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Existence of solutions of fractional boundary value problems with p-Laplacian operator

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Abstract

In this paper, the existence of the solutions of the fractional differential equation with p-Laplacian operator and integral conditions is discussed. By Green's functions and the fixed point theorems, we state and prove the existence and uniqueness results of the problem. Two examples are given to illustrate the results.

Keywords: existence and uniqueness; fractional calculus; p-Laplacian

1 Introduction

Differential equations are useful in modern physics, engineering, and in various fields of science. In these days, the theory on existence and uniqueness of boundary value problems of linear and/or nonlinear fractional equations has attracted the attention of many authors. There are comprehensive studies in this area. At the same time, it is known that the p-Laplacian operator is also used in analyzing mechanics, physics and dynamic systems, and the related fields of mathematical modeling. However, there are few studies of the existence and uniqueness of boundary conditions of fractional differential equations with the p-Laplacian operator, see [1–27] and the references therein.

Zhang *et al.* [4] studied the eigenvalue problem for a class of singular p-Laplacian fractional differential equations involving a Riemann-Stieltjes integral boundary condition:

$$\begin{aligned}
&-D_t^{\beta}\big(\phi_p\big(D_t^{\alpha}x\big)\big)(t) = \lambda f\big(t, x(t)\big), \quad t \in (0,1),\\
&x(0) = 0, \qquad D_t^{\alpha}x(0) = 0,\\
&x(1) = \int_0^1 x(s)\,dA(s),
\end{aligned}$$

where D_t^{β} and D_t^{α} are standard Riemann-Liouville derivatives with $1 < \alpha \le 2$, $0 < \beta \le 1$, A is a function of the bounded variation, and $\int_0^1 x(s)\,dA(s)$ is the standard Riemann-Stieltjes integral. In their study, the results are based on upper and lower solution methods and the Schauder fixed point theorem.

In [5], Su *et al.* studied the existence criteria of non-negative solutions of nonlinear p-Laplacian fractional differential equations with first order derivative,

$$\begin{cases} \varphi_p({}^cD^\alpha u(t)) = \varphi_p(\lambda)f(t,u(t),u'(t)), & \text{for } t \in (0,1), \\ k_0 u(0) - k_1 u(1) = 0, \\ m_0 u(0) - m_1 u(1) = 0, \\ x^{(r)}(0) = 0, & r = 2,3,\ldots,[\alpha], \end{cases}$$

where φ_p is p-Laplacian operator, *i.e.* $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, and $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, ${}^cD^\alpha$ is the Caputo derivative and we have the function $f(t,u,u') : [0,1] \times [0,\infty) \times (-\infty,+\infty) \to [0,\infty)$ which satisfies the Carathéodory type conditions. Moreover, the nonlinear alternative of Leray-Schauder type and Banach fixed point theorems are used.

Han *et al.* [6] studied nonlinear fractional differential equations with p-Laplacian operator and boundary value conditions,

$$\begin{aligned} &D_{0+}^\alpha\big(\varphi_p\big(D_{0+}^\alpha u(t)\big)\big) + a(t)f(u) = 0, \quad \text{for } 0 < t < 1, \\ &u(0) = \gamma u(\xi) + \lambda, \\ &\varphi_p\big(D_{0+}^\alpha u(0)\big) = \big(\varphi_p\big(D_{0+}^\alpha u(1)\big)\big)' = \big(\varphi_p\big(D_{0+}^\alpha u(0)\big)\big), \end{aligned}$$

where $0 < \alpha \le 1$, $2 < \beta \le 3$, and D_{0+}^α, D_{0+}^β are Caputo fractional derivatives, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, and $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, and the parameters are $0 \le \gamma < 1$, $0 \le \xi \le 1$, $\lambda > 0$. The continuous functions $a : (0,1) \to [0,\infty)$ and $f : [0,\infty) \to [0,\infty)$ are given. The Green's function properties and the Schauder fixed point theorem are used.

In [2], Liu *et al.* studied the solvability of the Caputo fractional differential equation with boundary value conditions involving the p-Laplacian operator. The existence and uniqueness of the problem is found by the Banach fixed point theorem. The problem is given in the following:

$$\big(\varphi_p\big(D_{0+}^\alpha x(t)\big)\big)' = f\big(t,x(t)\big), \quad \text{for } t \in (0,1),$$

with boundary value conditions

$$\begin{aligned} &x(0) = r_0 x(1), \\ &x'(0) = r_1 x'(1), \\ &x^{(j)}(0) = 0, \end{aligned}$$

where $i = 2,3,\ldots,[\alpha] - 1$. Here, φ_p is the p-Laplacian operator and D_{0+}^α is the Caputo fractional derivative, $1 < \alpha \in R$, and the nonlinear function $f \in C([0,1] \times R, R)$ is given.

In [7], Lu *et al.* studied the existence of nonnegative solutions of a nonlinear fractional boundary value problem with the p-Laplacian operator:

$$\begin{aligned} &D_{0+}^\beta\big(\varphi_p\big(D_{0+}^\alpha u(t)\big)\big) = f\big(t,u(t)\big), \quad \text{for } 0 < t < 1, \\ &u(0) = u'(0) = u'(1) = 0, \\ &D_{0+}^\alpha u(0) = D_{0+}^\alpha u(1) = 0, \end{aligned}$$

where $2 < \alpha \le 3$, $1 < \beta \le 2$, and D_{0+}^α, D_{0+}^β are the standard Riemann-Liouville fractional derivatives. Green's functions, the Guo-Krasnoselskii theorem, and the Leggett-Williams fixed point theorems are used.

In [1], Wang and Xiang used upper and lower solutions method to find the existence results of at least one non-negative solution of the p-Laplacian fractional boundary value problem, which is given in the following:

$$
\begin{aligned}
&D_{0+}^{\gamma}\big(\phi_p\big(D_{0+}^{\alpha}u(t)\big)\big) = f\big(t,u(t)\big), \quad \text{for } 0<t<1,\\
&u(0)=0, \qquad u'(1)=au(\xi),\\
&D_{0+}^{\alpha}u(0)=0, \qquad D_{0+}^{\alpha}u(1)=bD_{0+}^{\alpha}u(\eta),
\end{aligned}
$$

where $1<\alpha,\gamma\leq 2$, $0\leq a,b\leq 1$, $0<\xi,\eta<1$, and also D_{0+}^{α}, D_{0+}^{γ} are Riemann-Liouville fractional operators.

