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Permanence of the periodic predator-prey-mutualist system

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Abstract

In this paper, we study the permanence and the periodic solution of the periodic predator-prey-mutualist system. It is well known that mutualist species can reduce the capture rate of the predator species to the prey species. By further developing the analysis technique of Teng, a set of conditions which ensure the permanence of the system are obtained. In addition, sufficient conditions are derived for the existence of positive periodic solutions to the system. An example together with its numerical simulation shows the feasibility of the main results.

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Keywords: predator-prey-mutualist system; permanence; periodic solution

1 Introduction

As was pointed out by Berryman [1], the dynamic relationship between predator and prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Already the predator-prey model has been studied by several scholars [2–10]. For example, Das *et al.* [8] investigated a three species ecosystem consisting of a prey, a predator, and a top predator. They derived the criteria for local and global stability of all the eight equilibrium points by using a Routh-Hurwitz and Lyapunov function. Wu and Li [9] studied the permanence and global attractivity of the discrete predator-prey system with Hassell-Varley-Holling type III functional response. Chen and Chen [10] proposed a ratio-dependent predator-prey model incorporating a prey refuge. They studied the global stability, limit cycle, and Hopf bifurcation of the system.

Mutualism is one of the most important relationships in the real world, for instance, ants prevent herbivores from feeding on plants (see [11]) and ants prevent predators from feeding on aphids (see [12, 13]). As was pointed out by Murray [14]: 'this area has not been as widely studied as the others even though its importance is comparable to that of predator-prey and competition interactions.' To this end, Rai and Krawcewicz [15] proposed the following three species predator-prey-mutualist system:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x\left(1-\frac{x}{K}\right)-\frac{\beta xz}{1+my}, \\ \frac{dy}{dt} &= \gamma y\left(1-\frac{y}{lx+L_0}\right), \end{aligned} \tag{1.1}$$

$$\frac{dz}{dt} = z\left(-s + \frac{c\beta x}{1 + my}\right),$$

where $x(t)$, $y(t)$, $z(t)$ denote the densities of prey, mutualist and predator population at any time t, respectively. They applied the equivariant degree method to study the Hopf bifurcation phenomenon of the system.

In this paper, we will study the non-autonomous case of system (1.1), *i.e.*,

$$\begin{aligned} \dot{x} &= x\left(a_1(t) - b_1(t)x - \frac{c_1(t)z}{d_1(t) + d_2(t)y}\right), \\ \dot{y} &= y\left(a_2(t) - \frac{y}{d_3(t) + d_4(t)x}\right), \\ \dot{z} &= z\left(-a_3(t) + \frac{c_2(t)x}{d_1(t) + d_2(t)y}\right), \end{aligned} \tag{1.2}$$

where x is the density of the prey at time t, y is the density of the mutualist and z is the density of the predator at time t, respectively. $a_i(t)$ $(i = 1, 2, 3)$, $d_j(t)$ $(j = 1, 2, 3, 4)$, and $c_k(t)$ $(k = 1, 2)$ are all continuously positive w-periodic functions. The assumption of periodicity of the parameters is a way to incorporate the periodicity of the environment (*e.g.* seasonal effects of weather condition, food supplies, temperature, mating habits, harvesting, *etc.*). For this system, due to the lack of density restriction of the predator species, one could not investigate the stability property of the system by constructing the suitable Lyapunov function. Hence, to investigate the persistent property of the system has become one of the most important topics, and we try to push forward this topic.

We arrange the rest of the paper as follows: In Section 2, we introduce one lemma and state the main results of this paper. The results are proved in Section 3. In Section 4, a suitable example together with its numeric simulation is present to show the feasibility of the main results. We end this paper by a briefly conclusion. For more works on the non-autonomous predator-prey system, one could refer to [16–19] and the references cited therein.

2 Statement of the main results

Let us first consider the logistic equation,

$$\frac{du(t)}{dt} = u(t)\bigl(\alpha(t) - \beta(t)u(t)\bigr), \tag{2.1}$$

where $\alpha(t)$ and $\beta(t)$ are periodic continuous functions on R with common periodic $w > 0$.

Lemma 2.1 [20] *If $\beta(t) \geq 0$ for all $t \in R$ and $\int_0^w \beta(t)\,dt > 0$, then* (2.1) *has a unique nonnegative w-periodic solution $u^*(t)$ which is globally asymptotically stable, that is, $u(t) - u^*(t) \to 0$ as $t \to \infty$ for any positive solution $u(t)$ of* (2.1). *Moreover, if $\int_0^w \alpha(t)\,dt > 0$, then $u^*(t) > 0$ for all $t \in R$ and if $\int_0^w \alpha(t)\,dt \leq 0$ then $u^*(t) \equiv 0$.*

Definition 2.2 System (1.2) is said to be permanent if there exist positive constants η_i, M_i, $i = 1, 2, 3$, such that

$$\eta_1 \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq M_1,$$

$$\eta_2 \le \liminf_{t\to\infty} y(t) \le \limsup_{t\to\infty} y(t) \le M_2,$$

$$\eta_3 \le \liminf_{t\to\infty} z(t) \le \limsup_{t\to\infty} z(t) \le M_3$$

for any positive solution $(x(t), y(t), z(t))$ of system (1.2).

