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Initial value problem for second-order random fuzzy differential equations

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Abstract

In this paper the second-order random fuzzy differential equations (SRFDEs) under generalized Hukuhara differentiability are introduced. Under suitable conditions we obtain the existence and uniqueness results of solutions to an SRFDE. To prove this assertion we use the idea of successive approximations. Some examples are given to illustrate these results.

Keywords: fuzzy random variables; random fuzzy differential equations; second-order random fuzzy differential equations; generalized Hukuhara derivative

1 Introduction

The study of fuzzy differential equations (FDEs) forms a suitable setting for the mathematical modeling of real-world problems in which uncertainties or vagueness pervade. Most practical problems can be modeled as FDEs [1, 2]. Therefore, FDEs are a very important topic both in theory and application, for example, in population models, in engineering, in chaotic systems and in modeling hydraulics. Differentiability of fuzzy-valued functions was first introduced by Chang and Zadeh [3], and followed by Dubois and Prade [4], who defined and used the extension principle [5]. Other approaches have been discussed by Puri and Ralescu [6], which generalized and extended the concept of Hukuhara differentiability for set-valued mappings to the class of fuzzy mappings. In this setting the fuzzy differential equations can be viewed as an abstract differential equation via embedding the fuzzy number space into Banach space. In this framework, many papers concerned with the existence and uniqueness problems. The problem of the existence and uniqueness begins with the investigations of Kaleva (see [7]) for the fuzzy Volterra integral equation that is equivalent to the initial value problem for fuzzy differential equations, where the Lipschitz condition and the Banach fixed point theorem and the method of successive approximations are applied in the problem of the existence and uniqueness of the solution. Wu *et al.* [8, 9] and Song and Wu [10] changed the initial value problem of fuzzy differential equations into abstract differential equations on a closed convex cone in a Banach space by the operator j, that is, the isometric embedding from (E^d, D_0) onto its range in the Banach space X. They established the relationship between a solution and its approximate type and dissipative-type conditions. Lupulescu [11] established a new concept of inner product on the fuzzy space. By help of these concepts author formulated some dissipative conditions for fuzzy initial value problem and, under these conditions, author established the global existence and uniqueness of a solution of fuzzy differential equations. In the last few years, many researchers have worked on the theoretical of fuzzy differential equations [12–14] and other recent works such as the study of some topological properties and structure of the solutions to the Cauchy problem for fuzzy differential systems (see [15, 16]). Subsequently, some very important extensions of the fuzzy differential equations based on H-derivative are the fuzzy functional differential equations [17], the random fuzzy differential equations [18], the fuzzy neutral differential equations [19], and the fuzzy fractional differential equations [20, 21]. However, the approach using Hukuhara differentiation suffers a grave disadvantage, *i.e.*, the solution has the property that the diameter $\mathrm{diam}[x(t)]^\alpha$ is nondecreasing in t, and so it is very hard to get any deep results on qualitative theory for fuzzy differential equations, such as asymptotic property, periodicity, bifurcation. Furthermore, Bede [22] proved that a large class of two-point boundary value problems have no solutions at all under H-differentiability.

Recently, Bede *et al.* [23–25] and Stefanini and Bede [26] solved the above mentioned approach under strongly generalized differentiability of fuzzy-number-valued functions and studied fuzzy initial valued for the fuzzy differential equations involving strongly generalized differentiability. In this case the derivative exists and the solutions of fuzzy differential equations may have decreasing diameters, but the uniqueness is lost. Thus, almost all important discussions on the qualitative problems for FDEs are deduced in the framework of this approach (see [22, 27, 28]). Therefore, our point is that the generalization of the concept of H-differentiability can be of great help in the dynamic study of fuzzy differential equations and random fuzzy problems. In [24], first-order linear fuzzy differential equation under generalized differentiability concept are considered and solutions of this problem in some especial cases were presented. See also [29, 30] Malinowski studied two kinds of solutions to random fuzzy initial value problem under strongly generalized differentiability. In [31] a linear fuzzy nuclear decay equation under generalized differentiability is studied and numerical solutions are found. Meanwhile, Allahviranloo *et al.* [32–35] and Khastan *et al.* [36] have solved these FDEs in the sense of generalized derivatives. Subsequently, some extensions of the fuzzy differential equations based on generalized differentiability are the fuzzy functional integro-differential equations [37] and the random fuzzy integro-differential equations [38–40].

Random fuzzy differential equations (RFDEs) deal with the real phenomena, not only with randomness but also with fuzziness. Puri and Ralescu introduced a fuzzy-set-valued random variable in [41], and gave the concept of differentiability by Hukuhara difference in [6]. In the literature, one can find various definitions of fuzzy random variables. For the first time the concept of a fuzzy random variable was proposed by Kwakernaak [42]. Further, it was used by Kruse and Meyer [43]. In [41, 44], there appear two notions of measurability of fuzzy mappings. The relations between different concepts of measurability for fuzzy random variables are contained in the papers of Colubi *et al.* [45], Terán Agraz [46], López-Díaz and Ralescu [47]. In this paper, we will use a definition of fuzzy random variable which was introduced by Puri and Ralescu [48]. This definition is currently the one most often used in probabilistic and statistical aspects of the theory of fuzzy random variables.

In [18, 49], the authors considered the random fuzzy differential equation with initial value

$$x'(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} f_\omega\bigl(t,x(t,\omega)\bigr), \qquad x(t_0,\omega) \overset{\mathbb{P}.1}{=} x_0(\omega) \in E^d, \tag{1.1}$$

where $f : \Omega \times [t_0, t_0 + p] \times E^d \to E^d$ and the symbol $'$ denotes the fuzzy derivative is understood in the sense of Puri and Ralescu [6]. Malinowski also showed that if f is continuous and $f_\omega(t, x)$ satisfies the Lipschitz condition with respect to x, then there exists a unique local solution for the random fuzzy initial value problem (1.1). In [49] the existence and uniqueness of the solution for RFDEs with non-Lipschitz coefficients is proven. Furthermore, using generalized Hukuhara differentiability, Malinowski [29, 30] studied two kinds of solutions to (1.1) under condition that the right-hand side of equation is Lipschitzian and generalized Lipschitz. Author established the local and global existence and uniqueness results for (1.1) by using the method of successive approximations. Besides, in fact, a large class of physically important problem is described by fuzzy random differential systems. We believed that mathematical models of physical phenomena should have the properties that existence and uniqueness of solution and the solution's behavior changes continuously with the initial conditions. The importance of existence and uniqueness theorems in the study of initial value problems is well known due to their relevance in establishing the well-posedness of the real-world problems arising in physical and engineering systems. Uniqueness results play a significant role in the continuation of solutions and in the theory of autonomous systems. While the uniqueness results almost always come at the cost of stringent conditions, they are valuable, for without such uniqueness results it is impossible to make predictions about the behavior of physical systems. Therefore, in this paper, we consider the second-order random fuzzy differential equation initial value problem of the form

$$\begin{cases} D_{\mathrm{H}}^{2,g} x(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} f_\omega(t, x(t,\omega), D_{\mathrm{H}}^{1,g} x(t,\omega)), \\ x(t_0,\omega) \overset{\mathbb{P}.1}{=} I_1(\omega), \qquad D_{\mathrm{H}}^{1,g} x(t_0,\omega) \overset{\mathbb{P}.1}{=} I_2(\omega) \in E^d, \end{cases} \tag{1.2}$$

where $f : \Omega \times [t_0, t_0 + p] \times E^d \times E^d \to E^d$ and the symbol $D_{\mathrm{H}}^{2,g}$ denotes the second-order generalized Hukuhara derivative. The purpose of this article is to discuss the behaviors of solutions to the second-order random fuzzy differential equations under generalized Hukuhara differentiability, such as the existence and uniqueness of solutions, and that the solution's behavior changes continuously with the initial conditions, which are important in the theory of fuzzy stochastic dynamical system analysis.

In this paper, we study four kinds of solutions to SRFDEs. The different types of solutions to SRFDEs are generated by the usage of two different concepts of the fuzzy derivative. We were inspired and motivated by the results of Bede and Gal [23], Malinowski [29, 30, 50], and Allahviranloo *et al.* [51] concerning deterministic FDEs with generalized fuzzy derivative and recently by the paper of Stefanini and Bede [26] where two types of solutions to interval differential equations were investigated.

The paper is organized as follows: In Section 2, we collect the fundamental notions and facts about fuzzy set space, fuzzy differentiation and integration. We recall the notions of fuzzy random variable and fuzzy stochastic process. In Section 3, we discuss the SRFDEs with two kinds of fuzzy derivatives. For both cases, under suitable conditions we prove the existence and uniqueness of solutions to SRFDEs by using a contraction principle and the method of successive approximations. We carry out an analysis of the behavior of the solutions when data of the equation are subject to errors. In Section 4, we provide some examples to illustrate these results.

2 Preliminaries

In this section, we give some notations and properties related to fuzzy set space, and summarize the major results for integration and differentiation of fuzzy-set-valued mappings. We recall also the notations of fuzzy random variable and fuzzy stochastic process. Let $K_c(\mathbb{R}^d)$ denote the family of all nonempty, compact and convex subsets of $\mathbb{R}^d$. The addition and scalar multiplication in $K_c(\mathbb{R}^d)$ are defined as usual, *i.e.*, for $A, B \in K_c(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$,

$$A + B = \{a + b \mid a \in A, b \in B\}, \qquad \lambda A = \{\lambda a \mid a \in A\}.$$

The Hausdorff distance or Pompeiu-Hausdorff distance d_{H} in $K_c(\mathbb{R}^d)$ is defined as follows:

$$d_{\mathrm{H}}(A, B) = \max\Big\{\sup_{a\in A}\inf_{b\in B}\|a - b\|, \sup_{b\in B}\inf_{a\in A}\|a - b\|\Big\},$$

where $A, B \in K_c(\mathbb{R}^d)$, and $\|\cdot\|$ denotes usual Euclidean norm in $\mathbb{R}^d$. It is well known (see [52]) that $K_c(\mathbb{R}^d)$ is a complete, separable, and locally compact metric space with respect to d_{H}. Define $E^d = \{u : \mathbb{R}^d \to [0, 1]$ such that $u(z)$ satisfies (i)-(iv) stated below$\}$:

(i) u is normal, that is, there exists $z_0 \in \mathbb{R}^d$ such that $u(z_0) = 1$;
(ii) u is fuzzy convex, *i.e.*, $u(\lambda z_1 + (1 - \lambda)z_2) \geq \min\{u(z_1), u(z_2)\}$ for any $z_1, z_2 \in \mathbb{R}^d$ and $\lambda \in [0, 1]$;
(iii) u is upper semicontinuous;
(iv) $[u]^0 = \mathrm{cl}\{z \in \mathbb{R}^d : u(z) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}^d, \|\cdot\|)$.

