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On a close to symmetric system of difference equations of second order

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Abstract

Closed form formulas of the solutions to the following system of difference equations:

$$x_n = \frac{y_{n-1}y_{n-2}}{x_{n-1}(a_n + b_n y_{n-1}y_{n-2})}, \qquad y_n = \frac{x_{n-1}x_{n-2}}{y_{n-1}(\alpha_n + \beta_n x_{n-1}x_{n-2})}, \qquad n \in \mathbb{N}_0,$$

where a_n, b_n, α_n, β_n, $n \in \mathbb{N}_0$, and initial values x_{-i}, y_{-i}, $i \in \{1,2\}$ are real numbers, are found. The domain of undefinable solutions to the system is described. The long-term behavior of its solutions is studied in detail for the case of constant a_n, b_n, α_n and β_n, $n \in \mathbb{N}_0$.

MSC: Primary 39A10; 39A20

Keywords: system of difference equations; closed form solution; long-term behavior; periodic solutions

1 Introduction

Studying concrete nonlinear difference equations and systems is a topic of a great recent interest (see, *e.g.*, [1–46] and the references therein). Studying systems of difference equations, especially symmetric and close to symmetric ones, is a topic of considerable interest (see, *e.g.*, [2, 6, 7, 10, 12–16, 18, 19, 23, 24, 26–29, 31–38, 40, 41, 44, 46]). Another topic of interest is solvable difference equations and systems and their applications (see, *e.g.*, [1–5, 7, 17, 20, 21, 23–27, 29–37, 39–46]). Renewed interest in the area started after the publication of [20] where a formula for a solution of a difference equation was theoretically explained. The most interesting thing in [20] was a change of variables which reduced the equation to a linear one with constant coefficients. Related ideas were later used, *e.g.*, in [1, 4, 7, 17, 21, 23–27, 29–37, 39–45].

Quite recently in [2] the following systems of difference equations were presented:

$$x_n = \frac{y_{n-1}y_{n-2}}{x_{n-1}(\pm 1 \pm y_{n-1}y_{n-2})}, \qquad y_n = \frac{x_{n-1}x_{n-2}}{y_{n-1}(\pm 1 \pm x_{n-1}x_{n-2})}, \quad n \in \mathbb{N}_0, \tag{1}$$

where x_{-i}, y_{-i}, $i \in \{1,2\}$ are real numbers, and some formulas for their solutions are given, some of which are proved by induction.

The next system of difference equations

$$x_n = \frac{y_{n-1}y_{n-2}}{x_{n-1}(a_n + b_n y_{n-1}y_{n-2})},$$
$$y_n = \frac{x_{n-1}x_{n-2}}{y_{n-1}(\alpha_n + \beta_n x_{n-1}x_{n-2})}, \quad n \in \mathbb{N}_0, \tag{2}$$

where a_n, b_n, α_n, β_n, $n \in \mathbb{N}_0$, and initial values x_{-i}, y_{-i}, $i \in \{1,2\}$, are real numbers, is a generalization of the system in (1). Our aim is to show that more general system (2) is solvable by giving a natural method for getting its solutions. The domain of undefinable solutions to the system is also described. For the case when a_n, b_n, α_n, β_n, $n \in \mathbb{N}_0$, are constant, the long-term behavior of its solutions is investigated in detail.

A solution $(x_n, y_n)_{n\ge -2}$ of system (2) is called *periodic*, or *eventually periodic*, with period p if there is $n_0 \ge -2$ such that

$$x_{n+p} = x_n \quad \text{and} \quad y_{n+p} = y_n \quad \text{for } n \ge n_0.$$

For some results in the area, see, *e.g.*, [6, 9–11, 19, 21, 22, 28].

2 Solutions to system (2) in closed form

Assume first that $x_{-i} \ne 0$, $y_{-i} \ne 0$, $i \in \{1,2\}$. Then, by the method of induction and the equations in (2), it follows that for every well-defined solution to system (2), $x_n \ne 0$ and $y_n \ne 0$, for every $n \in \mathbb{N}_0$. On the other hand, if $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, then the first equation in (2) implies that $y_{n_0-1} = 0$ or $y_{n_0-2} = 0$. If $y_{n_0-1} = 0$, then $x_{n_0-2} = 0$ or $x_{n_0-3} = 0$, while if $y_{n_0-2} = 0$, then $x_{n_0-3} = 0$ or $x_{n_0-4} = 0$. Repeating this procedure, we get that $x_{-i} = 0$ or $y_{-i} = 0$ for some $i \in \{1,2\}$. Similarly, if $y_{n_1} = 0$ for some $n_1 \in \mathbb{N}$, we get $x_{-i} = 0$ or $y_{-i} = 0$ for some $i \in \{1,2\}$. Hence, for a well-defined solution $(x_n, y_n)_{n\ge -2}$ of system (2), we have that

$$x_n y_n \ne 0, \quad n \ge -2 \tag{3}$$

if and only if $x_{-i}y_{-i} \ne 0$, $i \in \{1,2\}$.

Assume now that $(x_n, y_n)_{n\ge -2}$ is a solution to system (2) such that (3) holds. Then, by multiplying the first equation in (2) by x_{n-1} and the second one by y_{n-1}, and using the following changes of variables

$$u_n = \frac{1}{x_n x_{n-1}}, \qquad v_n = \frac{1}{y_n y_{n-1}}, \tag{4}$$

$n \ge -1$, system (2) is transformed in the following one:

$$u_n = a_n v_{n-1} + b_n, \qquad v_n = \alpha_n u_{n-1} + \beta_n, \quad n \in \mathbb{N}_0. \tag{5}$$

From (5) it follows that

$$u_n = a_n\alpha_{n-1}u_{n-2} + a_n\beta_{n-1} + b_n, \tag{6}$$

$$v_n = \alpha_n a_{n-1}v_{n-2} + \alpha_n b_{n-1} + \beta_n, \quad n \in \mathbb{N}. \tag{7}$$

This means that $(u_{2n})_{n\in\mathbb{N}_0}$, $(u_{2n-1})_{n\in\mathbb{N}_0}$, $(v_{2n})_{n\in\mathbb{N}_0}$, and $(v_{2n-1})_{n\in\mathbb{N}_0}$ are solutions to two linear first-order difference equations, which are solvable.