We first consider the following system:

$$\dot{u_1} = u_1\big(a_1(t) - b_1(t)u_1\big), \tag{2.2}$$

from Lemma 2.1, (2.2) has a unique nonnegative w-periodic solution $u_{10}(t)$ which is globally asymptotically stable.

Second, we consider the following system:

$$\dot{u_2} = u_2\left(a_2(t) - \frac{u_2}{d_3(t) + d_4(t)M^*}\right), \tag{2.3}$$

where $M^* = \max_{0\le t\le w}\{u_{10}(t) + 1\}$, from Lemma 2.1, (2.3) also has a unique nonnegative w-periodic solution $u_{20}(t)$ which is globally asymptotically stable.

As concerns the persistent property of the system (1.2), we have the following result.

Theorem 2.3 *System* (1.2) *is permanent if*

$$\int_0^w \left(-a_3(t) + \frac{c_2(t)u_{10}(t)}{d_1(t) + d_2(t)u_{20}(t)}\right) dt > 0 \tag{2.4}$$

holds, where $u_{10}(t)$ *and* $u_{20}(t)$ *are the unique positive periodic solution of systems* (2.2) *and* (2.3), *respectively.*

As a direct corollary of Theorem 2 in [21], from Theorem 2.3, we have the following.

Corollary 2.4 *Under the assumption that* (2.4) *holds, system* (1.2) *admits of at least one positive w-periodic solution.*

3 Proof of the main result

We need the following propositions to prove Theorem 2.3.

Proposition 3.1 *There exist positive constants* M_1 *and* M_2, *such that*

$$\limsup_{t\to\infty} x(t) \le M_1, \qquad \limsup_{t\to\infty} y(t) \le M_2 \tag{3.1}$$

for all solutions of system (1.2).

Proof Obviously, $R^3_+ = \{(x, y, z) | x \ge 0, y \ge 0, z \ge 0\}$ is a positively invariant set of system (1.2). Given any solution $(x(t), y(t), z(t))$ of system (1.2), we have

$$\dot{x} \le x\big(a_1(t) - b_1(t)x\big). \tag{3.2}$$

By Lemma 2.1, the auxiliary equation

$$\dot{u_1} = u_1\big(a_1(t) - b_1(t)u_1\big) \tag{3.3}$$

has a unique globally attractive positive w-periodic solution $u_{10}(t)$. Let $u_1(t)$ is the solution of (3.3) with $u_1(0) = x(0)$, by the comparison theorem, we have

$$x(t) \leq u_1(t), \quad t \geq 0. \tag{3.4}$$

Moreover, from the global attractivity of $u_{10}(t)$, for every given ε ($0 < \varepsilon < 1$), there exists a $T_1 > 0$, such that

$$\left|u_1(t) - u_{10}(t)\right| < \varepsilon, \quad t \geq T_1. \tag{3.5}$$

Equation (3.4) combined with (3.5) leads to

$$x(t) < u_{10}(t) + \varepsilon, \quad t > T_1. \tag{3.6}$$

Let $M_1 = \max_{0\leq t\leq w}\{u_{10}(t) + \varepsilon\}$, we have

$$\limsup_{t\to\infty} x(t) \leq M_1. \tag{3.7}$$

Since $M^* = \max_{0\leq t\leq w}\{u_{10}(t) + 1\}$, there exists a large enough $T_2 \geq T_1$ such that for all $t > T_2$, one has

$$x(t) < M^*. \tag{3.8}$$

From (3.8), we have

$$\dot{y} \leq y\left(a_2(t) - \frac{y}{d_3(t) + d_4(t)M^*}\right), \quad t \geq T_2. \tag{3.9}$$

By Lemma 2.1, the auxiliary equation

$$\dot{u_2} = u_2\left(a_2(t) - \frac{u_2}{d_3(t) + d_4(t)M^*}\right) \tag{3.10}$$

has a unique globally attractive positive w-periodic solution $u_{20}(t)$. Similarly, we find that there is a constant $T_3 > T_2$ such that

$$y(t) < u_{20}(t) + \varepsilon, \quad t > T_3. \tag{3.11}$$

Let $M_2 = \max_{0\leq t\leq w}\{u_{20}(t) + \varepsilon\}$, we have

$$\limsup_{t\to\infty} y(t) \leq M_2. \tag{3.12}$$

This completes the proof of Proposition 3.1. □

Proposition 3.2 *There is a universal constant $\alpha > 0$ such that*

$$\limsup_{t\to\infty} z(t) > \alpha. \tag{3.13}$$

Proof If (2.4) holds, we can choose the constant $\varepsilon_1 > 0$ such that

$$\int_0^w \left(-a_3(t) + \frac{c_2(t)(u_{10}(t) - \varepsilon_1)}{d_1(t) + d_2(t)(u_{20}(t) + \varepsilon_1)} \right) dt > 0. \tag{3.14}$$

