Z}[\pi]$ either $V' := V$ if $\delta = 1$ or $V' := V \mod 2$ if $\delta = -1$. Consider the actions on π of the groups $\tilde G \subset \tilde G_L \subset \hat G_L$, $L \in \mathbb{Z}$, in the first two "mixed" cases,

Case	δ	ε	conditions on v		faithful solution (x,y)
(4) a	$-$	$-$	$v=\beta^{2n}\prod B_{v_i}^{n_i}$	$n=0, p_{Q'}(V')\neq 0$	$\emptyset^{\text{(iii)}}$
b				$n\in\mathbb{Z}\setminus\{0\}$, V' does not satisfy (iv) of Remarks 6.9(A)	$\emptyset^{\text{(iii)}}$
c			$v=u^2$	$w_-(u)=-1$	$([u^2B^{-1},u^{-1}],u^{-1})$, $([u,B^{-1}],B^{-1}uB)^{\text{(i)}}$
d			$v=(\alpha\beta)^{2n}$	$n\in\mathbb{Z}$	$([(\alpha\beta)^{2n},\beta],\beta)^{\text{(i)}}$
e			$v=(\alpha\beta\alpha\beta^{-1})^m$	$w_-(u)=-1$	$(1,\ u)^{\text{(i)}}$

Table 4: Mixed cases for faithful solutions of $xyxy^{-1}=vB^{-1}v^{-1}B$ where $B=\alpha\beta\alpha\beta^{-1}$, $p_K(v)=\bar\beta^{2n}$

Case	δ	ε	conditions on v		non-faithful solution (x,y)
(2) a	$+$	$-$	$v=\beta^{2n}\prod B_{v_i}^{n_i}$	$n=0, p_Q(V)\neq 0$	$\emptyset^{\text{(iii)}}$
b				$n\in\mathbb{Z}\setminus\{0\}$, V does not satisfy (iv) of 6.9(A)	$\emptyset^{\text{(iii)}}$
c			$v=u^{2k}$	$w_-(u)=-1, k\in\mathbb{Z}$	$(u^{2k}(uB)^{-2k},B^{-1}u^{-1})^{\text{(i)}}$
d			$v=B\beta^{2n}$	$n\in\mathbb{Z}$	$((\alpha\beta\alpha)^{2n}\beta^{-2n},\beta^{2n}(\alpha\beta\alpha)^{1-2n})^{\text{(i)}}$
e			$v=\beta^2B_\alpha$		$(B_{\beta^2\alpha}B_{\beta^2}^{-1}B_{\beta^2\alpha}^{-1}B_{\beta^2\alpha^2\beta^{-1}}^{-1}$, $B^{-2}B_\alpha^{-1}\alpha^2\beta^{-1})^{\text{(i)(ii)}}$
f			$v=\beta^2B_{\alpha^k}$	$k\in\mathbb{Z}, k\neq 0,1$	$\emptyset^{\text{(ii)(iii)}}$
g			$v=B_\alpha B_{\alpha^{-1}}$		$\emptyset^{\text{(ii)}}$
(3) a	$-$	$+$	$v=\prod B_{v_i}^{n_i}$	$p_{Q'}(V')\neq 0$	$\emptyset^{\text{(iii)}}$
b			$v=c^{2d}\prod B_{v_i}^{n_i}$	$\exists g\in\pi\setminus\hat{\mathcal{O}}_1$, $\hat\varepsilon_g(\hat V'_g)\neq 0$	$\emptyset^{\text{(iii)}}$
c			$v=u^2$		$([u^2B^{-1},u^{-1}],u^{-1})$, $([u,B^{-1}],B^{-1}uB)^{\text{(i)}}$
(4) a	$-$	$-$	$v=\prod B_{v_i}^{n_i}$	$p_{Q'}(V')\neq 0$	$\emptyset^{\text{(iii)}}$
b			$v=c^{2d}\prod B_{v_i}^{n_i}$	$\exists g\in\pi\setminus\hat{\mathcal{O}}_1$, $\hat\varepsilon_g(\hat V'_g)\neq 0$	$\emptyset^{\text{(iii)}}$
c			$v=B^m$	$m\in\mathbb{Z}, w_-(u)=1$	$(1,u)^{\text{(i)}}$
d			$v=u^2$	$w_-(u)=1$	$([u^2B^{-1},u^{-1}],u^{-1})$, $([u,B^{-1}],B^{-1}uB)^{\text{(i)}}$