Solving these equations, we get

$$u_{2n} = u_0 \prod_{j=1}^{n} a_{2j}\alpha_{2j-1} + \sum_{i=1}^{n}(a_{2i}\beta_{2i-1} + b_{2i}) \prod_{s=i+1}^{n} a_{2s}\alpha_{2s-1}, \tag{8}$$

$$u_{2n-1} = u_{-1} \prod_{j=1}^{n} a_{2j-1}\alpha_{2j-2} + \sum_{i=1}^{n}(a_{2i-1}\beta_{2i-2} + b_{2i-1}) \prod_{s=i+1}^{n} a_{2s-1}\alpha_{2s-2}, \tag{9}$$

$$v_{2n} = v_0 \prod_{j=1}^{n} \alpha_{2j}a_{2j-1} + \sum_{i=1}^{n}(\alpha_{2i}b_{2i-1} + \beta_{2i}) \prod_{s=i+1}^{n} \alpha_{2s}a_{2s-1}, \tag{10}$$

$$v_{2n-1} = v_{-1} \prod_{j=1}^{n} \alpha_{2j-1}a_{2j-2} + \sum_{i=1}^{n}(\alpha_{2i-1}b_{2i-2} + \beta_{2i-1}) \prod_{s=i+1}^{n} \alpha_{2s-1}a_{2s-2}. \tag{11}$$

Using (4) we obtain

$$x_{2n+i} = \frac{1}{u_{2n+i}x_{2n+i-1}} = \frac{u_{2n+i-1}}{u_{2n+i}} x_{2(n-1)+i}, \quad i \in \{0,1\},$$

and

$$y_{2n+i} = \frac{1}{v_{2n+i}y_{2n+i-1}} = \frac{v_{2n+i-1}}{v_{2n+i}} y_{2(n-1)+i}, \quad i \in \{0,1\},$$

for $2n + i \geq 0$, from which it follows that

$$x_{2m+i} = x_{i-2} \prod_{j=0}^{m} \frac{u_{2j+i-1}}{u_{2j+i}}, \tag{12}$$

$$y_{2m+i} = y_{i-2} \prod_{j=0}^{m} \frac{v_{2j+i-1}}{v_{2j+i}} \tag{13}$$

for every $m \in \mathbb{N}_0$, $i \in \{0,1\}$.

3 Case of constant coefficients

In this section we consider the case when all the coefficients in system (2) are constant, that is, when

$$a_n = a, \qquad b_n = b, \qquad \alpha_n = \alpha, \qquad \beta_n = \beta, \quad n \in \mathbb{N}_0.$$

Then (2) is

$$\begin{aligned} x_n &= \frac{y_{n-1}y_{n-2}}{x_{n-1}(a + by_{n-1}y_{n-2})}, \\ y_n &= \frac{x_{n-1}x_{n-2}}{y_{n-1}(\alpha + \beta x_{n-1}x_{n-2})}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{14}$$

Assume that $(x_n, y_n)_{n\geq -2}$ is a solution to system (2) such that (3) holds. Then we have

$$u_n = av_{n-1} + b, \qquad v_n = \alpha u_{n-1} + \beta, \quad n \in \mathbb{N}_0, \tag{15}$$

and

$$u_n = a\alpha u_{n-2} + a\beta + b, \tag{16}$$
$$v_n = a\alpha v_{n-2} + \alpha b + \beta, \quad n \in \mathbb{N}. \tag{17}$$

From (8)-(11), we obtain

$$\begin{aligned} u_{2n-l} &= u_{-l}(a\alpha)^n + (a\beta + b)\frac{1-(a\alpha)^n}{1-a\alpha} \\ &= \frac{a\beta + b + (a\alpha)^n(u_{-l}(1-a\alpha) - a\beta - b)}{1-a\alpha} \end{aligned} \tag{18}$$

for $n \in \mathbb{N}_0$, $l \in \{0,1\}$ when $a\alpha \neq 1$, while if $a\alpha = 1$, we have

$$u_{2n-l} = u_{-l} + (a\beta + b)n, \quad n \in \mathbb{N}_0, l \in \{0,1\}, \tag{19}$$

and we also have

$$\begin{aligned} v_{2n-l} &= v_{-l}(a\alpha)^n + (\alpha b + \beta)\frac{1-(a\alpha)^n}{1-a\alpha} \\ &= \frac{\alpha b + \beta + (a\alpha)^n(v_{-l}(1-a\alpha) - \alpha b - \beta)}{1-a\alpha}, \end{aligned} \tag{20}$$

$n \in \mathbb{N}_0$, $l \in \{0,1\}$ if $a\alpha \neq 1$, while if $a\alpha = 1$, we have

$$v_{2n-l} = v_{-l} + (\alpha b + \beta)n, \quad n \in \mathbb{N}_0, l \in \{0,1\}. \tag{21}$$

Now we present formulae for solutions to system (14).

Case $a\alpha \neq 1$. We have

$$x_{2m} = x_{-2}\prod_{j=0}^{m}\frac{u_{2j-1}}{u_{2j}} = x_{-2}\prod_{j=0}^{m}\frac{a\beta + b + (a\alpha)^j(u_{-1}(1-a\alpha) - a\beta - b)}{a\beta + b + (a\alpha)^j(u_0(1-a\alpha) - a\beta - b)}, \tag{22}$$

$$x_{2m+1} = x_{-1}\prod_{j=0}^{m}\frac{u_{2j}}{u_{2j+1}} = x_{-1}\prod_{j=0}^{m}\frac{a\beta + b + (a\alpha)^j(u_0(1-a\alpha) - a\beta - b)}{a\beta + b + (a\alpha)^{j+1}(u_{-1}(1-a\alpha) - a\beta - b)}, \tag{23}$$

$$y_{2m} = y_{-2}\prod_{j=0}^{m}\frac{v_{2j-1}}{v_{2j}} = y_{-2}\prod_{j=0}^{m}\frac{\alpha b + \beta + (a\alpha)^j(v_{-1}(1-a\alpha) - \alpha b - \beta)}{\alpha b + \beta + (a\alpha)^j(v_0(1-a\alpha) - \alpha b - \beta)}, \tag{24}$$

$$y_{2m+1} = y_{-1}\prod_{j=0}^{m}\frac{v_{2j}}{v_{2j+1}} = y_{-1}\prod_{j=0}^{m}\frac{\alpha b + \beta + (a\alpha)^j(v_0(1-a\alpha) - \alpha b - \beta)}{\alpha b + \beta + (a\alpha)^{j+1}(v_{-1}(1-a\alpha) - \alpha b - \beta)} \tag{25}$$

for every $m \in \mathbb{N}_0$.

Case $a\alpha = 1$. We have

$$x_{2m} = x_{-2}\prod_{j=0}^{m}\frac{u_{2j-1}}{u_{2j}} = x_{-2}\prod_{j=0}^{m}\frac{u_{-1}+(a\beta+b)j}{u_0+(a\beta+b)j}, \tag{26}$$

$$x_{2m+1} = x_{-1}\prod_{j=0}^{m}\frac{u_{2j}}{u_{2j+1}} = x_{-1}\prod_{j=0}^{m}\frac{u_0+(a\beta+b)j}{u_{-1}+(a\beta+b)(j+1)}, \tag{27}$$

$$y_{2m} = y_{-2}\prod_{j=0}^{m}\frac{v_{2j-1}}{v_{2j}} = y_{-2}\prod_{j=0}^{m}\frac{v_{-1}+(\alpha b+\beta)j}{v_0+(\alpha b+\beta)j}, \tag{28}$$

$$y_{2m+1} = y_{-1}\prod_{j=0}^{m}\frac{v_{2j}}{v_{2j+1}} = y_{-1}\prod_{j=0}^{m}\frac{v_0+(\alpha b+\beta)j}{v_{-1}+(\alpha b+\beta)(j+1)} \tag{29}$$

for every $m \in \mathbb{N}_0$.

