Chen and Zhang *Journal of Inequalities and Applications* (2017) 2017:205
DOI 10.1186/s13660-017-1479-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH
Open Access

Padé approximant related to inequalities involving the constant e and a generalized Carleman-type inequality

Chao-Ping Chen* and Hui-Jie Zhang

*Correspondence:
chenchaoping@sohu.com
School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province 454000, China

Abstract

Based on the Padé approximation method, in this paper we determine the coefficients a_j and b_j ($1 \leq j \leq k$) such that

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^x = \frac{x^k + a_1x^{k-1} + \cdots + a_k}{x^k + b_1x^{k-1} + \cdots + b_k} + O\left(\frac{1}{x^{2k+1}}\right), \quad x \to \infty,$$

where $k \geq 1$ is any given integer. Based on the obtained result, we establish new upper bounds for $(1 + 1/x)^x$. As an application, we give a generalized Carleman-type inequality.

MSC: 26D15; 41A60

Keywords: Carleman's inequality; weight coefficient; Padé approximant

1 Introduction

Let $a_n \geq 0$ for $n \in \mathbb{N} := \{1, 2, \ldots\}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$. Then

$$\sum_{n=1}^{\infty} (a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty} a_n. \tag{1.1}$$

The constant e is the best possible. The inequality (1.1) was presented in 1922 in [1] by Carleman and it is called Carleman's inequality. Carleman discovered this inequality during his important work on quasi-analytical functions.

Carleman's inequality (1.1) was generalized by Hardy [2] (see also [3, p.256]) as follows: If $a_n \geq 0$, $\lambda_n > 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$ for $n \in \mathbb{N}$, and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_n \left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e\sum_{n=1}^{\infty} \lambda_n a_n. \tag{1.2}$$

Note that inequality (1.2) is usually referred to as a Carleman-type inequality or weighted Carleman-type inequality. In [2], Hardy himself said that it was Pólya who pointed out this inequality to him.

In [4–20], some strengthened and generalized results of (1.1) and (1.2) have been given by estimating the weight coefficient $(1+1/n)^n$. For example, Yang [17] proved that, for $n \in \mathbb{N}$,

$$e\left(1-\frac{1}{2(n+\frac{5}{6})}\right) < \left(1+\frac{1}{n}\right)^n < e\left(1-\frac{1}{2(n+1)}\right), \tag{1.3}$$

and then used it to obtain the following strengthened Carleman inequality:

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\frac{1}{2(n+1)}\right)a_n. \tag{1.4}$$

Xie and Zhong [15] proved that, for $x \geq 1$,

$$e\left(1-\frac{7}{14x+12}\right) < \left(1+\frac{1}{x}\right)^x < e\left(1-\frac{6}{12x+11}\right), \tag{1.5}$$

and then used it to improve the Carleman-type inequality (1.2) as follows. If $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n}\lambda_m$, $a_n \geq 0$ for $n \in \mathbb{N}$, and $0 < \sum_{n=1}^{\infty}\lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e\sum_{n=1}^{\infty}\left(1-\frac{6}{12(\frac{\Lambda_n}{\lambda_n})+11}\right)\lambda_n a_n. \tag{1.6}$$

Taking $\lambda_n \equiv 1$ in (1.6) yields

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\frac{6}{12n+11}\right)a_n, \tag{1.7}$$

which improves (1.4).

Recently, Mortici and Hu [14] proved that, for $x \geq 1$,

$$\begin{aligned} &\frac{x+\frac{5}{12}}{x+\frac{11}{12}} - \frac{5}{288x^3} + \frac{343}{8{,}640x^4} - \frac{2{,}621}{41{,}472x^5} \\ &\quad < \frac{1}{e}\left(1+\frac{1}{x}\right)^x < \frac{x+\frac{5}{12}}{x+\frac{11}{12}} - \frac{5}{288x^3} + \frac{343}{8{,}640x^4} - \frac{2{,}621}{41{,}472x^5} + \frac{300{,}901}{3{,}483{,}648x^6}, \end{aligned} \tag{1.8}$$

and then they used it to establish the following improvement of Carleman's inequality:

$$\begin{aligned} &\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \\ &\quad < e\sum_{n=1}^{\infty}\left(\frac{12n+5}{12n+11} - \frac{5}{288n^3} + \frac{343}{8{,}640n^4} - \frac{2{,}621}{41{,}472n^5} + \frac{300{,}901}{3{,}483{,}648n^6}\right)a_n, \end{aligned}$$

which can be written as

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}(1-\varepsilon_n)a_n, \tag{1.9}$$

where

$$\varepsilon_n = \frac{104{,}509{,}440n^6 + 3{,}628{,}800n^4 - 4{,}971{,}456n^3 + 5{,}603{,}472n^2 - 5{,}945{,}040n - 16{,}549{,}555}{17{,}418{,}240n^6(12n+11)}. \quad (1.10)$$