Elements of E^d are often called fuzzy sets of $\mathbb{R}^d$. For $\alpha \in (0, 1]$, define $[u]^\alpha = \{z \in \mathbb{R}^d \mid u(z) \geq \alpha\}$. We will call this set an α-cut (α-level set) of the fuzzy set u. For $u \in E^d$ one has $[u]^\alpha \in K_c(\mathbb{R}^d)$ for every $\alpha \in [0, 1]$. For two fuzzy sets $u_1, u_2 \in E^d$, we denote $u_1 \leq u_2$ if and only if $[u_1]^\alpha \subset [u_2]^\alpha$. If $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a function then, according to Zadeh's extension principle, one can extend (*cf.* [7]) g to $E^d \times E^d \to E^d$ by the formula $g(u_1, u_2)(z) = \sup_{z=g(z_1,z_2)} \min\{u_1(z_1), u_2(z_2)\}$. It is well known (see [53]) that if g is continuous then $[g(u_1, u_2)]^\alpha = g([u_1]^\alpha, [u_2]^\alpha)$ for all $u_1, u_2 \in E^d$, $\alpha \in [0, 1]$. Especially, for addition and scalar multiplication in fuzzy set space E^d, we have (*cf.* [12]): $[u_1 + u_2]^\alpha = [u_1]^\alpha + [u_2]^\alpha$, $[\lambda u_1]^\alpha = \lambda[u_1]^\alpha$. In the case $d = 1$, the α-cut set of a fuzzy number u is a closed bounded interval $[\underline{u}(\alpha), \overline{u}(\alpha)]$, where $\underline{u}(\alpha)$ denotes the left-hand endpoint of $[u]^\alpha$ and $\overline{u}(\alpha)$ denotes the right-hand endpoint of $[u]^\alpha$. It should be noted that, for $a \leq b \leq c$, $a, b, c \in \mathbb{R}$, a triangular fuzzy number $u = (a, b, c)$ is given such that $\underline{u}(\alpha) = a + (b - a)\alpha$ and $\overline{u}(\alpha) = c - (c - b)\alpha$ are the endpoints of the α-cut for all $\alpha \in [0, 1]$. Let us denote by

$$D_0[u_1, u_2] = \sup\big\{d_{\mathrm{H}}\big([u_1]^\alpha, [u_2]^\alpha\big) : 0 \leq \alpha \leq 1\big\}$$

the distance between u_1 and u_2 in E^d, where $d_{\mathrm{H}}([u_1]^\alpha, [u_2]^\alpha)$ is the Pompeiu-Hausdorff distance between two sets $[u_1]^\alpha, [u_2]^\alpha$ of $K_c(\mathbb{R}^d)$. In fact (E^d, D_0) is a complete metric space. Some properties of metric D_0 are as follows (see *e.g.* [41]):

$$D_0[u_1 + u_3, u_2 + u_3] = D_0[u_1, u_2],$$

$$D_0[\lambda u_1, \lambda u_2] = |\lambda| D_0[u_1, u_2],$$

$$D_0[u_1 + u_3, u_2 + u_4] \leq D_0[u_1, u_2] + D_0[u_3, u_4]$$

for all $u_1, u_2, u_3, u_4 \in E^d$ and $\lambda \in \mathbb{R}$. It is also known that (E^d, D_0) is not separable and is not locally compact (*cf.* [44, 54]). Let $u, v \in E^d$. If there exists $w \in E^d$ such that $u = v + w$, then w is called the H-difference of u, v and it is denoted by $u \ominus v$. Let us remark that $u \ominus v \neq u + (-1)v$. Let us denote $\hat{0} \in E^d$ the zero element of E^d as follows: $\hat{0}(z) = 1$ if $z = 0$ and $\hat{0}(z) = 0$ if $z \neq 0$, where 0 is the zero element of $\mathbb{R}^d$.

One can verify the following remark (*cf.* [29, 30]).

Remark 2.1 Let $u_1, u_2, u_3, u_4 \in E^d$.

(P1) If $u_1 \ominus u_2$, $u_1 \ominus u_3$ exist, then $D_0[u_1 \ominus u_2, \hat{0}] = D_0[u_1, u_2]$ and $D_0[u_1 \ominus u_2, u_1 \ominus u_3] = D_0[u_2, u_3]$.

(P2) If $u_1 \ominus u_2$, $u_3 \ominus u_4$ exist, then $D_0[u_1 \ominus u_2, u_3 \ominus u_4] = D_0[u_1 + u_4, u_2 + u_3]$.

(P3) If $u_1 \ominus u_2$, $u_1 \ominus (u_2 + u_3)$ exist, then there exist $(u_1 \ominus u_2) \ominus u_3$ and $(u_1 \ominus u_2) \ominus u_3 = u_1 \ominus (u_2 + u_3)$.

(P4) If $u_1 \ominus u_2$, $u_1 \ominus u_3$, $u_3 \ominus u_2$ exist, then there exist $(u_1 \ominus u_2) \ominus (u_1 \ominus u_3)$ and $(u_1 \ominus u_2) \ominus (u_1 \ominus u_3) = u_3 \ominus u_2$.

Further we want to introduce the notions of integrability and differentiability which will be used in the paper. Let $[a, b] \subset \mathbb{R}$ be a compact interval, $-\infty < a < b < +\infty$. We recall some measurability and integrability properties for the fuzzy mappings in [41, 52].

A fuzzy mapping $x : I = [a, b] \to E^d$ is called strongly measurable if for all $\alpha \in [0, 1]$ the set-valued mapping $x_\alpha : I \to K_c(\mathbb{R}^d)$ defined by $x_\alpha(t) = [x(t)]^\alpha$ is Lebesgue measurable. A fuzzy mapping $x : I \to E^d$ is called integrably bounded, if there exists an integrable function $h : I \to \mathbb{R}^+$ such that $\|\varphi(t)\| \leq h(t)$, for all $\varphi \in [x(t)]^0$.

Definition 2.1 (see Puri and Ralescu [41]) Let $x : I \to E^d$. The integral of x over I, denoted by $\int_I x(t)\,dt$, is defined levelwise by the expression

$$\left[\int_I x(t)\,dt\right]^\alpha = \int_I x_\alpha(t)\,dt = \left\{\int_I \varphi(t)\,dt \;\middle|\; \varphi : I \to \mathbb{R}^d \text{ is a measurable selection for } x_\alpha\right\}$$

for all $\alpha \in (0, 1]$.

By virtue of Remark 4.1 in [7] we have $[\int_I x(t)\,dt]^0 = \int_I [x(t)]^0\,dt$. A strongly measurable and integrably bounded mapping $x : I \to E^d$ is said to be integrable over I if $\int_I x(t)\,dt \in E^d$. We recall (see [7, 10, 52]) some properties of integrability for fuzzy mappings.

(P5) If $x : I \to E^d$ is strongly measurable and integrably bounded, then x is integrable.

(P6) If $x : I \to E^d$ is continuous, then it is integrable.

(P7) If $x : I \to E^d$ is continuous, then $u(t) = \int_a^t x(s)\,ds$ is Lipschitz continuous on $[a, b]$.

(P8) Let $x : I \to E^d$ be integrable over I. Then, for any $c \in (a, b)$, x is integrable over $[a, c]$ and $[c, b]$, and

$$\int_a^b x(s)\,ds = \int_a^c x(s)\,ds + \int_c^b x(s)\,ds.$$

Proposition 2.1 can also be found in [9].

Proposition 2.1 *Let $x, y : I \to E^d$ be integrable and $\lambda \in \mathbb{R}$. Then*

(i) $\int_a^b (x(t) + y(t))\,dt = \int_a^b x(t)\,dt + \int_a^b y(t)\,dt$;

(ii) $\int_a^b \lambda x(t)\,dt = \lambda \int_a^b x(t)\,dt$;

(iii) $D_0[x,y]$ *is integrable*;

(iv) $D_0[\int_a^b x(t)\,dt, \int_a^b y(t)\,dt] \le \int_a^b D_0[x(t),y(t)]\,dt$.

It is well known that the strongly generalized differentiability was introduced in [23] and studied in [24, 29–31, 34, 50].

Definition 2.2 Let $x : [a,b] \to E^d$ and $t \in [a,b]$. We say that x is strongly generalized differentiable of the first-order differential at t, if there exists $D_{\mathrm{H}}^{1,g}x(t) \in E^d$, such that

(i) for all $h > 0$ sufficiently small, $\exists x(t+h) \ominus x(t)$, $\exists x(t) \ominus x(t-h)$ and

$$\lim_{h\searrow 0} D_0\left[\frac{x(t+h)\ominus x(t)}{h}, D_{\mathrm{H}}^{1,g}x(t)\right] = 0, \qquad \lim_{h\searrow 0} D_0\left[\frac{x(t)\ominus x(t-h)}{h}, D_{\mathrm{H}}^{1,g}x(t)\right] = 0$$

or

(ii) for all $h > 0$ sufficiently small, $\exists x(t) \ominus x(t+h)$, $\exists x(t-h) \ominus x(t)$, and

$$\lim_{h\searrow 0} D_0\left[\frac{x(t)\ominus x(t+h)}{-h}, D_{\mathrm{H}}^{1,g}x(t)\right] = 0, \qquad \lim_{h\searrow 0} D_0\left[\frac{x(t-h)\ominus x(t)}{-h}, D_{\mathrm{H}}^{1,g}x(t)\right] = 0$$

or

(iii) for all $h > 0$ sufficiently small, $\exists x(t+h) \ominus x(t)$, $\exists x(t-h) \ominus x(t)$, and

$$\lim_{h\searrow 0} D_0\left[\frac{x(t+h)\ominus x(t)}{h}, D_{\mathrm{H}}^{1,g}x(t)\right] = 0, \qquad \lim_{h\searrow 0} D_0\left[\frac{x(t-h)\ominus x(t)}{-h}, D_{\mathrm{H}}^{1,g}x(t)\right] = 0$$

or

(iv) for all $h > 0$ sufficiently small, $\exists x(t) \ominus x(t+h)$, $\exists x(t) \ominus x(t-h)$, and

$$\lim_{h\searrow 0} D_0\left[\frac{x(t)\ominus x(t+h)}{-h}, D_{\mathrm{H}}^{1,g}x(t)\right] = 0, \qquad \lim_{h\searrow 0} D_0\left[\frac{x(t)\ominus x(t-h)}{h}, D_{\mathrm{H}}^{1,g}x(t)\right] = 0.$$

We say that a function is (i)-differentiable if it is strongly generalized differentiable as in case (i) of the definition above, *etc.*

Lemma 2.1 (Bede and Gal [23]) *If* $x(t) = (x_1(t), x_2(t), x_3(t))$ *is a triangular fuzzy-valued function*, *then*

(i) *if* x *is* (i)-*differentiable* (*i.e.*, *Hukuhara differentiable*), *then* $D_{\mathrm{H}}^{1,g}x(t) = (x_1'(t), x_2'(t), x_3'(t))$;

(ii) *if* x *is* (ii)-*differentiable*, *then* $D_{\mathrm{H}}^{1,g}x(t) = (x_3'(t), x_2'(t), x_1'(t))$.