4 Long-term behavior of solutions to system (14)

Before we formulate and prove the main results regarding the long-term behavior of well-defined solutions to system (14), we quote the following well-known asymptotic formula which will be used in the proofs of the main results:

$$(1+x)^{-1} = 1 - x + O(x^2), \quad \text{as } x \to 0. \tag{30}$$

We also define the following quantities:

$$L_1 := \frac{u_{-1}(1-a\alpha)-a\beta-b}{u_0(1-a\alpha)-a\beta-b}, \qquad L_2 := \frac{u_0(1-a\alpha)-a\beta-b}{a\alpha(u_{-1}(1-a\alpha)-a\beta-b)},$$

$$L_3 := \frac{v_{-1}(1-a\alpha)-\alpha b-\beta}{v_0(1-a\alpha)-\alpha b-\beta}, \qquad L_4 := \frac{v_0(1-a\alpha)-\alpha b-\beta}{a\alpha(v_{-1}(1-a\alpha)-\alpha b-\beta)}.$$

Finally, we give another auxiliary result.

Lemma 1 *If* $a\alpha \neq 1$, $a\beta + b \neq 0 \neq \alpha b + \beta$. *Then system* (14) *has two-periodic solutions.*

Proof The equilibrium solution to system (15) is

$$u_n = \bar{u} = \frac{a\beta+b}{1-a\alpha} \neq 0, \qquad v_n = \bar{v} = \frac{\alpha b+\beta}{1-a\alpha} \neq 0, \quad n \in \mathbb{N}_0. \tag{31}$$

From (4) and (31) it follows that

$$x_n = \frac{1-a\alpha}{(a\beta+b)x_{n-1}} = x_{n-2}, \quad n \in \mathbb{N}_0, \tag{32}$$

and

$$y_n = \frac{1-a\alpha}{(\alpha b+\beta)y_{n-1}} = y_{n-2}, \quad n \in \mathbb{N}_0, \tag{33}$$

as desired. □

The next three results are devoted to the long-term behavior of well-defined solutions to system (14).

Theorem 1 *Assume that $|a\alpha| \neq 1$ and $(x_n, y_n)_{n\geq-2}$ is a well-defined solution to system* (14). *Then the following statements are true.*

(a) *If $a\beta + b \neq 0 \neq \alpha b + \beta$ and $|a\alpha| < 1$, then (x_n, y_n) converges to a, not necessarily prime, two-periodic solution.*
(b) *If $u_{-1} = u_0 = (a\beta + b)/(1 - a\alpha)$, then the sequences $(x_{2m})_{m\geq-1}$ and $(x_{2m+1})_{m\geq-1}$ are constant.*
(c) *If $v_{-1} = v_0 = (\alpha b + \beta)/(1 - a\alpha)$, then the sequences $(y_{2m})_{m\geq-1}$ and $(y_{2m+1})_{m\geq-1}$ are constant.*
(d) *If $|a\alpha| > 1$ and $u_{-1} = (a\beta + b)/(1 - a\alpha) \neq u_0$, then $x_{2m} \to 0$ and $|x_{2m+1}| \to \infty$, as $m \to \infty$.*
(e) *If $|a\alpha| > 1$ and $u_{-1} \neq (a\beta + b)/(1 - a\alpha) = u_0$, then $x_{2m+1} \to 0$ and $|x_{2m}| \to \infty$, as $m \to \infty$.*
(f) *If $|a\alpha| > 1$ and $v_{-1} = (a\beta + b)/(1 - a\alpha) \neq v_0$, then $y_{2m} \to 0$ and $|y_{2m+1}| \to \infty$, as $m \to \infty$.*
(g) *If $|a\alpha| > 1$ and $v_{-1} \neq (a\beta + b)/(1 - a\alpha) = v_0$, then $y_{2m+1} \to 0$ and $|y_{2m}| \to \infty$, as $m \to \infty$.*
(h) *If $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ and $|L_1| < 1$, then $x_{2m} \to 0$, as $m \to \infty$.*
(i) *If $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ and $|L_1| > 1$, then $|x_{2m}| \to \infty$, as $m \to \infty$.*
(j) *If $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ and $L_1 = 1$, then $(x_{2m})_{m\geq-1}$ is constant.*
(k) *If $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ and $L_1 = -1$, then $(x_{4m})_{m\geq-1}$ and $(x_{4m+2})_{m\geq-1}$ are convergent.*
(l) *If $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ and $|L_2| < 1$, then $x_{2m+1} \to 0$, as $m \to \infty$.*
(m) *If $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ and $|L_2| > 1$, then $|x_{2m+1}| \to \infty$, as $m \to \infty$.*
(n) *If $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ and $L_2 = 1$, then $(x_{2m+1})_{m\geq-1}$ is constant.*
(o) *If $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ and $L_2 = -1$, then $(x_{4m+1})_{m\geq-1}$ and $(x_{4m+3})_{m\geq-1}$ are convergent.*
(p) *If $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ and $|L_3| < 1$, then $y_{2m} \to 0$, as $m \to \infty$.*
(q) *If $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ and $|L_3| > 1$, then $|y_{2m}| \to \infty$, as $m \to \infty$.*
(r) *If $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ and $L_3 = 1$, then $(y_{2m})_{m\geq-1}$ is constant.*
(s) *If $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ and $L_3 = -1$, then $(y_{4m})_{m\geq-1}$ and $(y_{4m+2})_{m\geq-1}$ are convergent.*
(t) *If $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ and $|L_4| < 1$, then $y_{2m+1} \to 0$, as $m \to \infty$.*
(u) *If $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ and $|L_4| > 1$, then $|y_{2m+1}| \to \infty$, as $m \to \infty$.*
(v) *If $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ and $L_4 = 1$, then $(y_{2m+1})_{m\geq-1}$ is constant.*
(w) *If $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ and $L_4 = -1$, then $(y_{4m+1})_{m\geq-1}$ and $(y_{4m+3})_{m\geq-1}$ are convergent.*

Proof Let

$$p_m = \frac{a\beta + b + (a\alpha)^m(u_{-1}(1 - a\alpha) - a\beta - b)}{a\beta + b + (a\alpha)^m(u_0(1 - a\alpha) - a\beta - b)},$$
$$\hat{p}_m = \frac{a\beta + b + (a\alpha)^m(u_0(1 - a\alpha) - a\beta - b)}{a\beta + b + (a\alpha)^{m+1}(u_{-1}(1 - a\alpha) - a\beta - b)},$$

$$q_m = \frac{\alpha b + \beta + (a\alpha)^m (v_{-1}(1 - a\alpha) - \alpha b - \beta)}{\alpha b + \beta + (a\alpha)^m (v_0(1 - a\alpha) - \alpha b - \beta)},$$
$$\hat{q}_m = \frac{\alpha b + \beta + (a\alpha)^m (v_0(1 - a\alpha) - \alpha b - \beta)}{\alpha b + \beta + (a\alpha)^{m+1} (v_{-1}(1 - a\alpha) - \alpha b - \beta)}$$

for $m \in \mathbb{N}_0$.

