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Stresses in Thin, Multi-Layer Pipes in Large Radial Vibrations

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Abstract – *Free, large radial oscillations of multi-layered, thin, long, pipes are investigated using the theory of finite elastic deformations. The material of each layer is assumed to be homogeneous, isotropic, hyperelastic and incompressible. Closed form solutions are obtained for the nonlinear, ordinary differential equation governing the motion of the inner surface of the cylinder pipe. The motions of the other material points can then be obtained using the incompressibility condition. It is shown that the radial stress is negligible throughout the thickness of the pipe. Tangential stress distributions at different times are given as a function of the radial distance for one, two and three layer pipes.*

***Keywords**: Stress, Thin-Walled, Multi-Layer Pipes, Large, Radial Vibration*

Nomenclature

$\bar{p}(t)$	$2p_1(t) / \rho_1 R_1^2$
$p_1(t)$	Internal pressure
t	Time
ρ_1	Density of the first layer
ϕ_i	Elastic constant of the i-th layer
R_1	Un-deformed inner radius of the first layer
r_1	Deformed inner radius of the first layer
N	Number of layers
η	r_1 / R_1
S_i	$\left(R_{i+1} / R_1\right)^2 - 1$
m_1	ρ_i / ρ_1
$(\dot{})$	$d(\ ) / dt$

I. Introduction

Several authors studied the large amplitude oscillations of single layer tubes and hollow spherical shells (Knowles [1]), (Nowinski [2]), (Beniste [3]), (Roussos [4]), and (Mason [5]). In these papers, finite amplitude radial motions of long, single layered, thin or thick walled cylindrical pipes were investigated and the exact second order, non linear differential equation governing the forced oscillations due to a uniform inter internal pressure obeying the idea gas law were obtained.

This equation, however, could be solved in closed form only partially to yield the frequency vs internal pressure relationship.

A complete solution was supplied by an approximate solution technique (Knowles [1]).

This paper was confined to the study of the variation of the frequency with initial internal pressure for single layer tubes.

In most works, the problem studied in were extended to cover the radial stress distribution along the thickness of the tube for one, two or three layer tubes caused by a suddenly applied internal gas pressure obeying the ideal gas law. The differential equation governing the free, large amplitude radial oscillations of thin-walled, layered tubes and its solution were also given by (Shahinpoor [6]). However, this paper does not contain any study of stress field in thin, multi-layer tubes.

In this study, free, large amplitude radial motions of thin-walled, long, perfectly bonded, multi-layer cylindrical pipes are investigated. The material of each layer is assumed to be homogeneous, isotropic, perfectly elastic and incompressible. The theory of finite elastic deformations (Green [7]) is governing the radial motions of the inner surface is solved in closed form to give the used in the formulation of the problem. The non linear, ordinary differential equation governing the radial motions of the inner surface is solved in closed form to give the used in the formulation of the problem to give the coordinate function of the inner surface as a function of time.

The motions of the other material points are obtained by the use of the incompressibility condition.

Calculations indicate that radial stress at any point is of negligible order at all times even for very large amplitudes. The distribution of the tangential stress as a function of the radial distance is given for one, two and three layer tubes for different times.

As a special case, free, infinitesimal radial oscillation of thin-walled, multi layered pipes of neo Hookean materials are considered and an expression for the

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frequency of oscillations is obtained. This expression reduces to that obtained by (Knowles [8]) for single layer tube made of a neo Hookean material.

II. Formulation of the Problem

Consider a long, circular cylindrical pipe of N perfectly bonded concentric layers of arbitrary thickness, each made of a neo-Hookean material.

The exact, second order, ordinary, non linear differential equation governing the finite radial motions of such a tube due to a suddenly applied internal pressure obeying the idea gas law in given by:

$$\bar{p}(t)=\frac{1}{\rho_1 R_1^2}\sum_{i=1}^{N}\phi_i\left[\begin{array}{l}\frac{S_i+1}{S_i+\eta^2}-\frac{S_{i-1}+1}{S_{i-1}+\eta^2}+\\ -Ln\frac{(S_{i-1}+1)(S_i+\eta^2)}{(S_i+1)(S_{i-1}+\eta^2)}\end{array}\right]+$$
$$+\left(\eta\ddot{\eta}+\dot{\eta}^2\right)\sum_{i=1}^{N}m_i Ln\frac{S_i+\eta^2}{S_{i-1}+\eta^2}-\eta^2\dot{\eta}^2 \qquad (1)$$
$$\sum_{i=1}^{N}m_i\frac{S_1-S_{i-1}}{\left(S_{i-1}+\eta^2\right)\left(S_i+\eta^2\right)}$$

