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Balls in generalizations of metric spaces

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Abstract

This paper discusses balls in partial b-metric spaces and cone metric spaces, respectively. Let (X, p_b) be a partial b-metric space in the sense of Mustafa *et al.* For the family $\triangle$ of all p_b-open balls in (X, p_b), this paper proves that there are $x, y \in B \in \triangle$ such that $B' \nsubseteq B$ for all $B' \in \triangle$, where B and B' are with centers x and y, respectively. This result shows that $\triangle$ is not a base of any topology on X, which shows that a proposition and a claim on partial b-metric spaces are not true. By some relations among $\ll$, $<$, and $\leq$ in cone metric spaces, this paper also constructs a cone metric space (X, d) and shows that $\overline{\{y \in X : d(x,y) \ll \varepsilon\}} \neq \{y \in X : d(x,y) \leq \varepsilon\}$ in general, which corrects an error on cone metric spaces. However, it must be emphasized that these corrections do not affect the rest of the results in the relevant papers.

MSC: 54A10; 54E35

Keywords: ball; partial b-metric space; cone metric space

1 Introduction

Partial b-metric spaces and cone metric spaces are important generalizations of metric spaces, which were introduced and investigated by Shukla in [1] and Huang-Zhang in [2], respectively.

Recently, Mustafa *et al.* introduced a new concept of partial b-metric by modifying partial b-metric in the sense of [1] in order to guarantee that each partial b-metric p_b can induce a b-metric ([3]). Furthermore, they proved the following proposition.

Proposition 1.1 ([3]) *Let (X, p_b) be a partial b-metric space (in the sense of [3]). For each $x \in X$ and $\varepsilon > 0$, the p_b-open ball with center x and radius ε is*

$$B_{p_b}(x, \varepsilon) = \{y \in X : p_b(x,y) < p_b(x,x) + \varepsilon\}.$$

Then for each $B_{p_b}(x, \varepsilon)$ and each $y \in B_{p_b}(x, \varepsilon)$, there is $\delta > 0$ such that $B_{p_b}(y, \delta) \subseteq B_{p_b}(x, \varepsilon)$.

Thus, from Proposition 1.1, the following claim arose naturally ([3]).

Claim 1.2 ([3]) *Let (X, p_b) be a partial b-metric space (in the sense of [3]). Put $\triangle = \{B_{p_b}(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$, i.e., $\triangle$ is the family of all p_b-open balls. Then $\triangle$ is a base of some topology on X.*

It is also worthy noting that Proposition 1.1 and Claim 1.2 were cited in [4].

For balls in a cone metric space (X, d), Turkoglu and Abuloha gave the following equality (see [5], Proposition 2), where $x \in X$ and $\varepsilon \gg \theta$:

Equality 1.3 $\overline{\{y \in X : d(x,y) \ll \varepsilon\}} = \{y \in X : d(x,y) \leq \varepsilon\}$.

In this paper, we discuss Proposition 1.1, Claim 1.2, and Equality 1.3. For Proposition 1.1 and Claim 1.2, we construct a partial b-metric space (X, p_b) in the sense of [3], and show that there are a p_b-open ball $B_{p_b}(x, \varepsilon)$ and $y \in B_{p_b}(x, \varepsilon)$ such that $B_{p_b}(y, \delta) \nsubseteq B_{p_b}(x, \varepsilon)$ for all $\delta > 0$, and hence $\triangle$ is not a base of any topology on X, which shows that Proposition 1.1 (including its proof) and Claim 1.2 are not true. For Equality 1.3, we establish some relations between balls and their closures in cone metric spaces by $\ll$, $<$, and $\leq$, and we give an example to show that Equality 1.3 is not true. However, it must be emphasized that these corrections do not affect the rest of the results in [3, 5].

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ denote the set of all natural numbers, the set of all real numbers and the set of all nonnegative real numbers, respectively. For a subset A of a space X, $\overline{F}$ denotes the closure of F in X. For undefined notations and terminology, one can refer to [3, 5].

2 Results and discussion

We give the main results of this paper by the following two subsections.