In this paper, we focus on the existence of solutions of the fractional differential equation

$$
D_{0+}^{\beta}\phi_p\big(D_{0+}^{\alpha}u(t)\big) = f\big(t,u(t),D_{0+}^{\gamma}u(t)\big), \tag{1}
$$

with the p-Laplacian operator and integral boundary conditions,

$$
\begin{aligned}
&u(0)+\mu_1 u(1)=\sigma_1\int_0^1 g\big(s,u(s)\big)\,ds, \qquad (2)\\
&u'(0)+\mu_2 u'(1)=\sigma_2\int_0^1 h\big(s,u(s)\big)\,ds,\\
&D_{0+}^{\alpha}u(0)=0,\\
&D_{0+}^{\alpha}u(1)=\upsilon D_{0+}^{\alpha}u(\eta),
\end{aligned}
$$

where $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are for the Caputo fractional differential equation with $1<\alpha\leq 2, 1<\beta\leq 2$, ν, μ_i, σ_i $(i=1,2)$ are non-negative parameters. f, g, h are continuous functions. By the Green's functions and fixed point theorems, we state and prove the existence and uniqueness results of the solutions. Two examples are given to illustrate the results.

2 Preliminaries

The basic definitions are given in the following.

Definition 1 The Riemann-Liouville fractional integral of order $\alpha>0$ for a function $f:(0,+\infty)\to R$ is defined as

$$
I_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds,
$$

provided that the right hand side of the integral is pointwise defined on $(0,+\infty)$ and Γ is the gamma function.

Definition 2 The *Caputo* derivative of order $\alpha>0$ for a function $f:(0,+\infty)\to R$ is written as

$$
D_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)\,ds,
$$

where $n=[\alpha]+1$, $[\alpha]$ is the integral part of α.

Lemma 3 *Let $u \in C(0,1) \cap L^1(0,1)$ with the fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I^{\alpha}_{0+} D^{\alpha}_{0+} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for $c_i \in R$ ($i = 1, 2, \ldots, n$), where n is the smallest integer greater than or equal to α.

Lemma 4 *Let $\alpha > 0$. Then the differential equation $D^{\alpha}_{0+} f(t) = 0$ has solutions*

$$f(t) = k_0 + k_1 t + k_2 t^2 + \cdots + k_{n-1} t^{n-1}$$

and

$$I^{\alpha}_{0+} D^{\alpha}_{0+} f(t) = f(t) + k_0 + k_1 t + k_2 t^2 + \cdots + k_{n-1} t^{n-1},$$

where $k_i \in R$ and $i = 1, 2, \ldots, n = [\alpha] + 1$.

The Caputo fractional derivative of order $n - 1 < \alpha < n$ for t^{γ} is given by

$$D^{\alpha}_{0+} t^{\gamma} = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & \gamma \in N \text{ and } \gamma \geq n \text{ or } \gamma \notin N \text{ and } \gamma > n-1, \\ 0, & \gamma \in \{0, 1, \ldots, n-1\}. \end{cases} \tag{3}$$

Also, for brevity, we set

$$\omega_1 = \frac{\sigma_1}{1+\mu_1} - \frac{\sigma_2 \mu_1}{(1+\mu_1)(1+\mu_2)}, \qquad \omega_2 = \frac{\sigma_2}{1+\mu_2},$$
$$c_1(\eta) = \frac{\upsilon^{p-1} \eta^{\beta-1}}{(1-\upsilon^{p-1}\eta)\Gamma(\beta+1)}, \qquad L = ct^{\beta-1} c_1(\eta).$$

We use the following properties of the p-Laplacian operator: $\phi_p(u) = |u|^{p-2}u$, $p > 1$, and $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

(L1) If $1 < p < 2$, $uv > 0$, $|u|, |v| \geq r > 0$, then

$$\left|\phi_p(u) - \phi_p(v)\right| \leq (p-1) r^{p-2} |u - v|.$$

(L2) If $p > 2$, $|u|, |v| \leq \mathbb{R}$ then

$$\left|\phi_p(u) - \phi_p(v)\right| \leq (p-1) R^{p-2} |u - v|.$$

We define two Green's functions $G(t,s)$ and $H(t,s)$,

$$G(t,s) = \begin{cases} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} - \left(\frac{\mu_1(1+\mu_2)+t\mu_2(1+\mu_1)}{(1+\mu_1)(1+\mu_2)}\right) \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\ \quad + \frac{\mu_1\mu_2(1-\tau)^{\alpha-2}}{(1+\mu_1)(1+\mu_2)\Gamma(\alpha-1)}, & t \geq \tau, \\ -\left(\frac{\mu_1(1+\mu_2)+t\mu_2(1+\mu_1)}{(1+\mu_1)(1+\mu_2)}\right) \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\ \quad + \frac{\mu_1\mu_2(1-\tau)^{\alpha-2}}{(1+\mu_1)(1+\mu_2)\Gamma(\alpha-1)}, & t \leq \tau, \end{cases}$$

and

$$H(t,s)=\begin{cases}\frac{[(t-s)]^{\beta-1}}{\Gamma(\beta)}-\frac{t(1-s)^{\beta-1}}{(1-\upsilon^{p-1}\eta)\Gamma(\beta)}, & 0\le s\le t\le 1,\eta\le s,\\ \frac{[(t-s)]^{\beta-1}}{\Gamma(\beta)}-\frac{t(1-s)^{\beta-1}}{(1-\upsilon^{p-1}\eta)\Gamma(\beta)}+\frac{t\upsilon^{p-1}(\eta-s)^{\beta-1}}{(1-\upsilon^{p-1}\eta)\Gamma(\beta)}, & 0\le s\le t\le 1,\eta\ge s,\\ \frac{-t(1-s)^{\beta-1}}{(1-\upsilon^{p-1}\eta)\Gamma(\beta)}, & 0\le t\le s\le 1,\eta\le s,\\ \frac{-t(1-s)^{\beta-1}}{(1-\upsilon^{p-1}\eta)\Gamma(\beta)}+\frac{t\upsilon^{p-1}(\eta-s)^{\beta-1}}{(1-\upsilon^{p-1}\eta)\Gamma(\beta)}, & 0\le t\le s\le 1,\eta\ge s.\end{cases}$$