For information as regards the history of Carleman-type inequalities, please refer to [21–24].

It follows from (1.8) that

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^x = \frac{x+\frac{5}{12}}{x+\frac{11}{12}} + O\left(\frac{1}{x^3}\right), \quad x \to \infty. \quad (1.11)$$

Using the Padé approximation method, in Section 3 we derive (1.11) and the following approximation formula:

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^x = \frac{x^2+\frac{87}{100}x+\frac{37}{240}}{x^2+\frac{137}{100}x+\frac{457}{1{,}200}} + O\left(\frac{1}{x^5}\right), \quad x \to \infty. \quad (1.12)$$

Equation (1.12) motivates us to present the following inequality:

$$\left(1+\frac{1}{n}\right)^n < e\left(\frac{n^2+\frac{87}{100}n+\frac{37}{240}}{n^2+\frac{137}{100}n+\frac{457}{1{,}200}}\right) = e\left(1-\frac{8(75n+34)}{1{,}200n^2+1{,}644n+457}\right), \quad n \in \mathbb{N}. \quad (1.13)$$

Following the same method used in the proof of Theorem 3.2, we can prove the inequality (1.13). We here omit it.

According to Pólya's proof of (1.1) in [25],

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \le \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n a_n, \quad (1.14)$$

and then the following strengthened Carleman's inequality is derived directly from (1.13):

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\frac{8(75n+34)}{1{,}200n^2+1{,}644n+457}\right)a_n, \quad (1.15)$$

which improves (1.7).

Based on the Padé approximation method, we determine the coefficients a_j and b_j ($1 \le j \le k$) such that

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^x = \frac{x^k+a_1x^{k-1}+\cdots+a_k}{x^k+b_1x^{k-1}+\cdots+b_k} + O\left(\frac{1}{x^{2k+1}}\right), \quad x \to \infty, \quad (1.16)$$

where $k \ge 1$ is any given integer. Based on the obtained result, we establish new upper bounds for $(1+1/x)^x$. As an application, we give a generalization to the Carleman-type inequality.

The numerical values given have been calculated using the computer program MAPLE 13.

2 A useful lemma

For later use, we introduce the following set of partitions of an integer $n \in \mathbb{N} = \mathbb{N}_0 \setminus \{0\} := \{1, 2, 3, \ldots\}$:

$$\mathcal{A}_n := \{(k_1, k_2, \ldots, k_n) \in \mathbb{N}_0^n : k_1 + 2k_2 + \cdots + nk_n = n\}. \tag{2.1}$$

In number theory, the partition function $p(n)$ represents the number of possible partitions of $n \in \mathbb{N}$ (*e.g.*, the number of distinct ways of representing n as a sum of natural numbers regardless of order). By convention, $p(0) = 1$ and $p(n) = 0$ if n is a negative integer. For more information on the partition function $p(n)$, please refer to [26] and the references therein. The first values of the partition function $p(n)$ are (starting with $p(0) = 1$) (see [27]):

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, \ldots.$$

It is easy to see that the cardinality of the set $\mathcal{A}_n$ is equal to the partition function $p(n)$. Now we are ready to present a formula which determines the coefficients a_j in (2.2) with the help of the partition function given by the following lemma.

Lemma 2.1 ([28]) *The following approximation formula holds true*:

$$\left(1 + \frac{1}{x}\right)^x = e \sum_{j=0}^{\infty} \frac{c_j}{x^j} \quad \textit{as } x \to \infty, \tag{2.2}$$

where the coefficients c_j $(j \in \mathbb{N})$ *are given by*

$$c_0 = 1 \quad \textit{and} \quad c_j = (-1)^j \sum_{(k_1, k_2, \ldots, k_j) \in \mathcal{A}_j} \frac{1}{k_1! k_2! \cdots k_j!} \left(\frac{1}{2}\right)^{k_1} \left(\frac{1}{3}\right)^{k_2} \cdots \left(\frac{1}{j+1}\right)^{k_j}, \tag{2.3}$$

where the $\mathcal{A}_j$ (*for* $j \in \mathbb{N}$) *are given in* (2.1).

