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Young measure solutions for a fourth-order wave equation with variable growth

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Abstract

In this paper, we study the existence of Young measure solutions to a fourth-order wave equation with variable exponent nonlinearity on a bounded domain. The asymptotic behavior of the Young measure solutions is also investigated by applying a lemma developed by Nakao.

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Keywords: nonlinear wave equation; Young measure solution: variable exponent space; nonlinear elastodynamics; Galerkin's approximation; asymptotic stability

1 Introduction

In this paper, we consider the initial boundary value problem of the following model:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} + \Delta\big(|\Delta u|^{p(x)-2}\Delta u\big) + |u|^{q(x)-2}u + a\left(\int_\Omega |u|^{q(x)}\,dx\right)\frac{\partial u}{\partial t} = f(x,t), \quad (x,t)\in Q_T,\\
&u = \Delta u = 0, \quad (x,t)\in \partial\Omega\times(0,T), \qquad (1.1)\\
&u(x,0) = u_0(x), \qquad \frac{\partial u(x,0)}{\partial t} = u_1(x), \quad x\in\Omega,
\end{aligned}
$$

where $\Omega\subset\mathbb{R}^N$ ($N\geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, $0<T<\infty$ is a given constant, and $Q_T=\Omega\times(0,T)$. The coefficient $a:[0,\infty)\to(0,\infty)$ and the exponents $p,q:\overline{\Omega}\to(1,\infty)$ are given continuous functions and $f:Q_T\to\mathbb{R}$. PDEs with variable exponent growth conditions are usually called equations with nonstandard growth conditions. After Kováčik and Rákosník first discussed the variable exponent Lebesgue space $L^{p(x)}$ and Sobolev space $W^{k,p(x)}$ in [1], a lot of research has been done concerning these kinds of variable exponent spaces; see for example [2, 3] for the properties of such spaces and [4–13] for the applications of variable exponent spaces on partial differential equations. In [14] Růžička presented the mathematical theory for the application of variable exponent spaces in electro-rheological fluids. Problems with variable exponent growth conditions also appear in the mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids [15] and nonlinear elastics [16–18]. Another field of application of equations with variable exponent growth conditions is image restoration [19].

We claim that the Young measure solutions of problem (1.1) can be approximated by the following problem with a viscosity term $\varepsilon\Delta^2\frac{\partial u}{\partial t}$ ($\varepsilon > 0$):

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} + \Delta\bigl(|\Delta u|^{p(x)-2}\Delta u\bigr) + |u|^{q(x)-2}u \\
&\quad + a\biggl(\int_\Omega |u|^{q(x)}\,dx\biggr)\frac{\partial u}{\partial t} + \varepsilon\Delta^2\frac{\partial u}{\partial t} = f(x,t), \quad (x,t)\in Q_T, \\
&u = \Delta u = 0, \quad (x,t)\in\partial\Omega\times(0,T), \\
&u(x,0) = u_0(x), \qquad \frac{\partial u(x,0)}{\partial t} = u_1(x), \quad x\in\Omega.
\end{aligned} \tag{1.2}$$

When $p(x)\equiv 2$ and the space dimension $N = 1$, problems of the type (1.2) are a class of essential fourth-order wave equations appearing in elastoplastic-microstructure models. They govern the longitudinal motion of an elastoplastic bar and antiplane shearing deformation; see [20]. For $p(x)\equiv 2$ and the multidimensional case, Chen and Yang [21] discussed the global existence, asymptotic behavior and blow-up of solutions to the initial boundary problem of the equation with weak damping term $\frac{\partial u}{\partial t}$; see also Messaoudi [22] for wave equations with nonlinear damping. For the analysis of nonlinear second-order hyperbolic equations with damping, we refer to the seminal work of Lions and Strauss [23]; see also Friedman and Nečas [24], and Emmrich and Thalhammer [25, 26]. In recent years, hyperbolic equations with variable exponent growth conditions were studied by Antontsev in [27], Haehnle and Prohl in [28], Pinasco in [29]; see also Autuori *et al.* in [30, 31] for the Kirchhoff equations with $p(x)$-growth. It is to be noted here that the viscosity term $\Delta^2\frac{\partial u}{\partial t}$ plays a key role in the proof of the global existence. The global existence results of weak solutions for second-order wave equations (even if $p(x)\equiv$ constant $\neq 2$) without the viscosity term $\Delta^2\frac{\partial u}{\partial t}$ have been found only in one space dimension; see DiPerna [32] and Shearer [33]. To the best of our knowledge, the equations without the viscosity term are studied only in [34–37]. In that work, the concept of Young measure solutions has been introduced and applied to dynamic problems and wave equations.

Thus motivated, in the present paper, we prove the global existence of Young measure solutions of problem (1.1), we first construct Young measure solutions as the limit of the sequence of solutions of problem (1.2). Then we give a decay estimate to the Young measure solutions of problem (1.1).

Our work is organized as follows. In Section 2, we give some necessary definitions and properties of variable exponent Lebesgue spaces and Sobolev spaces. In Section 3, we obtain the existence of weak solutions of problem (1.2) by Galerkin's approximation method. In Section 4, under some conditions, from the sequence of solutions of problem (1.2) and some *a priori* estimates, we get the existence of Young measure solutions by letting $\varepsilon\to 0$. In Section 5, we investigate the decay property of Young measure solutions and get a decay rate estimate by using Nakao's lemma.

2 Preliminaries

In this section, we first recall some necessary properties of variable exponent Lebesgue spaces and Sobolev spaces; see [1–3] for the details.

Let $\Omega\subset\mathbb{R}^N$ be a domain. A measurable function $p:\Omega\to[1,\infty)$ is called a variable exponent and we define $p^- = \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p^+ = \operatorname{ess\,sup}_{x\in\Omega} p(x)$. If p^+ is finite, then the exponent p is said to be bounded. The variable exponent Lebesgue space is

$$L^{p(x)}(\Omega)=\left\{u:\Omega\to\mathbb{R}\text{ is a measurable function; }\rho_{p(x)}(u)=\int_\Omega|u(x)|^{p(x)}\,dx<\infty\right\}$$

with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)}=\inf\{\lambda>0:\rho_{p(x)}(\lambda^{-1}u)\le 1\},$$

then $L^{p(x)}(\Omega)$ is a Banach space, and when p is bounded, we have the following relations:

$$\min\{\|u\|^{p^-}_{L^{p(x)}(\Omega)},\|u\|^{p^+}_{L^{p(x)}(\Omega)}\}\le\rho_{p(x)}(u)\le\max\{\|u\|^{p^-}_{L^{p(x)}(\Omega)},\|u\|^{p^+}_{L^{p(x)}(\Omega)}\}.$$

That is, if p is bounded, then norm convergence is equivalent to convergence with respect to the modular $\rho_{p(x)}$. For a bounded exponent the dual space $(L^{p(x)}(\Omega))'$ can be identified with $L^{p'(x)}(\Omega)$, where the conjugate exponent $p'(x)$ is defined by $p'(x)=\frac{p(x)}{p(x)-1}$ for each $x\in\Omega$. If $1<p^-\le p^+<\infty$, then $L^{p(x)}(\Omega)$ is separable and reflexive.

In the variable exponent Lebesgue space, Hölder's inequality is still valid; see [1], Theorem 2.1. For all $u\in L^{p(x)}(\Omega)$, $v\in L^{p'(x)}(\Omega)$ with $p(x)\in(1,\infty)$ the following inequality holds:

$$\int_\Omega|uv|\,dx\le\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}\le 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}.$$

If $0<|\Omega|<\infty$ and p, q are variable exponents such that $p(x)\le q(x)$ for each $x\in\Omega$, then there exists a continuous embedding $L^{p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$.

Definition 2.1 (see [2]) We say that a bounded exponent $p:\Omega\to\mathbb{R}$ is log-Hölder continuous if there is a constant $C>0$ such that

$$|p(y)-p(z)|\log|y-z|\le C$$

for all points $y,z\in\Omega$.

The variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega):D^\alpha u\in L^{p(x)}(\Omega),|\alpha|\le k\}$$

and equipped with the norm

$$\|u\|_{W^{k,p(x)}(\Omega)}=\sum_{|\alpha|\le k}\|D^\alpha u\|_{L^{p(x)}(\Omega)},$$

then the space $W^{k,p(x)}(\Omega)$ is a Banach space. The space $W^{k,p(x)}_0(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with the above norm. If $1<p^-\le p^+<\infty$, then the space $W^{k,p(x)}(\Omega)$ is separable and reflexive; If $p:\Omega\to(1,\infty)$ is a bounded log-Hölder continuous function, then $C_0^\infty(\Omega)$ is dense in $W^{k,p(x)}_0(\Omega)$.

Theorem 2.1 (see [2]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and assume that $p : \mathbb{R}^N \to (1,\infty)$ is a bounded log-Hölder continuous exponent such that $p^- > 1$, then for any $u \in W_0^{1,p(x)}(\Omega)$ we have*

$$\|u\|_{L^{p(x)}(\Omega)} \leq c\|\nabla u\|_{L^{p(x)}(\Omega)},$$

where the constant c only depends on the dimension N, $|\Omega|$ and the log-Hölder constant of p.

Theorem 2.2 (see [2]) *Let Ω be a bounded domain with smooth boundary. Assume that $p : \Omega \to (1,\infty)$ is a bounded log-Hölder continuous function with $p^+ < \frac{N}{k}$ and $q : \Omega \to (1,\infty)$ is a bounded measurable function with $q(x) \leq p^* = \frac{Np(x)}{N-kp(x)}$. Then there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega),$$

where the embedding constant depends on $|\Omega|$, N, q^+ and the log-Hölder constant of p.

Theorem 2.3 (see [38]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose that p is a bounded log-Hölder continuous functions in Ω, with $p^- > 1$. Then there exists a constant $C > 0$ depending only on N, Ω and the log-Hölder constant of p such that for each $u \in W_0^{2,p(x)}(\Omega)$, the following inequality holds:*

$$\|u\|_{W_0^{2,p(x)}(\Omega)} \leq C\|\Delta u\|_{L^{p(x)}(\Omega)}.$$

Proposition 2.1 (see [29, 39]) *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $\{\omega_i\}_{i=1}^{\infty}$ be an orthogonal basis in $L^2(\Omega)$. Then for any $\varepsilon > 0$, there exists a positive number N_ε such that*

$$\|u\|_{L^2(\Omega)} \leq \left(\sum_{i=1}^{N_\varepsilon}\left(\int_\Omega u\omega_i\,dx\right)^2\right)^{\frac{1}{2}} + \varepsilon\|u\|_{W_0^{1,p}(\Omega)}$$

for all $u \in W_0^{1,p}(\Omega)$ where $2 \leq p < \infty$.

The following theorem gives a relation between almost everywhere convergence and weak convergence.