Lemma 5 *Let $f,g,h\in C(0,1)$, and with $1<\alpha\le 2$ we have the following fractional boundary value problem*:

$$D^{\beta}_{0+}\phi_p\bigl(D^{\alpha}_{0+}u(t)\bigr)=f(t), \tag{4}$$

$$\begin{cases}u(0)+\mu_1u(1)=\sigma_1\int_0^1 g(s)\,ds,\\ u'(0)+\mu_2u'(1)=\sigma_2\int_0^1 h(s)\,ds,\end{cases} \tag{5}$$

$$\begin{aligned}&D^{\alpha}_{0+}u(0)=0,\\ &D^{\alpha}_{0+}u(1)=\nu D^{\alpha}_{0+}u(\eta),\end{aligned} \tag{6}$$

it has a unique solution which is given by

$$(\mathcal{T}u)(t)=\int_0^t G(t,s)\phi_q\left(\int_0^1 H(t,\tau)f(\tau)\,d\tau\right)ds+\omega_1+\omega_2t,$$

with

$$\omega_1=\frac{\sigma_1}{1+\mu_1}-\frac{\sigma_2\mu_1}{(1+\mu_1)(1+\mu_2)}\quad \textit{and}\quad \omega_2=\frac{\sigma_2}{1+\mu_2}.$$

Proof By applying I^{β}_{0+} to both sides of (4), we get

$$\begin{aligned}&\phi_p\bigl(D^{\alpha}_{0+}u(t)\bigr)=\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}f(s)\,ds-b_1-b_2t,\quad b_1,b_2\in R,\\ &D^{\alpha}_{0+}u(t)=\phi_q\left(\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}f(s)\,ds-b_1-b_2t\right).\end{aligned}$$

Using the boundary conditions $D^{\alpha}_{0+}u(0)=0$ and $D^{\alpha}_{0+}u(1)=\nu D^{\alpha}_{0+}u(\eta)$, we have

$$\phi_q(-b_1)=0\quad\Longrightarrow\quad b_1=0,$$

and secondly,

$$\begin{aligned}\phi_q\left(\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}f(s)\,ds-b_2\right)&=\nu\phi_q\left(\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}f(s)\,ds-b_2\eta\right)\\ &=\phi_q\left(\nu^{\frac{1}{q-1}}\left(\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}f(s)\,ds-b_2\eta\right)\right).\end{aligned}$$

Moreover, since ϕ_p is one-to-one,

$$I^{\beta}_{0+}f(1)-b_2=\nu^{p-1}\bigl(I^{\beta}_{0+}f(\eta)-b_2\eta\bigr)=\nu^{p-1}I^{\beta}_{0+}f(\eta)-\nu^{p-1}b_2\eta,$$

$$I_{0+}^{\beta}f(1)-\nu^{p-1}I_{0+}^{\beta}f(\eta)=\bigl(1-\nu^{p-1}\eta\bigr)b_2.$$

Then

$$\begin{aligned}b_2&=\frac{1}{(1-\nu^{p-1}\eta)}I_{0+}^{\beta}f(1)-\frac{\nu^{p-1}}{(1-\nu^{p-1}\eta)}I_{0+}^{\beta}f(\eta)\\&=\frac{1}{(1-\nu^{p-1}\eta)}\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}f(s)\,ds-\frac{\nu^{p-1}}{(1-\nu^{p-1}\eta)}\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}f(s)\,ds.\end{aligned}$$

Since $\phi_p(D_{0+}^{\alpha}u(t))=I_{0+}^{\beta}f(t)-b_1-b_2t$,

$$\begin{aligned}\phi_p\bigl(D_{0+}^{\alpha}u(t)\bigr)&=\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}f\bigl(s,u(s)\bigr)\,ds-\frac{t}{(1-\nu^{p-1}\eta)}\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}f\bigl(s,u(s)\bigr)\,ds\\&\quad+\frac{t\nu^{p-1}}{(1-\nu^{p-1}\eta)}\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}f\bigl(s,u(s)\bigr)\,ds\\&=\int_0^1H(t,s)f(s)\,ds,\\D_{0+}^{\alpha}u(t)&=\phi_q\biggl(\int_0^1H(t,s)f(s)\,ds\biggr),\\u(t)&=\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\biggl(\int_0^1H(t,s)f(s)\,ds\biggr)\,d\tau-c_1-c_2t.\end{aligned}\tag{7}$$

By the boundary conditions (5), we get

$$\begin{aligned}&-c_1+\mu_1\biggl(\int_0^1\frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_p\biggl(\int_0^1H(\tau,s)f(s)\,ds\biggr)\,d\tau-c_1-c_2\biggr)=\sigma_1\int_0^1g(s)\,ds,\\&\mu_1\int_0^1\frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_p\biggl(\int_0^1H(\tau,s)f(s)\,ds\biggr)\,d\tau-c_2\mu_1-\sigma_1\int_0^1g(s)\,ds=c_1(1+\mu_1),\\&c_1=\frac{\mu_1}{(1+\mu_1)}\int_0^1\frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_p\biggl(\int_0^1H(\tau,s)f(s)\,ds\biggr)\,d\tau-c_2\frac{\mu_1}{(1+\mu_1)}\\&\qquad-\frac{\sigma_1}{(1+\mu_1)}\int_0^1g(s)\,ds,\\&c_2=\frac{\mu_2}{(1+\mu_2)}\int_0^1\frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi_p\biggl(\int_0^1H(\tau,s)f(s)\,ds\biggr)\,d\tau-\frac{\sigma_2}{(1+\mu_2)}\int_0^1h(s)\,ds.\end{aligned}\tag{8}$$

Inserting c_2 into (8), we get the values of c_1, and inserting c_1 and c_2 into (7), we have

$$\begin{aligned}u(t)&=\int_0^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_p\biggl(\int_0^1H(t,s)f\bigl(s,u(s)\bigr)\,ds\biggr)\,d\tau\\&\quad-\biggl(\frac{\mu_1(1+\mu_2)+t\mu_2(1+\mu_1)}{(1+\mu_1)(1+\mu_2)}\biggr)\int_0^1\frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_p\biggl(\int_0^1H(\tau,s)f\bigl(s,u(s)\bigr)\,ds\biggr)\,d\tau\\&\quad+\frac{\mu_1\mu_2}{(1+\mu_1)(1+\mu_2)}\int_0^1\frac{(1-\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\phi_p\biggl(\int_0^1H(\tau,s)f\bigl(s,u(s)\bigr)\,ds\biggr)\,d\tau\\&\quad+\frac{\sigma_1}{1+\mu_1}\int_0^1g\bigl(s,u(s)\bigr)\,ds-\biggl(\frac{\sigma_2\mu_1-t\sigma_2(1+\mu_1)}{(1+\mu_1)(1+\mu_2)}\biggr)\int_0^1h\bigl(s,u(s)\bigr)\,ds.\end{aligned}$$

□

Lemma 6 *The functions $G(t,s)$ and $H(t,s)$ are continuous on $[0,1] \times [0,1]$ and $H(t,s)$ satisfies the following properties*:

(1) $H(t,s) \leq 0$, *for* $t,s \in [0,1]$,

(2) $H(t,s) \geq H(s,s)$, *for* $t,s \in [0,1]$,

(3) *the Green's function $H(t,s)$ satisfies the following condition*:

$$0 \leq \int_0^1 \left|H(t,s)\right| ds \leq \frac{B(\beta,\beta)}{(1-\upsilon^{p-1}\eta)\Gamma(\beta)},$$

where B is the Beta function.