3 Padé approximant related to asymptotics for the constant *e*

For later use, we introduce the Padé approximant (see [29–34]). Let f be a formal power series

$$f(t) = c_0 + c_1 t + c_2 t^2 + \cdots. \tag{3.1}$$

The Padé approximation of order (p, q) of the function f is the rational function, denoted by

$$[p/q]_f(t) = \frac{\sum_{j=0}^{p} a_j t^j}{1 + \sum_{j=1}^{q} b_j t^j}, \tag{3.2}$$

where $p \geq 0$ and $q \geq 1$ are two given integers, the coefficients a_j and b_j are given by (see [29–31, 33, 34])

$$\begin{cases} a_0 = c_0, \\ a_1 = c_0 b_1 + c_1, \\ a_2 = c_0 b_2 + c_1 b_1 + c_2, \\ \vdots \\ a_p = c_0 b_p + \cdots + c_{p-1} b_1 + c_p, \\ 0 = c_{p+1} + c_p b_1 + \cdots + c_{p-q+1} b_q, \\ \vdots \\ 0 = c_{p+q} + c_{p+q-1} b_1 + \cdots + c_p b_q, \end{cases} \tag{3.3}$$

and the following holds:

$$[p/q]_f(t) - f(t) = O\bigl(t^{p+q+1}\bigr). \tag{3.4}$$

Thus, the first $p + q + 1$ coefficients of the series expansion of $[p/q]_f$ are identical to those of f. Moreover, we have (see [32])

$$[p/q]_f(t) = \frac{\begin{vmatrix} t^q f_{p-q}(t) & t^{q-1} f_{p-q+1}(t) & \cdots & f_p(t) \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} t^q & t^{q-1} & \cdots & 1 \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}, \tag{3.5}$$

with $f_n(x) = c_0 + c_1 x + \cdots + c_n x^n$, the nth partial sum of the series f (f_n is identically zero for $n < 0$).

Let

$$f(x) = \frac{1}{e}\left(1 + \frac{1}{x}\right)^x. \tag{3.6}$$

It follows from (2.2) that, as $x \to \infty$,

$$f(x) = \sum_{j=0}^{\infty} \frac{c_j}{x^j} = 1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2{,}447}{5{,}760x^4} - \frac{959}{2{,}304x^5} + \frac{238{,}043}{580{,}608x^6} - \cdots, \tag{3.7}$$

with the coefficients c_j given by (2.3). In what follows, the function f is given in (3.6).

We now give a derivation of equation (1.11). To this end, we consider

$$[1/1]_f(x) = \frac{\sum_{j=0}^{1} a_j x^{-j}}{1 + \sum_{j=1}^{1} b_j x^{-j}}.$$

Noting that

$$c_0 = 1, \qquad c_1 = -\frac{1}{2}, \qquad c_2 = \frac{11}{24}, \qquad c_3 = -\frac{7}{16}, \qquad c_4 = \frac{2{,}447}{5{,}760} \tag{3.8}$$

holds, we have, by (3.3),

$$\begin{cases} a_0 = 1, \\ a_1 = b_1 - \frac{1}{2}, \\ 0 = \frac{11}{24} - \frac{1}{2}b_1, \end{cases}$$

that is,

$$a_0 = 1, \qquad a_1 = \frac{5}{12}, \qquad b_1 = \frac{11}{12}.$$

We thus obtain

$$[1/1]_f(x) = \frac{1 + \frac{5}{12x}}{1 + \frac{11}{12x}} = \frac{x + \frac{5}{12}}{x + \frac{11}{12}}, \tag{3.9}$$

and we have, by (3.4),

$$\frac{1}{e}\left(1 + \frac{1}{x}\right)^x - \frac{x + \frac{5}{12}}{x + \frac{11}{12}} = O\left(\frac{1}{x^3}\right), \quad x \to \infty. \tag{3.10}$$