(a) By using (30) we have

$$p_m = \frac{1 + (a\alpha)^m (u_{-1}(1 - a\alpha) - a\beta - b)(a\beta + b)^{-1}}{1 + (a\alpha)^m (u_0(1 - a\alpha) - a\beta - b)(a\beta + b)^{-1}}$$
$$= 1 + (u_{-1} - u_0)(1 - a\alpha)(a\beta + b)^{-1}(a\alpha)^m + o\big((a\alpha)^m\big), \tag{34}$$
$$\hat{p}_m = \frac{1 + (a\alpha)^m (u_0(1 - a\alpha) - a\beta - b)(a\beta + b)^{-1}}{1 + (a\alpha)^{m+1} (u_{-1}(1 - a\alpha) - a\beta - b)(a\beta + b)^{-1}}$$
$$= 1 + \frac{(1 - a\alpha)(u_0 - a\alpha u_{-1} - a\beta - b)}{a\beta + b}(a\alpha)^m + o\big((a\alpha)^m\big), \tag{35}$$
$$q_m = \frac{1 + (a\alpha)^m (v_{-1}(1 - a\alpha) - \alpha b - \beta)(\alpha b + \beta)^{-1}}{1 + (a\alpha)^m (v_0(1 - a\alpha) - \alpha b - \beta)(\alpha b + \beta)^{-1}}$$
$$= 1 + (v_{-1} - v_0)(1 - a\alpha)(\alpha b + \beta)^{-1}(a\alpha)^m + o\big((a\alpha)^m\big), \tag{36}$$
$$\hat{q}_m = \frac{1 + (a\alpha)^m (v_0(1 - a\alpha) - \alpha b - \beta)(\alpha b + \beta)^{-1}}{1 + (a\alpha)^{m+1} (v_{-1}(1 - a\alpha) - \alpha b - \beta)(\alpha b + \beta)^{-1}}$$
$$= 1 + \frac{(1 - a\alpha)(v_0 - a\alpha v_{-1} - \alpha b - \beta)}{\alpha b + \beta}(a\alpha)^m + o\big((a\alpha)^m\big) \tag{37}$$

for sufficiently large m.

From (34)-(37), by using the condition $|a\alpha| < 1$ and a well-known criterion for the convergence of products, the statement easily follows.

(b) By using the condition $u_{-1} = u_0 = (a\beta + b)/(1 - a\alpha)$ in (22) and (23), the statement immediately follows.

(c) By using the condition $v_{-1} = v_0 = (\alpha b + \beta)/(1 - a\alpha)$ in (24) and (25), the statement immediately follows.

(d) By using the condition $u_{-1} = (a\beta + b)/(1 - a\alpha) \neq u_0$, we get

$$p_m = \frac{a\beta + b}{a\beta + b + (a\alpha)^m (u_0(1 - a\alpha) - a\beta - b)}, \tag{38}$$
$$\hat{p}_m = \frac{a\beta + b + (a\alpha)^m (u_0(1 - a\alpha) - a\beta - b)}{a\beta + b}. \tag{39}$$

Letting $m \to \infty$ in (38) and (39) and using the condition $|a\alpha| > 1$, we have $p_m \to 0$ and $|\hat{p}_m| \to \infty$, from which along with (22) and (23) the statement easily follows.

(e) By using the condition $u_{-1} \neq (a\beta + b)/(1 - a\alpha) = u_0$, we get

$$p_m = \frac{a\beta + b + (a\alpha)^m (u_{-1}(1 - a\alpha) - a\beta - b)}{a\beta + b}, \tag{40}$$
$$\hat{p}_m = \frac{a\beta + b}{a\beta + b + (a\alpha)^{m+1} (u_{-1}(1 - a\alpha) - a\beta - b)}. \tag{41}$$

Letting $m \to \infty$ in (40) and (41) and using the condition $|a\alpha| > 1$, we have $|p_m| \to \infty$ and $\hat{p}_m \to 0$, from which along with (22) and (23) the statement easily follows.

(f) By using the condition $v_{-1} = (a\beta + b)/(1 - a\alpha) \neq v_0$, we get

$$q_m = \frac{\alpha b + \beta}{\alpha b + \beta + (a\alpha)^m(v_0(1 - a\alpha) - \alpha b - \beta)}, \tag{42}$$

$$\hat{q}_m = \frac{\alpha b + \beta + (a\alpha)^m(v_0(1 - a\alpha) - \alpha b - \beta)}{\alpha b + \beta}. \tag{43}$$

Letting $m \to \infty$ in (42) and (43) and using the condition $|a\alpha| > 1$, we have $q_m \to 0$ and $|\hat{q}_m| \to \infty$, from which along with (24) and (25) the statement easily follows.

(g) By using the condition $v_{-1} \neq (a\beta + b)/(1 - a\alpha) = v_0$, we get

$$q_m = \frac{\alpha b + \beta + (a\alpha)^m(v_{-1}(1 - a\alpha) - \alpha b - \beta)}{\alpha b + \beta}, \tag{44}$$

$$\hat{q}_m = \frac{\alpha b + \beta}{\alpha b + \beta + (a\alpha)^{m+1}(v_{-1}(1 - a\alpha) - \alpha b - \beta)}. \tag{45}$$

Letting $m \to \infty$ in (44) and (45) and using the condition $|a\alpha| > 1$, we have $|q_m| \to \infty$ and $\hat{q}_m \to 0$, from which along with (24) and (25) the statement easily follows.

(h), (i) Note that $\lim_{m\to\infty} p_m = L_1$. Hence, from the assumptions $|L_1| < 1$, that is, $|L_1| > 1$ along with (22), the statements easily follow.

(j) The statement immediately follows by using the condition $L_1 = 1$ in (22).

(k) Since $L_1 = -1$ and by using (30), we have that

$$\begin{aligned} p_m &= \frac{a\beta + b + (a\alpha)^m(u_{-1}(1 - a\alpha) - a\beta - b)}{a\beta + b - (a\alpha)^m(u_{-1}(1 - a\alpha) - a\beta - b)} \\ &= -\frac{1 + \frac{a\beta+b}{(a\alpha)^m(u_{-1}(1-a\alpha)-a\beta-b)}}{1 - \frac{a\beta+b}{(a\alpha)^m(u_{-1}(1-a\alpha)-a\beta-b)}} \\ &= -\left(1 + \frac{2(a\beta + b)}{(a\alpha)^m(u_{-1}(1 - a\alpha) - a\beta - b)} + o\left(\frac{1}{(a\alpha)^m}\right)\right). \end{aligned} \tag{46}$$

From (46), by using the condition $|a\alpha| > 1$ and a well-known criterion for the convergence of products, the statement easily follows.

(l), (m) Note that $\lim_{m\to\infty} \hat{p}_m = L_2$. Hence, from the assumptions $|L_2| < 1$, that is, $|L_2| > 1$ along with (23), the statements easily follow.