Lemma 2.2 (Chalco-Cano and Román-Flores [55]) *Let* $x : I \to E^1$ *be a fuzzy-valued function and denote* $[x(t)]^\alpha = [\underline{x}(t,\alpha), \overline{x}(t,\alpha)]$ *for each* $\alpha \in [0,1]$. *Then*:

(i) *If* x *is* (i)-*differentiable*, *then* $\underline{x}(t,\alpha)$ *and* $\overline{x}(t,\alpha)$ *are differentiable functions and* $[D_{\mathrm{H}}^{1,g}x(t)]^\alpha = [\underline{x}'(t,\alpha), \overline{x}'(t,\alpha)]$.

(ii) *If* x *is* (ii)-*differentiable*, *then* $\underline{x}(t,\alpha)$ *and* $\overline{x}(t,\alpha)$ *are differentiable functions and* $[D_{\mathrm{H}}^{1,g}x(t)]^\alpha = [\overline{x}'(t,\alpha), \underline{x}'(t,\alpha)]$.

Theorem 2.1 *Let* $x : (a,b) \to E^d$ *be* (i)-*differentiable or* (ii)-*differentiable on* (a,b), *and assume that the derivative* $D_{\mathrm{H}}^{1,g}x$ *is integrable over* (a,b). *We have*

(a) *if* x *is* (i)-*differentiable on* (a,b), *then* $\int_a^b D_{\mathrm{H}}^{1,g}x(t)\,dt = x(b) \ominus x(a)$;

(b) *if* x *is* (ii)-*differentiable on* (a,b), *then* $\int_a^b D_{\mathrm{H}}^{1,g}x(t)\,dt = (-1)(x(a) \ominus x(b))$.

One can obtain a formulation of equivalence between solutions of first-order random fuzzy differential equations and random fuzzy integral equations (*see* [29, 30]).

Lemma 2.3 *The first-order random fuzzy differential equation*

$$D_{\mathrm{H}}^{1,g}x(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} g_\omega\bigl(t,x(t,\omega)\bigr), \qquad x(t_0,\omega) \overset{\mathbb{P}.1}{=} x_0(\omega) \in E^d, \tag{2.1}$$

where $g_\omega(\cdot,\cdot) : [t_0,t_0+p] \times E^d \to E^d$ *is supposed to be continuous with* $\mathbb{P}.1$, *is equivalent to one of the integral equations*

$$x(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} x_0(\omega) + \int_{t_0}^{t} g_\omega\bigl(s,x(s,\omega)\bigr)\,ds \tag{2.2}$$

for case (i)-*differentiability, or*

$$x(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} x_0(\omega) \ominus (-1)\int_{t_0}^{t} g_\omega\bigl(s,x(s,\omega)\bigr)\,ds \tag{2.3}$$

for case (ii)-*differentiability* (*where* $0 < r \le p$ *such that equation* (2.3) *is well defined, i.e., the foregoing Hukuhara difference does exist*). *Moreover, if* $x : I \times \Omega \to E^d$ *is a solution to random fuzzy integral equation* (2.2) (*random fuzzy integral equation* (2.3)), *then the function* $t \mapsto \operatorname{diam}[x(t,\omega)]^\alpha$ *is nondecreasing* (*nonincreasing*) *for* $\mathbb{P}$-*a.a.* $\omega \in \Omega$ *and for every* $\alpha \in [0,1]$, *where* $\operatorname{diam}[x(t,\omega)]^\alpha$ *denotes the diameter of the set* $[x(t,\omega)]^\alpha \in K_c(\mathbb{R}^d)$.

In the sequel, we express the definition of second-order strongly generalized differentiability which is proposed in [51].

Definition 2.3 Let $x : (a,b) \to E^d$ and $t \in (a,b)$. We say that x is strongly generalized differentiable of the second-order differential at t, if there exists $D_{\mathrm{H}}^{2,g}x(t) \in E^d$, such that

(i) for all $h > 0$ sufficiently small, $\exists D_{\mathrm{H}}^{1,g}x(t+h) \ominus D_{\mathrm{H}}^{1,g}x(t)$, $\exists D_{\mathrm{H}}^{1,g}x(t) \ominus D_{\mathrm{H}}^{1,g}x(t-h)$ and the following limits hold (in the metric D_0):

$$\lim_{h\searrow 0}\frac{D_{\mathrm{H}}^{1,g}x(t+h) \ominus D_{\mathrm{H}}^{1,g}x(t)}{h} = \lim_{h\searrow 0}\frac{D_{\mathrm{H}}^{1,g}x(t) \ominus D_{\mathrm{H}}^{1,g}x(t-h)}{h} = D_{\mathrm{H}}^{2,g}x(t)$$

or

(ii) for all $h > 0$ sufficiently small, $\exists D_{\mathrm{H}}^{1,g}x(t) \ominus D_{\mathrm{H}}^{1,g}x(t+h)$, $\exists D_{\mathrm{H}}^{1,g}x(t-h) \ominus D_{\mathrm{H}}^{1,g}x(t)$, and the following limits hold (in the metric D_0):

$$\lim_{h\searrow 0}\frac{D_{\mathrm{H}}^{1,g}x(t) \ominus D_{\mathrm{H}}^{1,g}x(t+h)}{-h} = \lim_{h\searrow 0}\frac{D_{\mathrm{H}}^{1,g}x(t-h) \ominus D_{\mathrm{H}}^{1,g}x(t)}{-h} = D_{\mathrm{H}}^{2,g}x(t)$$

or

(iii) for all $h > 0$ sufficiently small, $\exists D_{\mathrm{H}}^{1,g}x(t+h) \ominus D_{\mathrm{H}}^{1,g}x(t)$, $\exists D_{\mathrm{H}}^{1,g}x(t-h) \ominus D_{\mathrm{H}}^{1,g}x(t)$, and the following limits hold (in the metric D_0):

$$\lim_{h\searrow 0}\frac{D_{\mathrm{H}}^{1,g}x(t+h) \ominus D_{\mathrm{H}}^{1,g}x(t)}{h} = \lim_{h\searrow 0}\frac{D_{\mathrm{H}}^{1,g}x(t-h) \ominus D_{\mathrm{H}}^{1,g}x(t)}{-h} = D_{\mathrm{H}}^{2,g}x(t)$$

or

(iv) for all $h > 0$ sufficiently small, $\exists D_H^{1,g}x(t) \ominus D_H^{1,g}x(t+h)$, $\exists D_H^{1,g}x(t) \ominus D_H^{1,g}x(t-h)$, and the following limits hold (in the metric D_0):

$$\lim_{h\searrow 0} \frac{D_H^{1,g}x(t) \ominus D_H^{1,g}x(t+h)}{-h} = \lim_{h\searrow 0} \frac{D_H^{1,g}x(t) \ominus D_H^{1,g}x(t-h)}{h} = D_H^{2,g}x(t).$$

In this paper we consider only the two first of Definition 2.3. Further, we say that x is (i-i)-differentiable ((ii-ii)-differentiable) on I, if x and its derivative are differentiable in the sense (i) (in the sense (ii)) of Definition 2.2 and (i) ((ii)) of Definition 2.3, respectively. Similarly, we say that x is (i-ii)-differentiable ((ii-i)-differentiable) on I, if x and its derivative are differentiable in the sense (i) (in the sense (ii)) of Definition 2.2 and (ii) ((i)) of Definition 2.3, respectively.

Similar to Lemma 2.2, we have the following result for second-order derivative under generalized Hukuhara differentiability.

Theorem 2.2 [35] *Let $x : [a,b] \to E^1$ and $D_H^{1,g}x : [a,b] \to E^1$ are two differentiable fuzzy-valued functions. Moreover, we denote the α-cut representation of the fuzzy-valued function $x(t)$ by $[x(t)]^\alpha = [\underline{x}(t,\alpha), \overline{x}(t,\alpha)]$, then*:

(a) *Let $x(t)$ and $D_H^{1,g}x(t)$ be* (i)*-differentiable, or let $x(t)$ and $D_H^{1,g}x(t)$ be* (ii)*-differentiable; then: $\underline{x}(t,\alpha)$, $\overline{x}(t,\alpha)$ have first-order and second-order derivatives and*

$$\left[D_H^{2,g}x(t)\right]^\alpha = \left[\left(\underline{x}(t,\alpha)\right)'', \left(\overline{x}(t,\alpha)\right)''\right].$$

(b) *Let $x(t)$ be* (i)*-differentiable and $D_H^{1,g}x(t)$ be* (ii)*-differentiable, or, let $x(t)$ be* (ii)*-differentiable and $D_H^{1,g}x(t)$ be* (i)*-differentiable; then $\underline{x}(t,\alpha)$, $\overline{x}(t,\alpha)$ have first-order and second-order derivatives and*

$$\left[D_H^{2,g}x(t)\right]^\alpha = \left[\left(\overline{x}(t,\alpha)\right)'', \left(\underline{x}(t,\alpha)\right)''\right].$$

For $I = [a,b] \subset \mathbb{R}$ let $C(I,E^d)$ denote the space of continuous mappings form I to E^d. Define a metric H in $C(I,E^d)$ by $H[z,w] = \sup_{t\in[a,b]} D_0[z(t),w(t)]$, where $z,w \in C(I,E^d)$. It is well known that $(C(I,E^d),H)$ is a complete metric space. Moreover, in vector form, for $Z, W \in C(I,E^d \times E^d)$, we define $\mathcal{H}[Z,W] = \sup_{t\in[a,b]} \mathcal{D}_0[Z(t),W(t)]$, where $\mathcal{D}_0[Z,W] = \max\{D_0[z_1,w_1], D_0[z_2,w_2]\}$, $Z = (z_1,z_2)$, $W = (w_1,w_2) \in E^d \times E^d$. Obviously, the metric space $(C(I,E^d \times E^d),\mathcal{H})$ is a complete space. In addition, throughout this paper, we shall use the notation

$$\begin{aligned} C^l\left(I,E^d\right) = \bigl\{&x : I \to E^d; D_H^{i,g}x \text{ is strongly generalized differentiable, differentiable} \\ &\text{and continuous for } i = 0,1,2, \text{ where } D_H^{0,g}x = x\bigr\}, \end{aligned}$$

where strongly generalized differentiability at the endpoints a and b, is interpreted right and left differentiability at these points, respectively.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space. A function $x : \Omega \to E^d$ is called a fuzzy random variable, if the set-valued mapping $[x(\cdot)]^\alpha : \Omega \to K_c(\mathbb{R}^d)$ is a measurable multifunction for all $\alpha \in [0,1]$, *i.e.*,

$$\left\{\omega \in \Omega \mid \left[x(\omega)\right]^\alpha \cap B \neq \emptyset\right\} \in \mathcal{F}$$

for every closed set $B \subset \mathbb{R}^d$.

Definition 2.4 (see [18, 29, 30, 56]) A mapping $x : [a,b] \times \Omega \to E^d$ is said to be a fuzzy stochastic process if $x(\cdot,\omega)$ is a fuzzy-set-valued function with any fixed $\omega \in \Omega$ (this function will be called a trajectory), and $x(t,\cdot)$ is a fuzzy random variable for any fixed $t \in [a,b]$, *i.e.*, x can be thought of as a family $\{x(t), t \in [a,b]\}$ of fuzzy random variables.