If free, large amplitude radial vibrations of thin-walled, layered tubes are considered, Eq. (1) reduces to:

$$\ddot{\eta}+\frac{1}{\rho_1 R_1^2}\frac{\sum_{i=1}^{N}\phi_i\left(S_i-S_{i-1}\right)}{\sum_{i=1}^{N}m_i\left(S_i-S_{i-1}\right)}\left(\eta-\eta^{-3}\right)=0 \qquad (2)$$

or:

$$\ddot{\eta}+\frac{1}{\rho_1 R_1^2}\frac{\sum_{i=1}^{N}\phi_i V_i}{\sum_{i=1}^{N}m_i V_i}\left(\eta-\eta^{-3}\right)=0 \qquad (3)$$

where $V_i =$ the volume of the i^{th} layer, and the following assumptions and approximations have been used:

$$S_N\le\alpha\xrightarrow{for}\eta\ge 1$$
$$S_N\le\eta^3\alpha\xrightarrow{for}0<\eta\le 1$$
$$\alpha=\sqrt{\zeta}$$

$\zeta = a$ (small number depending on the desired accuracy)

$$1/(S_i+1)\cong 1-S_i$$

$$Ln\frac{(S_{i-1}+1)(S_i+\eta^2)}{(S_i+1)(S_{i-1}+\eta^2)}\cong\frac{1-\eta^2}{\eta^2}\left(S_i-S_{i-1}\right)$$
$$Ln\frac{\left(S_i+\eta^2\right)}{\left(S_{i-1}+\eta^2\right)}\cong\frac{S_i-S_{i-1}}{\eta^2} \qquad (4)$$
$$\frac{S_i-S_{i-1}}{\left(S_i+\eta^2\right)\left(S_{i-1}+\eta^2\right)}\cong\frac{S_i-S_{i-1}}{\eta^4}$$

For single layer pipes, Eq. (3) reduces to the one obtained by Nowinski [8].

This implies that the law of mixtures holds for this non linear problem. For small, free, radial vibrations of thin, layered pipes $\eta=1+\varepsilon$, $\varepsilon<<1$, and Eq. (3) becomes:

$$\ddot{\varepsilon}+\frac{4}{\rho_1 R_1^2}\frac{\sum_{i=1}^{N}\phi_i\left(S_i-S_{i-1}\right)}{\sum_{i=1}^{N}m_i\left(S_i-S_{i-1}\right)}\varepsilon=0 \qquad (5)$$

and the frequency of vibrations is given by:

$$\omega_0^2=\frac{4}{\rho_1 R_1^2}\frac{\sum_{i=1}^{N}\phi_i\left(S_i-S_{i-1}\right)}{\sum_{i=1}^{N}m_i\left(S_i-S_{i-1}\right)} \qquad (6)$$

which coincides with the equation obtained by (Knowles [1]) for single layer pipes made of a neo-Hookean material.

In the case of the free, large amplitude radial oscillations with the initial conditions $\eta(0)=\eta_0$ and $\dot{\eta}(0)=\dot{\eta}_0$ the solution of Eq. (3) is:

$$\eta(t)=\left\{\frac{1}{2}\left[\begin{array}{l}\frac{\eta^2{}_0}{k^2}+\dot{\eta}_0{}^2+\dot{\eta}_0{}^{-2}+\\ +A\sin 2kt+B\cos 2kt\end{array}\right]\right\}^{1/2} \qquad (7)$$

where:

$$k^2=\frac{1}{\rho_1 R_1^2}\frac{\sum_{i=1}^{N}\phi_i\left(S_i-S_{i-1}\right)}{\sum_{i=1}^{N}m_i\left(S_i-S_{i-1}\right)}$$
$$A=2\eta_0\dot{\eta}_0/k \qquad (8)$$
$$B=\eta_0{}^2-\eta_0{}^{-2}-\frac{\dot{\eta}_0{}^2}{k^2}$$