(n) The statement immediately follows by using the condition $L_2 = 1$ in (23).

(o) Since $L_2 = -1$ and by using (30), we have that

$$\begin{aligned} \hat{p}_m &= \frac{a\beta + b + (a\alpha)^m(u_0(1 - a\alpha) - a\beta - b)}{a\beta + b - (a\alpha)^m(u_0(1 - a\alpha) - a\beta - b)} \\ &= -\frac{1 + \frac{a\beta+b}{(a\alpha)^m(u_0(1-a\alpha)-a\beta-b)}}{1 - \frac{a\beta+b}{(a\alpha)^m(u_0(1-a\alpha)-a\beta-b)}} \\ &= -\left(1 + \frac{2(a\beta + b)}{(a\alpha)^m(u_0(1 - a\alpha) - a\beta - b)} + o\left(\frac{1}{(a\alpha)^m}\right)\right). \end{aligned} \tag{47}$$

From (47), by using the condition $|a\alpha| > 1$ and a well-known criterion for the convergence of products, the statement easily follows.

(p), (q) Note that $\lim_{m\to\infty} q_m = L_3$. Hence, from the assumptions $|L_3| < 1$, that is, $|L_3| > 1$ along with (24), the statements easily follow.

(r) The statement immediately follows by using the condition $L_3 = 1$ in (24).

(s) Since $L_3 = -1$ and by using (30), we have that

$$q_m = \frac{\alpha b + \beta + (a\alpha)^m(v_{-1}(1-a\alpha) - \alpha b - \beta)}{\alpha b + \beta - (a\alpha)^m(v_{-1}(1-a\alpha) - \alpha b - \beta)}$$
$$= -\frac{1 + \frac{\alpha b+\beta}{(a\alpha)^m(v_{-1}(1-a\alpha)-\alpha b-\beta)}}{1 - \frac{\alpha b+\beta}{(a\alpha)^m(v_{-1}(1-a\alpha)-\alpha b-\beta)}}$$
$$= -\left(1 + \frac{2(\alpha b + \beta)}{(a\alpha)^m(v_{-1}(1-a\alpha) - \alpha b - \beta)} + o\left(\frac{1}{(a\alpha)^m}\right)\right). \tag{48}$$

From (48), by using the condition $|a\alpha| > 1$ and a well-known criterion for the convergence of products, the statement easily follows.

(t), (u) Note that $\lim_{m\to\infty} \hat{q}_m = L_4$. Hence, from the assumptions $|L_4| < 1$, that is, $|L_4| > 1$ along with (25), the statements easily follow.

(v) The statement immediately follows by using the condition $L_4 = 1$ in (25).

(w) Since $L_4 = -1$ and by using (30), we have that

$$\hat{q}_m = \frac{\alpha b + \beta + (a\alpha)^m(v_0(1-a\alpha) - \alpha b - \beta)}{\alpha b + \beta - (a\alpha)^m(v_0(1-a\alpha) - \alpha b - \beta)}$$
$$= -\frac{1 + \frac{\alpha b+\beta}{(a\alpha)^m(v_0(1-a\alpha)-\alpha b-\beta)}}{1 - \frac{\alpha b+\beta}{(a\alpha)^m(v_0(1-a\alpha)-\alpha b-\beta)}}$$
$$= -\left(1 + \frac{2(\alpha b + \beta)}{(a\alpha)^m(v_0(1-a\alpha) - \alpha b - \beta)} + o\left(\frac{1}{(a\alpha)^m}\right)\right). \tag{49}$$

From (49), by using the condition $|a\alpha| > 1$ and a well-known criterion for the convergence of products, the statement easily follows. □

Let

$$M_1 := \frac{u_{-1}(u_{-1} - b - a\beta)}{u_0(u_0 - b - a\beta)}, \qquad M_2 := \frac{v_{-1}(v_{-1} - \beta - \alpha b)}{v_0(v_0 - \beta - \alpha b)}.$$

Theorem 2 *Assume that $a\alpha = -1$ and $(x_n, y_n)_{n\geq -2}$ is a well-defined solution to system* (14). *Then the following statements are true.*

(a) *If $|M_1| < 1$, then $x_{2m} \to 0$ and $|x_{2m+1}| \to \infty$, as $m \to \infty$.*
(b) *If $|M_1| > 1$, then $x_{2m+1} \to 0$ and $|x_{2m}| \to \infty$, as $m \to \infty$.*
(c) *If $M_1 = 1$, then $(x_n)_{n\geq -2}$ is four-periodic.*
(d) *If $M_1 = -1$, then $(x_n)_{n\geq -2}$ is eight-periodic.*
(e) *If $|M_2| < 1$, then $y_{2m} \to 0$ and $|y_{2m+1}| \to \infty$, as $m \to \infty$.*
(f) *If $|M_2| > 1$, then $y_{2m+1} \to 0$ and $|y_{2m}| \to \infty$, as $m \to \infty$.*
(g) *If $M_2 = 1$, then $(y_n)_{n\geq -2}$ is four-periodic.*
(h) *If $M_2 = -1$, then $(y_n)_{n\geq -2}$ is eight-periodic.*

Proof First, note that since $a\alpha = -1$, from (22)-(25) we have

$$x_{4m} = x_0 M_1^m, \qquad x_{4m+2} = x_{-2} M_1^{m+1}, \qquad x_{4m+1} = \frac{x_1}{M_1^m}, \qquad x_{4m+3} = \frac{x_{-1}}{M_1^{m+1}}, \tag{50}$$

$$y_{4m} = y_0 M_2^m, \qquad y_{4m+2} = y_{-2} M_2^{m+1}, \qquad y_{4m+1} = \frac{y_1}{M_2^m}, \qquad y_{4m+3} = \frac{y_{-1}}{M_2^{m+1}}, \tag{51}$$

for $m \in \mathbb{N}_0$. From (50) and (51) all the statements easily follow. □

Let

$$N_1 := \frac{u_{-1}}{u_0}, \qquad N_2 := \frac{v_{-1}}{v_0}.$$

Theorem 3 *Assume that $a\alpha = 1$ and $(x_n, y_n)_{n\ge -2}$ is a well-defined solution to system* (14). *Then the following statements hold true.*