Proof The proofs of properties (1)-(2) are given in [1]. Thus we will prove property (3) for any $t,s \in [0,1]$. The Green's function $H(t,s)$ is negative. Therefore,

$$0 \leq \int_0^1 \left|H(t,s)\right| ds \leq \int_0^1 \left|H(s,s)\right| ds \leq \frac{B(\beta,\beta)}{(1-\upsilon^{p-1}\eta)\Gamma(\beta)}. \qquad \square$$

3 Existence and uniqueness results

In this section, we state and prove existence and uniqueness results of the fractional BVP (1)-(2) by using the Banach fixed point theorem. Our study concerns the space

$$C_\gamma\bigl([0,1],R\bigr) = \bigl\{u \in C\bigl([0,1],R\bigr), D_{0+}^{\gamma}u \in C\bigl([0,1],R\bigr)\bigr\},$$

which is shown in the form

$$\|u\|_\gamma = \|u\|_c + \bigl\|D_{0+}^{\gamma}u\bigr\|_c,$$

where $\|\cdot\|_c$ is the sup norm in $C([0,1],R)$.

The following notations will be used throughout this paper:

$$\Delta_1 = \frac{1}{\Gamma(\alpha+1)}\left[1 + \frac{|\mu_1||1+\mu_2| + |\mu_2||1+\mu_1|}{|1+\mu_1||1+\mu_2|}\right] + \frac{1}{\Gamma(\alpha)}\left[\frac{|\mu_1||\mu_2|}{|1+\mu_1||1+\mu_2|}\right],$$

$$\Delta_2 = \frac{1}{\Gamma(\alpha-\gamma+1)}\left[1 + \frac{|\mu_2|}{\Gamma(2-\gamma)|1+\mu_2|}\right],$$

$$\Delta_g = \frac{|\sigma_1|}{|1+\mu_1|},$$

$$\Delta_{h_1} = \frac{|\sigma_2||\mu_1 + |1+\mu_1||}{|1+\mu_1||1+\mu_2|}, \qquad \Delta_{h_2} = \frac{|\sigma_2|}{\Gamma(2-\gamma)|1+\mu_2|}.$$

To state and prove our first result, we pose the following conditions:

(A1) The function $f : [0,1] \times R \times R \to R$ is jointly continuous.

(A2) There exists a function $l_f \in L^{\frac{1}{\tau}}([0,1],R^+)$ such that

$$\bigl|f(t,u_1,u_2) - f(t,v_1,v_2)\bigr| \leq l_f(t)\bigl(|u_1-v_1| + |u_2-v_2|\bigr),$$

for all $(t,u_1,u_2),(t,v_1,v_2) \in [0,1] \times R \times R$.

(A3) The functions g and h are jointly continuous and there exists $l_g, l_h \in L^1([0,1], R^+)$ such that

$$\left|g(t,u)-g(t,v)\right| \leq l_g(t)|u-v|$$

and

$$\left|h(t,u)-h(t,v)\right| \leq l_h(t)|u-v|,$$

for each $(t,u),(t,v) \in [0,1] \times R$.

Next, we define an operator, $\mathcal{T}_0$ which is $\mathcal{T}_0 : C[0,1] \to C[0,1]$ as follows:

$$\mathcal{T}_0 x(t) = \phi_q\left(\int_0^1 H(t,s) f\left(s, x(s), D_{0+}^{\gamma} x(s)\right) ds\right).$$

Lemma 7 *Assume* (A1)-(A3) *hold and* $q > 2$. *There exists a constant* $l_{T_0} > 0$ *such that*

$$\left|\mathcal{T}_0 u(t) - \mathcal{T}_0 v(t)\right| \leq l_{\mathcal{T}_0} \|u-v\|_{\gamma},$$

for all $u, v \in B_r$. *We have*

$$l_{\mathcal{T}_0} = (q-1) L_H^{q-2} \|l_f\|_\infty \int_0^1 \left|H(s,s)\right| ds \leq (q-1) L_H^{q-2} \|l_f\|_\infty \frac{B(\beta,\beta)}{(1-\upsilon^{p-1}\eta)\Gamma(\beta)}.$$

Proof If $p > 2$ and $t > 0$ we have the following estimation:

$$\begin{aligned}
\left|\int_0^1 H(t,s) f\left(s,u(s),D_{0+}^{\gamma}u(s)\right) ds\right| &\leq \int_0^1 \left|H(t,s)\right|\left|f\left(s,u(s),D_{0+}^{\gamma}u(s)\right)\right| ds \\
&\leq \int_0^1 \left|H(t,s)\right| l_f(s)\left(\left|u(s)\right| + \left|D_{0+}^{\gamma}u(s)\right| + \left|f(s,0,0)\right|\right) ds \\
&\leq \left(\|l_f\|_\infty \|u\|_\gamma + M\right) \int_0^1 \left|H(s,s)\right| ds \\
&\leq \left(\|l_f\|_\infty r + M\right) \int_0^1 \left|H(s,s)\right| ds \\
&= L_H,
\end{aligned}$$

where $M = \max_{s\in[0,1]} |f(s,0,0)|$. Now using the property (L2), we get the desired inequality,

$$\begin{aligned}
&\left|(\mathcal{T}_0 u)(t) - (\mathcal{T}_0 v)(t)\right| \\
&\quad = \left|\phi_q\left(\int_0^1 H(t,s) f\left(s,u(s),D_{0+}^{\gamma}u(s)\right) ds\right) - \phi_q\left(\int_0^1 H(t,s) f\left(s,v(s),D_{0+}^{\gamma}v(s)\right) ds\right)\right| \\
&\quad \leq (q-1) L_H^{q-2} \left|\int_0^1 H(t,s)\left(f\left(s,u(s),D_{0+}^{\gamma}u(s)\right) - f\left(s,v(s),D_{0+}^{\gamma}v(s)\right)\right) ds\right| \\
&\quad \leq (q-1) L_H^{q-2} \|l_f\|_\infty \|u-v\|_\gamma \int_0^1 \left|H(s,s)\right| ds
\end{aligned}$$

$$\leq (q-1)L_H^{q-2}\|l_f\|_\infty \frac{B(\beta,\beta)}{(1-\upsilon^{p-1}\eta)\Gamma(\beta)}\|u-v\|_\gamma$$
$$= l_{\mathcal{T}_0}\|u-v\|_\gamma. \qquad \square$$

Theorem 8 *Assume* (A1)-(A3) *hold. If*

$$\left\{l_{\mathcal{T}_0}\left(\sum_{i=1}^{2}\Delta_i\right) + \Delta_g\|l_g\|_1 + \left(\sum_{i=1}^{2}\Delta h_i\right)\|l_h\|_1\right\} < 1, \tag{9}$$

then our BVP (1)-(2) *has a unique solution on* $[0,1]$.