For any constant $\alpha > 0$, we consider the following equation:

$$\frac{dv_1}{dt} = v_1(t)\left(a_1(t) - \frac{c_1(t)2\alpha}{d_1(t)} - b_1(t)v_1(t) \right). \tag{3.15}$$

Owing to $\int_0^w a_1(t)\,dt > 0$, $\int_0^w (a_1(t) - \frac{c_1(t)2\alpha}{d_1(t)})\,dt > 0$ for small enough $\alpha > 0$. By Lemma 2.1, (3.15) has a unique positive w-periodic solution $v_{1\alpha}^*(t)$ which is globally asymptotically stable. Let $\bar{v}_{1\alpha}(t)$ be the solution of (3.15) with initial condition $\bar{v}_{1\alpha}(0) = u_{10}(0)$, where $u_{10}(t)$ is the unique positive periodic solution of (2.2). Hence, for above ε_1, there exists a sufficiently large $T_4 > T_3$ such that

$$\left| \bar{v}_{1\alpha}(t) - v_{1\alpha}^*(t) \right| < \frac{\varepsilon_1}{4} \quad \text{for } t \geq T_4.$$

By the continuity of the solution in the parameter, we have $\bar{v}_{1\alpha}(t) \to u_{10}(t)$ uniformly in $[T_4, T_4 + w]$ as $\alpha \to 0$. Hence, for $\varepsilon_1 > 0$, there exists a $\alpha_0 = \alpha_0(\varepsilon_1) > 0$ such that

$$\left| \bar{v}_{1\alpha}(t) - u_{10}(t) \right| < \frac{\varepsilon_1}{4} \quad \text{for } t \in [T_4, T_4 + w], 0 < \alpha < \alpha_0.$$

So, we have

$$\left| v_{1\alpha}^*(t) - u_{10}(t) \right| < \frac{\varepsilon_1}{2} \quad \text{for } t \in [T_4, T_4 + w].$$

Note that $v_{1\alpha}^*(t)$ and $u_{10}(t)$ are all w-periodic, hence

$$\left| v_{1\alpha}^*(t) - u_{10}(t) \right| < \frac{\varepsilon_1}{2} \quad \text{for } t \geq 0, 0 < \alpha < \alpha_0.$$

Choosing a constant α_1 ($0 < \alpha_1 < \alpha_0$, $2\alpha_1 < \varepsilon_1$), we have

$$v_{1\alpha_1}^*(t) \geq u_{10}(t) - \frac{\varepsilon_1}{2}, \quad t \geq 0. \tag{3.16}$$

If $\limsup_{t\to\infty} z(t) \leq \alpha_1$, then there exists $\phi \in R_+^3$ such that

$$\limsup_{t\to\infty} z(t,\phi) < \alpha_1,$$

where $(x(t,\phi), y(t,\phi), z(t,\phi))$ is the solution of system (1.2) with $\phi(0) = (\phi_1(0), \phi_2(0), \phi_3(0)) > 0$. So, there exists $T_5 > T_4$ such that

$$z(t) < 2\alpha_1 < \varepsilon_1, \quad t \geq T_5. \tag{3.17}$$

We have

$$\dot{x} \geq x\left(a_1(t) - \frac{c_1(t)2\alpha_1}{d_1(t)} - b_1(t)x \right) \quad \text{for } t \geq T_5. \tag{3.18}$$

Let $v_1(t)$ be the solution of (3.15) with $\alpha = \alpha_1$ and $v_1(T_5) = x(T_5)$, then

$$x(t) \geq v_1(t), \quad t \geq T_5.$$

By the global asymptotic stability of $v^*_{1\alpha_1}(t)$, for the given $\varepsilon = \frac{\varepsilon_1}{2}$, there exists $T_6 \geq T_5$, such that

$$\left|v_1(t) - v^*_{1\alpha_1}(t)\right| < \frac{\varepsilon_1}{2}, \quad t \geq T_6.$$

So,

$$x(t) \geq v_1(t) > v^*_{1\alpha_1}(t) - \frac{\varepsilon_1}{2}, \quad t \geq T_6,$$

and hence, by using (3.16), it follows that

$$x(t) > u_{10}(t) - \varepsilon_1, \quad t \geq T_6. \tag{3.19}$$

From (3.8), there exists $T_7 \geq T_6$ such that $y(t) < u_{20}(t) + \varepsilon_1$. Therefore, by using (3.17) and (3.19), for $t \geq T_7$ it follows that

$$\dot{z} \geq z(t)\left(-a_3(t) + \frac{c_2(t)(u_{10}(t) - \varepsilon_1)}{d_1(t) + d_2(t)(u_{20}(t) + \varepsilon_1)}\right).$$