Definition 2.5 (see [18, 29, 30, 56]) A fuzzy stochastic process $x(t,\omega) \in E^d$ is called continuous if there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and such that for every $\omega \in \Omega_0$ the trajectory $x(\cdot,\omega)$ is a continuous function on $[a,b]$ with respect to the metric D_0.

For convenience, from now on, we shall write $x(\omega) \overset{\mathbb{P}.1}{=} y(\omega)$ to replace $\mathbb{P}(\{\omega \mid x(\omega) = y(\omega)\}) = 1$ for short, where x, y are random elements, and similarly for inequalities. Also we shall write $x(t,\omega) \overset{[a,b],\mathbb{P}.1}{=} y(t,\omega)$ to replace $\mathbb{P}(\{\omega \mid x(t,\omega) = y(t,\omega)\}, \forall t \in [a,b]) = 1$ for short, where x, y are some stochastic processes, and similarly for inequalities.

3 Main results

Let $t_0 \in \mathbb{R}$, $p > 0$. In this section, we shall consider again the following initial value problem for the second-order random fuzzy differential equation:

$$\begin{cases} D_{\mathrm{H}}^{2,g} x(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} f_\omega(t, x(t,\omega), D_{\mathrm{H}}^{1,g} x(t,\omega)), \\ x(t_0,\omega) \overset{\mathbb{P}.1}{=} I_1(\omega), \qquad D_{\mathrm{H}}^{1,g} x(t_0,\omega) \overset{\mathbb{P}.1}{=} I_2(\omega) \in E^d, \end{cases} \tag{3.1}$$

where the symbol $D_{\mathrm{H}}^{2,g}$ denotes the second-order strongly generalized differentiable from Definition 2.3, $t \in I = [t_0, t_0 + p]$, $f : \Omega \times I \times E^d \times E^d \to E^d$. A solution for problem (3.1) is a fuzzy stochastic process $x \in C^2([t_0, t_0 + p] \times \Omega, E^d)$ satisfying (3.1). We say that fuzzy stochastic process $x \in C^2([t_0, t_0 + p] \times \Omega, E^d)$ is a (i-i)-solution (respectively, (ii-ii)-solution, (i-ii)-solution and (ii-i)-solution) of (3.1), if x and $D_{\mathrm{H}}^{1,g} x$ are (i)-differentiable (respectively, x and $D_{\mathrm{H}}^{1,g} x$ are (ii)-differentiable, x is (i)-differentiable and $D_{\mathrm{H}}^{1,g} x$ is (ii)-differentiable, x is (ii)-differentiable and $D_{\mathrm{H}}^{1,g} x$ is (i)-differentiable) on the entire $[t_0, t_0 + p]$ and also x and $D_{\mathrm{H}}^{1,g} x$ satisfy (3.1). A solution x to (3.1) is unique, if $D_0[x(t,\omega), \hat{x}(t,\omega)] \overset{\mathbb{P}.1}{=} 0$ for any fuzzy stochastic process $\hat{x} : [t_0, t_0 + p] \times \Omega \to E^d$ that is a solution to (3.1).

In the sequel, a similar result can be found in [51]. One can obtain a formulation of equivalence between solutions of second-order random fuzzy differential equations and random fuzzy integral equations.

Theorem 3.1 *Assume that $f_\omega(\cdot,\cdot,\cdot) : [t_0, t_0 + p] \times E^d \times E^d \to E^d$ is continuous with* $\mathbb{P}$.1. *A fuzzy stochastic process $x : [t_0, t_0 + p] \times \Omega \to E^d$ is a solution to the problem* (3.1) *if and only if $x \in C^2([t_0, t_0 + p] \times \Omega, E^d)$ and x satisfies one of the following random fuzzy integral equations*:

$$\text{(S1)} \quad x(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} I_1(\omega) + I_2(\omega)(t - t_0) + \int_{t_0}^{t} \left(\int_{t_0}^{s} f_\omega\bigl(\tau, x(\tau,\omega), D_{\mathrm{H}}^{1,g} x(\tau,\omega)\bigr)\, d\tau \right) ds$$

if x and $D_{\mathrm{H}}^{1,g} x$ are (i)-*differentiable*;

$$\text{(S2)} \quad x(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} I_1(\omega) \ominus (-1)\biggl(I_2(\omega)(t - t_0) + \int_{t_0}^{t} \left(\int_{t_0}^{s} f_\omega\bigl(\tau, x(\tau,\omega), D_{\mathrm{H}}^{1,g} x(\tau,\omega)\bigr)\, d\tau \right) ds \biggr)$$

if x is (i)*-differentiable and $D_{\mathrm{H}}^{1,g}x$ is* (ii)*-differentiable*;

$$\text{(S3)}\quad x(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} I_1(\omega) + I_2(\omega)(t-t_0) \ominus (-1)\int_{t_0}^{t}\left(\int_{t_0}^{s} f_\omega\big(\tau, x(\tau,\omega), D_{\mathrm{H}}^{1,g}x(\tau,\omega)\big)\,d\tau\right)ds$$

if x is (ii)*-differentiable and $D_{\mathrm{H}}^{1,g}x$ is* (i)*-differentiable*;

$$\text{(S4)}\quad x(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} I_1(\omega) \ominus (-1)\left(I_2(\omega)(t-t_0) \ominus (-1)\int_{t_0}^{t}\left(\int_{t_0}^{s} f_\omega\big(\tau, x(\tau,\omega), D_{\mathrm{H}}^{1,g}x(\tau,\omega)\big)\,d\tau\right)ds\right)$$

if x and $D_{\mathrm{H}}^{1,g}x$ are (ii)*-differentiable*.

Remark 3.1 We can reduce (3.1) to the following systems of two first-order random fuzzy differential equations:

$$\begin{cases} D_{\mathrm{H}}^{1,g}z_1(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} z_2(t,\omega), \\ D_{\mathrm{H}}^{1,g}z_2(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} f_\omega(t, z_1(t,\omega), z_2(t,\omega)), \end{cases} \tag{3.2}$$

together with the initial conditions

$$z_1(t_0,\omega) \overset{\mathbb{P}.1}{=} I_1(\omega), \qquad z_2(t_0,\omega) \overset{\mathbb{P}.1}{=} I_2(\omega). \tag{3.3}$$

For convenience, we apply the vector notation $Z(t,\omega) = \left[\begin{smallmatrix} z_1(t,\omega) \\ z_2(t,\omega)\end{smallmatrix}\right]$, $D_{\mathrm{H}}^{1,g}Z(t,\omega) = \left[\begin{smallmatrix} D_{\mathrm{H}}^{1,g}z_1(t,\omega) \\ D_{\mathrm{H}}^{1,g}z_2(t,\omega)\end{smallmatrix}\right]$, and we rewrite the problem (3.2) and (3.3) as

$$\begin{aligned} &D_{\mathrm{H}}^{1,g}Z(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \begin{bmatrix} z_2(t,\omega) \\ f_\omega(t, z_1(t,\omega), z_2(t,\omega)) \end{bmatrix}, \\ &Z(t_0,\omega) = \begin{bmatrix} z_1(t_0,\omega) \\ z_2(t_0,\omega) \end{bmatrix} \overset{\mathbb{P}.1}{=} \begin{bmatrix} I_1(\omega) \\ I_2(\omega) \end{bmatrix}. \end{aligned} \tag{3.4}$$

We note that problems (3.1) and (3.2) are equivalent. Similarly to Lemma 2.3, one can obtain a formulation of equivalence between solutions of system of two first-order random fuzzy differential equations and system of random fuzzy integral equations.

Lemma 3.1 *Let $f_\omega(\cdot,\cdot,\cdot) : [t_0, t_0+p] \times E^d \times E^d \to E^d$ be continuous with $\mathbb{P}.1$. The problem* (3.2) *is equivalent to one of the following random fuzzy integral equations systems*:

$$\text{(K1)}\quad Z(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \begin{bmatrix} z_1(t_0,\omega) + \int_{t_0}^{t} z_2(s,\omega)\,ds \\ z_2(t_0,\omega) + \int_{t_0}^{t} f_\omega(s, z_1(s,\omega), z_2(s,\omega))\,ds \end{bmatrix} \tag{3.5}$$

if z_1 and z_2 are (i)*-differentiable on* $[t_0, t_0+p]$;

$$\text{(K2)}\quad Z(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \begin{bmatrix} z_1(t_0,\omega) + \int_{t_0}^{t} z_2(s,\omega)\,ds \\ z_2(t_0,\omega) \ominus (-1)\int_{t_0}^{t} f_\omega(s, z_1(s,\omega), z_2(s,\omega))\,ds \end{bmatrix} \tag{3.6}$$

if z_1 *is* (i)-*differentiable and* z_2 (ii)-*differentiable on* $[t_0, t_0 + p]$;

$$\text{(K3)}\quad Z(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \begin{bmatrix} z_1(t_0,\omega) \ominus (-1)\int_{t_0}^t z_2(s,\omega)\,ds \\ z_2(t_0,\omega) + \int_{t_0}^t f_\omega(s, z_1(s,\omega), z_2(s,\omega))\,ds \end{bmatrix} \tag{3.7}$$

if z_1 *is* (ii)-*differentiable and* z_2 (i)-*differentiable on* $[t_0, t_0 + p]$;

$$\text{(K4)}\quad Z(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \begin{bmatrix} z_1(t_0,\omega) \ominus (-1)\int_{t_0}^t z_2(s,\omega)\,ds \\ z_2(t_0,\omega) \ominus (-1)\int_{t_0}^t f_\omega(s, z_1(s,\omega), z_2(s,\omega))\,ds \end{bmatrix} \tag{3.8}$$

if z_1 *and* z_2 *are* (ii)-*differentiable on* $[t_0, t_0 + p]$. *Provided these requirements, the above Hukuhara differences exist.*