2.1 p_b-Open balls in partial b-metric spaces

The following partial b-metric spaces were introduced by Shukla in [1].

Definition 2.1 [1] Let X be a non-empty set. A mapping $p_b : X \times X \longrightarrow \mathbb{R}^+$ is called a partial b-metric with coefficient $s \geq 1$ and (X, p_b) is called a partial b-metric space with coefficient $s \geq 1$ if the following are satisfied for all $x, y, z \in X$:

(1) $x = y \Longleftrightarrow p_b(x,x) = p_b(y,y) = p_b(x,y)$.
(2) $p_b(x,y) = p_b(y,x)$.
(3) $p_b(x,x) \leq p_b(x,y)$.
(4) $p_b(x,y) \leq s(p_b(x,z) + p_b(z,y)) - p_b(z,z)$.

Remark 2.2 If $s = 1$ in Definition 2.1, then (X, p_b) is a partial metric space, which was introduced by Matthews (for example, see [3]). Further, put $d_{p_b} : X \times X \longrightarrow \mathbb{R}^+$ by $d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y)$ for all $x, y \in X$, then d_{p_b} is a metric on X and (X, d_p) is a metric space.

However, if $s > 1$, then we cannot guarantee that each partial b-metric can induce a b-metric by the method in Remark 2.2. So Mustafa *et al.* gave the following partial b-metric p_b by modifying Definition 2.1(4) and proved that the p_b induces a b-metric by the method in Remark 2.2.

Definition 2.3 ([3]) Let X be a non-empty set. A mapping $p_b : X \times X \longrightarrow \mathbb{R}^+$ is called a partial b-metric with coefficient $s \geq 1$ and (X, p_b) is called a partial b-metric space with coefficient $s \geq 1$ if the following are satisfied for all $x, y, z \in X$:

(1) $x = y \Longleftrightarrow p_b(x,x) = p_b(y,y) = p_b(x,y)$.
(2) $p_b(x,y) = p_b(y,x)$.

(3) $p_b(x,x) \leq p_b(x,y)$.
(4) $p_b(x,y) \leq s(p_b(x,z) + p_b(z,y) - p_b(z,z)) + \frac{1-s}{2}(p_b(x,x) + p_b(y,y))$.

Remark 2.4 If x, y, z satisfy Definition 2.3(1), (2), (3) and are different from each other, then it is easy to check that x, y, z Definition 2.3(4) holds.

As a known fact, Proposition 1.1 and Claim 1.2 are not true if (X,p_b) is a partial b-metric space in the sense of Definition 2.1 ([6]). So it is important to check whether Proposition 1.1 and Claim 1.2 are true if (X,p_b) is a partial b-metric space in the sense of Definition 2.3. The following example shows that the result of the check is negative, which comes from [6]. In the following, all partial b-metric spaces are in the sense of Definition 2.3.

Example 2.5 Let $X = \{u,v,w\}$ and put $p_b : X \times X \longrightarrow \mathbb{R}^+$ as follows:
(i) $p_b(u,u) = p_b(w,w) = 1$ and $p_b(v,v) = 0.5$.
(ii) $p_b(u,w) = p_b(w,u) = 1.5$.
(iii) $p_b(v,w) = p_b(w,v) = 1$.
(iv) $p_b(u,v) = p_b(v,u) = 3$.
Let $B_{p_b}(u,\varepsilon)$ be described in Proposition. Then the following hold:
(1) p_b is a partial b-metric with coefficient $s = 3$.
(2) $w \in B_{p_b}(u,1)$ and for any $\varepsilon > 0$, $B_{p_b}(w,\varepsilon) \not\subseteq B_{p_b}(u,1)$.