Integrating the above inequality from T_7 to t yields

$$z(t) \geq z(T_7)\exp\int_{T_7}^{t}\left(-a_3(s) + \frac{c_2(s)(u_{10}(s) - \varepsilon_1)}{d_1(s) + d_2(s)(u_{20}(s) + \varepsilon_1)}\right)ds.$$

Thus, from (3.14) it follows that $z(t) \to \infty$ as $t \to \infty$. This is a contradiction. This completes the proof of Proposition 3.2. □

Proposition 3.3 *There is a universal constant* $\eta_2 > 0$ *such that*

$$\liminf_{t\to\infty} y(t) \geq \eta_2. \tag{3.20}$$

Proof From the second equation of system (1.2) it follows that

$$\dot{y} \geq y\left(a_2(t) - \frac{y}{d_3(t)}\right), \tag{3.21}$$

and we consider the following equation:

$$\dot{u}_3 = u_3\left(a_2(t) - \frac{u_3}{d_3(t)}\right). \tag{3.22}$$

By Lemma 2.1, (3.22) has a unique positive w-periodic solution $u_{30}(t)$. Similar to the analysis of (3.4)-(3.7), for ε enough small, without loss of generality, $\varepsilon < \min_{0\leq t\leq w} u_{30}(t)$, we find that there is a constant $T_8 > T_7$ such that

$$y(t) \geq u_{30}(t) - \varepsilon > 0, \quad t > T_8. \tag{3.23}$$

Letting $\eta_2 = \min_{0\le t\le w}\{u_{30}(t) - \varepsilon\}$, we have

$$\liminf_{t\to\infty} y(t) \ge \eta_2. \tag{3.24}$$

This completes the proof of Proposition 3.3. □

Proposition 3.4 *There is a universal constant* $\eta_3 > 0$ *such that*

$$\liminf_{t\to\infty} z(t) \ge \eta_3. \tag{3.25}$$

Proof Suppose that (3.25) is not true, then there is a sequence $\{\phi_m\} \in R_+^3$, such that

$$\liminf_{t\to\infty} z(t,\phi_m) < \frac{\alpha}{m+1}, \quad m = 1,2,\ldots. \tag{3.26}$$

On the other hand, by Proposition 3.2, we have

$$\limsup_{t\to\infty} z(t,\phi_m) > \alpha, \quad m = 1,2,\ldots. \tag{3.27}$$

Hence, there are time sequences $\{s_q^m\}$ and $\{t_q^m\}$ satisfying

$$\begin{aligned} &0 < s_1^m < t_1^m < s_2^m < t_2^m < \cdots < s_q^m < t_q^m < \cdots, \\ &s_q^m \to +\infty, \qquad t_q^m \to +\infty \quad \text{as } q \to +\infty \end{aligned}$$

and

$$\begin{aligned} &z\bigl(s_q^m,\phi_m\bigr) = \alpha, \qquad z\bigl(t_q^m,\phi_m\bigr) = \frac{\alpha}{m+1}, \\ &\frac{\alpha}{m+1} < z(t,\phi_m) < \alpha, \quad t \in \bigl(s_q^m, t_q^m\bigr). \end{aligned}$$

From the third equation of system (1.2) it follows that

$$\dot{z} \ge -a_3(t)z. \tag{3.28}$$

Obviously, by integrating (3.28) from s_q^m to t_q^m,

$$z\bigl(t_q^m,\phi_m\bigr) \ge z\bigl(s_q^m,\phi_m\bigr)\exp\int_{s_q^{(m)}}^{t_q^{(m)}} \bigl(-a_3(t)\bigr)\,dt$$

or

$$\int_{s_q^{(m)}}^{t_q^{(m)}} \bigl(a_3(t)\bigr)\,dt \ge \ln(m+1).$$

Thus, from the boundedness of $a_3(t)$, we have

$$t_q^{(m)} - s_q^{(m)} \to +\infty \quad \text{as } m \to +\infty. \tag{3.29}$$

By (3.14), there are constants $P > 0$ and $\gamma > 0$, such that, for any $t \geq P$ and $a \geq 0$,

$$\int_a^{a+t} \left(-a_3(s) + \frac{c_2(s)(u_{10}(s) - \varepsilon_1)}{d_1(s) + d_2(s)(u_{20}(s) + \varepsilon_1)} \right) ds > \gamma. \tag{3.30}$$

For any m, q, and $t \in [s_q^{(m)}, t_q^{(m)}]$, we have

$$\dot{x}(t, \phi_m) \geq x(t, \phi_m) \left(a_1(t) - \frac{c_1(t)\alpha}{d_1(t)} - b_1(t)x(t, \phi_m) \right). \tag{3.31}$$