Theorem 2.4 (see [7]) *Let $p : \Omega \to \mathbb{R}$ be a bounded log-Hölder continuous function with $p^- > 1$. If $\{u_n\}_{n=1}^{\infty}$ is bounded in $L^{p(x)}(Q_T)$ and $u_n \to u$ a.e. in Q_T as $n \to \infty$, then there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that $u_n \rightharpoonup u$ weakly in $L^{p(x)}(Q_T)$ as $n \to \infty$.*

Denote by $C_0(\mathbb{R}^N)$ $(N \geq 1)$ the closure of continuous functions in $\mathbb{R}^N$ with compact support. The dual of $C_0(\mathbb{R}^N)$ can be identified with the space $\mathcal{M}(\mathbb{R}^N)$ of signed Radon measures with finite mass via the pairing

$$\langle \mu, f\rangle = \int_{\mathbb{R}^N} f\,d\mu.$$

A map $\mu : E \to \mathcal{M}(\mathbb{R}^N)$ $(E \subset \mathbb{R}^N)$ is called weak $*$ measurable if the functions $x \to \langle \mu(x), f\rangle$ are measurable for all $f \in C_0(\mathbb{R}^N)$. We write μ_x instead of $\mu(x)$.

Theorem 2.5 (see [40], Theorem 3.1) *Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure and let $z_j : E \to \mathbb{R}^N$ be a sequence of measurable functions. Then there exist a subsequence z_{j_k} and a weak $*$ measurable map $\nu : E \to \mathcal{M}(\mathbb{R}^N)$ such that*

(i) $\nu_x \geq 0$, $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^N)} = \int_{\mathbb{R}^d} d\nu_x \leq 1$, *for a.e.* $x \in E$.

(ii) *For all* $f \in C_0(\mathbb{R}^N)$

$$f(z_{j_k}) \rightharpoonup \langle \nu_x, f\rangle \quad \textit{weakly} * \textit{ in } L^\infty(E).$$

(iii) *Let $K \subset \mathbb{R}^N$ be compact. Then*

$$\operatorname{supp} \nu_x \subset K \quad \textit{if } \operatorname{dist}(z_{j_k}, K) \to 0 \textit{ in measure}.$$

(iv) *Furthermore one has*

$$(\mathrm{i}') \quad \|\nu_x\|_{\mathcal{M}(\mathbb{R}^N)} = 1, \quad \textit{for a.e. } x \in E$$

if and only if the sequence does not escape to infinity, i.e. if

$$\lim_{L\to\infty} \sup_k \bigl|\{x \in E : |z_{j_k}| \geq L\}\bigr| = 0. \tag{2.1}$$

(v) *If* (i′) *holds, if $A \subset E$ is measurable and $f \in C(\mathbb{R}^N)$ and if $f(z_{j_k})$ is relatively weakly compact in $L^1(A)$, then $f(z_{j_k}) \rightharpoonup \langle \nu_x, f\rangle$ weakly in $L^1(A)$.*

(vi) *If* (i′) *holds, then in* (iii) *one can replace 'if' by 'if and only if'.*

Remark 2.1 If for some $s > 0$ and all $j \in \mathbb{N}$

$$\int_E |z_j|^s \leq C$$

then (2.1) holds.

3 Existence of weak solutions of problem (1.2)

In this section, let $\varepsilon \in (0,1)$ fixed, we prove the existence of weak solutions for problem (1.2). Our main hypotheses are the following:

(H1) $p, q : \overline{\Omega} \to (1,\infty)$ are two log-Hölder continuous functions satisfying

$$\max\left\{1, \frac{2N}{N+4}\right\} < p^- = \inf_{\overline{\Omega}} p(x) \leq p^+ = \sup_{\overline{\Omega}} p(x) < \frac{N}{2}$$

and

$$1 < q^- = \inf_{\overline{\Omega}} q(x) \leq q(x) < \frac{Np(x)}{N-2p(x)}, \quad \text{for } x \in \overline{\Omega}.$$

Here $\overline{\Omega}$ denotes the closure of Ω.

(H2) $a \in C([0,\infty))$ and there exists a constant $a_0 > 0$ such that

$$a(s) \geq a_0 > 0, \quad \text{for } s \in [0,\infty).$$

(H3) $u_0 \in W^{2,p(x)}(\Omega) \cap W_0^{2,2}(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^2(Q_T)$.

Definition 3.1 A function $u_\varepsilon : Q_T \to \mathbb{R}$ is called a weak solution of (1.2), if

$$\begin{cases} u_\varepsilon \in L^\infty(0,T;W_0^{2,p(x)}(\Omega)) \cap L^\infty(0,T;L^{q(x)}(\Omega)) \cap C(0,T;W_0^{2,2}(\Omega)), \\ \frac{\partial u_\varepsilon}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{2,2}(\Omega)), \end{cases}$$

and

$$\begin{aligned} &-\int_\Omega \frac{\partial u_\varepsilon(x,\tau)}{\partial t}\varphi(x,\tau)\,dx - \int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}\frac{\partial \varphi}{\partial t}\,dx\,dt + \int_{Q_T} |\Delta u_\varepsilon|^{p(x)-2}\Delta u_\varepsilon \Delta\varphi \\ &\quad + \varepsilon\Delta\frac{\partial u_\varepsilon}{\partial t}\Delta\varphi\,dx\,dt + \int_{Q_T} a\left(\int_\Omega |u_\varepsilon|^{q(x)}\,dx\right)\frac{\partial u_\varepsilon}{\partial t}\varphi\,dx\,dt \\ &= \int_{Q_T} f\varphi\,dx\,dt + \int_\Omega u_1\varphi(x,0)\,dx, \end{aligned}$$

for all $\varphi \in C^1(0,T;C_0^\infty(\Omega))$ and $\tau \in (0,T]$.

We choose a sequence $\{\omega_j\}_{j=1}^\infty \subset C_0^\infty(\Omega)$ such that $C_0^\infty(\Omega) \subset \overline{\bigcup_{n=1}^\infty V_n}^{C^2(\bar{\Omega})}$ and $\{\omega_j\}_{j=1}^\infty$ is a complete orthogonal basis in $L^2(\Omega)$, where $V_n = \operatorname{span}\{\omega_1,\omega_2,\ldots,\omega_n\}$; see [7, 39].

Since $\bigcup_{n=1}^\infty V_n$ is dense in $C^2(\bar{\Omega})$, we have the following lemma.

Lemma 3.1 (see [41]) *For the function $u_0 \in W^{2,p(x)}(\Omega) \cap W_0^{2,2}(\Omega)$, there exists a sequence ψ_n with $\psi_n \in V_n$ such that $\psi_n \to u_0$ in $W^{2,p(x)}(\Omega) \cap W_0^{2,2}(\Omega)$ as $n \to \infty$.*

Proof For $u_0 \in W^{2,p(x)}(\Omega) \cap W_0^{2,2}(\Omega)$, there exists a sequence $\{v_n\}$ in $C_0^\infty(\Omega)$ such that $v_n \to u_0$ in $W^{2,p(x)}(\Omega) \cap W_0^{2,2}(\Omega)$. Since $\{v_n\} \subset C_0^\infty(\Omega) \subset \overline{\bigcup_{m=1}^\infty V_m}^{C^2(\bar{\Omega})}$, we can find a sequence $\{v_n^k\} \subset \bigcup_{m=1}^\infty V_m$ such that, for each $n \in \mathbb{N}$, we have $v_n^k \to u_n$ in $C^2(\overline{\Omega})$ as $k \to \infty$. For $\frac{1}{2^n}$, there exists $k_n \geq 1$ such that $\|v_n^{k_n} - u_n\|_{C^2(\overline{\Omega})} \leq \frac{1}{2^n}$. Thus

$$\left\|v_n^{k_n} - u_0\right\|_{W^{2,p(x)}(\Omega)\cap W_0^{2,2}(\Omega)} \leq C\left\|v_n^{k_n} - v_n\right\|_{C^2(\overline{\Omega})} + \|v_n - u_0\|_{W^{2,p(x)}(\Omega)\cap W_0^{2,2}(\Omega)}.$$

That is, $v_n^{k_n} \to u_0$ in $W^{2,p(x)}(\Omega) \cap W_0^{2,2}(\Omega)$ as $n \to \infty$. Denote $u_n = v_n^{k_n}$. Since $u_n \in \bigcup_{m=1}^\infty V_m$, there exists V_{m_n} such that $u_n \in V_{m_n}$; without loss of generality, we assume that $V_{m_1} \subset V_{m_2}$ as $m_1 \leq m_2$. We assume that $m_1 > 1$ and define ψ_n as follows: $\psi_n(x) = 0$, $n = 1,\ldots,m_1 - 1$; $\psi_n = u_1$, $n = m_1,\ldots,m_2 - 1$; $\psi_n = u_2$, $n = m_2,\ldots,m_3 - 1$; …, then we obtain the sequence $\{\psi_n\}$ and $\psi_n \to u_0$ in $W^{2,p(x)}(\Omega) \cap W_0^{2,2}(\Omega)$ as $n \to \infty$. □

The existence of weak solutions of problem (1.1) is proved by Galerkin's approximation. We shall find the sequence of approximate solutions in the form

$$u_n(x,t) = \sum_{j=1}^n \bigl(\eta_n(t)\bigr)_j \omega_j(x).$$

The unknown functions $(\eta_n(t))_j$ are determined by ordinary differential equations in the following.

We first define a vector-valued function $P_n(t,\mu,\nu):[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ as follows:

$$\begin{aligned}
&\big(P_n(t,\mu,\nu)\big)_i\\
&\quad=\int_\Omega\left|\sum_{j=1}^n\mu_j\Delta\omega_j\right|^{p(x)-2}\left(\sum_{j=1}^n\mu_j\Delta\omega_j\right)\Delta\omega_i+\left|\sum_{j=1}^n\mu_j\Delta\omega_j\right|^{q(x)-2}\left(\sum_{j=1}^n\mu_j\Delta\omega_j\right)\Delta\omega_i\,dx\\
&\qquad+\int_\Omega\varepsilon\sum_{j=1}^n\nu_j\Delta\omega_j\Delta\omega_i+a\left(\int_\Omega\left|\sum_{j=1}^n\nu_j\omega_j\right|^{q(x)}dx\right)\omega_i\,dx,\quad i=1,\dots,n,
\end{aligned}$$

where $\mu=(\mu_1,\dots,\mu_n)$ and $\nu=(\nu_1,\dots,\nu_n)$. Now we consider the following Cauchy problem of second-order ordinary differential equations:

$$\begin{cases}\eta''(t)+P_n(t,\eta(t),\eta'(t))=F_n(t),\\ \eta(0)=U_{0n},\qquad \eta'(0)=U_{1n},\end{cases}\tag{3.1}$$

where $(U_{0n})_i=\int_\Omega\psi_n\omega_i\,dx$, $(U_{1n})_i=\int_\Omega\phi_n\omega_i\,dx$, $F_n=\int_\Omega f_n\omega_i\,dx$, $\psi_n\in V_n$, $\phi_n\in V_n$, $f_n\in C_0^\infty(Q_T)$, and $\psi_n\to u_0$ strongly in $W^{2,p(x)}(\Omega)\cap W_0^{1,2}(\Omega)$ (ψ_n from Lemma 3.1), $\phi_n\to u_1$ strongly in $L^2(\Omega)$, $f_n\to f$ strongly in $L^2(Q_T)$ (since $C_0^\infty(Q_T)$ is dense in $L^2(Q_T)$).