We now give a derivation of equation (1.12). To this end, we consider

$$[2/2]_f(x) = \frac{\sum_{j=0}^{2} a_j x^{-j}}{1 + \sum_{j=1}^{2} b_j x^{-j}}.$$

Noting that (3.8) holds, we have, by (3.3),

$$\begin{cases} a_0 = 1, \\ a_1 = b_1 - \frac{1}{2}, \\ a_2 = b_2 - \frac{1}{2}b_1 + \frac{11}{24}, \\ 0 = -\frac{7}{16} + \frac{11}{24}b_1 - \frac{1}{2}b_2, \\ 0 = \frac{2{,}447}{5{,}760} - \frac{7}{16}b_1 + \frac{11}{24}b_2, \end{cases}$$

that is,

$$a_0 = 1, \qquad a_1 = \frac{87}{100}, \qquad a_2 = \frac{37}{240}, \qquad b_1 = \frac{137}{100}, \qquad b_2 = \frac{457}{1{,}200}.$$

We thus obtain

$$[2/2]_f(x) = \frac{1 + \frac{87}{100x} + \frac{37}{240x^2}}{1 + \frac{137}{100x} + \frac{457}{1{,}200x^2}} = \frac{x^2 + \frac{87}{100}x + \frac{37}{240}}{x^2 + \frac{137}{100}x + \frac{457}{1{,}200}} \tag{3.11}$$

and we have, by (3.4),

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^{x}-\frac{x^{2}+\frac{87}{100}x+\frac{37}{240}}{x^{2}+\frac{137}{100}x+\frac{457}{1,200}}=O\left(\frac{1}{x^{5}}\right),\quad x\to\infty. \tag{3.12}$$

Using the Padé approximation method and the expansion (3.7), we now present a general result given by Theorem 3.1. As a consequence, we obtain (1.16).

Theorem 3.1 *The Padé approximation of order* (p,q) *of the asymptotic formula of the function* $f(x)=\frac{1}{e}(1+\frac{1}{x})^{x}$ *(at the point* $x=\infty$*) is the following rational function*:

$$[p/q]_{f}(x)=\frac{1+\sum_{j=1}^{p}a_{j}x^{-j}}{1+\sum_{j=1}^{q}b_{j}x^{-j}}=x^{q-p}\left(\frac{x^{p}+a_{1}x^{p-1}+\cdots+a_{p}}{x^{q}+b_{1}x^{q-1}+\cdots+b_{q}}\right), \tag{3.13}$$

where $p\geq 1$ *and* $q\geq 1$ *are two given integers, the coefficients* a_j *and* b_j *are given by*

$$\begin{cases} a_{1}=b_{1}+c_{1}, \\ a_{2}=b_{2}+c_{1}b_{1}+c_{2}, \\ \vdots \\ a_{p}=b_{p}+\cdots+c_{p-1}b_{1}+c_{p}, \\ 0=c_{p+1}+c_{p}b_{1}+\cdots+c_{p-q+1}b_{q}, \\ \vdots \\ 0=c_{p+q}+c_{p+q-1}b_{1}+\cdots+c_{p}b_{q}, \end{cases} \tag{3.14}$$

c_j *is given in* (2.3), *and the following holds*:

$$f(x)-[p/q]_{f}(x)=O\left(\frac{1}{x^{p+q+1}}\right),\quad x\to\infty. \tag{3.15}$$

Moreover, we have

$$[p/q]_{f}(x)=\frac{\begin{vmatrix} \frac{1}{x^{q}}f_{p-q}(x) & \frac{1}{x^{q-1}}f_{p-q+1}(x) & \cdots & f_{p}(x) \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p} & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} \frac{1}{x^{q}} & \frac{1}{x^{q-1}} & \cdots & 1 \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p} & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}, \tag{3.16}$$

with $f_{n}(x)=\sum_{j=0}^{n}\frac{c_{j}}{x^{j}}$, *the nth partial sum of the asymptotic series* (3.7).