(a) *If $a\beta + b = 0$ and $|N_1| < 1$, then $x_{2m} \to 0$ and $|x_{2m+1}| \to \infty$, as $m \to \infty$;*
(b) *If $a\beta + b = 0$ and $|N_1| > 1$, then $|x_{2m}| \to \infty$ and $x_{2m+1} \to 0$, as $m \to \infty$;*
(c) *If $a\beta + b = 0$ and $N_1 = 1$, then $(x_{2m})_{m\ge -1}$ and $(x_{2m+1})_{m\ge -1}$ are constant;*
(d) *If $a\beta + b = 0$ and $N_1 = -1$, then $(x_{4m+i})_{m\ge -1}$, $i = \overline{0,3}$, are constant.*
(e) *If $a\beta + b \ne 0$ and $(u_{-1} - u_0)/(a\beta + b) > 0$, then $|x_{2m}| \to \infty$, as $m \to \infty$;*
(f) *If $a\beta + b \ne 0$ and $(u_{-1} - u_0)/(a\beta + b) < 0$, then $x_{2m} \to 0$, as $m \to \infty$;*
(g) *If $a\beta + b \ne 0$ and $u_{-1} = u_0$, then $(x_{2m})_{m\ge -1}$ is constant;*
(h) *If $a\beta + b \ne 0$ and $(u_0 - u_{-1})/(a\beta + b) > 1$, then $|x_{2m+1}| \to \infty$, as $m \to \infty$;*
(i) *If $a\beta + b \ne 0$ and $(u_0 - u_{-1})/(a\beta + b) < 1$, then $x_{2m+1} \to 0$, as $m \to \infty$;*
(j) *If $a\beta + b \ne 0$ and $u_{-1} - u_0 = a\beta + b$, then $(x_{2m+1})_{m\ge -1}$ is constant;*
(k) *If $\alpha b + \beta = 0$ and $|N_2| < 1$, then $y_{2m} \to 0$ and $|y_{2m+1}| \to \infty$, as $m \to \infty$;*
(l) *If $\alpha b + \beta = 0$ and $|N_2| > 1$, then $|y_{2m}| \to \infty$ and $y_{2m+1} \to 0$, as $m \to \infty$;*
(m) *If $\alpha b + \beta = 0$ and $N_2 = 1$, then $(y_{2m})_{m\ge -1}$ and $(y_{2m+1})_{m\ge -1}$ are constant;*
(n) *If $\alpha b + \beta = 0$ and $N_2 = -1$, then $(y_{4m+i})_{m\ge -1}$, $i = \overline{0,3}$, are constant.*
(o) *If $\alpha b + \beta \ne 0$ and $(v_{-1} - v_0)/(\alpha b + \beta) > 0$, then $|y_{2m}| \to \infty$, as $m \to \infty$;*
(p) *If $\alpha b + \beta \ne 0$ and $(v_{-1} - v_0)/(\alpha b + \beta) < 0$, then $y_{2m} \to 0$, as $m \to \infty$;*
(q) *If $\alpha b + \beta \ne 0$ and $v_{-1} = v_0$, then $(y_{2m})_{m\ge -1}$ is constant.*
(r) *If $\alpha b + \beta \ne 0$ and $(v_0 - v_{-1})/(\alpha b + \beta) < 1$, then $y_{2m+1} \to 0$, as $m \to \infty$;*
(s) *If $\alpha b + \beta \ne 0$ and $(v_0 - v_{-1})/(\alpha b + \beta) > 1$, then $|y_{2m+1}| \to \infty$, as $m \to \infty$;*
(t) *If $\alpha b + \beta \ne 0$ and $v_{-1} - v_0 = \alpha b + \beta$, then $(y_{2m+1})_{m\ge -1}$ is constant.*

Proof Let

$$r_m = \frac{u_{-1} + (a\beta + b)m}{u_0 + (a\beta + b)m}, \qquad \hat{r}_m = \frac{u_0 + (a\beta + b)m}{u_{-1} + (a\beta + b)(m+1)},$$
$$s_m = \frac{v_{-1} + (\alpha b + \beta)m}{v_0 + (\alpha b + \beta)m}, \qquad \hat{s}_m = \frac{v_0 + (\alpha b + \beta)m}{v_{-1} + (\alpha b + \beta)(m+1)}, \quad m \in \mathbb{N}_0.$$

(a)-(d) Since in this case we have

$$x_{2m} = x_{-2}\left(\frac{u_{-1}}{u_0}\right)^{m+1}, \qquad x_{2m+1} = x_{-1}\left(\frac{u_0}{u_{-1}}\right)^{m+1}, \quad m \in \mathbb{N}_0,$$

these statements easily follow.

(e), (f) By using (30) we have

$$\begin{aligned} r_m &= \frac{u_{-1} + (a\beta + b)m}{u_0 + (a\beta + b)m} = \left(1 + \frac{u_{-1}}{(a\beta + b)m}\right)\left(1 + \frac{u_0}{(a\beta + b)m}\right)^{-1} \\ &= \left(1 + \frac{u_{-1}}{(a\beta + b)m} + O\left(\frac{1}{m^2}\right)\right)\left(1 - \frac{u_0}{(a\beta + b)m} + O\left(\frac{1}{m^2}\right)\right) \\ &= 1 + \frac{u_{-1} - u_0}{(a\beta + b)m} + O\left(\frac{1}{m^2}\right) \end{aligned} \tag{52}$$

for sufficiently large m.

From (52), by using the fact that for every $k \in \mathbb{N}$

$$\sum_{j=k}^{m} \frac{1}{j} \to \infty, \quad \text{as } m \to \infty, \tag{53}$$

and a known criterion for convergence of products, the statements easily follow.

(g) Using the condition $u_{-1} = u_0$ in (26), the statement immediately follows.

(h), (i) By using (30) we have

$$\begin{aligned} \hat{r}_m &= \frac{u_0 + (a\beta + b)m}{u_{-1} + (a\beta + b)(m+1)} = \left(1 + \frac{u_0}{(a\beta + b)m}\right)\left(1 + \frac{u_{-1} + a\beta + b}{(a\beta + b)m}\right)^{-1} \\ &= \left(1 + \frac{u_0}{(a\beta + b)m}\right)\left(1 - \frac{u_{-1} + a\beta + b}{(a\beta + b)m} + O\left(\frac{1}{m^2}\right)\right) \\ &= 1 + \frac{u_0 - u_{-1} - a\beta - b}{(a\beta + b)m} + O\left(\frac{1}{m^2}\right) \end{aligned} \tag{54}$$

for sufficiently large m.

From (54), (53), (27) and a known criterion for convergence of products, the statements easily follow.

(j) Using the condition $u_0 = u_{-1} + a\beta + b$ in (27), the statement immediately follows.

(k)-(n) Since in this case we have

$$y_{2m} = y_{-2}\left(\frac{v_{-1}}{v_0}\right)^{m+1}, \qquad y_{2m+1} = y_{-1}\left(\frac{v_0}{v_{-1}}\right)^{m+1}, \quad m \in \mathbb{N}_0,$$

these statements easily follow.

(o), (p) By using (30) we have

$$\begin{aligned} s_m &= \frac{v_{-1} + (\alpha b + \beta)m}{v_0 + (\alpha b + \beta)m} = \left(1 + \frac{v_{-1}}{(\alpha b + \beta)m}\right)\left(1 + \frac{v_0}{(\alpha b + \beta)m}\right)^{-1} \\ &= \left(1 + \frac{v_{-1}}{(\alpha b + \beta)m}\right)\left(1 - \frac{v_0}{(\alpha b + \beta)m} + O\left(\frac{1}{m^2}\right)\right) \\ &= 1 + \frac{v_{-1} - v_0}{(\alpha b + \beta)m} + O\left(\frac{1}{m^2}\right) \end{aligned} \tag{55}$$

for sufficiently large m.