Proof (1) It is not difficult to check that p_b satisfies Definition 2.3(1), (2), (3). In order to check that p_b satisfies Definition 2.3(4), we only need to consider the following three cases by Remark 2.4.
(1) $x = u$, $y = v$, $z = w$:

$$p_b(u,v) = 3,$$
$$3\bigl(p_b(u,w) + p_b(w,v) - p_b(w,w)\bigr) + \frac{1-3}{2}\bigl(p_b(u,u) + p_b(v,v)\bigr) = 3.$$

So $p_b(u,v) \leq 3(p_b(u,w) + p_b(w,v) - p_b(w,w)) + \frac{1-3}{2}(p_b(u,u) + p_b(v,v))$.
(2) $x = u$, $y = w$, $z = v$:

$$p_b(u,w) = 1.5,$$
$$3\bigl(p_b(u,v) + p_b(v,w) - p_b(v,v)\bigr) + \frac{1-3}{2}\bigl(p_b(u,u) + p_b(w,w)\bigr) = 8.5.$$

So $p_b(u,w) \leq 3(p_b(u,v) + p_b(v,w) - p_b(v,v)) + \frac{1-3}{2}(p_b(u,u) + p_b(w,w))$.
(3) $x = v$, $y = w$, $z = u$:

$$p_b(v,w) = 1,$$
$$3\bigl(p_b(v,u) + p_b(u,w) - p_b(u,u)\bigr) + \frac{1-3}{2}\bigl(p_b(v,v) + p_b(w,w)\bigr) = 9.$$

So $p_b(v,w) \leq 3(p_b(v,u) + p_b(u,w) - p_b(u,u)) + \frac{1-3}{2}(p_b(v,v) + p_b(w,w))$.
Thus, p_b is a partial b-metric with coefficient $s = 3$.

(2) Since $p_b(u,w) = 1.5 < 1 + 1 = p_b(u,u) + 1$, $w \in B_{p_b}(u,1)$. In addition, for any $\varepsilon > 0$, $p_b(w,v) = 1 < 1 + \varepsilon = p_b(w,w) + \varepsilon$, so $v \in B_{p_b}(w,\varepsilon)$. On the other hand, $p_b(u,v) = 3 \not< 2 = 1 + 1 = p_b(u,u) + 1$, so $v \notin B_{p_b}(u,1)$. This shows that $B_{p_b}(w,\varepsilon) \not\subseteq B_{p_b}(u,1)$. □

Remark 2.6 Example 2.5 shows that Proposition 1.1 and Claim 1.2 are not true if (X,p_b) is a partial b-metric space.

However, we have the following.

Proposition 2.7 ([7]) *Let (X,p_b) be a partial b-metric space and $\triangle$ be described in Claim* 1.2. *Then $\triangle$ is a subbase for some topology on X. We denote the topology by $\mathcal{T}_{p_b}$.*

It is well known that the space $(X,\mathcal{T}_{p_b})$ is T_0 but does not need to be T_1 ([7]). The following proposition give a sufficient and necessary such that $(X,\mathcal{T}_{p_b})$ is a T_1-space.

Proposition 2.8 *Let (X,p_b) be a partial b-metric space in the sense of Definition* 2.3. *Then the following are equivalent*:

(1) *$(X,\mathcal{T}_{p_b})$ is a T_1-space.*

(2) *$p_b(x,y) > \max\{p_b(x,x),p_b(y,y)\}$ for each pair of distinct points $x,y \in X$.*

Proof (1) $\Longrightarrow$ (2): Let $(X,\mathcal{T}_{p_b})$ be a T_1-space. If $x,y \in X$ and $x \neq y$, then there is a neighborhood U of x such that $y \notin U$. Since $\triangle$ is a subbase of $(X,\mathcal{T}_{p_b})$ from Proposition 2.7, there are $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k > 0$ such that $y \notin \bigcap\{B_{p_b}(x,\varepsilon_i) : i = 1,2,\ldots,k\}$, and hence there is $i_0 \in \{1,2,\ldots,k\}$ such that $y \notin B_{p_b}(x,\varepsilon_{i_0})$. So $p_b(x,y) \geq p_b(x,x) + \varepsilon_{i_0} > p_b(x,x)$. In the same way, $p_b(x,y) > p_b(y,y)$. So $p_b(x,y) > \max\{p_b(x,x),p_b(y,y)\}$.