Proof It is obtained immediately by Theorem 2.1 and Lemma 2.3. Indeed, in the sequel we only prove this for the case z_1 and z_2 are (ii)-differentiable, the proof of the other case being similar. Assume that $Z : [t_0, t_0 + r] \times \Omega \to E^d \times E^d$ is a solution to the problem (3.2). Hence z_1, z_2 are (ii)-differentiable on $[t_0, t_0 + r]$ and $D_{\mathrm{H}}^{1,g} Z$ is integrable as a continuous function. Applying Theorem 2.1 we obtain

$$Z(t_0,\omega) \overset{[t_0,t_0+r],\mathbb{P}.1}{=} Z(t,\omega) + (-1)\int_{t_0}^t D_{\mathrm{H}}^{1,g} Z(s,\omega)\,ds$$

or

$$\begin{bmatrix} z_1(t_0,\omega) \\ z_2(t_0,\omega) \end{bmatrix} \overset{[t_0,t_0+r],\mathbb{P}.1}{=} \begin{bmatrix} z_1(t,\omega) \\ z_2(t,\omega) \end{bmatrix} + (-1) \begin{bmatrix} \int_{t_0}^t D_{\mathrm{H}}^{1,g} z_1(s,\omega)\,ds \\ \int_{t_0}^t D_{\mathrm{H}}^{1,g} z_2(s,\omega)\,ds \end{bmatrix}$$

for every $(t,\omega) \in [t_0, t_0 + r] \times \Omega$. Since

$$D_H^{1,g} z_1(s,\omega) \overset{[t_0,t],\mathbb{P}.1}{=} z_2(t,\omega) \quad \text{and} \quad D_H^{1,g} z_2(s,\omega) \overset{[t_0,t],\mathbb{P}.1}{=} f_\omega\bigl(s, z_1(s,\omega), z_2(s,\omega)\bigr)$$

for $t \in [t_0, t_0 + r]$ we obtain (from Lemma 2.3)

$$\begin{bmatrix} z_1(t,\omega) \\ z_2(t,\omega) \end{bmatrix} \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \begin{bmatrix} z_1(t_0,\omega) \ominus (-1)\int_{t_0}^t z_2(s,\omega)\,ds \\ z_2(t_0,\omega) \ominus (-1)\int_{t_0}^t f_\omega(s, z_1(s,\omega), z_2(s,\omega))\,ds \end{bmatrix}.$$

To show that the opposite implication is true let us assume that $z_1, z_2 : [t_0, t_0 + r] \times \Omega \to E^d$ are continuous fuzzy stochastic processes and they satisfy equation (3.8). Equation (3.8) allows us to claim that there exist Hukuhara differences

$$z_1(t_0,\omega) \ominus (-1)\int_{t_0}^t z_2(s,\omega)\,ds \quad \text{and} \quad z_2(t_0,\omega) \ominus (-1)\int_{t_0}^t f_\omega\bigl(s, z_1(s,\omega), z_2(s,\omega)\bigr)\,ds$$

for every $(t,\omega) \in [t_0, t_0 + r] \times \Omega$. Now, let $t \in [t_0, t_0 + r)$ and small positive h such that $(t + h) \in [t_0, t_0 + r]$ and $(t - h) \in (t_0, t_0 + r]$. By Remark 2.1(P4), we observe that

$$\begin{aligned} z_1(t-h,\omega) \ominus z_1(t,\omega) \overset{(t_0,t_0+r],\mathbb{P}.1}{=} & \left(z_1(t_0,\omega) \ominus (-1)\int_{t_0}^{t-h} z_2(s,\omega)\,ds\right) \\ & \ominus \left(z_1(t_0,\omega) \ominus (-1)\int_{t_0}^{t} z_2(s,\omega)\,ds\right) \end{aligned}$$

$$\overset{(t_0,t_0+r],\mathbb{P}.1}{=} (-1)\int_{t-h}^{t} z_2(s,\omega)\,ds, \tag{3.9}$$

$$\begin{aligned} z_2(t-h,\omega)\ominus z_2(t,\omega) &\overset{(t_0,t_0+r],\mathbb{P}.1}{=} \left(z_2(t_0,\omega)\ominus(-1)\int_{t_0}^{t-h} f_\omega\big(s,z_1(s,\omega),z_2(s,\omega)\big)\,ds\right) \\ &\quad\ominus\left(z_2(t_0,\omega)\ominus(-1)\int_{t_0}^{t} f_\omega\big(s,z_1(s,\omega),z_2(s,\omega)\big)\,ds\right) \\ &\overset{(t_0,t_0+r],\mathbb{P}.1}{=} (-1)\int_{t-h}^{t} f_\omega\big(s,z_1(s,\omega),z_2(s,\omega)\big)\,ds, \end{aligned} \tag{3.10}$$

and

$$z_1(t,\omega)\ominus z_1(t+h,\omega) \overset{(t_0,t_0+r],\mathbb{P}.1}{=} (-1)\int_{t}^{t+h} z_2(s,\omega)\,ds, \tag{3.11}$$

$$z_2(t,\omega)\ominus z_2(t+h,\omega) \overset{(t_0,t_0+r],\mathbb{P}.1}{=} (-1)\int_{t}^{t+h} f_\omega\big(s,z_1(s,\omega),z_2(s,\omega)\big)\,ds. \tag{3.12}$$

Therefore, from (3.9)-(3.12) we infer that

$$\begin{aligned} Z(t-h,\omega)\ominus Z(t,\omega) &= \begin{bmatrix} z_1(t-h,\omega)\ominus z_1(t,\omega) \\ z_2(t-h,\omega)\ominus z_2(t,\omega) \end{bmatrix} \\ &\overset{[t_0,t_0+p],\mathbb{P}.1}{=} \begin{bmatrix} (-1)\int_{t-h}^{t} z_2(s,\omega)\,ds \\ (-1)\int_{t-h}^{t} f_\omega(s,z_1(s,\omega),z_2(s,\omega))\,ds \end{bmatrix} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} Z(t,\omega)\ominus Z(t-h,\omega) &= \begin{bmatrix} z_1(t,\omega)\ominus z_1(t+h,\omega) \\ z_2(t,\omega)\ominus z_2(t+h,\omega) \end{bmatrix} \\ &\overset{[t_0,t_0+p],\mathbb{P}.1}{=} \begin{bmatrix} (-1)\int_{t}^{t+h} z_2(s,\omega)\,ds \\ (-1)\int_{t}^{t+h} f_\omega(s,z_1(s,\omega),z_2(s,\omega))\,ds \end{bmatrix}. \end{aligned} \tag{3.14}$$

Multiplying (3.13) by $\frac{1}{-h}$ and passing to the limit with $h \searrow 0$ we have

$$\begin{aligned} &D_0\left[\frac{z_1(t,\omega)\ominus z_1(t+h,\omega)}{-h}, z_2(t,\omega)\right] \\ &\quad\overset{[t_0,t_0+p],\mathbb{P}.1}{=} D_0\left[\frac{1}{h}\int_t^{t+h} z_2(s,\omega)\,ds, \frac{1}{h}\int_t^{t+h} z_2(t,\omega)\,ds\right] \\ &\quad\overset{[t_0,t_0+p],\mathbb{P}.1}{\le} \frac{1}{h}\int_t^{t+h} D_0\big[z_2(s,\omega), z_2(t,\omega)\big]\,ds \\ &\quad\overset{(t_0,t_0+r],\mathbb{P}.1}{\le} \max_{s\in[t,t+h]} D_0\big[z_2(s,\omega), z_2(t,\omega)\big] \to 0 \end{aligned}$$

and

$$\begin{aligned} &D_0\left[\frac{z_2(t,\omega)\ominus z_2(t+h,\omega)}{-h}, f_\omega\big(t,z_1(t,\omega),z_2(t,\omega)\big)\right] \\ &\quad\overset{[t_0,t_0+p],\mathbb{P}.1}{=} D_0\left[\frac{1}{h}\int_t^{t+h} f_\omega\big(s,z_1(s,\omega),z_2(s,\omega)\big)\,ds, \frac{1}{h}\int_t^{t+h} f_\omega\big(t,z_1(t,\omega),z_2(t,\omega)\big)\,ds\right] \end{aligned}$$

$$\overset{[t_0,t_0+p],\mathbb{P}.1}{\leq} \frac{1}{h}\int_t^{t+h} D_0\big[f_\omega\big(s,z_1(s,\omega),z_2(s,\omega)\big),f_\omega\big(t,z_1(t,\omega),z_2(t,\omega)\big)\big]\,ds$$
$$\overset{(t_0,t_0+r],\mathbb{P}.1}{\leq} \max_{s\in[t,t+h]} D_0\big[f_\omega\big(s,z_1(s,\omega),z_2(s,\omega)\big),f_\omega\big(t,z_1(t,\omega),z_2(t,\omega)\big)\big] \to 0.$$

Similar to (3.14) we obtain

$$D_0\left[\frac{z_1(t-h,\omega)\ominus z_1(t,\omega)}{-h},z_2(t,\omega)\right] \overset{(t_0,t_0+r],\mathbb{P}.1}{\to} 0$$

and

$$D_0\left[\frac{z_2(t-h,\omega)\ominus z_2(t,\omega)}{-h},f_\omega\big(t,z_1(t,\omega),z_2(t,\omega)\big)\right] \overset{(t_0,t_0+r],\mathbb{P}.1}{\to} 0.$$

By Definition 2.2, it follows that z_1 and z_2 are (ii)-differentiable, and consequently

$$D_{\mathrm{H}}^{1,g}Z(t,\omega) = \begin{bmatrix} D_{\mathrm{H}}^{1,g}z_1(t,\omega) \\ D_{\mathrm{H}}^{1,g}z_2(t,\omega)\end{bmatrix} \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \begin{bmatrix} z_2(t,\omega) \\ f_\omega(t,z_1(t,\omega),z_2(t,\omega))\end{bmatrix}.$$

The proof is complete. □

The following theorems present the existence and uniqueness results for problem (3.2). For the existence and uniqueness, we use the method of successive approximations.

Let us consider the mappings $f:\Omega\times[t_0,t_0+p]\times E^d\times E^d\to E^d$ that satisfy the following assumptions:

(H1) the mapping $f_\cdot(t,u,v):\Omega\to E^d$ is a fuzzy random variable for every $(t,u,v)\in[t_0,t_0+p]\times E^d\times E^d$;

(H2) the mapping $f_\omega(\cdot,\cdot,\cdot):[t_0,t_0+p]\times E^d\times E^d\to E^d$ is continuous with $\mathbb{P}$.1;

(H3) there exist two stochastic processes $L_1,L_2:[t_0,t_0+p]\times\Omega\to\mathbb{R}^+$ such that $L_1(\cdot,\omega)$, $L_2(\cdot,\omega)$ are continuous with $\mathbb{P}$.1 and

$$D_0\big[f_\omega(t,u_1,v_1),f_\omega(t,u_2,v_2)\big]\leq L_1(t,\omega)D_0[u_1,u_2]+L_2(t,\omega)D_0[v_1,v_2]$$

with $\mathbb{P}$.1 for every $t\in[t_0,t_0+p]$.