Let $\eta'(t)=X(t)$, $Y(t)=(\eta(t),X(t))$, and $H_n(t,Y)=(X,F_n-P_n(t,\eta,X))$. Then the problem (3.1) is transformed into the following problem:

$$\begin{cases}Y'(t)=H_n\big(t,Y(t)\big),\\ Y(0)=(U_{0n},U_{1n}).\end{cases}\tag{3.2}$$

The assumption (H2) implies

$$\begin{aligned}
P_n(t,\eta,X)X&=P_n\big(t,\eta,\eta'\big)\eta'\\
&=\int_\Omega|\Delta u_n|^{p(x)-2}\Delta u_n\Delta\frac{\partial u_n}{\partial t}+|u_n|^{q(x)-2}u_n\frac{\partial u_n}{\partial t}\,dx+\varepsilon\int_\Omega\Delta\frac{\partial u_n}{\partial t}\Delta\frac{\partial u_n}{\partial t}\,dx\\
&\quad+\int_\Omega a\left(\int_\Omega|u_n|^{q(x)}\,dx\right)\left|\frac{\partial u_n}{\partial t}\right|^2dx\\
&\geq\frac{d}{dt}\int_\Omega\frac{|\Delta u_n|^{p(x)}}{p(x)}+\frac{|u_n|^{q(x)}}{q(x)}\,dx+\varepsilon\int_\Omega\left|\Delta\frac{\partial u_n}{\partial t}\right|^2dx+a_0\int_\Omega\left|\frac{\partial u_n}{\partial t}\right|^2dx.
\end{aligned}$$

From (3.2) and Young's inequality, we obtain

$$\begin{aligned}
&Y'Y+\frac{d}{dt}\int_\Omega\frac{|\Delta u_n|^{p(x)}}{p(x)}+\frac{|u_n|^{q(x)}}{q(x)}\,dx+\varepsilon\int_\Omega\left|\Delta\frac{\partial u_n}{\partial t}\right|^2dx+\frac{a_0}{2}\int_\Omega\left|\frac{\partial u_n}{\partial t}\right|^2dx\\
&\quad\leq C\left(\frac{1}{2}|Y|^2+\int_\Omega\frac{|\Delta u_n|^{p(x)}}{p(x)}\,dx+\int_\Omega f_n^2\,dx\right).
\end{aligned}\tag{3.3}$$

Thus,

$$\frac{d}{dt}\left(\frac{1}{2}|Y|^2+\int_\Omega\frac{|\Delta u_n|^{p(x)}}{p(x)}+\frac{|u_n|^{q(x)}}{q(x)}\,dx\right)\le C\left(\frac{1}{2}|Y|^2+\int_\Omega\frac{|\Delta u_n|^{p(x)}}{p(x)}\,dx+\int_\Omega f_n^2\,dx\right).$$

Gronwall's inequality and $f_n\to f$ strongly in $L^2(Q_T)$ imply

$$|Y|^2+\int_\Omega\frac{|\Delta u_n|^{p(x)}}{p(x)}\,dx\le C, \tag{3.4}$$

where C is a constant independent of n and ε. Thus, $|Y-Y(0)|\le 2\sqrt{C}$. We denote

$$L_n=\max_{(t,Y)\in[0,T]\times B(Y(0),2\sqrt{C})}\bigl|H_n(t,Y)\bigr|,\qquad \tau_n=\min\left\{T,\frac{2\sqrt{C}}{L_n}\right\},$$

where $B(Y(0),2\sqrt{C})$ is the ball of radius $2\sqrt{C}$ with center at the point $Y(0)$ in $\mathbb{R}^{2n}$. From the definition of $H(t,Y)$, $H(t,Y)$ is continuous with respect to (t,Y). By Peano's theorem, we know that (3.2) admits a C^1 solution on $[0,\tau_n]$, that is, (3.1) has a C^2 solution on $[0,\tau_n]$ denoted by $\eta_n^1(t)$. Let $\eta(\tau_n)$, $\frac{\partial\eta(\tau_n)}{\partial t}$ be the initial value of problem (3.1), then we can repeat the above process and get a C^2 solution $\eta_n^2(t)$ on $[\tau_n,2\tau_n]$. Without loss of generality, we assume that $T=[\frac{T}{\tau_n}]\tau_n+(\frac{T}{\tau_n})\tau_n$, $0<(\frac{T}{\tau_n})<1$, where $[\frac{T}{\tau_n}]$ is the integer part of $\frac{T}{\tau_n}$, $(\frac{T}{\tau_n})$ is the decimal part of $\frac{T}{\tau_n}$. We can divide $[0,T]$ into $[(i-1)\tau_n,i\tau_n]$, $i=1,\dots,L$, and $[L\tau_n,T]$ where $L=[\frac{T}{\tau_n}]$, then there exists a C^2 solution $\eta_n^i(t)$ in $[(i-1)\tau_n,i\tau_n]$, $i=1,\dots,L$, and $\eta_n^{L+1}(t)$ in $[L\tau_n,T]$. Therefore, we get a solution $\eta_n(t)\in C^2([0,T])$ defined by

$$\eta_n(t)=\begin{cases}\eta_n^1(t), & \text{if } t\in[0,\tau_n],\\ \eta_n^2(t), & \text{if } t\in(\tau_n,2\tau_n],\\ \dots & \\ \eta_n^L(t), & \text{if } t\in((L-1)\tau_n,L\tau_n],\\ \eta_n^{L+1}(t), & \text{if } t\in(L\tau_n,T].\end{cases}$$

Lemma 3.2 (*A priori* estimates) *The estimates*

$$\int_\Omega\left|\frac{\partial u_n(x,t)}{\partial t}\right|^2dx+\int_\Omega\bigl|\Delta u_n(x,t)\bigr|^{p(x)}+\bigl|u_n(x,t)\bigr|^{q(x)}\,dx+\varepsilon\int_\Omega\bigl|\Delta u_n(x,t)\bigr|^2\,dx\le C,$$
$$\forall t\in[0,T],$$
$$\int_{Q_T}\left|\frac{\partial u_n}{\partial t}\right|^2dx\,dt+\int_{Q_T}|\Delta u_n|^{p(x)}+|u_n|^{q(x)}\,dx\,dt+\varepsilon\int_{Q_T}\left|\Delta\frac{\partial u_n}{\partial t}\right|^2dx\,dt\le C$$

hold uniformly with respect to n.

Proof By (3.4), we have

$$\int_\Omega\bigl|u_n(x,t)\bigr|^2+\left|\frac{\partial u_n(x,t)}{\partial t}\right|^2dx+\int_\Omega\bigl|\Delta u_n(x,t)\bigr|^{p(x)}+\bigl|u_n(x,t)\bigr|^{q(x)}\,dx\le C,$$
$$\text{for } t\in[0,T].$$

Further, integrating the inequality (3.3) with respect to t over $[0,T]$, we obtain

$$\int_{Q_T} \varepsilon \left| \Delta \frac{\partial u_n}{\partial t} \right|^2 + \left| \frac{\partial u_n}{\partial t} \right|^2 dx\,dt \le C.$$

Moreover, for each $t \in [0,T]$,

$$\int_\Omega \left| \Delta u_n(x,t) \right|^2 dx \le 2T \int_\Omega \int_0^T \left| \Delta \frac{\partial u_n}{\partial t} \right|^2 dx\,dt + 2 \int_\Omega \left| \Delta u_n(x,0) \right|^2 dx \le \frac{C}{\varepsilon}.$$

Thus, this lemma is proved □

By Lemma 3.2, we have the following.

Lemma 3.3 *The estimate*

$$\|u_n\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))} + \big\| |\Delta u_n|^{p(x)-2} \Delta u_n \big\|_{L^{p'(x)}(Q_T)} + \big\| |u_n|^{q(x)-2} u_n \big\|_{L^{p'(x)}(Q_T)} + \left\| a\left(\int_\Omega |u_n|^{q(x)}\,dx \right) \frac{\partial u_n}{\partial t} \right\|_{L^2(Q_T)} \le C$$

holds uniformly with respect to n and ε.

Proof By Theorem 2.3, we have

$$\|u_n\|_{W_0^{2,p(x)}(\Omega)} \le C \|\Delta u_n\|_{L^{p(x)}(\Omega)} \le C.$$

Thus, $\|u_n\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))} \le C$. By Lemma 3.2, we obtain

$$\int_{Q_T} \big| |\Delta u_n|^{p(x)-2} \nabla u_n \big|^{p'(x)} dx\,dt \le \int_{Q_T} |\nabla u_n|^{p(x)}\,dx\,dt \le C.$$

Thus,

$$\begin{aligned} &\big\| |\Delta u_n|^{p(x)-2} \nabla u_n \big\|_{L^{p'(x)}(Q_T)} \\ &\quad \le \max\left\{ \left(\int_\Omega |\Delta u_n|^{p(x)}\,dx \right)^{\frac{p^- -1}{p^-}}, \left(\int_\Omega |\Delta u_n|^{p(x)}\,dx \right)^{\frac{p^+ -1}{p^+}} \right\} \le C. \end{aligned}$$

Similarly, $\||u_n|^{q(x)-2} u_n\|_{L^{q'(x)}(Q_T)} \le C$. Since $a \in C([0,\infty))$ and $\int_\Omega |u_n(x,t)|^{q(x)}\,dx \le C$, we have

$$\left\| a\left(\int_\Omega |u_n|^{q(x)}\,dx \right) \frac{\partial u_n}{\partial t} \right\|_{L^2(Q_T)} \le C.$$

This lemma is proved. □

Theorem 3.1 *Assume* (H1)-(H3). *Then for each $\varepsilon \in (0,1)$ problem* (1.2) *has a weak solution.*

Proof By Lemma 3.2 and Lemma 3.3, there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), u_ε, ξ, η, and ζ such that

$$\begin{cases} \frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u_\varepsilon}{\partial t} & \text{weakly } * \text{ in } L^\infty(0,T;L^2(\Omega)), \\ u_n \rightharpoonup u_\varepsilon & \text{weakly} * \text{ in } L^\infty(0,T;W_0^{2,p(x)}(\Omega)\cap L^{q(x)}(\Omega)) \\ & \cap L^\infty(0,T;W_0^{2,2}(\Omega)), \\ \frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u_\varepsilon}{\partial t} & \text{weakly in } L^2(0,T;W_0^{2,2}(\Omega)), \\ |\Delta u_n|^{p(x)-2}\Delta u_n \rightharpoonup \xi & \text{weakly in } L^{p'(x)}(Q_T), \\ |u_n|^{q(x)-2}u_n \rightharpoonup \eta & \text{weakly in } L^{q'(x)}(Q_T), \\ a(\int_\Omega |u_n|^{q(x)}\,dx)\frac{\partial u_n}{\partial t} \rightharpoonup \zeta & \text{weakly in } L^2(Q_T). \end{cases}$$

Since $u_n \in L^\infty(0,T;W_0^{1,2}(\Omega))$ and $\frac{\partial u_n}{\partial t} \in L^2(Q_T)$, by the Lions-Aubin lemma, there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that $u_n \to u_\varepsilon$ strongly in $L^2(Q_T)$ and a.e. on Q_T. Further, $|u_n|^{q(x)-2}u_n \to |u_\varepsilon|^{q(x)-2}u_\varepsilon$ a.e. on Q_T. In view of Theorem 2.4, we obtain $\eta = |u_\varepsilon|^{q(x)-2}u_\varepsilon$.