Let $v_1(t)$ be the solution of (3.15) with the initial condition $v_1(s_q^{(m)}) = \alpha$, then by (3.31), we have $x(t, \phi_m) \geq v_1(t)$ for all $t \in [s_q^{(m)}, t_q^{(m)}]$. By the periodicity of (3.15), it follows that the periodic solution $v_{1\alpha}^*(t)$ is globally uniformly asymptotically stable. Hence, from (3.16), we find that there is a constant $T_0 > P$, and T_0 is independent on any q and m, such that

$$v_1(t) \geq u_{10}(t) - \varepsilon_1 \quad \text{for all } t \geq T_0 + s_q^{(m)}.$$

By (3.29), there is a $N_0 > 0$ such that $t_q^{(m)} > s_q^{(m)} + 2T_0$ for all $m \geq N_0$. Hence,

$$\begin{aligned} &x(t, \phi_m) u_{10}(t) - \varepsilon_1, \\ &y(t, \phi_m) < u_{20}(t) + \varepsilon_1 \quad \text{for all } t \in \left[s_q^{(m)} + T_0, t_q^{(m)}\right], m \geq N_0. \end{aligned}$$

Since

$$\dot{z}(t, \phi_m) \geq z(t, \phi_m) \left(-a_3(t) + \frac{c_2(t)(u_{10}(t) - \varepsilon_1)}{d_1(t) + d_2(t)(u_{20}(t) + \varepsilon_1)} \right)$$

for all $t \in [s_q^{(m)} + T_0, t_q^{(m)}]$ and $m \geq N_0$, by integrating from $s_q^{(m)} + T_0$ to $t_q^{(m)}$, we obtain

$$\begin{aligned} z\big(t_q^{(m)}, \phi_m\big) &\geq z\big(s_q^{(m)} + T_0, \phi_m\big) \\ &\quad \times \exp \int_{s_q^{(m)} + T_0}^{t_q^{(m)}} \left(-a_3(t) + \frac{c_2(t)(u_{10}(t) - \varepsilon_1)}{d_1(t) + d_2(t)(u_{20}(t) + \varepsilon_1)} \right) dt. \end{aligned}$$

By (3.30), we have

$$\begin{aligned} \frac{\alpha}{m+1} &\geq \frac{\alpha}{m+1} \exp \int_{s_q^{(m)} + T_0}^{t_q^{(m)}} \left(-a_3(t) + \frac{c_2(t)(u_{10}(t) - \varepsilon_1)}{d_1(t) + d_2(t)(u_{20}(t) + \varepsilon_1)} \right) dt \\ &> \frac{\alpha}{m+1}, \end{aligned}$$

which is a contradiction. This completes the proof of Proposition 3.4. □

Proposition 3.5 *There is a universal constant* $M_3 > 0$ *such that*

$$\limsup_{t \to \infty} z(t) \leq M_3. \tag{3.32}$$

Proof Choose the constant $M > 0$ and small enough ε such that

$$\int_0^w \left(a_1(t) - \frac{c_1(t)M}{d_1(t) + d_2(t)(M_2 + \varepsilon)} \right) dt < 0. \tag{3.33}$$

We first prove that

$$\liminf_{t\to\infty} z(t) \le M. \tag{3.34}$$

If (3.34) is not true, then there is a $T_9 \ge T_8$ such that $z(t) > M$ and $y(t) \le M_2 + \varepsilon$ (ε is small enough) for all $t \ge T_9$. Since

$$\dot{x} \le x\left(a_1(t) - \frac{c_1(t)M}{d_1(t) + d_2(t)(M_2 + \varepsilon)} - b_1(t)x \right) \quad \text{for all } t \ge T_9.$$

By Lemma 2.1 we easily obtain $x(t) \to 0$ as $t \to \infty$. Choose a constant $0 < \varepsilon_2 < \eta_2$ such that

$$\int_0^w \left(-a_3(t) + \frac{c_2(t)\varepsilon_2}{d_1(t) + d_2(t)(\eta_2 - \varepsilon_2)} \right) dt < 0. \tag{3.35}$$

Then there is a $T_{10} > T_9 > 0$ such that $x(t) \le \varepsilon_2$ and $y(t) > \eta_2 - \varepsilon_2$ for all $t \ge T_{10}$, and so

$$\dot{z} \le z\left(-a_3(t) + \frac{c_2(t)\varepsilon_2}{d_1(t) + d_2(t)(\eta_2 - \varepsilon_2)} \right). \tag{3.36}$$

Integrating the above inequality from T_{10} to t leads to

$$z(t) \le z(T_{10}) \exp \int_{T_{10}}^t \left(-a_3(s) + \frac{c_2(s)\varepsilon_2}{d_1(s) + d_2(s)(\eta_2 - \varepsilon_2)} \right) ds.$$