Proof Let us define the operator $\mathcal{T}: C_\gamma([0,1],R) \to C_\gamma([0,1],R)$ to transform our BVP (1)-(2) into a fixed point problem,

$$\begin{aligned}
&(\mathcal{T}u)(t)\\
&\quad = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathcal{T}_0\bigl(f\bigl(s,u(s),D_{0+}^\gamma u(s)\bigr)\bigr)\,ds\\
&\qquad - \frac{\mu_1}{(1+\mu_1)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathcal{T}_0\bigl(f\bigl(s,u(s),D_{0+}^\gamma u(s)\bigr)\bigr)\,ds\\
&\qquad + \frac{\mu_1\mu_2}{(1+\mu_1)(1+\mu_2)}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathcal{T}_0\bigl(f\bigl(s,u(s),D_{0+}^\gamma u(s)\bigr)\bigr)\,ds\\
&\qquad - \frac{\mu_2 t}{(1+\mu_2)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathcal{T}_0\bigl(f\bigl(s,u(s),D_{0+}^\gamma u(s)\bigr)\bigr)\,ds\\
&\qquad + \frac{\sigma_1}{(1+\mu_1)}\int_0^1 g\bigl(s,u(s)\bigr)\,ds - \frac{\sigma_2(\mu_1-(1+\mu_1)t)}{(1+\mu_2)(1+\mu_1)}\int_0^1 h\bigl(s,u(s)\bigr)\,ds.
\end{aligned} \tag{10}$$

Taking the γth fractional derivative, we get

$$\begin{aligned}
&D_{0+}^\gamma(\mathcal{T}u)(t)\\
&\quad = \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\mathcal{T}_0\bigl(f\bigl(s,u(s),D_{0+}^\gamma u(s)\bigr)\bigr)\,ds\\
&\qquad - \frac{\mu_2}{(1+\mu_2)}\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\mathcal{T}_0\bigl(f\bigl(s,u(s),D_{0+}^\gamma u(s)\bigr)\bigr)\,ds\\
&\qquad + \frac{\sigma_2}{(1+\mu_2)}\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\int_0^1 h\bigl(s,u(s)\bigr)\,ds
\end{aligned} \tag{11}$$

for $t \in [0,1]$. Since f, g, h are continuous, the expression (10) and (11) are well defined. Clearly, the fixed point of the operator $\mathcal{T}$ is the solution of the problem (1)-(2). To show the existence and uniqueness of the solution, the Banach fixed point theorem is used and then we show $\mathcal{T}$ is contraction. We have

$$\begin{aligned}
&\bigl|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)\bigr|\\
&\quad \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} l_{\mathcal{T}_0}\|u-v\|_\gamma\,ds\\
&\qquad + \frac{|\mu_1|}{|1+\mu_1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} l_{\mathcal{T}_0}\|u-v\|_\gamma\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\mu_1||\mu_2|}{|1+\mu_1||1+\mu_2|}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}l_{\mathcal{T}_0}\|u-v\|_\gamma\,ds\\
&+\frac{|\mu_2|}{|1+\mu_2|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}l_{\mathcal{T}_0}\|u-v\|_\gamma\,ds\\
&+\frac{|\sigma_1|}{|1+\mu_1|}\int_0^1 l_g(s)\bigl(\bigl|u(s)-v(s)\bigr|+\bigl|D_{0+}^\gamma u(s)-D_{0+}^\gamma v(s)\bigr|\bigr)\,ds\\
&+\frac{|\sigma_2||\mu_1+|1+\mu_1||}{|1+\mu_2||1+\mu_1|}\int_0^1 l_h(s)\bigl(\bigl|u(s)-v(s)\bigr|+\bigl|D_{0+}^\gamma u(s)-D_{0+}^\gamma v(s)\bigr|\bigr)\,ds\\
\le{}&\biggl\{l_{\mathcal{T}_0}\biggl(\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds+\frac{|\mu_1||1+\mu_2|+|\mu_2||1+\mu_1|}{|1+\mu_1||1+\mu_2|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\\
&+\frac{|\mu_1||\mu_2|}{|1+\mu_1||1+\mu_2|}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,ds\biggr)\\
&+\frac{|\sigma_1|}{|1+\mu_1|}\int_0^1 l_g(s)\,ds+\frac{|\sigma_2||\mu_1+|1+\mu_1||}{|1+\mu_2||1+\mu_1|}\int_0^1 l_h(s)\,ds\biggr\}\|u-v\|_\gamma\\
\le{}&\biggl\{l_{\mathcal{T}_0}\biggl(\frac{1}{\Gamma(\alpha+1)}+\frac{|\mu_1||1+\mu_2|+|\mu_2||1+\mu_1|}{\Gamma(\alpha+1)|1+\mu_1||1+\mu_2|}+\frac{|\mu_1||\mu_2|}{\Gamma(\alpha)|1+\mu_1||1+\mu_2|}\biggr)\\
&+\frac{|\sigma_1|}{|1+\mu_1|}\|l_g\|_1+\frac{|\sigma_2||\mu_1+|1+\mu_1||}{|1+\mu_2||1+\mu_1|}\|l_h\|_1\biggr\}\|u-v\|_\gamma. \qquad (12)
\end{aligned}
$$