Next, we prove that there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $\frac{\partial u_n}{\partial t} \to \frac{\partial u_\varepsilon}{\partial t}$ strongly in $L^2(Q_T)$.

Since $(\eta_n'(t))_j = \int_\Omega \frac{\partial u_n}{\partial t}\omega_j\,dx$, by Lemma 3.2, $(\eta_n'(t))_j$ is uniformly bounded on $[0,T]$. For $\forall 0 \le t_1 < t_2 \le T$, integrating (3.1) with respect to t from t_1 to t_2, we have

$$\begin{aligned} &\int_\Omega \frac{\partial u_n(x,t_1)}{\partial t}\omega_j\,dx - \int_\Omega \frac{\partial u_n(x,t_2)}{\partial t}\omega_j\,dx + \int_{t_1}^{t_2}\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n\Delta\omega_j + |u_n|^{q(x)-2}u_n\omega_j \\ &\quad + \varepsilon\Delta\frac{\partial u_n}{\partial t}\Delta\omega_j + a\left(\int_\Omega |u_n|^{q(x)}\,dx\right)\frac{\partial u_n}{\partial t}\omega_j\,dx\,dt = \int_{t_1}^{t_2}\int_\Omega f_n\omega_j\,dx\,dt. \end{aligned}$$

Hölder's inequality, Lemma 3.2, and Lemma 3.3 imply

$$\begin{aligned} &\bigl|\bigl(\eta_n(t_1)\bigr)_j - \bigl(\eta_n(t_2)\bigr)_j\bigr| \\ &\quad\le 2\biggl(\bigl\||\Delta u_n|^{p(x)-2}\Delta u_n\bigr\|_{L^{p'(x)}(Q_T)}\|\Delta\omega_j\|_{L^{p(x)}(Q_{t_1}^{t_2})} + \bigl\||u_n|^{q(x)-2}u_n\bigr\|_{L^{q'(x)}(Q_T)}\|\omega_j\|_{L^{q(x)}(Q_{t_1}^{t_2})} \\ &\qquad + \left\|\Delta\frac{\partial u_n}{\partial t}\right\|_{L^2(Q_T)}\|\Delta\omega_j\|_{L^2(Q_{t_1}^{t_2})} + \left\|a\left(\int_\Omega |u_n|^{q(x)}\,dx\right)\frac{\partial u_n}{\partial t}\right\|_{L^2(Q_T)}\|\omega_j\|_{L^2(Q_{t_1}^{t_2})}\biggr) \\ &\quad\le C\bigl(\|\Delta\omega_j\|_{L^{p(x)}(Q_{t_1}^{t_2})} + \|\Delta\omega_j\|_{L^2(Q_{t_1}^{t_2})} + \|\omega_j\|_{L^2(Q_{t_1}^{t_2})} + \|\omega_j\|_{L^{q(x)}(Q_{t_1}^{t_2})}\bigr) \\ &\quad\le \max\bigl\{|t_1-t_2|^{\frac{1}{p^-}}, |t_1-t_2|^{\frac{1}{p^+}}, |t_1-t_2|^{\frac{1}{2}}, |t_1-t_2|^{\frac{1}{q^-}}, |t_1-t_2|^{\frac{1}{q^+}}\bigr\} \\ &\qquad\times\left(\left(\int_\Omega |\Delta\omega_j|^{p(x)}\,dx\right)^{\frac{1}{p^-}} + \left(\int_\Omega |\Delta\omega_j|^{p(x)}\,dx\right)^{\frac{1}{p^+}} + \left(\int_\Omega |\Delta\omega_j|^2\,dx\right)^{\frac{1}{2}}\right. \\ &\qquad\left. + \left(\int_\Omega |\omega_j|^{q(x)}\,dx\right)^{\frac{1}{q^-}} + \left(\int_\Omega |\omega_j|^{q(x)}\,dx\right)^{\frac{1}{q^+}} + \left(\int_\Omega |\omega_j|^2\,dx\right)^{\frac{1}{2}}\right) \\ &\quad\le C(j)\max\bigl\{|t_1-t_2|^{\frac{1}{p^-}}, |t_1-t_2|^{\frac{1}{p^+}}, |t_1-t_2|^{\frac{1}{2}}, |t_1-t_2|^{\frac{1}{q^-}}, |t_1-t_2|^{\frac{1}{q^+}}\bigr\}, \end{aligned}$$

where $Q_{t_1}^{t_2} = \Omega \times (t_1,t_2)$. Thus, the sequence $\{(\eta_n(t))_j\}_{n=1}^\infty$ is uniformly bounded and equicontinuous for fixed j and arbitrary $n \ge j$. By the Ascoli-Arzela theorem and the usual diagonal procedure, there exists a subsequence of $\{(\eta_n)_j\}$ still denoted by $\{(\eta_n)_j\}$ such that $(\eta_n(t))_j$ converges uniformly on $[0,T]$ to some continuous function $\lambda_j^\varepsilon(t)$ for each fixed $j = 1, 2, \ldots$.

For $r \le n$ with $r \in \mathbb{N}$, by Lemma 3.2, we have

$$\sum_{j=1}^{r} \left| \left(\eta_n'(t)\right)_j \right|^2 \le \int_\Omega \left| \frac{\partial u_n}{\partial t} \right|^2 dx \le C, \quad \forall t \in [0,T].$$

Letting $n \to \infty$, we get

$$\sum_{j=1}^{r} \left| \lambda_j^\varepsilon(t) \right|^2 \le C, \quad \forall t \in [0,T].$$

Then letting $r \to \infty$, we obtain

$$\sum_{j=1}^{\infty} \left| \lambda_j^\varepsilon(t) \right|^2 \le C, \quad \forall t \in [0,T].$$

Set $\overline{u}_\varepsilon(x,t) = \sum_{j=1}^\infty \lambda_j^\varepsilon(t)\omega_j(x)$, then $\sup_{0\le t\le T} \|\overline{u}_\varepsilon(x,t)\|_{L^2(\Omega)} \le C(T)$ and, for each $j \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \int_\Omega \frac{\partial u_n}{\partial t} \omega_j \, dx = \int_\Omega \overline{u}_\varepsilon \omega_j \, dx \tag{3.5}$$

uniformly on $[0,T]$. For each $\delta_1 > 0$ and $\phi \in L^2(\Omega)$, by the completeness of $\{\omega_j\}$, there exists a $m_0 > 0$ such that $\|\phi - \sum_{i=1}^{m_0} (\int_\Omega \phi\omega_i \, dx)\omega_i\|_{L^2(\Omega)} \le \delta_1$. Thus,

$$\begin{aligned} \left| \int_\Omega \left(\frac{\partial u_n}{\partial t} - \bar{u}_\varepsilon \right) \phi \, dx \right| &\le \left\| \frac{\partial u_n}{\partial t} - \bar{u}_\varepsilon \right\|_{L^2(\Omega)} \left\| \phi - \sum_{i=1}^{m_0} \left(\int_\Omega \phi\omega_i \, dx \right) \omega_i \right\|_{L^2(\Omega)} \\ &\quad + \left| \int_\Omega \left(\frac{\partial u}{\partial t} - \bar{u}_\varepsilon \right) \sum_{i=1}^{m_0} \left(\int_\Omega \phi\omega_i \, dx \right) \omega_i \, dx \right| \\ &\le C\delta_1 + \left| \int_\Omega \left(\frac{\partial u_n}{\partial t} - \bar{u}_\varepsilon \right) \sum_{i=1}^{m_0} \left(\int_\Omega \phi\omega_i \, dx \right) \omega_i \, dx \right|. \end{aligned} \tag{3.6}$$

For $\delta_1 > 0$, by (3.5), there exists a $M > 0$ such that

$$\left| \int_\Omega \left(\frac{\partial u_n}{\partial t} - \bar{u}_\varepsilon \right) \omega_i \, dx \right| \le \frac{\delta_1}{m_0}, \quad \text{for } n > M \text{ and } i = 1, \ldots, m_0.$$

By (3.6) and the Hölder inequality, we have

$$\begin{aligned} \left| \int_\Omega \left(\frac{\partial u_n}{\partial t} - \bar{u}_\varepsilon \right) \phi \, dx \right| &\le \left\| \frac{\partial u_n}{\partial t} - \bar{u}_\varepsilon \right\|_{L^2(\Omega)} \left\| \phi - \sum_{i=1}^{m_0} \left(\int_\Omega \phi\omega_i \, dx \right) \omega_i \right\|_{L^2(\Omega)} \\ &\quad + \left| \int_\Omega \left(\frac{\partial u}{\partial t} - \bar{u}_\varepsilon \right) \sum_{i=1}^{m_0} \left(\int_\Omega \phi\omega_i \, dx \right) \omega_i \, dx \right| \end{aligned}$$

$$\leq C\delta_1 + \sum_{i=1}^{m_0} \left| \int_\Omega \phi\omega_i \, dx \right| \left| \int_\Omega \left(\frac{\partial u_n}{\partial t} - \bar{u}_\varepsilon \right) \omega_i \, dx \right|$$
$$\leq \left(C + \|\phi\|_{L^2(\Omega)} \right) \delta_1, \quad \text{for } n > M. \tag{3.7}$$

It follows from (3.7) and the arbitrariness of δ_1 that

$$\frac{\partial u_n}{\partial t} \rightharpoonup \overline{u}_\varepsilon \quad \text{weakly in } L^2(\Omega). \tag{3.8}$$

uniformly on $[0, T]$ as $n \to \infty$. For each $\varphi \in C_0^\infty(Q_T)$, by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_{Q_T} \left(\frac{\partial u_n}{\partial t} - \overline{u}_\varepsilon \right) \varphi \, dx \, dt = 0.$$

Hence,

$$\lim_{n\to\infty} \int_{Q_T} \frac{\partial u_n}{\partial t} \varphi \, dx \, dt = \int_{Q_T} \overline{u}_\varepsilon \varphi \, dx \, dt.$$

On the other hand, by integration by parts, we get

$$\int_{Q_T} \frac{\partial u_n}{\partial t} \varphi \, dx \, dt = -\int_{Q_T} u_n \frac{\partial \varphi}{\partial t} \, dx \, dt.$$

Letting $n \to \infty$ in above equality, we have

$$\int_{Q_T} \overline{u}_\varepsilon \varphi \, dx \, dt = -\int_{Q_T} u_\varepsilon \frac{\partial \varphi}{\partial t} \, dx \, dt, \quad \text{for } \varphi \in C_0^\infty(Q_T).$$

Thus, we obtain $\overline{u} = \frac{\partial u_\varepsilon}{\partial t}$. Moreover, for each $j \in \mathbb{N}$, Lemma 3.2, and Lebesgue's dominated convergence theorem yield

$$\lim_{n\to\infty} \int_0^T \left(\int_\Omega \left(\frac{\partial u_n}{\partial t} - \frac{\partial u_\varepsilon}{\partial t} \right) \omega_j \, dx \right)^2 dt = 0.$$

Thus, for $\delta > 0$, by Proposition 2.1, there exists a positive number N_δ independent of n such that

$$\left\| \frac{\partial u_n}{\partial t} - \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^2(Q_T)}$$
$$\leq 2 \sum_{i=1}^{N_\delta} \int_0^T \left(\int_\Omega \left(\frac{\partial u_n}{\partial t} - \frac{\partial u_\varepsilon}{\partial t} \right) \omega_i \, dx \right)^2 dt + 2\delta^2 \int_0^T \left\| \frac{\partial u_n}{\partial t} - \frac{\partial u_\varepsilon}{\partial t} \right\|_{W_0^{1,2}(\Omega)}^2 dt.$$

A similar discussion to (3.7) shows that there is a $\widetilde{M}(\delta) > 0$ such that

$$\left\| \frac{\partial u_n}{\partial t} - \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^2(Q_T)} \leq C\delta^2, \quad \text{for } n > \widetilde{M}(\delta).$$

Thus, $\frac{\partial u_n}{\partial t} \to \frac{\partial u_\varepsilon}{\partial t}$ strongly in $L^2(Q_T)$. Further, there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that $\frac{\partial u_n}{\partial t} \to \frac{\partial u_\varepsilon}{\partial t}$ a.e. on Q_T.