Hence, we have $z(t) \to 0$ as $t \to \infty$ which is a contradiction. Now, if (3.32) is not true, then there is a sequence $\{\phi_m\} \subset R_+^3$ such that $\limsup_{t\to\infty} z(t, \phi_m) > (2M + 1)m$ for all $m = 1, 2, \ldots$. By (3.34), there are time sequences $\{s_q^m\}$ and $\{t_q^m\}$ satisfying

$$0 < s_1^m < t_1^m < s_2^m < t_2^m < \cdots < s_q^m < t_q^m < \cdots,$$
$$s_q^m \to +\infty, \qquad t_q^m \to +\infty \quad \text{as } q \to +\infty$$

and

$$z\left(s_q^m, \phi_m\right) = 2M, \qquad z\left(t_q^m, \phi_m\right) = (2M + 1)m,$$
$$2M < z(t, \phi_m) < (2M + 1)m \quad \text{for all } t \in \left(s_q^m, t_q^m\right).$$

Since there is a $T_1^{(m)} > 0$ such that $x(t, \phi_m) < M_1$ for all $t \ge T_1^{(m)}$, we have

$$\dot{z}(t, \phi_m) \le z(t, \phi_m)\left(-a_3(t) + \frac{c_2(t)M_1}{d_1(t)} \right) \quad \text{for all } t \ge T_1^{(m)}.$$

Obviously, there is a $K_1(m) > 0$ such that $s_q^m > T_1^{(m)}$ for all $q \geq K_1(m)$. Hence, we obtain

$$z\big(t_q^m, \phi_m\big) \leq z\big(s_q^m, \phi_m\big) \exp \int_{s_q^{(m)}}^{t_q^{(m)}} \left(-a_3(t) + \frac{c_2(t)M_1}{d_1(t)}\right) dt \quad \text{for all } q \geq K_1(m).$$

Consequently,

$$t_q^{(m)} - s_q^{(m)} \to \infty \quad \text{as } m \to \infty, q \geq K_1(m). \tag{3.37}$$

By (3.35), there is a $p_1 > 0$ such that for any $t \geq p_1$ and $a \geq 0$,

$$\int_a^{a+t} \left(-a_3(s) + \frac{c_2(s)\varepsilon_2}{d_1(s) + d_2(s)(\eta_2 - \varepsilon_2)}\right) ds < 0.$$

By (3.33), there is a $p_2 > 0$ such that for all $t \geq p_2$ and $b \geq 0$,

$$\int_b^{b+t} \left(a_1(s) - \frac{c_1(s)M}{d_1(s) + d_2(s)(M_2 + \varepsilon)}\right) ds < \ln \frac{\varepsilon_2}{2M}.$$

By (3.37), there is a $L > 0$ such that $t_q^{(m)} - s_q^{(m)} > p_1 + p_2$ for all $m \geq L$, $q \geq K_1(m)$. For any $m \geq L$, $q \geq K_1(m)$ and $t \in [s_q^{(m)} + p_1 + p_2, t_q^{(m)}]$, we have

$$\begin{aligned} \dot{x}(t, \phi_m) &\leq x(t, \phi_m)\left(a_1(t) - \frac{c_1(t)2M}{d_1(t) + d_2(t)(M_2 + \varepsilon)}\right) \\ &\leq x(t, \phi_m)\left(a_1(t) - \frac{c_1(t)M}{d_1(t) + d_2(t)(M_2 + \varepsilon)}\right). \end{aligned}$$

Hence,

$$x(t, \phi_m) \leq x\big(s_q^{(m)}, \phi_m\big) \exp \int_{s_q^{(m)}}^{t} \left(a_1(s) - \frac{c_1(s)M}{d_1(s) + d_2(s)(M_2 + \varepsilon)}\right) ds < \varepsilon_2.$$

Since

$$\dot{z}(t, \phi_m) \leq z(t, \phi_m)\left(-a_3(t) + \frac{c_2(t)\varepsilon_2}{d_1(t) + d_2(t)(\eta_2 - \varepsilon_2)}\right), \quad t \in \big[s_q^{(m)} + p_1 + p_2, t_q^{(m)}\big].$$

we obtain

$$\begin{aligned} z\big(t_q^{(m)}, \phi_m\big) &\leq z\big(s_q^{(m)} + p_1 + p_2, \phi_m\big) \\ &\quad \times \exp \int_{s_q^{(m)} + p_1 + p_2}^{t_q^{(m)}} \left(-a_3(t) + \frac{c_2(t)\varepsilon_2}{d_1(t) + d_2(t)(\eta_2 - \varepsilon_2)}\right) dt \\ &< (2M + 1)m, \end{aligned}$$

which is contradictory with $z(t_q^{(m)}, \phi_m) = (2M + 1)m$. Therefore, there is a constant $M_3 > 0$ such that (3.32) holds. □

Proposition 3.6 *There is a universal constant* $0 < \eta < \eta_2$ *such that*

$$\limsup_{t\to\infty} x(t) > \eta \tag{3.38}$$

and

$$\int_0^t \left(-a_3(s) + \frac{c_2(s)2\eta}{d_1(s) + d_2(s)\eta_2}\right) ds < 0$$

hold, where η_2 *is obtained in Proposition* 3.3.