Table 5: Mixed cases for non-faithful solutions of $xyx^{-\delta}y^{-1}=vB^{-1}v^{-1}B$ where $B=\alpha\beta\alpha^{-\varepsilon}\beta^{-1}$ and (due to Table 3) $p_K(v)=\bar\beta^{2n}$ in Case (2), $p_T(v)=\bar\alpha^{2m}\bar\beta^{2n}$ in Case (3), $p_K(v)=\bar\alpha^{2m}\bar\beta^{4n}$ in Case (4); if $|m|+|n|>0$ in Case (3) or (4), one denotes $d:=\gcd(m,n)$ and $c:=\alpha^{m/d}\beta^{n/d}$ or $c:=\alpha^{m/d}\beta^{2n/d}$ (respectively).

(i) Direct calculation.

(ii) Using the Wicks forms (see Wicks [46], Vdovina [42, 44], Culler [8] and Example 6.11).

(iii) There is no solution of the second derived equation, see Theorems 5.10, 6.4, 6.8, and Remarks 6.9.

see (55) and (56), and the action of the group $\hat{G}$ in the remaining two "mixed" cases, see (52). Consider the corresponding orbits $\tilde{\mathcal{O}}_g$, $\tilde{\mathcal{O}}_{g,L}$, $\hat{\mathcal{O}}_{g,L}$, and $\hat{\mathcal{O}}_g$, $g \in \pi$, see Propositions 6.5 and 6.10. Consider the corresponding augmentations (or the twisted augmentations in the case of the equation (2)

$$\tilde{\varepsilon}_g\colon \mathcal{Z}[\tilde{\mathcal{O}}_g] \to \mathcal{Z}, \qquad \tilde{\varepsilon}_{g,L}\colon \mathcal{Z}[\tilde{\mathcal{O}}_{g,L}] \to \mathcal{Z},$$
$$\hat{\varepsilon}_{g,L}\colon \mathcal{Z}[\hat{\mathcal{O}}_{g,L}] \to \mathcal{Z} \text{ or } \mathbb{Z}_2, \qquad \hat{\varepsilon}_g\colon \mathbb{Z}_2[\hat{\mathcal{O}}_g] \to \mathbb{Z}_2,$$

see (58), (59), (60), (61) and (53). Here $g \in \pi$ as in (iv$_\mathrm{e}$), (iv$_\mathrm{o}$) of Remarks 6.9(A) in the first two "mixed" cases, while $g \in \pi \setminus \hat{\mathcal{O}}_1$ in the other two "mixed" cases. Consider the quotients Q and Q' as in (7) and (50), and the projections $p_Q\colon \mathbb{Z}[\pi] \to Q$ and $p_{Q'}\colon \mathbb{Z}_2[\pi] \to Q'$.

Many of the non-existence results in Tables 4 and 5 follow from the non-existence of a solution of the corresponding second derived equation, see Theorems 5.10, 6.4 and 6.8, and Remarks 6.9.

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Departamento de Matemática, IME-USP
Caixa Postal 66281, Agência Cidade de São Paulo, 05314-970 São Paulo SP, Brasil

Department of Mathematics and Mechanics, Moscow State University
Moscow 119992, Russia

Fakultät für Mathematik, Ruhr-Universität Bochum, 44780 Bochum, Germany

dlgoncal@ime.usp.br, ekudr@gmx.de, marlene.schwarz@ruhr-uni-bochum.de

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