By using the Hölder inequality, we get

$$
\begin{aligned}
&\bigl|\mathcal{T}u(t)-\mathcal{T}v(t)\bigr|\le\bigl\{l_{\mathcal{T}_0}\Delta_1+\Delta_g\|l_g\|_1+\Delta_{h_1}\|l_h\|_1\bigr\}\|u-v\|_\gamma, \qquad (13)\\
&\bigl|D_{0+}^\gamma(\mathcal{T}u)(t)-D_{0+}^\gamma(\mathcal{T}v)(t)\bigr|\\
&\quad\le\int_0^t\frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}l_{\mathcal{T}_0}\|u-v\|_\gamma\,ds\\
&\qquad+\frac{|\mu_2|}{|1+\mu_2|}\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\int_0^1\frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}l_{\mathcal{T}_0}\|u-v\|_\gamma\,ds\\
&\qquad+\frac{|\sigma_2|}{|1+\mu_2|}\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\int_0^1 l_h(s)\bigl(\bigl|u(s)-v(s)\bigr|+\bigl|D_{0+}^\gamma u(s)-D_{0+}^\gamma v(s)\bigr|\bigr)\,ds\\
&\quad\le\biggl\{\frac{l_{\mathcal{T}_0}}{\Gamma(\alpha-\gamma)}\int_0^t(t-s)^{\alpha-\gamma-1}\,ds\\
&\qquad+\frac{l_{\mathcal{T}_0}t^{1-\gamma}|\mu_2|}{\Gamma(\alpha-\gamma)\Gamma(2-\gamma)|1+\mu_2|}\int_0^1(1-s)^{\alpha-\gamma-1}\,ds\\
&\qquad+\frac{|\sigma_2|t^{1-\gamma}}{|1+\mu_2|\Gamma(2-\gamma)}\int_0^1 l_h(s)\,ds\biggr\}\|u-v\|_\gamma\\
&\quad\le\biggl\{l_{\mathcal{T}_0}\biggl(\frac{1}{\Gamma(\alpha-\gamma+1)}+\frac{|\mu_2|}{\Gamma(\alpha-\gamma+1)\Gamma(2-\gamma)|1+\mu_2|}\biggr)\\
&\qquad+\frac{|\sigma_2|}{|1+\mu_2|\Gamma(2-\gamma)}\int_0^1 l_h(s)\,ds\biggr\}\|u-v\|_\gamma\\
&\quad\le\biggl\{\frac{l_{\mathcal{T}_0}}{\Gamma(\alpha-\gamma+1)}\biggl(1+\frac{|\mu_2|}{\Gamma(2-\gamma)|1+\mu_2|}\biggr)\\
&\qquad+\frac{|\sigma_2|}{|1+\mu_2|\Gamma(2-\gamma)}\|l_h\|_1\biggr\}\|u-v\|_\gamma. \qquad (14)
\end{aligned}
$$

Similarly,

$$\left|D_{0+}^{\gamma}\left(\mathcal{T}u(t)\right)-D_{0+}^{\gamma}\left(\mathcal{T}v(t)\right)\right| \leq \left\{l_{\mathcal{T}_0}\Delta_2+\Delta_{h_2}\|l_h\|_1\right\}\|u-v\|_{\gamma}. \tag{15}$$

With the help of (13)-(15), we find that

$$\begin{aligned}
&\|Tu-Tv\|_{\gamma} \\
&\quad\leq \left\{l_{\mathcal{T}_0}(\Delta_1+\Delta_2)+\Delta_g\|l_g\|_1+(\Delta_{h_1}+\Delta_{h_2})\|l_h\|_1\right\}\|u-v\|_{\gamma} \\
&\quad= \left\{l_{\mathcal{T}_0}\left(\sum_{i=1}^{2}\Delta_i\right)+\Delta_g\|l_g\|_1+\left(\sum_{i=1}^{2}\Delta_{h_i}\right)\|l_h\|_1\right\}\|u-v\|_{\gamma}.
\end{aligned}$$

Thus $\mathcal{T}$ is a contraction mapping by condition (9). By the Banach fixed point theorem, $\mathcal{T}$ has a fixed point which is the solution of the BVP. □

4 Existence results

Theorem 9 *Assume*:

(iv) *There exist non-decreasing functions* $\varphi:[0,\infty)\times[0,\infty)\to[0,\infty)$ *and* $\psi_i:[0,\infty)\to[0,\infty)$, $i=1,2$, *and the functions* $l_f\in L^{\frac{1}{\tau}}([0,1],R^+)$ *and* $l_g, l_h\in L^1([0,1],R^+)$ *such that*

$$\begin{aligned}
&\left|f(t,u,v)\right|\leq l_f(t)\varphi\left(|u|+|v|\right), \\
&\left|g(t,u)\right|\leq l_g(t)\psi_1\left(|u|\right), \\
&\left|h(t,u)\right|\leq l_h(t)\psi_2\left(|u|\right),
\end{aligned}$$

for all $t\in[0,1]$ *and* $u,v\in R$.

(v) *There exists a constant* $\mathcal{N}>0$ *such that*

$$\left[\frac{\mathcal{N}}{\varphi(\|u\|_{\gamma})l_{\mathcal{T}_0}\sum_{i=1}^{2}\Delta_i+\psi_1(\|u\|_{\gamma})\|l_g\|_1\Delta_g+\psi_2(\|u\|_{\gamma})\|l_h\|_1\sum_{i=1}^{2}\Delta_{h_i}}\right]>1. \tag{16}$$

Thus problem (1)-(2) *has at least one solution on* $[0,1]$.

Proof Let $B_r=\{u\in C_{\gamma}([0,1],R):\|u\|_{\gamma}\leq r\}$.

Step 1: Let the operator $\mathcal{T}:C_{\gamma}([0,1],R)\to C_{\gamma}([0,1],R)$ be given in (10) which defines B_r to be a bounded set. For all $u\in B_r$, we get

$$\begin{aligned}
&\left|(\mathcal{T}u)(t)\right| \\
&\quad\leq \frac{\varphi(r)}{\Gamma(\alpha)}l_{\mathcal{T}_0}\int_0^t(t-s)^{\alpha-1}\,ds \\
&\qquad+\frac{|\mu_1||1+\mu_2|+|\mu_2||1+\mu_1|}{|1+\mu_1||1+\mu_2|}\frac{\varphi(r)}{\Gamma(\alpha)}l_{\mathcal{T}_0}\int_0^1(1-s)^{\alpha-1}\,ds \\
&\qquad+\frac{|\mu_1||\mu_2|}{|1+\mu_1||1+\mu_2|}\frac{\varphi(r)}{\Gamma(\alpha-1)}l_{\mathcal{T}_0}\int_0^1(1-s)^{\alpha-2}\,ds \\
&\qquad+\frac{|\sigma_1|}{|1+\mu_1|}\psi_1(r)\int_0^1\left|l_g(s)\right|ds+\frac{|\sigma_2||\mu_1+|1+\mu_1||}{|1+\mu_2||1+\mu_1|}\psi_2(r)\int_0^1\left|l_h(s)\right|ds
\end{aligned}$$

and

$$\begin{aligned}
&\left|D_{0+}^{\gamma}(\mathcal{T}u)(t)\right| \\
&\quad\le \frac{\varphi(r)}{\Gamma(\alpha-\gamma)} l_{\mathcal{T}_0} \int_0^t (t-s)^{\alpha-\gamma-1}\,ds \\
&\qquad + \frac{|\mu_2|}{|1+\mu_2|} \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} \frac{\varphi(r)}{\Gamma(\alpha-\gamma)} l_{\mathcal{T}_0} \int_0^1 (1-s)^{\alpha-\gamma-1}\,ds \\
&\qquad + \frac{|\sigma_2|}{|1+\mu_2|} \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} \psi_2(r) \int_0^1 \left|l_h(s)\right|\,ds.
\end{aligned}$$