For $\forall \varphi \in C^1(0,T;C_0^\infty(\Omega))$, we can choose a sequence $\varphi_k \in C^1(0,T;V_k)$ such that $\varphi_k \to \varphi$ in $C^{1,2}(Q_T)$. Here for $v \in C^{1,2}(Q_T)$ equipped with the norm $\|v\| = \sup_{|\alpha|\le 2,(x,t)\in\overline{Q_T}}\{|D^\alpha v|, |\frac{\partial v}{\partial t}|\}$. For $\forall \tau \in (0,T]$, we have

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}\int_{Q_\tau}\frac{\partial^2 u_n}{\partial t^2}\varphi_k\,dx\,dt \\
&\quad= \lim_{k\to\infty}\lim_{n\to\infty}\biggl(\int_\Omega \frac{\partial u_n(x,\tau)}{\partial t}\varphi_k(x,\tau)\,dx - \int_\Omega \frac{\partial u_n(x,0)}{\partial t}\varphi_k(x,0)\,dx \\
&\qquad - \int_{Q_\tau}\frac{\partial u_n}{\partial t}\frac{\partial \varphi_k}{\partial t}\,dx\,dt\biggr) \\
&\quad= \lim_{k\to\infty}\biggl(\int_\Omega \frac{\partial u_\varepsilon(x,\tau)}{\partial t}\varphi_k(x,\tau)\,dx - \int_\Omega u_1\varphi(x,0)\,dx - \int_{Q_\tau}\frac{\partial u_\varepsilon}{\partial t}\frac{\partial \varphi_k}{\partial t}\,dx\,dt\biggr) \\
&\quad= \int_\Omega \frac{\partial u_\varepsilon(x,\tau)}{\partial t}\varphi(x,\tau)\,dx - \int_\Omega u_1\varphi(x,0)\,dx - \int_{Q_\tau}\frac{\partial u_\varepsilon}{\partial t}\frac{\partial \varphi}{\partial t}\,dx\,dt \\
&\quad= \lim_{n\to\infty}\int_{Q_\tau}\frac{\partial^2 u_n}{\partial t^2}\varphi\,dx\,dt,
\end{aligned}$$

where $Q_\tau = \Omega \times (0,\tau)$. Replacing ω_i in (3.1) by φ_k, we obtain

$$\begin{aligned}
&\int_{Q_\tau}\frac{\partial^2 u_n}{\partial t^2}\varphi_k\,dx\,dt + \int_{Q_\tau}|\Delta u_n|^{p(x)-2}\Delta u_n\Delta\varphi_k + |u_n|^{q(x)-2}u_n\varphi_k + \varepsilon\Delta\frac{\partial u_n}{\partial t}\Delta\varphi_k \\
&\quad + a\biggl(t,\int_\Omega |u_n|^{q(x)}\,dx\biggr)\frac{\partial u_n}{\partial t}\varphi_k\,dx\,dt = \int_{Q_\tau} f_n\varphi_k\,dx\,dt.
\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty}\int_{Q_\tau}\frac{\partial^2 u_n}{\partial t^2}\varphi\,dx\,dt = \int_{Q_\tau} f\varphi - \xi\Delta\varphi - |u_\varepsilon|^{q(x)-2}u_\varepsilon\varphi - \varepsilon\Delta\frac{\partial u_\varepsilon}{\partial t}\Delta\varphi - \zeta\varphi\,dx\,dt. \tag{3.9}$$

Furthermore, for any $\psi(x) \in C_0^\infty(\Omega)$, we get

$$\begin{aligned}
&\int_\Omega\biggl(\frac{\partial u_\varepsilon(x,\tau)}{\partial t} - u_1\biggr)\psi\,dx \\
&\quad= \lim_{n\to\infty}\int_\Omega\biggl(\frac{\partial u_n(x,\tau)}{\partial t} - \frac{\partial u_n(x,0)}{\partial t}\biggr)\psi(x)\,dx \\
&\quad= \lim_{n\to\infty}\int_0^\tau\int_\Omega \frac{\partial^2 u_n}{\partial t^2}\psi(x)\,dx\,dt \\
&\quad= \int_{Q_\tau} f\varphi - \xi\Delta\varphi - |u_\varepsilon|^{q(x)-2}u_\varepsilon\varphi - \varepsilon\Delta\frac{\partial u_\varepsilon}{\partial t}\Delta\varphi - \zeta\varphi\,dx\,dt \to 0
\end{aligned}$$

as $\tau \to 0$. Similarly, for $t_0 \in [0,T]$, we have

$$\lim_{\tau\to 0}\int_\Omega\biggl(\frac{\partial u_\varepsilon(x,\tau)}{\partial t} - \frac{\partial u_\varepsilon(x,t_0)}{\partial t}\biggr)\psi\,dx = 0, \quad \text{for } \psi \in C_0^\infty(\Omega).$$

Furthermore, we obtain $\frac{\partial u_\varepsilon(x,0)}{\partial t} = u_1$. Since $u_\varepsilon \in L^\infty(0,T;W_0^{2,2}(\Omega))$ and $\frac{\partial u_\varepsilon}{\partial t} \in L^2(0,T;W_0^{2,2}(\Omega))$, we can assume that $u_\varepsilon \in C(0,T;W_0^{2,2}(\Omega))$. Lemma 3.3 and the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ imply that $\int_\Omega u_n^2(x,T)\,dx \le C(T)$. Thus, there exist a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ and a function $\widehat{u}$ such that $u_n(x,T) \rightharpoonup \widehat{u}$ weakly in $L^2(\Omega)$. For each $\varphi \in C_0^\infty(\Omega)$ and $\eta \in C^1([0,T])$, we have

$$\int_{Q_T} \frac{\partial u_n}{\partial t}\varphi\eta\,dx\,dt = \int_\Omega u_n(x,T)\varphi\eta(T) - u_n(x,0)\varphi\eta(0)\,dx - \int_{Q_T} u_n\varphi\eta'(t)\,dx\,dt.$$

Letting $n \to \infty$, we get

$$\int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}\varphi\eta\,dx\,dt = \int_\Omega \widehat{u}\varphi\eta(T) - u_0\varphi\eta(0)\,dx - \int_{Q_T} u_\varepsilon\varphi\eta'(t)\,dx\,dt.$$

By integration by parts, we have

$$\int_\Omega \bigl(u_\varepsilon(x,T) - \widehat{u}\bigr)\varphi\eta(T)\,dx = \int_\Omega \bigl(u_\varepsilon(x,0) - u_0\bigr)\varphi\eta(0)\,dx.$$

Choosing $\eta(T) = 1$, $\eta(0) = 0$ or $\eta(T) = 0$, $\eta(0) = 1$, we obtain $\widehat{u} = u_\varepsilon(x,T)$ and $u_\varepsilon(x,0) = u_0(x)$ for $x \in \Omega$. Similarly, we can prove that $\Delta u_\varepsilon(x,0) = \Delta u_0$, $\Delta u_n(x,T) \rightharpoonup \Delta u_\varepsilon(x,T)$ weakly in $L^2(\Omega)$ (up to a subsequence) and

$$\int_\Omega \bigl|\Delta u_\varepsilon(x,T)\bigr|^2\,dx \le \liminf_{n\to\infty} \int_\Omega \bigl|\Delta u_n(x,T)\bigr|^2\,dx. \tag{3.10}$$

Further, by the compact embedding $W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, we get $u_n(x,T) \to u_\varepsilon(x,T)$ strongly in $L^2(\Omega)$.

Taking $\varphi = u_k$ in (3.9) and letting $k \to \infty$, we get

$$\begin{aligned}&\int_\Omega \frac{\partial u_\varepsilon(x,T)}{\partial t}u_\varepsilon(x,T)\,dx - \int_\Omega u_1u_0\,dx - \int_{Q_T}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2\,dx\,dt + \int_{Q_T} \xi\,\Delta u_\varepsilon + |u_\varepsilon|^{q(x)} \\ &\quad + \varepsilon\Delta\frac{\partial u_\varepsilon}{\partial t}\Delta u_\varepsilon + \zeta u_\varepsilon\,dx\,dt = \int_{Q_T} fu_\varepsilon\,dx\,dt.\end{aligned} \tag{3.11}$$

Multiplying (3.1) by $(\eta_n)_j$ and summing up j from 1 to n, then integrating with respect to t over $[0,T]$, we have

$$\begin{aligned}&\int_0^T\int_\Omega \frac{\partial^2 u_n}{\partial t^2}u_n\,dx\,dt + \int_0^T\int_\Omega |\Delta u_n|^{p(x)} + |u_n|^{q(x)} + \varepsilon\Delta\frac{\partial u_n}{\partial t}\Delta u_n \\ &\quad + a\left(\int_\Omega |u_n|^{q(x)}\,dx\right)\frac{\partial u_n}{\partial t}u_n\,dx\,dt = \int_0^T\int_\Omega f_nu_n\,dx\,dt.\end{aligned} \tag{3.12}$$

Thus,

$$\begin{aligned}0 &\le \int_0^T\int_\Omega \bigl(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u_\varepsilon|^{p(x)-2}\Delta u_\varepsilon\bigr)(\Delta u_n - \Delta u_\varepsilon)\,dx\,dt \\ &= \int_0^T\int_\Omega f_nu_n - |u_n|^{q(x)} - a\left(\int_\Omega |u_n|^{p(x)}\,dx\right)\frac{\partial u_n}{\partial t}u_n - \varepsilon\Delta\frac{\partial u_n}{\partial t}\Delta u_n\,dx\,dt\end{aligned}$$

$$- \int_\Omega \frac{\partial u_n(x,T)}{\partial t} u_n(x,T)\,dx + \int_\Omega \frac{\partial u_n(x,0)}{\partial t} u_n(x,0)\,dx + \int_0^T \int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^2 dx\,dt$$
$$- \int_0^T \int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n \Delta u_\varepsilon + |\Delta u_\varepsilon|^{p(x)-2}\Delta u_\varepsilon(\Delta u_n - \Delta u_\varepsilon)\,dx\,dt.$$