III. Stress Field

The non-zero stress components are obtained from the constitutive equation:

$$\tau^{ij} = \phi G^{ij} + \psi B^{ij} + p g^{ij}$$

as:

$$\tau^{11} = \phi \frac{1}{Q} + p,\ r^2 \tau^{22} = \phi Q + p,\ \tau^{33} = \phi + p \tag{9}$$

where $\psi = 0$ for a neo-Hookean material and 1, 2, and 3 denote, respectively, radial, tangential, and axial directions. The equation of equilibrium in the radial direction reduces to:

$$\frac{\partial \tau^{11}}{\partial Q} = \frac{1}{(1-Q)} \frac{\partial W}{\partial Q} + \frac{\rho}{2} \begin{bmatrix} \frac{1}{Q(1-Q)}\left(r_1 \ddot{r}_1 + \dot{r}_1^2\right) + \\ + \frac{1}{Q^2} \frac{r_1^2 \dot{r}_1^2}{R_1^2 (Q_i - 1)} \end{bmatrix} \tag{10}$$

where:

$$Q = r^2 / R^2,\ W = \frac{\phi}{2}(I - 3),\ I = 1 + Q + Q^{-1} \tag{11}$$

and I is the first strain invariant. It is now assumed that $\dot{\eta}_0 = 0$. Also introducing:

$$S = \left(R / R_1 \right)^2 - 1,\ \bar{\phi}_i = \phi_i / \phi_1 \tag{12}$$

The non- dimensionalized physical stress components $\sigma_{11} = \sigma_{11} / \phi_1,\ \sigma_{22} = \sigma_{22} / \phi_1$ in the radial and the tangential directions respectively, are given by:

$$\begin{aligned} \bar{\sigma}_{11}^{(i)} &= \tau^{11(i)} = \bar{\tau}^{11}(R_i, t) + \\ &+ \frac{1}{2}\left(1 - \bar{\eta}^4\right)\left(\bar{\phi}_i - m_i \bar{k}^2\right)\left(S - S_{i-1}\right) \end{aligned} \tag{13}$$

and:

$$\begin{aligned} \bar{\sigma}_{22}^{(i)} &= r^2 \tau^{22(i)} = \bar{\tau}^{11}(R_i, t) + \\ &+ \bar{\phi}_i \begin{bmatrix} \eta^2 - \bar{\eta}^2 + \\ + \left(1 - \eta^2\right)\left(1 + \bar{\eta}^4\right) S \end{bmatrix} + \\ &+ \frac{1}{2}\left(1 - \bar{\eta}^4\right)\left(\bar{\phi}_i - m_i \bar{k}^2\right)\left(S - S_{i-1}\right) \end{aligned} \tag{14}$$

where the superscript i denotes the layer and

$$S_{i-1} \le S \le S_i$$

It is noted that when a homogeneous thin pipe is considered, then $\sigma_{11} = 0$ throughout the pipe. This result coincides with the result obtained by the use of the linear theory:

$$\bar{k}^2 \le \frac{\sum_{i=1}^{N} \bar{\phi}_i \left(S_i - S_{i-1}\right)}{\sum_{i=1}^{N} m_i \left(S_i - S_{i-1}\right)}$$

IV. Illustrative Examples

Four examples are solved numerically to illustrate the distribution of tangential stress along the thickness of the pipe and expected results are obtained in all examples.

In the first example, it is assumed that the pipe is made of a single layer with $\eta_0 = 2,\ R_2 / R_1 = 1.006$.

Fig. 1 shows the variation of $\bar{\sigma}_{22}$ as a function of R / R_1 at:

i) $2kt = 0, 2\pi, 4\pi, ...$

ii) $2kt = \frac{\pi}{2}, \frac{5\pi}{2}, ...,$

iii) $2kt = \pi, 3\pi, ...$

It is seen that that $\bar{\sigma}_{22}$ is almost constant along the thickness of the pipe and that for $2kt = \pi, 3\pi, ...$ the tangential stress is compressive.

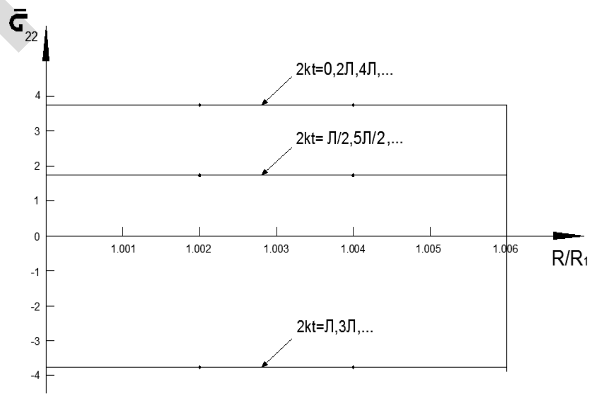

Fig. 1. Single Layer, $\eta_0 = 2,\ R_2 / R_1 = 1.006$

Although the examples given include pipes with at most three layers, the formulation allows the study of pipes with many layers. Fig. 2 shows $\bar{\sigma}_{22}$ vs R / R_1 for a two layer pipe with:

$$\phi_2 / \phi_1 = 2,\ m_1 = m_2 = 1,\ \eta_0 = 2,\ R_2 / R_1 = 1.0025$$

and $R_3 / R_1 = 1.005$.