By the Hölder inequality,

$$\begin{aligned}
&\left|(\mathcal{T}u)(t)\right| \\
&\quad\le \frac{\varphi(r)l_{\mathcal{T}_0}}{\Gamma(\alpha+1)} + \frac{(|\mu_1||1+\mu_2| + |\mu_2||1+\mu_1|)\varphi(r)l_{\mathcal{T}_0}}{|1+\mu_1||1+\mu_2|\Gamma(\alpha+1)} + \frac{|\mu_1||\mu_2|\varphi(r)l_{\mathcal{T}_0}}{|1+\mu_1||1+\mu_2|\Gamma(\alpha)} \\
&\qquad + \frac{|\sigma_1|\psi_1(r)\|l_g\|_1}{|1+\mu_1|} + \frac{|\sigma_2||\mu_1 + |1+\mu_1||\psi_2(r)\|l_h\|_1}{|1+\mu_2||1+\mu_1|} \\
&\quad\le \varphi(r)l_{\mathcal{T}_0}\Delta_1 + \Delta_g\psi_1(r)\|l_g\|_1 + \Delta_{h_1}\psi_2(r)\|l_h\|_1, \\
&\left|D_{0+}^{\gamma}(\mathcal{T}u)(t)\right| \\
&\quad\le \frac{\varphi(r)l_{\mathcal{T}_0}}{\Gamma(\alpha-\gamma+1)} + \frac{|\mu_2|\varphi(r)l_{\mathcal{T}_0}}{|1+\mu_2|\Gamma(2-\gamma)\Gamma(\alpha-\gamma+1)} + \frac{|\sigma_2|\psi_2(r)\|l_h\|_1}{|1+\mu_2|\Gamma(2-\gamma)} \\
&\quad\le \varphi(r)l_{\mathcal{T}_0}\Delta_2 + \Delta_{h_2}\psi_2(r)\|l_h\|_1.
\end{aligned}$$

Therefore,

$$\left\|(\mathcal{T}u)\right\|_{\gamma} \le \varphi(r)l_{\mathcal{T}_0}(\Delta_1+\Delta_2) + \Delta_g\psi_1(r)\|l_g\|_1 + (\Delta_{h_1}+\Delta_{h_2})\psi_2(r)\|l_h\|_1.$$

Step 2: The families $\{(\mathcal{T}u) : u \in B_r\}$ and $\{D_{0+}^{\gamma}(\mathcal{T}u) : u \in B_r\}$ are equicontinuous. For $t_1 < t_2$, we get

$$\begin{aligned}
&\left|(\mathcal{T}u)(t_2) - (\mathcal{T}u)(t_1)\right| \\
&\quad\le \frac{\varphi(r)l_{\mathcal{T}_0}}{\Gamma(\alpha)} \left[\int_0^{t_1} \left((t_1-s)^{\alpha-1} + (t_2-s)^{\alpha-1}\right)ds - \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\,ds\right] \\
&\qquad + \frac{|\mu_2||t_2-t_1|}{|1+\mu_2|} \frac{\varphi(r)l_{\mathcal{T}_0}}{\Gamma(\alpha)} \int_0^{t_1} (1-s)^{\alpha-1}\,ds \\
&\qquad + \frac{|\sigma_2||1+\mu_1||t_2-t_1|\psi_2(r)}{|\mu_2||1+\mu_1|} \int_0^1 \left|l_h(s)\right|\,ds \to 0 \quad \text{as } t_2 \to t_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left|D_{0+}^{\gamma}(\mathcal{T}u)(t_2) - D_{0+}^{\gamma}(\mathcal{T}u)(t_1)\right| \\
&\quad\le \frac{\varphi(r)l_{\mathcal{T}_0}}{\Gamma(\alpha-\gamma)} \left[\int_0^{t_1} \left((t_1-s)^{\alpha-\gamma-1} + (t_2-s)^{\alpha-\gamma-1}\right)ds - \int_{t_1}^{t_2} (t_2-s)^{\alpha-\gamma-1}\,ds\right]
\end{aligned}$$

$$+\frac{\varphi(r)l_{\mathcal{T}_0}|\mu_2||t_2^{1-\gamma}-t_1^{1-\gamma}|}{\Gamma(\alpha-\gamma)|1+\mu_2|\Gamma(2-\gamma)}\int_0^1(1-s)^{\alpha-\gamma-1}\,ds$$
$$+\frac{|\sigma_2||t_2^{1-\gamma}-t_1^{1-\gamma}|\psi_2(r)}{|1+\mu_2|\Gamma(2-\gamma)}\int_0^1\bigl|l_h(s)\bigr|\,ds\to 0\quad\text{as } t_2\to t_1.$$

Thus $\{(\mathcal{T}u):u\in B_r\}$ and $\{D_{0+}^{\gamma}(\mathcal{T}u):u\in B_r\}$ are equicontinuous and relatively compact in $C([0,1],R)$ by the Arzela-Ascoli theorem. Therefore $\mathcal{T}(B_r)$ is a relatively compact subset of $C_\gamma([0,1],R)$ and the operator $\mathcal{T}$ is compact.

Step 3: Let $u=\lambda(\mathcal{T}u)$ and $u=\lambda(D_{0+}^{\gamma}(\mathcal{T}u))$ for $0<\lambda<1$. For each $t\in[0,1]$, define $\overline{\mathcal{M}}=\{\|u\|_\gamma\in C_\gamma([0,1],R),\|u\|_\gamma<\mathcal{N}\}$. Then we get

$$\begin{aligned}\|u\|_c&=\bigl\|\lambda(\mathcal{T}u)\bigr\|_c\\&\le\varphi\bigl(\|u\|_\gamma\bigr)l_{\mathcal{T}_0}\Delta_1+\Delta_g\psi_1\bigl(\|u\|_\gamma\bigr)\|l_g\|_1+\Delta_{h_1}\psi_2\bigl(\|u\|_\gamma\bigr)\|l_h\|_1,\\\|u\|_c&=\bigl\|\lambda\bigl(D_{0+}^{\gamma}(\mathcal{T}u)\bigr)\bigr\|_c\\&\le\varphi\bigl(\|u\|_\gamma\bigr)l_{\mathcal{T}_0}\Delta_2+\Delta_{h_2}\psi_2\bigl(\|u\|_\gamma\bigr)\|l_h\|_1.\end{aligned}$$