From (55), (53), (28) and a known criterion for convergence of products, the statements easily follow.

(q) Using the condition $\nu_0 = \nu_{-1}$ in (28), the statement immediately follows.

(r), (s) By using (30) we have

$$\begin{aligned}\hat{s}_m &= \frac{\nu_0 + (\alpha b + \beta)m}{\nu_{-1} + (\alpha b + \beta)(m+1)} = \left(1 + \frac{\nu_0}{(\alpha b + \beta)m}\right)\left(1 + \frac{\nu_{-1} + \alpha b + \beta}{(\alpha b + \beta)m}\right)^{-1} \\ &= \left(1 + \frac{\nu_0}{(\alpha b + \beta)m}\right)\left(1 - \frac{\nu_{-1} + \alpha b + \beta}{(\alpha b + \beta)m} + O\left(\frac{1}{m^2}\right)\right) \\ &= 1 + \frac{\nu_0 - \nu_{-1} - \alpha b - \beta}{(\alpha b + \beta)m} + O\left(\frac{1}{m^2}\right) \qquad (56)\end{aligned}$$

for sufficiently large m.

From (56), (53), (29) and a known criterion for convergence of products, the statements easily follow.

(t) Using the condition $\nu_0 = \nu_{-1} + \alpha b + \beta$ in (29), the statement immediately follows. □

5 Domain of undefinable solutions to system (2)

In Section 2 we proved that solutions to system (2), for which $x_{-j} = 0$ or $y_{-j} = 0$ for some $j \in \{1, 2\}$, are not defined. The set of all such initial values is characterized here.

Definition 1 Consider the system of difference equations

$$\begin{aligned} x_n &= f(x_{n-1}, \dots, x_{n-s}, y_{n-1}, \dots, y_{n-s}, n), \\ y_n &= g(x_{n-1}, \dots, x_{n-s}, y_{n-1}, \dots, y_{n-s}, n), \quad n \in \mathbb{N}_0, \end{aligned} \qquad (57)$$

where $s \in \mathbb{N}$, and $x_{-i}, y_{-i} \in \mathbb{R}$, $i = \overline{1,s}$. The string of vectors

$$(x_{-s}, y_{-s}), \dots, (x_{-1}, y_{-1}), (x_0, y_0), \dots, (x_{n_0}, y_{n_0}),$$

where $n_0 \geq -1$, is called an *undefined solution* of system (57) if

$$x_j = f(x_{j-1}, \dots, x_{j-s}, y_{j-1}, \dots, y_{j-s}, j)$$

and

$$y_j = g(x_{j-1}, \dots, x_{j-s}, y_{j-1}, \dots, y_{j-s}, j)$$

for $0 \leq j < n_0 + 1$, and x_{n_0+1} or y_{n_0+1} is not a defined number, that is, the quantity

$$f(x_{n_0}, \dots, x_{n_0-s+1}, y_{n_0}, \dots, y_{n_0-s+1}, n_0 + 1)$$

or

$$g(x_{n_0}, \dots, x_{n_0-s+1}, y_{n_0}, \dots, y_{n_0-s+1}, n_0 + 1)$$

is not defined.

The set of all initial values $(x_{-s}, y_{-s}), \dots, (x_{-1}, y_{-1})$ which generate undefined solutions to system (57) is called *domain of undefinable solutions* of the system.

The next result characterizes the domain of undefinable solutions to system (2) when $a_n b_n \alpha_n \beta_n \neq 0$, $n \in \mathbb{N}_0$.

Theorem 4 *Assume that* $a_n b_n \alpha_n \beta_n \neq 0$, $n \in \mathbb{N}_0$. *Then the domain of undefinable solutions to system* (2) *is the following set*:

$$\begin{aligned}
\mathcal{U} = \bigcup_{m\in\mathbb{N}_0} \Bigg\{ &(x_{-2}, x_{-1}, y_{-2}, y_{-1}) \in \mathbb{R}^4 : \\
&\frac{1}{x_{-1}x_{-2}} = g_0^{-1} \circ f_1^{-1} \circ \cdots \circ g_{2m-2}^{-1} \circ f_{2m-1}^{-1} \circ g_{2m}^{-1} \circ f_{2m+1}^{-1}(0) \\
&\text{or } \frac{1}{x_{-1}x_{-2}} = g_0^{-1} \circ f_1^{-1} \circ \cdots \circ g_{2m-2}^{-1} \circ f_{2m-1}^{-1} \circ g_{2m}^{-1}(0) \\
&\text{or } \frac{1}{y_{-1}y_{-2}} = f_0^{-1} \circ g_1^{-1} \circ \cdots \circ f_{2m-2}^{-1} \circ g_{2m-1}^{-1} \circ f_{2m}^{-1}(0) \\
&\text{or } \frac{1}{y_{-1}y_{-2}} = f_0^{-1} \circ g_1^{-1} \circ \cdots \circ g_{2m-1}^{-1} \circ f_{2m}^{-1} \circ g_{2m+1}^{-1}(0) \Bigg\} \\
&\cup \big\{ (x_{-2}, x_{-1}, y_{-2}, y_{-1}) \in \mathbb{R}^4 : \\
&x_{-2} = 0 \text{ or } x_{-1} = 0 \text{ or } y_{-2} = 0 \text{ or } y_{-1} = 0 \big\},
\end{aligned} \tag{58}$$

where

$$f_n(t) = a_n t + b_n, \qquad g_n(t) = \alpha_n t + \beta_n, \quad n \in \mathbb{N}_0.$$

Proof We have already proved that the set

$$\big\{ (x_{-2}, x_{-1}, y_{-2}, y_{-1}) \in \mathbb{R}^4 : x_{-2} = 0 \text{ or } x_{-1} = 0 \text{ or } y_{-2} = 0 \text{ or } y_{-1} = 0 \big\}$$

belongs to the domain of undefinable solutions to system (2).

If $x_{-j} \neq 0 \neq y_{-j}$, $j = \overline{1,2}$ (*i.e.*, $x_n \neq 0 \neq y_n$ for every $n \geq -2$), then such a solution $(x_n, y_n)_{n\geq-2}$ is not defined if and only if

$$a_n + b_n y_{n-1} y_{n-2} = 0 \quad \text{or} \quad \alpha_n + \beta_n x_{n-1} x_{n-2} = 0 \tag{59}$$

for some $n \in \mathbb{N}_0$, which is equivalent to

$$v_{n-1} = -b_n/a_n \quad \text{or} \quad u_{n-1} = -\beta_n/\alpha_n \tag{60}$$

for some $n \in \mathbb{N}_0$.