Remark 3.1 Using (3.16), we can also derive (3.9) and (3.11). Indeed, we have

$$[1/1]_f(x) = \frac{\begin{vmatrix} \frac{1}{x}f_0(x) & f_1(x) \\ c_1 & c_2 \end{vmatrix}}{\begin{vmatrix} \frac{1}{x} & 1 \\ c_1 & c_2 \end{vmatrix}} = \frac{\begin{vmatrix} \frac{1}{x} & 1-\frac{1}{2x} \\ -\frac{1}{2} & \frac{11}{24} \end{vmatrix}}{\begin{vmatrix} \frac{1}{x} & 1 \\ -\frac{1}{2} & \frac{11}{24} \end{vmatrix}} = \frac{x+\frac{5}{12}}{x+\frac{11}{12}}$$

and

$$[2/2]_f(x) = \frac{\begin{vmatrix} \frac{1}{x^2}f_0(x) & \frac{1}{x}f_1(x) & f_2(x) \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}}{\begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} & 1 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}} = \frac{\begin{vmatrix} \frac{1}{x^2} & \frac{1}{x}(1-\frac{1}{2x}) & 1-\frac{1}{2x}+\frac{11}{24x^2} \\ -\frac{1}{2} & \frac{11}{24} & -\frac{7}{16} \\ \frac{11}{24} & -\frac{7}{16} & \frac{2,447}{5,760} \end{vmatrix}}{\begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} & 1 \\ -\frac{1}{2} & \frac{11}{24} & -\frac{7}{16} \\ \frac{11}{24} & -\frac{7}{16} & \frac{2,447}{5,760} \end{vmatrix}} = \frac{x^2+\frac{87}{100}x+\frac{37}{240}}{x^2+\frac{137}{100}x+\frac{457}{1,200}}.$$

Remark 3.2 Setting $(p,q)=(k,k)$ in (3.15), we obtain (1.16).

Setting

$$(p,q)=(3,3) \quad \text{and} \quad (p,q)=(4,4),$$

respectively, we obtain by Theorem 3.1, as $x\to\infty$,

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^x = \frac{x^3+\frac{162,713}{121,212}x^2+\frac{13,927}{26,936}x+\frac{41,501}{786,240}}{x^3+\frac{223,319}{121,212}x^2+\frac{237,551}{242,424}x+\frac{3,950,767}{29,090,880}} + O\left(\frac{1}{x^7}\right) \tag{3.17}$$

and

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^x = \frac{x^4+\frac{1,157,406,727}{634,301,284}x^3+\frac{8,452,872,239}{7,611,615,408}x^2+\frac{81,587,251,465}{319,687,847,136}x+\frac{15,842,677}{924,376,320}}{x^4+\frac{1,474,557,369}{634,301,284}x^3+\frac{13,811,559,391}{7,611,615,408}x^2+\frac{170,870,679,559}{319,687,847,136}x+\frac{1,724,393,461,793}{38,362,541,656,320}} + O\left(\frac{1}{x^9}\right). \tag{3.18}$$

Equations (3.17) and (3.18) motivate us to establish the following theorem.

Theorem 3.2 *For* $x>0$,

$$\left(1+\frac{1}{x}\right)^x < e\left(\frac{x^3+\frac{162,713}{121,212}x^2+\frac{13,927}{26,936}x+\frac{41,501}{786,240}}{x^3+\frac{223,319}{121,212}x^2+\frac{237,551}{242,424}x+\frac{3,950,767}{29,090,880}}\right) \tag{3.19}$$

and

$$\left(1+\frac{1}{x}\right)^x < e\left(\frac{x^4+\frac{1,157,406,727}{634,301,284}x^3+\frac{8,452,872,239}{7,611,615,408}x^2+\frac{81,587,251,465}{319,687,847,136}x+\frac{15,842,677}{924,376,320}}{x^4+\frac{1,474,557,369}{634,301,284}x^3+\frac{13,811,559,391}{7,611,615,408}x^2+\frac{170,870,679,559}{319,687,847,136}x+\frac{1,724,393,461,793}{38,362,541,656,320}}\right). \tag{3.20}$$

Proof We only prove the inequality (3.20). The proof of (3.19) is analogous. In order to prove (3.20), it suffices to show that

$$F(x) < 0 \quad \text{for } x > 0,$$

where

$$F(x) = x\ln\left(1+\frac{1}{x}\right) - 1 - \ln\left(\frac{x^4+\frac{1,157,406,727}{634,301,284}x^3+\frac{8,452,872,239}{7,611,615,408}x^2+\frac{81,587,251,465}{319,687,847,136}x+\frac{15,842,677}{924,376,320}}{x^4+\frac{1,474,557,369}{634,301,284}x^3+\frac{13,811,559,391}{7,611,615,408}x^2+\frac{170,870,679,559}{319,687,847,136}x+\frac{1,724,393,461,793}{38,362,541,656,320}}\right).$$