By (3.10) and (3.11), we get

$$\begin{aligned}
&\limsup_{n\to\infty} \int_0^T \int_\Omega \big(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u_\varepsilon|^{p(x)-2}\Delta u_\varepsilon\big)(\Delta u_n - \Delta u_\varepsilon)\,dx\,dt \\
&\quad \le \int_0^T \int_\Omega f u_\varepsilon - |u_\varepsilon|^{p(x)} - a\left(\int_\Omega |u_\varepsilon|^{q(x)}\,dx\right)\frac{\partial u_\varepsilon}{\partial t} u_\varepsilon - \xi\,\Delta u_\varepsilon\,dx\,dt \\
&\qquad - \frac{\varepsilon}{2}\int_\Omega \big|\Delta u_\varepsilon(x,T)\big|^2\,dx + \frac{\varepsilon}{2}\int_\Omega \big|\Delta u_\varepsilon(x,0)\big|^2\,dx - \int_\Omega \frac{u_\varepsilon(x,T)}{\partial t} u_\varepsilon(x,T)\,dx \\
&\qquad + \int_\Omega u_1 u_0\,dx + \int_0^T \int_\Omega \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx\,dt \\
&\quad = 0.
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \int_0^T \int_\Omega \big(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u_\varepsilon|^{p(x)-2}\Delta u_\varepsilon\big)(\Delta u_n - \Delta u_\varepsilon)\,dx\,dt = 0.$$

Following the ideas of [4], we set $Q_1 = \{(x,t) \in Q_T : p(x) \ge 2\}$ and $Q_2 = \{(x,t) \in Q_T : 1 < p(x) < 2\}$, then, as $n \to \infty$,

$$\begin{aligned}
&\int_{Q_1} |\Delta u_n - \Delta u_\varepsilon|^{p(x)}\,dx\,dt \\
&\quad \le C \int_{Q_1} \big(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u_\varepsilon|^{p(x)-2}\Delta u_\varepsilon\big)(\Delta u_n - \Delta u_\varepsilon)\,dx\,dt \\
&\quad \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\int_{Q_2} |\Delta u_n - \Delta u_\varepsilon|^{p(x)}\,dx\,dt \\
&\quad \le C \Big\| \big[\big(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u_\varepsilon|^{p(x)-2}\Delta u_\varepsilon\big)(\Delta u_n - \Delta u_\varepsilon)\big]^{\frac{p(x)}{2}} \Big\|_{L^{\frac{2}{p(x)}}(Q_2)} \\
&\qquad \times \Big\| \big(|\Delta u_n|^{p(x)} + |\Delta u_\varepsilon|^{p(x)}\big)^{\frac{2-p(x)}{2}} \Big\|_{L^{\frac{2}{2-p(x)}}(Q_2)} \\
&\quad \to 0.
\end{aligned}$$

Therefore, we obtain $\Delta u_n \to \Delta u_\varepsilon$ strongly in $L^{p(x)}(Q_\tau)$. Thus, there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that $\Delta u_n \to \Delta u_\varepsilon$ a.e. on Q_T. Further,

$$|\Delta u_n|^{p(x)-2}\Delta u_n \to |\Delta u_\varepsilon|^{p(x)-2}\Delta u_\varepsilon, \quad \text{for a.e. } (x,t) \in Q_T.$$

In view of Theorem 2.4, we get $\xi = |\Delta u_\varepsilon|^{p(x)-2}\Delta u_\varepsilon$. Similarly, we can prove that $u_n \to u_\varepsilon$ strongly in $L^{q(x)}(Q_T)$. Thus, there exists a subsequence of $\{u_n\}$ still denoted $\{u_n\}$ such that

$$\lim_{n\to 0}\int_\Omega \left|u_n(x,t)-u_\varepsilon(x,t)\right|^{q(x)}\,dx = 0, \quad \text{for a.e. } t\in[0,T].$$

Furthermore, we have

$$\lim_{n\to 0}\int_\Omega \left|u_n(x,t)\right|^{q(x)}\,dx = \int_\Omega \left|u_\varepsilon(x,t)\right|^{q(x)}\,dx, \quad \text{for a.e. } t\in[0,T].$$

Thus, $a(\int_\Omega |u_n|^{q(x)}\,dx)\frac{\partial u_n}{\partial t} \to a(\int_\Omega |u_\varepsilon|^{q(x)}\,dx)\frac{\partial u_\varepsilon}{\partial t}$ a.e. on Q_T. By Theorem 2.4, we obtain $\zeta = a(\int_\Omega |u_\varepsilon|^{q(x)}\,dx)\frac{\partial u_\varepsilon}{\partial t}$. It follows from (3.9) that the theorem is proved. □

Remark 3.1 Obviously, in this section, the two inequalities in (H2) can be replaced by $1 < p^- \le p^+ < \infty$ and $1 < q^- \le q^+ < \infty$, respectively.

4 Existence of Young measure solutions for problem (1.2)

In this section, from the sequence of approximate solutions $\{u_\varepsilon\}_{0<\varepsilon<1}$ of problem (1.2), we shall prove that the limit of u_ε (as $\varepsilon \to 0^+$) is a Young measure solution of problem (1.1).

Definition 4.1 A pair (u,ν) is called a Young measure solution of problem (1.1) if

$$\begin{aligned}&u\in L^\infty\bigl(0,T;W_0^{2,p(x)}(\Omega)\bigr)\cap L^\infty\bigl(0,T;L^{q(x)}(\Omega)\bigr)\cap W^{1,\infty}\bigl(0,T;L^2(\Omega)\bigr),\\&\nu=\{\nu_{x,t}\}_{x,t} \text{ is a probability measure,}\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega \frac{\partial u(x,\tau)}{\partial t}\varphi\,dx - \int_\Omega u_1\varphi(x,0)\,dx - \int_{Q_T}\frac{\partial u}{\partial t}\frac{\partial \varphi}{\partial t}\,dx\,dt + \int_{Q_T}\int_{\mathbb{R}} |A|^{p(x)-2}A\,d\nu(A)\Delta\varphi\\&\quad + |u|^{q(x)-2}u\varphi\,dx\,dt + \int_{Q_T} a\left(\int_\Omega |u|^{q(x)}\,dx\right)\frac{\partial u}{\partial t}\varphi\,dx\,dt = \int_{Q_T} f\varphi\,dx\,dt,\end{aligned}$$

for all $\varphi\in C^1(0,T;C_0^\infty(\Omega))$ and $\tau\in(0,T]$.

Theorem 4.1 *Under conditions* (H1)-(H3), *problem* (1.1) *has a Young measure solution.*

Proof For each $\tau\in(0,T]$ and $\varphi\in C^1(0,T;C_0^\infty(\Omega))$, we have by (3.9)

$$\begin{aligned}&\int_\Omega \frac{\partial u_\varepsilon(x,\tau)}{\partial t}\varphi(x,\tau)\,dx - \int_\Omega u_1\varphi(x,0)\,dx - \int_{Q_\tau}\frac{\partial u_\varepsilon}{\partial t}\frac{\partial \varphi}{\partial t}\,dx\,dt + \int_{Q_\tau} |\Delta u_\varepsilon|^{p(x)-2}\Delta u_\varepsilon\Delta\varphi\\&\quad + |u_\varepsilon|^{q(x)-2}u_\varepsilon\varphi + a\left(\int_\Omega |u_\varepsilon|^{q(x)}\,dx\right)\frac{\partial u_\varepsilon}{\partial t}\varphi + \varepsilon\Delta\frac{\partial u_\varepsilon}{\partial t}\Delta\varphi\,dx\,dt = \int_{Q_\tau} f\varphi\,dx\,dt. \qquad (4.1)\end{aligned}$$

Since the constant in Lemma 3.2 is independent of n and ε, by the convergence of u_n and $\frac{\partial u_n}{\partial t}$ in Section 3, we have

$$\int_\Omega \left|\frac{\partial u_\varepsilon(x,t)}{\partial t}\right|^2 dx + \int_\Omega \left|\Delta u_\varepsilon(x,t)\right|^{p(x)} + \left|u_\varepsilon(x,t)\right|^{q(x)}\,dx + \varepsilon\int_\Omega \left|\Delta u_\varepsilon(x,t)\right|^2\,dx \le C$$

for a.e $t \in [0,T]$ and

$$\int_{Q_T}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx\,dt + \int_{Q_T}|\Delta u_\varepsilon|^{p(x)} + |u_\varepsilon|^{q(x)}\,dx\,dt + \varepsilon\int_{Q_T}\left|\Delta\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx\,dt \le C.$$

Similarly, by Lemma 3.3, we have

$$\begin{aligned}&\|u_\varepsilon\|_{L^\infty(0,T;W_0^{2,p(x)}(\Omega))} + \left\||\Delta u_\varepsilon|^{p(x)-2}\Delta u_\varepsilon\right\|_{L^{p'(x)}(Q_T)}\\ &\quad + \left\||u_\varepsilon|^{q(x)-2}u_\varepsilon\right\|_{L^{q'(x)}(Q_T)} + \left\|a\left(\int_\Omega |u_\varepsilon|^{q(x)}\,dx\right)\frac{\partial u_\varepsilon}{\partial t}\right\|_{L^2(Q_T)} \le C.\end{aligned} \tag{4.2}$$

Thus, there exists a subsequence of $\{u_\varepsilon\}_{0<\varepsilon<1}$ still denoted by $\{u_\varepsilon\}_{0<\varepsilon<1}$ such that

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{weakly} * \text{ in } L^\infty(0,T;W_0^{2,p(x)}(\Omega)\cap L^{q(x)}(\Omega)),\\ \Delta u_\varepsilon \rightharpoonup \Delta u & \text{weakly in } L^{p(x)}(Q_T),\\ u_\varepsilon \rightharpoonup u & \text{weakly in } L^{q(x)}(Q_T),\\ \frac{\partial u_\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} & \text{weakly} * \text{ in } L^\infty(0,T;L^2(\Omega)),\\ |u_\varepsilon|^{q(x)-2}u_\varepsilon \rightharpoonup \alpha & \text{weakly in} L^{q'(x)}(Q_T),\\ a(\int_\Omega |u_\varepsilon|^{q(x)}\,dx)\frac{\partial u_\varepsilon}{\partial t} \rightharpoonup \beta & \text{weakly in} L^2(Q_T).\end{cases}$$