Theorem 3.2 *Let $I_1,I_2:\Omega\to E^d$ be fuzzy random variables. Let $f:\Omega\times[t_0,t_0+p]\times E^d\times E^d$ satisfies* (H1), (H2), *and* (H3). *Moreover, there exists a nonnegative constant M_f such that*

$$D_0\big[f_\omega(t,u,v),\hat{0}\big] \overset{[a,a+p],\mathbb{P}.1}{\leq} M_f \tag{3.15}$$

for $u,v\in E^d$. Then the successive approximations given by

$$\begin{aligned} Z_1^0(t,\omega) &= \begin{bmatrix} z_1^0(t_0,\omega) \\ z_2^0(t_0,\omega)\end{bmatrix}, \\ Z_1^{n+1}(t,\omega) &\overset{[t_0,t_0+p],\mathbb{P}.1}{=} \begin{bmatrix} z_1^0(t_0,\omega)+\int_{t_0}^t z_2^n(s,\omega)\,ds \\ z_2^0(t_0,\omega)+\int_{t_0}^t f_\omega(s,z_1^n(s,\omega),z_2^n(s,\omega))\,ds\end{bmatrix}\end{aligned} \tag{3.16}$$

for case (i-i)*-differentiability, and*

$$
\begin{aligned}
&Z_2^0(t,\omega) = \begin{bmatrix} z_1^0(t_0,\omega) \\ z_2^0(t_0,\omega) \end{bmatrix}, \\
&Z_2^{n+1}(t,\omega) \overset{[t_0,t_0+r_1],\mathbb{P}.1}{=} \begin{bmatrix} z_1^0(t_0,\omega) + \int_{t_0}^t z_2^n(s,\omega)\,ds \\ z_2^0(t_0,\omega) \ominus (-1)\int_{t_0}^t f_\omega(s, z_1^n(s,\omega), z_2^n(s,\omega))\,ds \end{bmatrix}
\end{aligned}
\tag{3.17}
$$

for case (i-ii)*-differentiability, and*

$$
\begin{aligned}
&Z_3^0(t,\omega) = \begin{bmatrix} z_1^0(t_0,\omega) \\ z_2^0(t_0,\omega) \end{bmatrix}, \\
&Z_3^{n+1}(t,\omega) \overset{[t_0,t_0+r_2],\mathbb{P}.1}{=} \begin{bmatrix} z_1^0(t_0,\omega) \ominus (-1)\int_{t_0}^t z_2^n(s,\omega)\,ds \\ z_2^0(t_0,\omega) + \int_{t_0}^t f_\omega(s, z_1^n(s,\omega), z_2^n(s,\omega))\,ds \end{bmatrix}
\end{aligned}
\tag{3.18}
$$

for case (ii-i)*-differentiability, and*

$$
\begin{aligned}
&Z_4^0(t,\omega) = \begin{bmatrix} z_1^0(t_0,\omega) \\ z_2^0(t_0,\omega) \end{bmatrix}, \\
&Z_4^{n+1}(t,\omega) \overset{[t_0,t_0+r_3],\mathbb{P}.1}{=} \begin{bmatrix} z_1^0(t_0,\omega) \ominus (-1)\int_{t_0}^t z_2^n(s,\omega)\,ds \\ z_2^0(t_0,\omega) \ominus (-1)\int_{t_0}^t f_\omega(s, z_1^n(s,\omega), z_2^n(s,\omega))\,ds \end{bmatrix}
\end{aligned}
\tag{3.19}
$$

for case (ii-ii)*-differentiability, converge uniformly to four unique solutions* Z_1, Z_2, Z_3, *and* Z_4 *of* (3.2), *respectively, provided that the above Hukuhara differences exist, on* $[t_0, t_0 + d]$ *where* $d = \min\{p, r_1, r_2, r_3\}$.

Proof We prove this for the case (ii-ii)-differentiability, the proof of the other cases being similar. To prove the theorem, we shall use the method of successive approximations. So, we define again the sequence $Z_4^n : [t_0, t_0 + r_3] \times \Omega \to E^d \times E^d$ as follows:

$$
Z_4^0(t,\omega) = \begin{bmatrix} z_1^0(t,\omega) \\ z_2^0(t,\omega) \end{bmatrix} = \begin{bmatrix} z_1(t_0,\omega) \\ z_2(t_0,\omega) \end{bmatrix} \overset{\mathbb{P}.1}{=} \begin{bmatrix} I_1(\omega) \\ I_2(\omega) \end{bmatrix},
$$

and, for $n \in \mathbb{N}$,

$$
\begin{aligned}
Z_4^{n+1}(t,\omega) &= \begin{bmatrix} z_1^{n+1}(t,\omega) \\ z_2^{n+1}(t,\omega) \end{bmatrix} \\
&\overset{[t_0,t_0+r_3],\mathbb{P}.1}{=} \begin{bmatrix} z_1(t_0,\omega) \ominus (-1)\int_{t_0}^t z_2^n(s,\omega)\,ds \\ z_2(t_0,\omega) \ominus (-1)\int_{t_0}^t f_\omega(s, z_1^n(s,\omega), z_2^n(s,\omega))\,ds \end{bmatrix}.
\end{aligned}
\tag{3.20}
$$

Then from (3.15) we have

$$
\begin{aligned}
D_0\big[z_1^1(t,\omega), z_1^0(t,\omega)\big] &= D_0\left[I_1(\omega) \ominus (-1)\int_{t_0}^t I_2(\omega)\,ds, I_1(\omega)\right] \le \int_{t_0}^t D_0\big[I_2(\omega), \hat{0}\big]\,ds \\
&\overset{[t_0,t_0+r_3],\mathbb{P}.1}{\le} M(t - t_0),
\end{aligned}
$$

$$D_0\big[z_2^1(t,\omega),z_2^0(t,\omega)\big]=D_0\left[I_2(\omega)\ominus(-1)\int_{t_0}^t f_\omega\big(s,z_1^0(s,\omega),z_2^0(s,\omega)\big)\,ds,I_2(\omega)\right]$$
$$\le\int_{t_0}^t D_0\big[f_\omega\big(s,z_1^0(s,\omega),z_2^0(s,\omega)\big),\hat{0}\big]\,ds\overset{[t_0,t_0+r_3],\mathbb{P}.1}{\le}M(t-t_0),$$

where $M=\max\{M_f,D_0[I_2(\omega),\hat{0}]\}$. Then we conclude

$$\mathcal{D}_0\big[Z_4^1(t,\omega),Z_4^0(t,\omega)\big]\overset{[t_0,t_0+r_3],\mathbb{P}.1}{\le}M(t-t_0).$$

Also, from (H3) and (3.20), we deduce that

$$D_0\big[z_1^2(t,\omega),z_1^1(t,\omega)\big]=D_0\left[I_1(\omega)\ominus(-1)\int_{t_0}^t z_2^1(\omega)\,ds,I_1(\omega)\ominus(-1)\int_{t_0}^t z_2^0(s,\omega)\,ds\right]$$
$$\overset{[t_0,t_0+r_3],\mathbb{P}.1}{\le}M\frac{(t-t_0)^2}{2},$$
$$D_0\big[z_2^2(t,\omega),z_2^1(t,\omega)\big]\le D_0\left[\int_{t_0}^t f_\omega\big(s,z_1^1(\omega),z_2^1(\omega)\big)\,ds,\int_{t_0}^t f_\omega\big(s,z_1^0(\omega),z_2^0(\omega)\big)\,ds\right]$$
$$\overset{[t_0,t_0+r_3],\mathbb{P}.1}{\le}L(\omega)M(t-t_0)^2,$$

where $L(\omega)=\max\{\sup_{t\in[t_0,t_0+r_3]}L_1(t,\omega),\sup_{t\in[t_0,t_0+r_3]}L_2(t,\omega)\}$. Then we conclude

$$\mathcal{D}_0\big[Z_4^2(t,\omega),Z_4^1(t,\omega)\big]\overset{[t_0,t_0+r_3],\mathbb{P}.1}{\le}M\big(1+L(\omega)\big)\frac{(t-t_0)^2}{2!}.$$

Continuing this way we get

$$\mathcal{D}_0\big[Z_4^{n+1}(t,\omega),Z_4^n(t,\omega)\big]\overset{[t_0,t_0+r_3],\mathbb{P}.1}{\le}M\big(1+L(\omega)\big)^n\frac{(t-t_0)^{n+1}}{(n+1)!}. \tag{3.21}$$

We observe that, for every $n\in\{0,1,2,\dots\}$, the function $Z_4^{n+1}(\cdot,\omega):[t_0,t_0+r_3]\to E^d\times E^d$ is continuous with $\mathbb{P}.1$. Indeed, for $t_0\le t_1\le t_2\le t_0+r_3$, $n\in\{1,2,3,\dots\}$, we see that

$$\mathcal{D}_0\big[Z_4^n(t_1,\omega),Z_4^n(t_2,\omega)\big]=\max\big\{D_0\big[z_1^n(t_1,\omega),z_1^n(t_2,\omega)\big],D_0\big[z_2^n(t_1,\omega),z_2^n(t_2,\omega)\big]\big\}$$
$$\overset{[t_0,t_0+r_3],\mathbb{P}.1}{\le}M|t_2-t_1|<\varepsilon,$$

provided $|t_2-t_1|<\delta$, where $\delta=\varepsilon\backslash M$, proving that Z_4^n is continuous with $\mathbb{P}.1$ on $[t_0,t_0+r_3]$. Now, let us fix $t\in[t_0,t_0+r_3]$ and consider successively, for $n\in\{0,1,2,3,\dots\}$ the functions $Z_4^n(t,\cdot):\Omega\to E^d\times E^d$ defined by (3.20) are fuzzy random variables for every $t\in[t_0,t_0+r_3]$. Indeed, since $z_1^0(t,\cdot)$, $z_2^0(t,\cdot)$ are random variables. It remains to show the same for the mappings $\omega\mapsto[\int_{t_0}^t f_\omega(s,z_1^{n-1}(s,\omega),z_2^{n-1}(s,\omega))\,ds]^\alpha$ is a measurable multifunction with $t\in[t_0,t_0+r_3]$, $n\in\{1,2,3,\dots\}$ and $\alpha\in[0,1]$. Let $\alpha\in[0,1]$ be fixed. By virtue of the definition of a fuzzy integral, the continuity assumption (H2) of f and the theorem of Nguyen [53], we derive that

$$\left[\int_{t_0}^t f_\omega\big(s,z_1^{n-1}(s,\omega),z_2^{n-1}(s,\omega)\big)\,ds\right]^\alpha=\int_{t_0}^t\big[f_\omega\big(s,z_1^{n-1}(s,\omega),z_2^{n-1}(s,\omega)\big)\big]^\alpha\,ds$$

for every $t \in [t_0, t_0 + r_3]$. As the integrand is a multifunction which is continuous in s and measurable in ω, the mapping $\omega \mapsto \int_{t_0}^{t} [f_\omega(s, z_1^{n-1}(s,\omega), z_2^{n-1}(s,\omega))]^\alpha \, ds$ is a measurable multifunction for each $\alpha \in [0,1]$. Hence $Z_4^n(t,\cdot) : \Omega \to E^d \times E^d$ is a fuzzy random variables for every $t \in [t_0, t_0 + r_3]$. Consequently, $\{Z_4^n\}_{n=0}^\infty$ is a sequence of fuzzy stochastic process. Now for any $n \in \{1,2,3,\ldots\}$ and $t \in [t_0, t_0 + r_3]$ we shall show that the sequence $\{Z_4^n(t,\omega)\}$ is a Cauchy sequence uniformly on the variable t with $\mathbb{P}.1$ and then $\{Z_4^n(\cdot,\omega)\}$ is uniformly convergent with $\mathbb{P}.1$. For $n > m > 0$, from (3.21) we obtain

$$\sup_{t\in[t_0,t_0+r_3]} \mathcal{D}_0\bigl[Z_4^n(t,\omega), Z_4^m(t,\omega)\bigr] \overset{\mathbb{P}.1}{\leq} M \sum_{i=m}^{n-1} \bigl(1 + L(\omega)\bigr)^i \frac{{r_3}^{i+1}}{(i+1)!}.$$