Differentiation yields

$$F'(x) = \ln\left(1+\frac{1}{x}\right) - \frac{P_8(x)}{P_9(x)},$$

where

$$\begin{aligned} P_8(x) = {} & 4,534,960,145,139,175,220,907,601 + 89,156,435,404,854,709,617,164,400x \\ & + 753,611,422,427,554,143,580,166,880x^2 \\ & + 3,400,732,641,706,885,239,015,784,320x^3 \\ & + 8,959,898,009,119,992,740,647,591,680x^4 \\ & + 14,212,846,466,921,911,377,490,790,400x^5 \\ & + 13,355,464,865,044,929,241,744,281,600x^6 \\ & + 6,842,437,276,900,714,847,214,796,800x^7 \\ & + 1,471,684,602,332,887,248,995,942,400x^8 \end{aligned}$$

and

$$\begin{aligned} P_9(x) = {} & \bigl(38,362,541,656,320x^4 + 69,999,958,848,960x^3 + 42,602,476,084,560x^2 \\ & + 9,790,470,175,800x + 657,486,938,177\bigr)\bigl(38,362,541,656,320x^4 \\ & + 89,181,229,677,120x^3 + 69,610,259,330,640x^2 + 20,504,481,547,080x \\ & + 1,724,393,461,793\bigr)(x+1). \end{aligned}$$

Differentiating $F'(x)$, we find

$$F''(x) = -\frac{Q_8(x)}{Q_{19}(x)},$$

where

$$\begin{aligned} Q_8(x) = {} & 1{,}285{,}425{,}745{,}031{,}439{,}744{,}924{,}351{,}944{,}181{,}267{,}498{,}830{,}297{,}392{,}321 \\ & + 28{,}378{,}097{,}964{,}665{,}213{,}870{,}448{,}253{,}775{,}917{,}974{,}735{,}833{,}555{,}915{,}520x \\ & + 247{,}639{,}239{,}538{,}550{,}650{,}618{,}428{,}925{,}475{,}351{,}177{,}418{,}903{,}828{,}519{,}360x^2 \\ & + 1{,}131{,}116{,}309{,}072{,}948{,}249{,}686{,}419{,}776{,}599{,}013{,}563{,}965{,}352{,}036{,}853{,}760x^3 \\ & + 2{,}998{,}129{,}273{,}934{,}033{,}621{,}834{,}452{,}343{,}529{,}577{,}599{,}070{,}175{,}646{,}117{,}120x^4 \\ & + 4{,}775{,}194{,}702{,}079{,}256{,}668{,}486{,}950{,}292{,}217{,}012{,}539{,}098{,}845{,}384{,}867{,}840x^5 \\ & + 4{,}503{,}188{,}365{,}939{,}207{,}771{,}317{,}966{,}173{,}833{,}346{,}921{,}724{,}385{,}791{,}590{,}400x^6 \\ & + 2{,}315{,}562{,}242{,}935{,}704{,}170{,}341{,}114{,}308{,}201{,}588{,}127{,}064{,}283{,}807{,}744{,}000x^7 \\ & + 500{,}009{,}489{,}498{,}922{,}911{,}594{,}629{,}442{,}997{,}057{,}334{,}195{,}586{,}408{,}448{,}000x^8 \end{aligned}$$

and

$$\begin{aligned} Q_{19}(x) = {} & x\bigl(38{,}362{,}541{,}656{,}320x^4 + 69{,}999{,}958{,}848{,}960x^3 + 42{,}602{,}476{,}084{,}560x^2 \\ & + 9{,}790{,}470{,}175{,}800x + 657{,}486{,}938{,}177\bigr)^2\bigl(38{,}362{,}541{,}656{,}320x^4 \\ & + 89{,}181{,}229{,}677{,}120x^3 + 69{,}610{,}259{,}330{,}640x^2 + 20{,}504{,}481{,}547{,}080x \\ & + 1{,}724{,}393{,}461{,}793\bigr)^2(x+1)^2. \end{aligned}$$