Proof If $\limsup_{t\to\infty} x(t) \le \eta$, then by Proposition 3.3, there is a $T_{11} > T_{10}$ such that $x(t) < 2\eta$ and $y(t) \ge \eta_2 - \varepsilon$ (ε is small enough) for all $t \ge T_{11}$. Since

$$\dot{z}(t) \le z(t)\left(-a_3(t) + \frac{c_2(t)2\eta}{d_1(t) + d_2(t)(\eta_2 - \varepsilon)}\right) \quad \text{for all } t \ge T_{11},$$

and since

$$\int_0^t \left(-a_3(s) + \frac{c_2(s)2\eta}{d_1(s) + d_2(s)\eta_2}\right) ds < 0,$$

and for a small enough ε,

$$\int_0^t \left(-a_3(s) + \frac{c_2(s)2\eta}{d_1(s) + d_2(s)(\eta_2 - \varepsilon)}\right) ds < 0,$$

by integrating from T_{11} to t, we can obtain

$$z(t) \le z(T_{11}) \exp \int_{T_{11}}^t \left(-a_3(s) + \frac{c_2(s)2\eta}{d_1(s) + d_2(s)(\eta_2 - \varepsilon)}\right) ds.$$

Hence, inequality (3.38) implies $z(t) \to 0$ as $t \to \infty$, which is a contradiction and Proposition 3.6 is proved. □

Proposition 3.7 *There is a universal constant* $\eta_1 > 0$ *such that*

$$\liminf_{t\to\infty} x(t) > \eta_1. \tag{3.39}$$

Proof We first choose the constant $0 < \varepsilon_3 < \eta$ such that

$$\int_0^w \left(-a_3(t) + \frac{c_2(t)\varepsilon_3}{d_1(t) + d_2(t)\eta_2}\right) dt < 0. \tag{3.40}$$

Here the constants $\eta_2 > 0$ and $\eta > 0$ are obtained in Propositions 3.3 and 3.6. Suppose that (3.39) is not true. Then there is a sequence $\{\phi_m\} \subset R^3_+$ such that $\liminf_{t\to\infty} x(t, \phi_m) < \frac{\varepsilon_3}{m+1}$ for all $m = 1, 2, \ldots$. By Proposition 3.6, there are two time sequences $\{s_q^{(m)}\}$ and $\{t_q^{(m)}\}$ satisfying the following conditions:

$$0 < s_1^m < t_1^m < s_2^m < t_2^m < \cdots < s_q^m < t_q^m < \cdots,$$

$$s_q^m \to +\infty, \qquad t_q^m \to +\infty \quad \text{as } q \to +\infty, m = 1, 2, \ldots$$

and

$$x\left(s_q^m,\phi_m\right)=\varepsilon_3,\qquad x\left(t_q^m,\phi_m\right)=\frac{\varepsilon_3}{m+1},$$
$$\frac{\varepsilon_3}{m+1}<x(t,\phi_m)<\varepsilon_3\quad\text{for all } t\in\left(s_q^m,t_q^m\right).$$

By Propositions 3.1 and 3.3, there is a $T^{(m)}>0$ such that $z(t,\phi_m)\le M_3$ and $x(t,\phi_m)\le M_1$ for all $t\ge T^{(m)}$, and further there is a $K^{(m)}>0$ such that $s_q^m\ge T^{(m)}$ for all $q\ge K^{(m)}$. Hence, for any $t\in[s_q^m,t_q^m]$ and $q\ge K^{(m)}$, we have

$$\dot{x}(t,\phi_m)\ge x(t,\phi_m)\left(a_1(t)-b_1(t)M_1-\frac{c_1(t)M_3}{d_1(t)}\right).$$

By integrating from s_q^m to t_q^m, we obtain

$$x\left(t_q^m,\phi_m\right)\ge x\left(s_q^m,\phi_m\right)\exp\int_{s_q^{(m)}}^{t_q^{(m)}}\left(a_1(t)-b_1(t)M_1-\frac{c_1(t)M_3}{d_1(t)}\right)dt.$$

Consequently,

$$\ln(m+1)\le\int_{s_q^{(m)}}^{t_q^{(m)}}\left(-a_1(t)+b_1(t)M_1+\frac{c_1(t)M_3}{d_1(t)}\right)dt\quad\text{for all } q\ge K^{(m)}.$$

Hence, we obtain

$$t_q^{(m)}-s_q^{(m)}\to\infty\quad\text{as } m\to\infty, q\ge K^{(m)}.\tag{3.41}$$

By (3.40), there is a constant $T_{12}>0$, and T_{12} is independent on any m and q, such that

$$M_3\exp\int_{s_q^{(m)}}^{t_q^{(m)}}\left(-a_3(t)+\frac{c_2(t)\varepsilon_3}{d_1(t)+d_2(t)(\eta_2-\varepsilon)}\right)dt<\eta_3.$$