The almost sure convergence of the series $\sum_{n=1}^\infty (1 + L(\omega))^{n-1} \frac{{r_3}^n}{n!}$ implies that for any $\varepsilon > 0$ we find $n_0 \in \mathbb{N}$ large enough such that, for $n, m > n_0$,

$$\mathcal{D}_0\bigl[Z_4^n(t,\omega), Z_4^m(t,\omega)\bigr] \overset{\mathbb{P}.1}{\leq} \varepsilon. \tag{3.22}$$

Then there exists $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and for every $\omega \in \Omega_0$ the sequence $\{Z_4^n(\cdot,\omega)\}$ is uniformly convergent with $\mathbb{P}.1$. For $\omega \in \Omega_0$ let $\hat{Z}_4(\cdot,\omega)$ denote its limit. Let us define a mapping $Z_4 : [t_0, t_0 + r_3] \times \Omega \to E^d \times E^d$ as

$$Z_4(t,\omega) = \begin{cases} \hat{Z}_4(t,\omega) & \text{for } t \in [t_0, t_0 + r_3] \times \Omega_0, \\ \hat{0} & \text{for } t \in [t_0, t_0 + r_3] \times (\Omega \backslash \Omega_0). \end{cases}$$

Then $\sup_{t\in[t_0,t_0+r_3]} \mathcal{D}_0[Z_4^n(t,\omega), Z_4(t,\omega)] \overset{\mathbb{P}.1}{\to} 0$ as $n \to \infty$. It is clear that $Z_4 : [t_0, t_0 + r_3] \times \Omega \to E^d \times E^d$ is in the form

$$Z_4(t,\omega) = \begin{bmatrix} z_1(t,\omega) \\ z_2(t,\omega) \end{bmatrix} \overset{[t_0,t_0+r_3],\mathbb{P}.1}{=} \begin{bmatrix} z_1(t_0,\omega) \ominus (-1) \int_{t_0}^{t} z_2(s,\omega)\, ds \\ z_2(t_0,\omega) \ominus (-1) \int_{t_0}^{t} f_\omega(s, z_1(s,\omega), z_2(s,\omega))\, ds \end{bmatrix}.$$

According to Lemma 3.1, Z_4 is a solution of the problem (3.2) for the case (ii-ii)-differentiability. To prove the uniqueness, let $W_4 : [t_0, t_0 + r_3] \times \Omega \to E^d \times E^d$ be a second solution of the problem (3.2) for the case (ii-ii)-differentiability on $[t_0, t_0 + r_3]$. Then for every $t \in [t_0, t_0 + r_3]$ we have

$$\begin{aligned} &\mathcal{D}_0\bigl[Z_4(t,\omega), W_4(t,\omega)\bigr] \\ &\quad = \max\bigl\{D_0\bigl[z_1(t,\omega), w_1(t,\omega)\bigr], D_0\bigl[z_2(t,\omega), w_2(t,\omega)\bigr]\bigr\} \\ &\quad \overset{[t_0,t_0+r_3],\mathbb{P}.1}{\leq} 2\bigl(1 + L(\omega)\bigr) \int_{t_0}^{t} \max\bigl\{D_0\bigl[z_1(s,\omega), w_1(s,\omega)\bigr], D_0\bigl[z_2(s,\omega), w_2(s,\omega)\bigr]\bigr\}\, ds. \end{aligned}$$

Applying Gronwall's inequality we can infer that $\mathcal{D}_0[Z_4(t,\omega), W_4(t,\omega)] \overset{[t_0,t_0+r_3],\mathbb{P}.1}{=} 0$, which leads us to the conclusion $Z_4(t,\omega) \overset{[a,a+p],\mathbb{P}.1}{=} W_4(t,\omega)$. This proves the uniqueness of the solution of the problem (3.2) for the case (ii-ii)-differentiability on $[t_0, t_0 + r_3]$. The proof is complete. □

In the sequel, we shall present some examples being simple illustrations of the theory of second-order random fuzzy differential equations. Let us start the illustrations by considering the following SRFDE:

$$\begin{cases} D_{\mathrm{H}}^{2,g}x(t,\omega) + aD_{\mathrm{H}}^{1,g}x(t,\omega) + bx(t,\omega) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} k(t,\omega), \\ x(t_0,\omega) \overset{\mathbb{P}.1}{=} I_1(\omega) \in E^1, \\ D_{\mathrm{H}}^{1,g}x(t_0,\omega) \overset{\mathbb{P}.1}{=} I_2(\omega) \in E^1, \end{cases} \tag{3.23}$$

where a, b are positive constants. Let us denote the α-cut ($\alpha \in [0,1]$) of I_1, I_2, and x as $[I_1(\omega)]^\alpha = [\underline{I}_1(\omega,\alpha), \overline{I}_1(\omega,\alpha)]$, $[I_2(\omega)]^\alpha = [\underline{I}_2(\omega,\alpha), \overline{I}_2(\omega,\alpha)]$, and $[x(t,\omega)]^\alpha = [\underline{x}(t,\omega,\alpha), \overline{x}(t,\omega,\alpha)]$, respectively. Obviously, $\underline{x}(\cdot,\cdot,\alpha), \overline{x}(\cdot,\cdot,\alpha) : [t_0, t+p] \times \Omega \to \mathbb{R}$ are crisp stochastic processes. In the sequel, we shall establish the explicit solution to (3.23). Our strategy of solving (3.23) is based on the choice of the derivative in the fuzzy differential equation. In order to solve (3.23) we have three steps: first we choose the type of derivative and change problem (3.23) to a system of ODE by using Theorem 2.2 and considering initial values. Second we solve the obtained ODE system. The final step is to find such a domain in which the solution and its derivatives have valid sets, *i.e.*, we ensure that $[\underline{x}(t,\omega,\alpha), \overline{x}(t,\omega,\alpha)]$, $[\underline{x}'(t,\omega,\alpha), \overline{x}'(t,\omega,\alpha)]$, and $[\underline{x}''(t,\omega,\alpha), \overline{x}''(t,\omega,\alpha)]$ are valid sets.

By using Lemma 2.2 and Theorem 2.2, we see that four ODE systems are possible for problem (3.23), as follows.

Case 1: x and $D_{\mathrm{H}}^{1,g}x$ are (i)-differentiable

$$\begin{cases} \underline{x}''(t,\omega,\alpha) + a\underline{x}'(t,\omega,\alpha) + b\underline{x}(t,\omega,\alpha) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \underline{k}(t,\omega,\alpha), \\ \overline{x}''(t,\omega,\alpha) + a\overline{x}'(t,\omega,\alpha) + b\overline{x}(t,\omega,\alpha) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \overline{k}(t,\omega,\alpha), \\ \underline{x}(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \underline{I}_1(\omega,\alpha), \qquad \overline{x}(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \overline{I}_1(\omega,\alpha), \\ \underline{x}'(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \underline{I}_2(\omega,\alpha), \qquad \overline{x}'(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \overline{I}_2(\omega,\alpha). \end{cases} \tag{3.24}$$

Case 2: x is (i)-differentiable and $D_{\mathrm{H}}^{1,g}x$ is (ii)-differentiable

$$\begin{cases} \overline{x}''(t,\omega,\alpha) + a\underline{x}'(t,\omega,\alpha) + b\underline{x}(t,\omega,\alpha) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \underline{k}(t,\omega,\alpha), \\ \underline{x}''(t,\omega,\alpha) + a\overline{x}'(t,\omega,\alpha) + b\overline{x}(t,\omega,\alpha) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \overline{k}(t,\omega,\alpha), \\ \underline{x}(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \underline{I}_1(\omega,\alpha), \qquad \overline{x}(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \overline{I}_1(\omega,\alpha), \\ \underline{x}'(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \underline{I}_2(\omega,\alpha), \qquad \overline{x}'(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \overline{I}_2(\omega,\alpha). \end{cases} \tag{3.25}$$

Case 3: x is (ii)-differentiable and $D_{\mathrm{H}}^{1,g}x$ is (i)-differentiable

$$\begin{cases} \overline{x}''(t,\omega,\alpha) + a\overline{x}'(t,\omega,\alpha) + b\underline{x}(t,\omega,\alpha) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \underline{k}(t,\omega,\alpha), \\ \underline{x}''(t,\omega,\alpha) + a\underline{x}'(t,\omega,\alpha) + b\overline{x}(t,\omega,\alpha) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \overline{k}(t,\omega,\alpha), \\ \underline{x}(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \underline{I}_1(\omega,\alpha), \qquad \overline{x}(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \overline{I}_1(\omega,\alpha), \\ \overline{x}'(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \underline{I}_2(\omega,\alpha), \qquad \underline{x}'(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \overline{I}_2(\omega,\alpha). \end{cases} \tag{3.26}$$

Case 4: x and $D_{\mathrm{H}}^{1,g}x$ are (ii)-differentiable

$$\begin{cases} \underline{x}''(t,\omega,\alpha) + a\overline{x}'(t,\omega,\alpha) + b\underline{x}(t,\omega,\alpha) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \underline{k}(t,\omega,\alpha), \\ \overline{x}''(t,\omega,\alpha) + a\underline{x}'(t,\omega,\alpha) + b\overline{x}(t,\omega,\alpha) \overset{[t_0,t_0+p],\mathbb{P}.1}{=} \overline{k}(t,\omega,\alpha), \\ \underline{x}(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \underline{I}_1(\omega,\alpha), \qquad \overline{x}(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \overline{I}_1(\omega,\alpha), \\ \underline{x}'(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \overline{I}_2(\omega,\alpha), \qquad \overline{x}'(t_0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \underline{I}_2(\omega,\alpha). \end{cases} \tag{3.27}$$

Remark 3.2 If we ensure that the solutions $(\underline{x}(t,\omega,\alpha),\overline{x}(t,\omega,\alpha))$ of the systems (3.24), (3.25), (3.26), and (3.27), respectively, are valid level sets of fuzzy-number-valued functions and if the first-order and second-order derivatives $(\underline{x}'(t,\omega,\alpha),\overline{x}'(t,\omega,\alpha))$, $(\underline{x}''(t,\omega,\alpha),\overline{x}''(t,\omega,\alpha))$ are valid level sets of fuzzy-number-valued functions with two kinds differentiability, respectively, then we can construct the solution of equation (3.23).