Hence, $F''(x) < 0$ for $x > 0$, and we have

$$F'(x) > \lim_{t\to\infty} F'(t) = 0 \quad\Longrightarrow\quad F(x) < \lim_{t\to\infty} F(t) = 0 \quad \text{for } x > 0.$$

The proof is complete. □

The inequality (3.20) can be written as

$$\left(1+\frac{1}{x}\right)^x < e\bigl(1-\mathcal{E}(x)\bigr), \quad x > 0, \tag{3.21}$$

where

$$\begin{aligned} \mathcal{E}(x) = {} & 48\bigl(399{,}609{,}808{,}920x^3 + 562{,}662{,}150{,}960x^2 \\ & + 223{,}208{,}570{,}235x + 22{,}227{,}219{,}242\bigr)/\bigl(38{,}362{,}541{,}656{,}320x^4 \\ & + 89{,}181{,}229{,}677{,}120x^3 + 69{,}610{,}259{,}330{,}640x^2 + 20{,}504{,}481{,}547{,}080x \\ & + 1{,}724{,}393{,}461{,}793\bigr). \end{aligned} \tag{3.22}$$

4 A generalized Carleman-type inequality

Theorem 4.1 *Let* $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$ $(\Lambda_n \geq 1)$, $a_n \geq 0$ $(n \in \mathbb{N})$ *and* $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$. *Then, for* $0 < p \leq 1$,

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < \frac{e^p}{p} \sum_{n=1}^{\infty} \left(1 - \mathcal{E}\left(\frac{\Lambda_n}{\lambda_n}\right)\right)^p \lambda_n a_n^p \Lambda_n^{p-1} \left(\sum_{k=1}^{n} \lambda_k (c_k a_k)^p\right)^{(1-p)/p}, \tag{4.1}$$

where $\mathcal{E}(x)$ *is given in* (3.22) *and*

$$c_n^{\lambda_n} = \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}.$$

Proof The inequality

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} \leq \frac{1}{p} \sum_{m=1}^{\infty} \left(1 + \frac{1}{\Lambda_m/\lambda_m}\right)^{p\Lambda_m/\lambda_m} \lambda_m a_m^p \Lambda_m^{p-1} \left(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p\right)^{(1-p)/p} \tag{4.2}$$

has been proved in Theorem 2.2 of [9] (see also [11, p.96]). From the above inequality and (3.20), we obtain (4.1). The proof is complete. □

Remark 4.1 In Theorem 2.2 of [9], $c_k^{\lambda_n} = \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}$ should be $c_n^{\lambda_n} = \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}$; see [9, p.44, line 3]. Likewise, $c_s^{\lambda_n} = \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}$ in Theorem 3.1 of [11] should be $c_n^{\lambda_n} = \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}$; see [11, p.96, equation (9)].

Remark 4.2 Taking $p = 1$ in (4.1) yields

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left(1 - \mathcal{E}\left(\frac{\Lambda_n}{\lambda_n}\right)\right) \lambda_n a_n, \tag{4.3}$$

which improves (1.6). Taking $\lambda_n \equiv 1$ in (4.3) yields

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \mathcal{E}(n)\right) a_n, \tag{4.4}$$

which improves (1.9).

Acknowledgements
The authors thank the referees for helpful comments.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Publisher's Note

Received: 9 June 2017 Accepted: 4 August 2017 Published online: 04 September 2017