Thus

$$\|u\|_\gamma\le\varphi\bigl(\|u\|_\gamma\bigr)l_{\mathcal{T}_0}\sum_{i=1}^{2}\Delta_i+\psi_1\bigl(\|u\|_\gamma\bigr)\|l_g\|_1\Delta_g+\psi_2\bigl(\|u\|_\gamma\bigr)\|l_h\|_1\sum_{i=1}^{2}\Delta_{h_i}.$$

That means

$$\frac{\|u\|_\gamma}{\varphi(\|u\|_\gamma)l_{\mathcal{T}_0}\sum_{i=1}^2\Delta_i+\psi_1(\|u\|_\gamma)\|l_g\|_1\Delta_g+\psi_2(\|u\|_\gamma)\|l_h\|_1\sum_{i=1}^2\Delta_{h_i}}\le 1.$$

For a non-negative $\mathcal{N}$ and $\|u\|_\gamma<\mathcal{N}$, the operator $\mathcal{T}$ which is defined in $\overline{\mathcal{M}}$ to be $C_\gamma([0,1],R)$ is continuous and compact. Therefore $\mathcal{T}$ has a fixed point in $\overline{\mathcal{M}}$. □

5 Examples

Example 10 Consider the following boundary value problem of a fractional differential equation:

$$\begin{cases}D_{0+}^{\frac{3}{2}}(\phi_pD_{0+}^{\frac{3}{2}}u)(t)=l_f(\frac{|u(t)|}{|u(t)|+1}+\frac{|D_{0+}^{\frac{3}{2}}u(t)|}{|D_{0+}^{\frac{3}{2}}u(t)|+1}),\\u(0)+0.1u(1)=0.01\int_0^1\frac{u(s)}{(1+s)^2}\,ds,\\u'(0)+0.1u'(1)=0.01\int_0^1(\frac{e^su(s)}{1+2e^s}+\frac{1}{2})\,ds.\end{cases}\qquad(17)$$

Here

$$\alpha,\beta=1.5,\qquad\mu_1,\mu_2=0.1,\qquad\sigma_1,\sigma_2=0.01,$$
$$\upsilon,\eta=0.3,\qquad\tau=0.4,\qquad\gamma=0.01,$$

and

$$f(t,u,v)=\frac{|u|}{|u|+1}+\frac{|v|}{|v|+1},$$

$$g(t,u)=\frac{u}{(1+s)^2},\qquad h(t,u)=\frac{e^s u}{(1+2e^s)}+\frac{1}{2}.$$

Since $0.88<\Gamma(1.5)<0.89$, $\Gamma(2)=1$, $\Gamma(2.5)=1$, we find

$$\Delta_1=0.89,\qquad \Delta_2=0.82,\qquad \Delta_g=0.009,$$
$$\Delta_{h_1}=0.0099,\qquad \Delta_{h_2}=0.009,\qquad l_g=l_h=1,$$

with simple calculations. Therefore

$$\begin{aligned}&\bigl\{l_{\mathcal{T}_0}(\Delta_1+\Delta_2)+2\Delta_g\|l_g\|_1+(\Delta_{h_1}+\Delta_{h_2})\|l_h\|_1\bigr\}\\&\quad<1.73l_{\mathcal{T}_0}+0.04\\&\quad<1.\end{aligned}$$

Then we can choose

$$l_{\mathcal{T}_0}<0.562.$$

Thus all assumptions of Theorem 8 satisfied. Therefore the problem has a unique solution on $[0,1]$.

Example 11 Consider the following boundary value problem of fractional differential equation:

$$\begin{cases}D_{0+}^{\frac{3}{2}}(\phi_p D_{0+}^{\frac{3}{2}}u)(t)=\frac{|u(t)|^3}{9(|u(t)|^3+3)}+\frac{|\sin D_{0+}^{\frac{3}{2}}u(t)|}{9(\sin D_{0+}^{\frac{3}{2}}u(t)+1)}+\frac{1}{12},\\ u(0)+0.1u(1)=0,01\int_0^1\frac{u(s)}{3(1+s)^2}\,ds,\\ u'(0)+0.1u'(1)=0,01\int_0^1\frac{e^s u(s)}{3(1+e^s)^2}\,ds,\\ D_{0+}^{\frac{3}{2}}u(0)=0,\\ D_{0+}^{\frac{3}{2}}u(1)=0,3D_{0+}^{\frac{3}{2}}u(0,3),\end{cases}\tag{18}$$

where f is given by

$$f(t,u,v)=\frac{|u|^3}{9(|u|^3+3)}+\frac{|\sin v|}{9(\sin v+1)}+\frac{1}{12}.$$

We have

$$\bigl|f(t,u,v)\bigr|\le\frac{|u|^3}{9(|u|^3+3)}+\frac{|\sin v|}{9(\sin v+1)}+\frac{1}{12},\quad u\in R.$$

Here

$$\alpha,\beta=1.5,\qquad \mu_1,\mu_2=0.1,\qquad \sigma_1,\sigma_2=0.01,$$
$$\upsilon,\eta=0.3,\qquad \tau=0.4,\qquad \gamma=0.01,$$
$$\Delta_1=0.89,\qquad \Delta_2=0.82,\qquad \Delta_g=0.009,$$

$$\Delta_{h_1} = 0.0099, \qquad \Delta_{h_2} = 0.009, \qquad l_g = l_h = 0.1,$$

and $g(t,u) = \frac{u(s)}{3(1+s)^2}$, $h(t,u) = \frac{e^s u(s)}{3(1+e^s)^2}$, $\varphi(\mathcal{N}) = \psi_1(\mathcal{N}) = \psi_2(\mathcal{N}) = \mathcal{N}$. If

$$\begin{aligned}
&\frac{\mathcal{N}}{\varphi(\mathcal{N})(0.561)(0.89+0.82) + \psi_1(\mathcal{N})(0.1)(0.009) + \psi_2(\mathcal{N})(0.1)(0.0099+0.009)} > 1, \\
&\frac{\mathcal{N}}{\mathcal{N}(0.96) + \mathcal{N}(0.0009) + \mathcal{N}(0.0019)} > 1, \\
&\frac{\mathcal{N}}{0.9628\mathcal{N}} > 1, \\
&1.04 > 1,
\end{aligned}$$

then (16) is satisfied. Then there exists at least one solution of the BVP on $[0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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