By (3.41), there is a $M_0>0$ such that $t_q^{(m)}>s_q^{(m)}+M_0$ for all $m\ge M_0$ and $q\ge K^{(m)}$. Hence, for any $t\in[s_q^m,t_q^m]$, $m\ge M_0$, and $q\ge K^{(m)}$, we have $y(t)\ge\eta_2-\varepsilon$ and

$$\dot{z}(t,\phi_m)\le z(t,\phi_m)\left(-a_3(t)+\frac{c_2(t)\varepsilon_3}{d_1(t)+d_2(t)(\eta_2-\varepsilon)}\right).$$

By integrating from $s_q^{(m)}$ to $s_q^{(m)}+M_0$, we obtain

$$\begin{aligned}\eta_3&<z\left(s_q^{(m)}+M_0,\phi_m\right)\\&\le z\left(s_q^{(m)},\phi_m\right)\exp\int_{s_q^{(m)}}^{s_q^{(m)}+M_0}\left(-a_3(t)+\frac{c_2(t)\varepsilon_3}{d_1(t)+d_2(t)(\eta_2-\varepsilon)}\right)dt<\eta_3,\end{aligned}$$

which is a contradiction, and Proposition 3.7 is proved. □

3.1 Proof of Theorem 2.3

The results of Theorem 2.3 now follow from Propositions 3.1-3.7.

4 Example

In this section, we shall give an example to illustrate the feasibility of the main results.

Example Consider the following predator-prey-mutualist system (see Figure 1):

$$\begin{aligned}\dot{x} &= x\left(3.2 - 0.3x - \frac{0.6z}{0.8 + 0.3y}\right),\\ \dot{y} &= y\left(3.0 - \frac{y}{0.5 + 0.5x}\right),\\ \dot{z} &= z\left(-(0.7 - 0.1\sin t) + \frac{0.5x}{0.8 + 0.3y}\right).\end{aligned} \tag{4.1}$$

Corresponding to system (1.2), one has

$$\begin{aligned}&a_1(t) = 3.2, \quad b_1(t) = 0.3, \quad c_1(t) = 0.6, \quad d_1(t) = 0.8,\\ &d_2(t) = 0.3, \quad a_2(t) = 3.0, \quad d_3(t) = 0.5, \quad d_4(t) = 0.5,\\ &a_3(t) = 0.7 - 0.1\sin t, \quad c_2(t) = 0.5.\end{aligned}$$

One easily sees that

$$\dot{u}_1(t) = u_1(t)\bigl(3.2 - 0.3u_1(t)\bigr)$$

has a unique positive periodic solution $u_{10}(t) \approx 10.667$. So $M^* = 11.667$,

$$\dot{u}_2(t) = u_2(t)\left(3.0 - \frac{u_2(t)}{0.5 + 0.5 \times 11.667}\right)$$

has a unique positive periodic solution $u_{20}(t) \approx 17.503$.

By simple computation, one has

$$\begin{aligned}&\int_0^w \left(-a_3(t) + \frac{c_2(t)u_{10}(t)}{d_1(t) + d_2(t)u_{20}(t)}\right) dt\\ &\quad = \int_0^{2\pi} \left(-(0.7 - 0.1\sin t) + \frac{0.5 \times 10.667}{0.8 + 0.3 \times 17.503}\right) dt \approx 1.1394 > 0.\end{aligned}$$

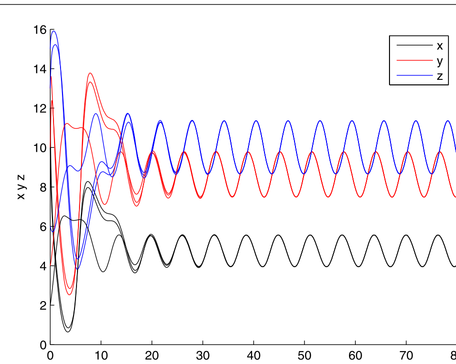

Figure 1 Dynamic behaviors of the solution ($x(t)$, $y(t), z(t)$) system (4.1) with initial conditions ($x(0)$, $y(0), z(0)$) = (10, 13, 15), (12, 10, 13) and (2, 4, 6), respectively.

Condition (2.4) is satisfied. Thus, corresponding to Theorem 2.3 and Corollary 2.4, we know that system (4.1) is permanent and admits at least one positive 2π-periodic solution.

5 Conclusion

In this paper, we studied a periodic predator-prey-mutualist system. From system (1.2), we see the mutualist species y can reduce the capture rate of the predator species z to the prey species x. By further developing the analysis technique of Teng [20], we obtain a set of conditions which ensure the permanence of system (1.2). Note that $u_{10}(t)$ and $u_{20}(t)$ are the globally attractive periodic solution of (2.2) and (2.3), respectively, which, as shown by Lemma 2.1, always exists. Hence, the left side of condition (2.4) implies that if the death rate of the predator species is enough small and the cooperation effect between species x and y is not very strong, the system is permanent.

Competing interests
The authors declare that there are no competing interests.

Authors' contributions
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