References

1. Carleman, T: Sur les fonctions quasi-analytiques. In: Comptes rendus du V^e Congres des Mathematiciens Scandinaves. Helsingfors, pp. 181-196 (1922)
2. Hardy, GH: Notes on some points in the integral calculus. Messenger Math. **54**, 150-156 (1925)
3. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities, 2nd edn. Cambridge University Press, Cambridge (1952)
4. Čižmešija, A, Pečarić, J, Persson, LE: On strengthened Carleman's inequality. Bull. Aust. Math. Soc. **68**, 481-490 (2003)
5. Chen, HW: On an infinite series for $(1 + 1/x)^x$ and its application. Int. J. Math. Math. Sci. **29**, 675-680 (2002)
6. Chen, CP: Generalization of weighted Carleman-type inequality. East J. Approx. **12**, 63-69 (2006)
7. Chen, CP, Qi, F: Generalization of Hardy's inequality. Proc. Jangjeon Math. Soc. **7**, 57-61 (2004)
8. Chen, CP, Cheung, WS, Qi, F: Note on weighted Carleman type inequality. Int. J. Math. Math. Sci. **3**, 475-481 (2005)
9. Dragomir, SS, Kim, YH: The strengthened Hardy inequalities and their new generalizations. Filomat **20**, 39-49 (2006)
10. Hu, Y: A strengthened Carleman's inequality. Commun. Math. Anal. **1**, 115-119 (2006)
11. Lü, Z, Gao, Y, Wei, Y: Note on the Carleman's inequality and Hardy's inequality. Comput. Math. Appl. **59**, 94-97 (2010)
12. Li, JL: Notes on an inequality involving the constant *e*. J. Math. Anal. Appl. **250**, 722-725 (2000)
13. Liu, HP, Zhu, L: New strengthened Carleman's inequality and Hardy's inequality. J. Inequal. Appl. **2007**, Article ID 84104 (2007). http://link.springer.com/article/10.1155/2007/84104/fulltext.html
14. Mortici, C, Hu, Y: On some convergences to the constant *e* and improvements of Carleman's inequality. Carpath. J. Math. **31**, 249-254 (2015)
15. Xie, Z, Zhong, Y: A best approximation for constant *e* and an improvement to Hardy's inequality. J. Math. Anal. Appl. **252**, 994-998 (2000)
16. Yan, P, Sun, GZ: A strengthened Carleman's inequality. J. Math. Anal. Appl. **240**, 290-293 (1999)
17. Yang, BC, Debnath, L: Some inequalities involving the constant *e*, and an application to Carleman's inequality. J. Math. Anal. Appl. **223**, 347-353 (1998)
18. Yang, BC: On Hardy's inequality. J. Math. Anal. Appl. **234**, 717-722 (1999)
19. Yang, XJ: On Carleman's inequality. J. Math. Anal. Appl. **253**, 691-694 (2001)
20. Yang, XJ: Approximations for constant *e* and their applications. J. Math. Anal. Appl. **262**, 651-659 (2001)
21. Johansson, M, Persson, LE, Wedestig, A: Carleman's inequality - history, proofs and some new generalizations. J. Inequal. Pure Appl. Math. **4**(3), Article ID 53 (2003). http://jipam.vu.edu.au/article.php?sid=291
22. Kaijser, S, Persson, LE, Öberg, A: On Carleman and Knopp's inequalities. J. Approx. Theory **117**, 140-151 (2002)
23. Kufner, A, Persson, LE: Weighted Inequalities of Hardy Type. World Scientific, Singapore (2003)
24. Pečarić, J, Stolarsky, KB: Carleman's inequality: history and new generalizations. Aequ. Math. **61**, 49-62 (2001)
25. Pólya, G: Proof of an inequality. Proc. Lond. Math. Soc. **24**, 57 (1926)
26. Wikipedia contributors: Partition (number theory), Wikipedia, the free encyclopedia. http://en.wikipedia.org/wiki/Partition_function_(number_theory)#Partition_function
27. Sloane, NJA: $a(n)$ = number of partitions of n (the partition numbers). Maintained by The OEIS Foundation. http://oeis.org/A000041
28. Chen, CP, Choi, J: Asymptotic formula for $(1 + 1/x)^x$ based on the partition function. Am. Math. Mon. **121**, 338-343 (2014)
29. Bercu, G: Padé approximant related to remarkable inequalities involving trigonometric functions. J. Inequal. Appl. **2016** 99 (2016). http://www.doc88.com/p-0037658479714.html
30. Bercu, G: The natural approach of trigonometric inequalities-Padé approximant. J. Math. Inequal. **11**, 181-191 (2017)
31. Bercu, G, Wu, S: Refinements of certain hyperbolic inequalities via the Padé approximation method. J. Nonlinear Sci. Appl. **9**, 5011-5020 (2016)
32. Brezinski, C, Redivo-Zaglia, M: New representations of Padé, Padé-type, and partial Padé approximants. J. Comput. Appl. Math. **284**, 69-77 (2015)
33. Li, X, Chen, CP: Padé approximant related to asymptotics for the gamma function. J. Inequal. Appl. **2017**, 53 (2017). http://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-017-1315-1
34. Liu, J, Chen, CP: Padé approximant related to inequalities for Gauss lemniscate functions. J. Inequal. Appl. **2016**, 320 (2016). http://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-016-1262-2