Since $p^- > \max\{1, \frac{2N}{N+4}\}$, the embedding $W_0^{2,p(x)}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Further, as $u_\varepsilon \in L^\infty(0,T;W_0^{2,p(x)}(\Omega))$ and $\frac{\partial u_\varepsilon}{\partial t} \in L^\infty(0,T;L^2(\Omega))$, by the Lions-Aubin lemma, there exists a subsequence of $\{u_\varepsilon\}_{0<\varepsilon<1}$ still denoted by $\{u_\varepsilon\}_{0<\varepsilon<1}$ such that $u_\varepsilon \to u$ strongly in $L^2(Q_T)$ and a.e. on Q_T. Thus, $|u_\varepsilon|^{q(x)-2}u_\varepsilon \to |u|^{q(x)-2}u$ a.e. on Q_T. In view of Theorem 2.4, we obtain $\alpha = |u|^{q(x)-2}u$. By assumption (H1), we have $\mu = \inf_{\overline{\Omega}}(\frac{Np(x)}{N-2p(x)} - q(x)) > 0$. For each measurable subset $U \subset Q_T$ with $|U| \le 1$, by Hölder's inequality, Theorem 2.2, and Theorem 2.3, we obtain

$$\int_\Omega |u_\varepsilon|^{q(x)} \le 2\left\||u_\varepsilon|^{q(x)}\right\|_{L^{\frac{Np(x)}{(N-2p(x))q(x)}}(U)}\|1\|_{L^{\frac{Np(x)}{Np(x)-Nq(x)+2p(x)q(x)}}(U)} \le C|U|^{\frac{\mu(N-2p^+)}{Np^+}}.$$

Thus, the sequence $\{|u_\varepsilon - u|^{q(x)}\}_{0<\varepsilon<1}$ is equi-integrable on $L^1(Q_T)$. The Vitali convergence theorem implies that $\int_{Q_T}|u_\varepsilon - u|^{q(x)}\,dx\,dt \to 0$, that is to say, we obtain $u_\varepsilon \to u$ strongly in $L^{q(x)}(Q_T)$. Thus, there exists a subsequence of $\{u_\varepsilon\}_{0<\varepsilon<1}$ still labeled by $\{u_\varepsilon\}_{0<\varepsilon<1}$ such that

$$\lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon - u|^{q(x)}\,dx = 0, \quad \text{for a.e. } t \in [0,T].$$

Furthermore,

$$\lim_{\varepsilon\to 0}\int_\Omega \left||u_\varepsilon|^{q(x)} - |u|^{q(x)}\right|\,dx = 0, \quad \text{for a.e. } t \in [0,T].$$

Hence, we find by the continuity of a that $a(\int_\Omega |u_\varepsilon|^{q(x)}\,dx) \to a(\int_\Omega |u|^{q(x)}\,dx)$ for a.e. $t \in [0,T]$. Since $\int_\Omega |u_\varepsilon|^{q(x)}\,dx \le C$ for a.e. $t \in [0,T]$ and $a \in C([0,\infty))$, for each $\varphi \in L^2(Q_T)$, by Lebesgue's dominated convergence theorem, we have

$$a\left(\int_\Omega |u_\varepsilon|^{q(x)}\,dx\right)\varphi \to a\left(\int_\Omega |u|^{q(x)}\,dx\right)\varphi \quad \text{strongly in } L^2(Q_T).$$

Further, by the weak convergence of $\frac{\partial u_\varepsilon}{\partial t}$ in $L^2(Q_T)$, we get

$$\lim_{\varepsilon\to 0}\int_{Q_T} a\left(\int_\Omega |u_\varepsilon|^{q(x)}\,dx\right)\frac{\partial u_\varepsilon}{\partial t}\varphi\,dx\,dt = \lim_{\varepsilon\to 0}\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\left(a\left(\int_\Omega |u_\varepsilon|^{q(x)}\,dx\right)\varphi\right)dx\,dt$$
$$= \int_{Q_T} a\left(\int_\Omega |u|^{q(x)}\,dx\right)\frac{\partial u}{\partial t}\varphi\,dx\,dt.$$

Thus, $a(\int_\Omega |u_\varepsilon|^{q(x)}\,dx)\frac{\partial u_\varepsilon}{\partial t} \rightharpoonup a(\int_\Omega |u|^{q(x)}\,dx)\frac{\partial u}{\partial t}$ weakly in $L^2(Q_T)$. The uniqueness of the limit implies that $\beta = a(\int_\Omega |u|^{q(x)}\,dx)\frac{\partial u}{\partial t}$.

Finally, we prove that the sequence $\{\Delta u_\varepsilon\}_{0<\varepsilon<1}$ generates a Young measure $\{\nu_{x,t}\}_{x,t}$ such that

$$|\Delta u_\varepsilon|^{p(x)-2}\Delta u_\varepsilon \rightharpoonup \int_{\mathbb{R}} |A|^{p(x)-2}A\,d\nu_{x,t}(A) \quad \text{weakly in } L^1(Q_T). \tag{4.3}$$

Following Theorem 2.5, we first verify that the Young measure $\nu_{x,t}$ generated by the sequence $\{\Delta u_\varepsilon\}_{0<\varepsilon<1}$ is a probability measure for a.e. $(x,t)\in Q_T$. Indeed, for $s\le p^-$, we have

$$\int_{Q_T} |\Delta u_\varepsilon|^s\,dx \le |\Omega|T + \int_{Q_T} |\Delta u_\varepsilon|^{p(x)}\,dx \le C.$$

It follows from (iv) in Theorem 2.5 that $\nu_{x,t}$ is a probability measure. Set $H(x,A) = |A|^{p(x)-2}A$. Next, we prove that the sequence $\{H(x,\Delta u_\varepsilon)\}_\varepsilon$ is weakly relatively compact in $L^1(Q_T)$. It is clear that $\{H(x,\Delta u_\varepsilon)\}_\varepsilon$ will be weakly relatively compact in $L^1(Q_T)$, if we prove that $\{H(x,\Delta u_\varepsilon)\}_\varepsilon$ is uniformly bounded and equi-integrable on $L^1(Q_T)$; see [42], Proposition 1.3. Indeed, for each measurable subset $U\subset Q_T$ with $|U|\le 1$, by (4.2) and Hölder's inequality, we have

$$\int_U \bigl|H(x,\Delta u_\varepsilon)\bigr|\,dx\,dt = \int_U |\Delta u_\varepsilon|^{p(x)-1}\,dx\,dt \le 2\|\Delta u_\varepsilon\|_{L^{p'(x)}(U)}\|1\|_{L^{p(x)}(U)} \le C|U|^{\frac{1}{p^+}}.$$

Thus, the sequence $\{H(x,\Delta u_\varepsilon)\}_\varepsilon$ is equi-integrable. Similarly, the sequence $\{H(x,\Delta u_\varepsilon)\}_\varepsilon$ is uniformly bounded on $L^1(Q_T)$. Therefore, the convergence property (4.3) holds.

The estimate (4.2) implies

$$\varepsilon\Delta\frac{\partial u_\varepsilon}{\partial t} \to 0 \quad \text{strongly in } L^2(Q_T).$$

From the same procedures as in Section 3, we can prove there exists a subsequence of $\{u_\varepsilon\}_{0<\varepsilon<1}$ still denoted by $\{u_\varepsilon\}_{0<\varepsilon<1}$ such that $\frac{\partial u_\varepsilon(x,t)}{\partial t} \rightharpoonup \frac{\partial u(x,t)}{\partial t}$ weakly in $L^2(\Omega)$ uniformly on $[0,T]$. Taking $\varepsilon\to 0$ in Definition 4.1, we obtain

$$\int_\Omega \frac{\partial u(x,\tau)}{\partial t}\varphi(x,\tau)\,dx - \int_\Omega u_1\varphi(x,0)\,dx - \int_{Q_T}\frac{\partial u}{\partial t}\frac{\partial\varphi}{\partial t}\,dx\,dt$$
$$+ \int_{Q_T}\int_{\mathbb{R}} |A|^{p(x)-2}A\,d\nu_{x,t}(A)\Delta\varphi + |u|^{q(x)-2}u\varphi + a\left(\int_\Omega |u|^{q(x)}\,dx\right)\frac{\partial u}{\partial t}\varphi\,dx\,dt$$
$$= \int_{Q_T} f\varphi\,dx\,dt,$$

for all $\varphi\in C^1(0,T;C_0^\infty(\Omega))$ and $\tau\in(0,T]$. □

By Theorem 4.1, we have the following corollary.

Corollary 4.1 *Suppose that* $f(x,t) \equiv 0$ *and* (H1) *and* (H2) *are satisfied. Then for a given* $u_0 \in W^{2,p(x)}(\Omega) \cap W_0^{2,2}(\Omega)$, *there exist a function* $u : \Omega \times (0,\infty) \to \mathbb{R}$ *and a Young measure* $\nu_{x,t}$ *such that for* $\forall T > 0$,

$$u \in L^\infty\big(0,T;W_0^{2,p(x)}(\Omega)\big) \cap L^\infty\big(0,T;L^{q(x)}(\Omega)\big) \cap W^{1,\infty}\big(0,T;L^2(\Omega)\big)$$

and

$$\int_\Omega \frac{\partial u(x,\tau)}{\partial t}\varphi(x,\tau)\,dx - \int_\Omega \frac{\partial u(x,0)}{\partial t}\varphi(x,0)\,dx - \int_{Q_T} \frac{\partial u}{\partial t}\frac{\partial \varphi}{\partial t}\,dx\,dt$$
$$+ \int_{Q_T}\int_{\mathbb{R}} |A|^{p(x)-2}A\,d\nu_{x,t}(A)\Delta\varphi + |u|^{q(x)-2}u\varphi + a\left(\int_\Omega |u|^{q(x)}\,dx\right)\frac{\partial u}{\partial t}\varphi\,dx\,dt = 0,$$

for all $\varphi \in C^1(0,T;C_0^\infty(\Omega))$ *and* $\tau \in (0,T]$.

5 Energy decay of Young measure solutions

In this section, we give the decay estimates of weak solutions obtained by Corollary 4.1. First we give a lemma by Nakao [43]

Lemma 5.1 (see [43]) *Let* $\Psi : (0,\infty) \to \mathbb{R}$ *be a bounded nonnegative function. If there exist two constants* $\alpha > 0$ *and* $\beta \geq 0$ *such that*

$$\sup_{t\leq s\leq t+1} \Psi^{1+\beta}(s) \leq \alpha\big(\Psi(t) - \Psi(t+1)\big), \quad \textit{for } \forall t \geq 0,$$

then there exist positive constants C *and* γ *such that*

$$\begin{cases} \Psi(t) \leq Ce^{-\gamma t}, & \forall t \geq 0, \textit{as } \beta = 0, \\ \Psi(t) \leq C(t+1)^{-\frac{1}{\beta}}, & \forall t \geq 0, \textit{as } \beta > 0. \end{cases}$$

Theorem 5.1 *Let* $p^- \geq 2$. *Then there exist constants* $C, \gamma > 0$ *such that the weak solutions obtained by Corollary* 4.1 *have the following estimates*: *If* $p^+ = 2$, *then*

$$\int_\Omega \left|\frac{\partial u(x,t)}{\partial t}\right|^2 dx + \int_\Omega \big|u(x,t)\big|^{q(x)}\,dx \leq Ce^{-\gamma t}, \quad \textit{for a.e. } t \geq 0.$$

If $p^+ > 2$, *then*

$$\int_\Omega \left|\frac{\partial u(x,t)}{\partial t}\right|^2 dx + \int_\Omega \big|u(x,t)\big|^{q(x)}\,dx \leq C(t+1)^{-\frac{p^+}{p^+-2}}, \quad \textit{for a.e. } t \geq 0.$$