Note that

$$f_n^{-1}(0) = -b_n/a_n \quad \text{and} \quad g_n^{-1}(0) = -\beta_n/\alpha_n, \quad n \in \mathbb{N}_0. \tag{61}$$

We have

$$v_{2m-1} = (g_{2m-1} \circ f_{2m-2} \circ \cdots \circ f_2 \circ g_1 \circ f_0)(v_{-1}), \tag{62}$$

$$v_{2m} = (g_{2m} \circ f_{2m-1} \circ \cdots \circ g_2 \circ f_1 \circ g_0)(u_{-1}), \tag{63}$$

$$u_{2m-1} = (f_{2m-1} \circ g_{2m-2} \circ \cdots \circ g_2 \circ f_1 \circ g_0)(u_{-1}), \tag{64}$$

$$u_{2m} = (f_{2m} \circ g_{2m-1} \circ \cdots \circ f_2 \circ g_1 \circ f_0)(v_{-1}) \tag{65}$$

for $m \in \mathbb{N}_0$.

From (61) and (62) we have that

$$-\frac{b_{2m}}{a_{2m}} = v_{2m-1} = (g_{2m-1} \circ f_{2m-2} \circ \cdots \circ f_2 \circ g_1 \circ f_0)(v_{-1})$$

for some $m \in \mathbb{N}_0$ if and only if

$$\frac{1}{y_{-1}y_{-2}} = f_0^{-1} \circ g_1^{-1} \circ \cdots \circ f_{2m-2}^{-1} \circ g_{2m-1}^{-1} \circ f_{2m}^{-1}(0). \tag{66}$$

From (61) and (63) we have that

$$-\frac{b_{2m+1}}{a_{2m+1}} = v_{2m} = (g_{2m} \circ f_{2m-1} \circ \cdots \circ g_2 \circ f_1 \circ g_0)(u_{-1})$$

for some $m \in \mathbb{N}_0$ if and only if

$$\frac{1}{x_{-1}x_{-2}} = g_0^{-1} \circ f_1^{-1} \circ \cdots \circ g_{2m-2}^{-1} \circ f_{2m-1}^{-1} \circ g_{2m}^{-1} \circ f_{2m+1}^{-1}(0). \tag{67}$$

From (61) and (64) we have that

$$-\frac{\beta_{2m}}{\alpha_{2m}} = u_{2m-1} = (f_{2m-1} \circ g_{2m-2} \circ \cdots \circ g_2 \circ f_1 \circ g_0)(u_{-1})$$

for some $m \in \mathbb{N}_0$ if and only if

$$\frac{1}{x_{-1}x_{-2}} = g_0^{-1} \circ f_1^{-1} \circ \cdots \circ g_{2m-2}^{-1} \circ f_{2m-1}^{-1} \circ g_{2m}^{-1}(0). \tag{68}$$

From (61) and (65) we have that

$$-\frac{\beta_{2m+1}}{\alpha_{2m+1}} = u_{2m} = (f_{2m} \circ g_{2m-1} \circ \cdots \circ f_2 \circ g_1 \circ f_0)(v_{-1})$$

for some $m \in \mathbb{N}_0$ if and only if

$$\frac{1}{y_{-1}y_{-2}} = f_0^{-1} \circ g_1^{-1} \circ \cdots \circ g_{2m-1}^{-1} \circ f_{2m}^{-1} \circ g_{2m+1}^{-1}(0). \tag{69}$$

From (66)-(69) we see that the first union in (58) also belongs to the domain of undefinable solutions, finishing the proof of the theorem. □

Remark 1 Quantities

$$g_0^{-1} \circ f_1^{-1} \circ \cdots \circ g_{2m-2}^{-1} \circ f_{2m-1}^{-1} \circ g_{2m}^{-1} \circ f_{2m+1}^{-1}(0), \tag{70}$$

$$g_0^{-1} \circ f_1^{-1} \circ \cdots \circ g_{2m-2}^{-1} \circ f_{2m-1}^{-1} \circ g_{2m}^{-1}(0), \tag{71}$$

$$f_0^{-1} \circ g_1^{-1} \circ \cdots \circ f_{2m-2}^{-1} \circ g_{2m-1}^{-1} \circ f_{2m}^{-1}(0), \tag{72}$$

$$f_0^{-1} \circ g_1^{-1} \circ \cdots \circ g_{2m-1}^{-1} \circ f_{2m}^{-1} \circ g_{2m+1}^{-1}(0) \tag{73}$$

can be calculated for every $m \in \mathbb{N}_0$.

Indeed, note that

$$g_0^{-1} \circ f_1^{-1} \circ \cdots \circ g_{2m-2}^{-1} \circ f_{2m-1}^{-1} \circ g_{2m}^{-1} \circ f_{2m+1}^{-1}(0) = \left(\prod_{j=0}^{m} \left(g_{2j}^{-1} \circ f_{2j+1}^{-1}\right)\right)(t)\Big|_{t=0}, \tag{74}$$

$$g_0^{-1} \circ f_1^{-1} \circ \cdots \circ g_{2m-2}^{-1} \circ f_{2m-1}^{-1} \circ g_{2m}^{-1}(0) = \left(\prod_{j=0}^{m-1} \left(g_{2j}^{-1} \circ f_{2j+1}^{-1}\right)\right)(t)\Big|_{t=g_{2m}^{-1}(0)}, \tag{75}$$

$$f_0^{-1} \circ g_1^{-1} \circ \cdots \circ f_{2m-2}^{-1} \circ g_{2m-1}^{-1} \circ f_{2m}^{-1}(0) = \left(\prod_{j=0}^{m-1} \left(f_{2j}^{-1} \circ g_{2j+1}^{-1}\right)\right)(t)\Big|_{t=f_{2m}^{-1}(0)}, \tag{76}$$

$$f_0^{-1} \circ g_1^{-1} \circ \cdots \circ g_{2m-1}^{-1} \circ f_{2m}^{-1} \circ g_{2m+1}^{-1}(0) = \left(\prod_{j=0}^{m} \left(f_{2j}^{-1} \circ g_{2j+1}^{-1}\right)\right)(t)\Big|_{t=0}, \tag{77}$$

and also that

$$\left(g_{2j}^{-1} \circ f_{2j+1}^{-1}\right)(t) = \frac{t}{\alpha_{2j}a_{2j+1}} - \frac{b_{2j+1}}{\alpha_{2j}a_{2j+1}} - \frac{\beta_{2j}}{\alpha_{2j}}, \quad j \in \mathbb{N}_0, \tag{78}$$

and

$$\left(f_{2j}^{-1} \circ g_{2j+1}^{-1}\right)(t) = \frac{t}{a_{2j}\alpha_{2j+1}} - \frac{\beta_{2j+1}}{a_{2j}\alpha_{2j+1}} - \frac{b_{2j}}{a_{2j}}, \quad j \in \mathbb{N}_0. \tag{79}$$

On the other hand, if

$$h_j(t) = c_j t + d_j, \quad j \in \mathbb{N}_0,$$

it is easy to see that

$$(h_0 \circ h_1 \circ \cdots \circ h_n)(t) = \left(\prod_{j=0}^{n} c_j\right) t + \sum_{i=0}^{n} d_i \prod_{j=0}^{i-1} c_j, \quad n \in \mathbb{N}_0. \tag{80}$$

From (74)-(80) explicit formulas for the quantities in (70)-(73) are easily obtained.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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