Example 3.1 Let $\Omega = (0,1)$, $\mathcal{F}$-Borel σ-field of subsets of Ω, $\mathbb{P}$-Lebesgue measure on $(\Omega,\mathcal{F})$. Let us consider the second-order random fuzzy differential equation as follows:

$$\begin{aligned} &D_{\mathrm{H}}^{2,g}x(t,\omega) \overset{[0,\pi/4],\mathbb{P}.1}{=} (-\omega,0,\omega),\\ &x(0,\omega) \overset{\mathbb{P}.1}{=} (-\omega,0,\omega),\\ &D_{\mathrm{H}}^{1,g}x(0,\omega) \overset{\mathbb{P}.1}{=} (-\omega,0,\omega). \end{aligned} \tag{3.28}$$

Case 1: From (3.24), we get

$$\begin{cases} \underline{x}''(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \omega(\alpha-1),\\ \overline{x}''(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \omega(1-\alpha),\\ \underline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}(0,\omega) \overset{\mathbb{P}.1}{=} \omega(1-\alpha),\\ \underline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha). \end{cases} \tag{3.29}$$

By solving (3.29), we obtain

$$\bigl[x(t,\omega)\bigr]^{\alpha} = \biggl[\omega(\alpha-1)+\omega(\alpha-1)t+\frac{\omega(\alpha-1)t^2}{2},\omega(1-\alpha)+\omega(1-\alpha)t+\frac{\omega(1-\alpha)t^2}{2}\biggr].$$

Clearly, x and $D_{\mathrm{H}}^{1,g}x$ are (i)-differentiable. Hence, there is an (i-i)-solution in this case. This solution is shown in Figure 1.

Case 2: From (3.25), we have

$$\begin{cases} \overline{x}''(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \omega(\alpha-1),\\ \underline{x}''(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \omega(1-\alpha),\\ \underline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha),\\ \underline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha). \end{cases} \tag{3.30}$$

Figure 1 (i-i)-solution of Example 3.1 in Case 1.

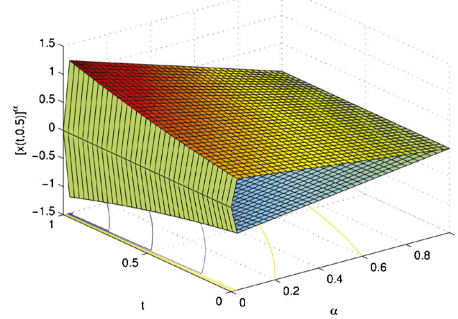
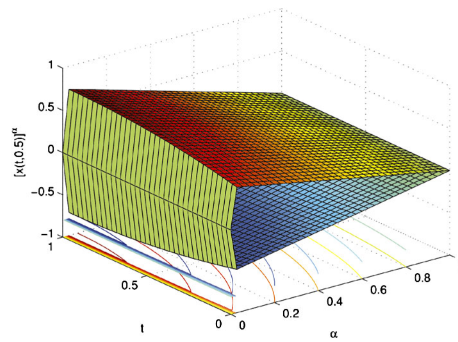


Figure 2 (i-ii)-solution of Example 3.1 in Case 2.



By solving (3.30), we get

$$\left[x(t,\omega)\right]^{\alpha} = \left[\omega(\alpha-1)+\omega(\alpha-1)t+\frac{\omega(1-\alpha)t^2}{2}, \omega(1-\alpha)+\omega(1-\alpha)t+\frac{\omega(\alpha-1)t^2}{2}\right].$$

Clearly, x is (i)-differentiable and $D_{\mathrm{H}}^{1,g}x$ is (ii)-differentiable. Hence, there is an (i-ii)-solution in this case. This solution is shown in Figure 2.

Case 3: From (3.26), we obtain

$$\begin{cases} \underline{x}''(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \omega(\alpha-1), \\ \overline{x}''(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \omega(1-\alpha), \\ \underline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha), \\ \underline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha), \qquad \overline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1). \end{cases} \tag{3.31}$$

By solving (3.31), we get

$$\left[x(t,\omega)\right]^{\alpha} = \left[\omega(\alpha-1)+\omega(1-\alpha)t+\frac{\omega(1-\alpha)t^2}{2}, \omega(1-\alpha)+\omega(\alpha-1)t+\frac{\omega(\alpha-1)t^2}{2}\right].$$

Since x is not (ii)-differentiable, there is no (ii-i)-solution in this case.

Case 4: From (3.27), we have

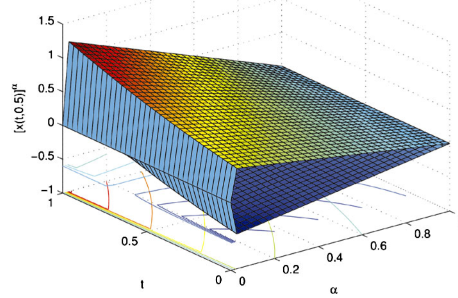
$$\begin{cases} \overline{x}''(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \omega(\alpha-1), \\ \underline{x}''(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \omega(1-\alpha), \\ \underline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha), \\ \underline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha), \qquad \overline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1). \end{cases} \tag{3.32}$$

By solving (3.32), we have

$$\left[x(t,\omega)\right]^{\alpha} = \left[\omega(\alpha-1)+\omega(1-\alpha)t+\frac{\omega(\alpha-1)t^2}{2}, \omega(1-\alpha)+\omega(\alpha-1)t+\frac{\omega(1-\alpha)t^2}{2}\right].$$

Notice that, in this case, since x is (ii)-differentiable and $D_{\mathrm{H}}^{1,g}x$ is (ii)-differentiable, such a solution is acceptable. This (ii-ii)-solution is shown in Figure 3.

Figure 3 **(ii-ii)-solution of Example 3.1 in Case 4.**



Example 3.2 Let $\Omega = (0,1)$, $\mathcal{F}$-Borel σ-field of subsets of Ω, $\mathbb{P}$-Lebesgue measure on $(\Omega, \mathcal{F})$. Let us consider the following second-order random fuzzy differential equation:

$$\begin{aligned} &D_{\mathrm{H}}^{2,g} x(t,\omega) + x(t,\omega) \overset{[0,\pi/4],\mathbb{P}.1}{=} (0,\omega,2\omega), \\ &x(0,\omega) \overset{\mathbb{P}.1}{=} (-\omega,0,\omega), \\ &D_{\mathrm{H}}^{1,g} x(0,\omega) \overset{\mathbb{P}.1}{=} (-\omega,0,\omega). \end{aligned} \tag{3.33}$$

Case 1: From (3.24), we get

$$\begin{cases} \underline{x}''(t,\omega,\alpha) + \underline{x}(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \alpha\omega, \\ \overline{x}''(t,\omega,\alpha) + \overline{x}(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \omega(2-\alpha), \\ \underline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha), \\ \underline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha). \end{cases} \tag{3.34}$$

By solving (3.34), we obtain

$$\big[x(t,\omega)\big]^{\alpha} = \big[\omega\alpha(1+\sin t) - \omega(\sin t + \cos t), \omega(2-\alpha)(1+\sin t) - \omega(\sin t + \cos t)\big].$$

Since $D_{\mathrm{H}}^{1,g} x$ is not (i)-differentiable, there is no solution in this case.

Case 2: From (3.25), we have

$$\begin{cases} \overline{x}''(t,\omega,\alpha) + \underline{x}(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \alpha\omega, \\ \underline{x}''(t,\omega,\alpha) + \overline{x}(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \omega(2-\alpha), \\ \underline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha), \\ \underline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha). \end{cases} \tag{3.35}$$

By solving (3.35), we get

$$\big[x(t,\omega)\big]^{\alpha} = \big[\omega\alpha(1+\sinh t) - \omega(\sinh t + \cos t), \omega(2-\alpha)(1+\sinh t) - \omega(\sinh t + \cos t)\big].$$

Since $D_{\mathrm{H}}^{1,g} x$ is not (ii)-differentiable, there is no solution in this case.

Figure 4 (ii-i)-solution of Example 3.2 in Case 3.

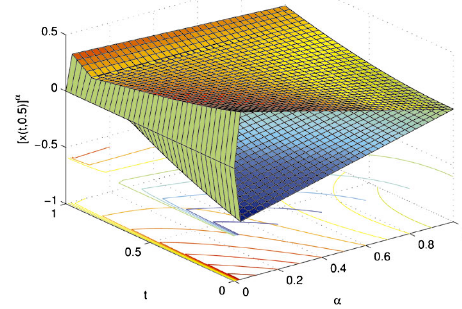
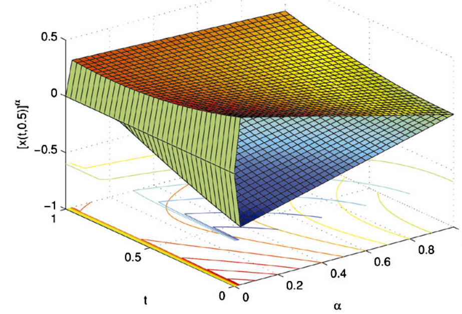


Figure 5 (ii-ii)-solution of Example 3.2 in Case 4.



Case 3: From (3.26), we obtain

$$\begin{cases} \underline{x}''(t,\omega,\alpha) + \underline{x}(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \alpha\omega, \\ \overline{x}''(t,\omega,\alpha) + \overline{x}(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \omega(2-\alpha), \\ \underline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha), \\ \underline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(1-\alpha), \qquad \overline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1). \end{cases} \tag{3.36}$$

By solving (3.36), we get

$$\bigl[x(t,\omega)\bigr]^\alpha = \bigl[\omega\alpha(1-\sinh t) + \omega(\sinh t - \cos t), \omega(2-\alpha)(1-\sinh t) + \omega(\sinh t - \cos t)\bigr].$$

Notice that, in this case, since x is (ii)-differentiable and $D_{\mathrm{H}}^{1,g}x$ is (i)-differentiable, such a solution is acceptable. This solution is shown in Figure 4.

Case 4: From (3.27), we have

$$\begin{cases} \overline{x}''(t,\omega,\alpha) + \underline{x}(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \alpha\omega, \\ \underline{x}''(t,\omega,\alpha) + \overline{x}(t,\omega,\alpha) \overset{[0,\pi/4],\mathbb{P}.1}{=} \omega(2-\alpha), \\ \underline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \\ \underline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1), \qquad \overline{x}'(0,\omega,\alpha) \overset{\mathbb{P}.1}{=} \omega(\alpha-1). \end{cases} \tag{3.37}$$

By solving (3.37), we have

$$\bigl[x(t,\omega)\bigr]^\alpha = \bigl[\omega\alpha(1-\sin t) + \omega(\sin t - \cos t), \omega(2-\alpha)(1-\sin t) + \omega(\sin t - \cos t)\bigr].$$

x and $D_{\mathrm{H}}^{1,g}x$ are (ii)-differentiable. Therefore, the obtained (ii-ii)-solution is valid. This solution is shown in Figure 5.

4 Conclusions

In this paper, we discussed the local existence and uniqueness results for the second-order random fuzzy differential equations. Under Lipschitz conditions we obtain the existence and uniqueness theorems of solution for SRFDE. In future work on SRFDEs, we would like to study the local and global existence and uniqueness results of solutions for second-order random fuzzy differential equation under weaker conditions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors contributed to each part of the work equally and read and proved the final version of the manuscript.

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