Proof We define

$$I_n(t) = \frac{1}{2}\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^2 dx + \int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)} + \frac{|u_n|^{q(x)}}{q(x)}\,dx.$$

The definition of $I_n(t)$ and equality (3.1) imply $I_n(t)$ is nonnegative and uniformly bounded. We assume that $I_n(t) \leq M$, $M > 0$ is a constant. For $\forall t > 0$ fixed, it follows from (3.9) and

(H2) that

$$\frac{d}{dt}I_n(t) + \varepsilon \int_\Omega \left|\Delta \frac{\partial u_n}{\partial t}\right|^2 dx + a_0 \int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^2 dx \le 0. \tag{5.1}$$

This implies $I_n(t)$ is a nonincreasing function. Putting $J_n^2(t) = I_n(t) - I_n(t+1)$ and integrating (5.1) over $(t, t+1)$, we get

$$\begin{aligned} J_n^2(t) &\ge \varepsilon \int_t^{t+1} \int_\Omega \left|\Delta \frac{\partial u_n(x,\tau)}{\partial t}\right|^2 + a_0 \left|\frac{\partial u_n(x,\tau)}{\partial t}\right|^2 dx\,d\tau \\ &\ge a_0 \int_t^{t+1} \int_\Omega \left|\frac{\partial u_n(x,\tau)}{\partial t}\right|^2 dx\,d\tau. \end{aligned} \tag{5.2}$$

By the mean value theorem and (5.2), there exist $t_1 \in [t, t+\frac{1}{3}]$ and $t_2 \in [t+\frac{2}{3}, t+1]$ such that

$$\int_\Omega \left|\frac{\partial u_n(x,t_i)}{\partial t}\right|^2 dx \le \frac{1}{a_0} J_n^2(t), \quad i = 1, 2. \tag{5.3}$$

From (3.1), we have

$$\begin{aligned} &\int_\Omega |\Delta u_n|^{p(x)} + |u_n|^{q(x)}\,dx \\ &\quad = -\int_\Omega \frac{\partial^2 u_n}{\partial t^2} u_n + \varepsilon \Delta \frac{\partial u_n}{\partial t} \Delta u_n + a\left(\int_\Omega |u_n|^{q(x)}\,dx\right) \frac{\partial u_n}{\partial t} u_n\,dx. \end{aligned} \tag{5.4}$$

Integrating (5.4) from t_1 to t_2, we obtain

$$\begin{aligned} &\int_{t_1}^{t_2} \int_\Omega \left|\Delta u_n(x,\tau)\right|^{p(x)} dx\,d\tau + \int_{t_1}^{t_2} \int_\Omega |u_n|^{q(x)}\,dx\,d\tau \\ &\quad = -\int_\Omega \frac{\partial u_n(x,t_2)}{\partial t} u_n(x,t_2)\,dx + \int_\Omega \frac{\partial u_n(x,t_1)}{\partial t} u_n(x,t_1)\,dx + \int_{t_1}^{t_2} \int_\Omega \left|\frac{\partial u_n(x,\tau)}{\partial t}\right|^2 dx\,d\tau \\ &\qquad - \varepsilon \int_{t_1}^{t_2} \int_\Omega \Delta \frac{\partial u_n(x,\tau)}{\partial t} \Delta u_n(x,\tau)\,dx\,d\tau \\ &\qquad - \int_{t_1}^{t_2} \int_\Omega a\left(\int_\Omega |u_n|^{q(x)}\,dx\right) \frac{\partial u_n}{\partial t} u_n\,dx\,d\tau. \end{aligned} \tag{5.5}$$

The Hölder inequality, (5.3), Theorem 2.2, Theorem 2.3, and $I_n(t)$ being decreasing imply

$$\begin{aligned} \left|\int_\Omega \frac{\partial u_n(x,t_i)}{\partial t} u_n(x,t_i)\,dx\right| &\le \left\|u_n(x,t_i)\right\|_{L^2(\Omega)} \left\|\frac{\partial u_n(x,t_i)}{\partial t}\right\|_{L^2(\Omega)} \\ &\le C_1 \left\|\Delta u_n(x,t_i)\right\|_{L^{p(x)}(\Omega)} J_n(t) \\ &\le C_2 \left(\int_\Omega \frac{|\Delta u_n(x,t_i)|^{p(x)}}{p(x)}\,dx\right)^{\frac{1}{p^+}} J_n(t) \\ &\le C_2 \big(I_n(t)\big)^{\frac{1}{p^+}} J_n(t), \quad i = 1, 2. \end{aligned} \tag{5.6}$$

Here the third inequality in (5.6) is obtained by

$$\begin{aligned}\|\Delta u_n(x,t_i)\|_{L^{p(x)}(\Omega)} &\le (p^+)^{\frac{1}{p^-}}\max\biggl\{\biggl(\int_\Omega \frac{|\Delta u_n(x,t_i)|^{p(x)}}{p(x)}\,dx\biggr)^{\frac{1}{p^-}},\\ &\quad \biggl(\int_\Omega \frac{|\Delta u_n(x,t_i)|^{p(x)}}{p(x)}\,dx\biggr)^{\frac{1}{p^+}}\biggr\}\\ &\le (p^+)^{\frac{1}{p^-}}\max\bigl\{M, M^{\frac{1}{p^-}-\frac{1}{p^+}}\bigr\}\biggl(\int_\Omega \frac{|\Delta u_n(x,t_i)|^{p(x)}}{p(x)}\,dx\biggr)^{\frac{1}{p^+}}, \quad i=1,2.\end{aligned}$$

Similarly,

$$\begin{aligned}&\biggl|\varepsilon\int_{t_1}^{t_2}\int_\Omega \Delta\frac{\partial u_n}{\partial t}\Delta u_n\,dx\,d\tau\biggr|\\ &\quad\le \int_{t_1}^{t_2}\biggl\|\varepsilon\Delta\frac{\partial u_n(x,\tau)}{\partial t}\biggr\|_{L^2(\Omega)}\sup_{t\le\tau\le t+1}\|\Delta u_n(x,\tau)\|_{L^2(\Omega)}\,d\tau\\ &\quad\le C_3\biggl(\varepsilon\int_{t_1}^{t_2}\int_\Omega \Delta\frac{\partial u_n(x,\tau)}{\partial t}\,dx\,d\tau\biggr)^{\frac12}\sup_{t\le\tau\le t+1}\|\Delta u_n(x,\tau)\|_{W^{2,p(x)}(\Omega)}\,d\tau\\ &\quad\le C_4 J_n(t)\bigl(I_n(t)\bigr)^{\frac{1}{p^+}}. \qquad (5.7)\end{aligned}$$

From the assumption (H2), Hölder's inequality, the second inequality in (5.2), Theorem 2.2, Theorem 2.3, and the boundedness of I_n, we have

$$\begin{aligned}&\biggl|\int_{t_1}^{t_2}\int_\Omega a\biggl(\int_\Omega |u_n|^{q(x)}\,dx\biggr)\frac{\partial u_n(x,\tau)}{\partial t}u_n(x,\tau)\,dx\,d\tau\biggr|\\ &\quad\le C_5\int_{t_1}^{t_2}\biggl\|\frac{\partial u_n}{\partial t}\biggr\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)}\,d\tau \le C_6 J_n(t)\bigl(I_n(t)\bigr)^{\frac{1}{p^+}}. \qquad (5.8)\end{aligned}$$

Gathering (5.5) with (5.6)-(5.8), we obtain

$$\int_{t_1}^{t_2}\int_\Omega \bigl|\Delta u_n(x,\tau)\bigr|^{p(x)} + \bigl|u_n(x,\tau)\bigr|^{q(x)}\,dx\,d\tau \le \frac{1}{a_0}J_n^2(t) + (2C_2+C_4+C_6)J_n(t)\bigl(I_n(t)\bigr)^{\frac{1}{p^+}}.$$

Thus,

$$\int_{t_1}^{t_2} I_n(\tau)\,d\tau \le \frac{3}{a_0}J_n^2(t) + (2C_2+C_4+C_6)J_n(t)\bigl(I_n(t)\bigr)^{\frac{1}{p^+}}.$$

By $I_n(t+1) \le 3\int_{t_1}^{t_2} I_n(\tau)\,d\tau$ and $I_n(t+1) = I_n(t) - J_n^2(t)$, we have

$$I_n(t) \le \biggl(1+\frac{9}{a_0}\biggr)J_n^2(t) + (6C_2+3C_4+3C_6)J_n(t)\bigl(I_n(t)\bigr)^{\frac{1}{p^+}}.$$

Further, Young's inequality yields

$$I_n(t) \le C_7 J_n^2(t) + C_8\bigl(J_n(t)\bigr)^{\frac{p^+}{p^+-1}}. \qquad (5.9)$$

Now we divide the proof in two cases: $p^+ = 2$ and $p^+ > 2$. We consider the case $p^+ = 2$ first. By the boundedness of $J_n(t)$, we have $I_n(t) \le C_9 J_n^2(t)$. Since $I_n(t)$ is nonincreasing, by Lemma 5.1, there exist constants $C > 0$ and $\gamma > 0$ such that

$$I_n(t) \le C\mathrm{e}^{-\gamma t}, \quad \forall t \ge 0.$$

Letting $n \to \infty$ in the above inequality, we arrive at

$$\int_\Omega \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx + \int_\Omega |\Delta u_\varepsilon|^{p(x)} + |u_\varepsilon|^{q(x)}\, dx \le C\mathrm{e}^{-\gamma t}, \quad \text{a.e. } t \ge 0.$$

Thus,

$$\int_\Omega \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx + \int_\Omega |u_\varepsilon|^{q(x)}\, dx \le C\mathrm{e}^{-\gamma t}, \quad \text{a.e. } t \ge 0.$$

Since $\frac{\partial u_\varepsilon(x,t)}{\partial t} \rightharpoonup \frac{\partial u(x,t)}{\partial t}$ weakly in $L^2(\Omega)$ uniformly on $[0,T]$ ($\forall T > 0$) and $u_\varepsilon \to u$ strongly in $L^{q(x)}(\Omega)$ for a.e. $t \in [0,T]$, we obtain

$$\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx + \int_\Omega |u|^{q(x)}\, dx \le C\mathrm{e}^{-\gamma t}, \quad \text{for a.e. } t \ge 0.$$

It remains to consider the case $p^+ > 2$. It follows from (5.9) that $I_n(t) \le C_{10}(J_n(t))^{\frac{p^+}{p^+-1}}$. Employing Lemma 5.1, we obtain

$$I_n(t) \le C(t+1)^{-\frac{p^+}{p^+-2}}.$$

Then letting $n \to \infty$, we deduce

$$\int_\Omega \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx + \int_\Omega |u_\varepsilon|^{q(x)}\, dx \le C(t+1)^{-\frac{p^+}{p^+-2}}, \quad \text{a.e. } t \ge 0.$$

Finally, letting $\varepsilon \to 0$, we conclude

$$\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx + \int_\Omega |u|^{q(x)}\, dx \le C(t+1)^{-\frac{p^+}{p^+-2}}, \quad \text{a.e. } t \ge 0.$$

Hence the theorem is proved. □

Competing interests
The author declares that they have no competing interests.

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