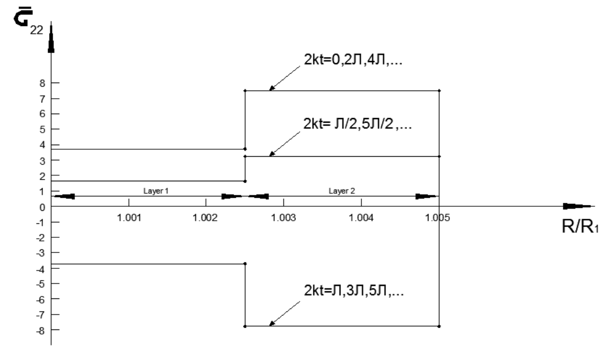

Fig. 2. Layer, $\phi_2/\phi_1 = 2, m_1 = m_2 = 1,$

$\eta_0 = R_2/R_1 = 1.0025, R_3/R_1 = 1.005$

A discontinuity of $\bar{\sigma}_{22}$ is observed at the interface. In each layer, $\bar{\sigma}_{22}$ is distributed almost uniformly.

$\bar{\sigma}_{22}$ vs R/R_1 curves are given for two-layer pipe with $\phi_2/\phi_1 = 0.2$ and all the other properties being the same as in the second example.

A similar behavior is observed except that $\bar{\sigma}_{22}$ is now lower in outer layer, (Fig. 3).

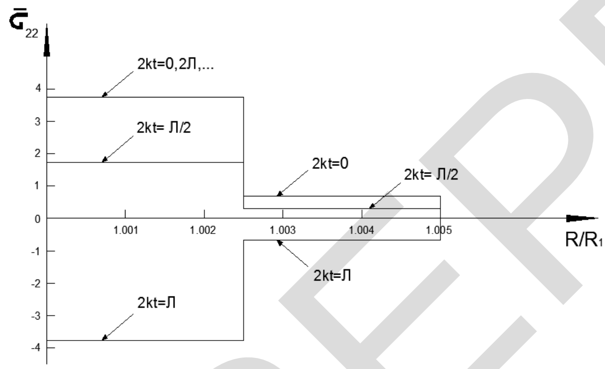

Fig. 3. Layer, $\phi_2/\phi_1 = 0.2, m_2 = m_1 = 1, \eta_0 = 2$

$R_2/R_1 = 1.0025, R_3/R_1 = 1.005,$

A sandwich pipe is studied in the last example. It is assumed that:

$$R_2/R_1 = 1.002, R_3/R_1 = 1.003, R_4/R_1 = 1.005,$$
$$\phi_2/\phi_1 = 0.1, \phi_3/\phi_1 = 1$$

$m_1 = m_2 = m_3 = 1$ and $\eta_0 = 2$.

Fig. 4 indicates that the tangential stress is very small in the mid-layer.

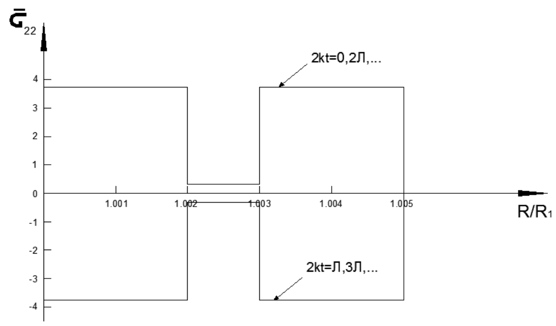

Fig. 4. Sandwich Pipe with

$R_2/R_1 = 1.002, R_3/R_1 = 1.003, R_4/R_1 = 1.005,$

$\phi_2/\phi_1 = 0.1, \phi_3/\phi_1 = 1, m_3 = m_2 = m_1 = 1, \eta_0 = 2$

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