(2) $\Longrightarrow$ (1): Let $x,y \in X$ and $x \neq y$. If $p_b(x,y) > \max\{p_b(x,x),p_b(y,y)\}$. Then $p_b(x,y) > p_b(x,x)$. Put $\varepsilon = p_b(x,y) - p_b(x,x) > 0$, then $p_b(x,y) = p_b(x,x) + \varepsilon$, and so $y \notin B_{p_b}(X,\varepsilon)$. In the same way, there is $\varepsilon' > 0$ such that $x \notin B_{p_b}(y,\varepsilon')$. Consequently, $(X,\mathcal{T}_{p_b})$ is a T_1-space. □

2.2 Balls in cone metric spaces

Definition 2.9 ([2, 5]) Let E be a real Banach space. A subset P of E is called a cone of E and (E,P) is called a cone space if the following are satisfied: where θ is zero element in E.

(1) P is closed, $P \neq \emptyset$, and $P \neq \{\theta\}$.

(2) $a,b \in \mathbb{R}^+$ and $\alpha,\beta \in P \Longrightarrow a\alpha + b\beta \in P$.

(3) $\alpha,-\alpha \in P \Longrightarrow \alpha = \theta$.

Definition 2.10 ([2, 5]) Let (E,P) be a cone space. Some partial orderings $\leq$, $<$, and $\ll$ on E with respect to P are defined as follows, respectively, where P° denotes the interior of P. Let $\alpha,\beta \in E$.

(1) $\alpha \leq \beta$ if $\beta - \alpha \in P$.

(2) $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.

(3) $\alpha \ll \beta$ if $\beta - \alpha \in P^\circ$.

Remark 2.11 Let (E,P) be a cone space. For the sake of conveniences, we also use notations '$\geq$', '$>$', and '$\gg$'' on E with respect to P. The meanings of these notations are clear and the following hold:

(1) $\alpha \geq \theta$ if and only if $\alpha \in P$.

(2) $\alpha \gg \theta$ if and only if $\alpha \in P^\circ$.
(3) $\alpha - \beta \gg \theta$ if and only if $\alpha \gg \beta$.
(4) $\alpha - \beta \geq \theta$ if and only if $\alpha \geq \beta$.
(5) $\alpha \gg \beta \Longrightarrow \alpha > \beta \Longrightarrow \alpha \geq \beta$.

In addition, in order to guarantee the existence of elements $\varepsilon \gg \theta$, we always assume that the cone P has non-empty interior ([5]).

Definition 2.12 ([2, 5]) Let X be a non-empty set and let (E,P) be a cone space. A mapping $d : X \times X \longrightarrow E$ is called a cone metric on X, and (X,d) is called a cone metric space if the following are satisfied:
(1) $d(x,y) \geq \theta$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$.
(2) $d(x,y) = d(y,x)$ for all $x,y \in X$.
(3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Notation 2.13 Let (X,d) be a cone metric space, $x \in X$, and $\varepsilon \gg \theta$. In this section, we use the following notations for balls in (X,d):
(1) $B(x,\varepsilon) = \{y \in X : d(x,y) \ll \varepsilon\}$.
(2) $B_1(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$.
(3) $B_2(x,\varepsilon) = \{y \in X : d(x,y) \leq \varepsilon\}$.

Proposition 2.14 ([5]) *Let (X,d) be a cone metric space. Put $\mathscr{B} = \{B(x,\varepsilon) : x \in X$ and $\varepsilon \gg \theta\}$. Then $\mathscr{B}$ is a base for some topology $\mathcal{T}$ on X.*

In this section, just as the investigation in [5], we always suppose that each cone metric space is a topological space described in Proposition 2.14.

Proposition 2.15 *Let (X,d) be a cone metric space. For each $x \in X$ and each $\varepsilon \gg \theta$, the following hold*:
(1) $B(x,\varepsilon) \subseteq B_1(x,\varepsilon) \subseteq B_2(x,\varepsilon)$.
(2) $\overline{B(x,\varepsilon)} \subseteq \overline{B_1(x,\varepsilon)} \subseteq \overline{B_2(x,\varepsilon)}$.
(3) $\overline{B_2(x,\varepsilon)} = B_2(x,\varepsilon)$.

Proof (1) It holds by Remark 2.11(5).
(2) It holds by the above item (1).
(3) Let $y \in \overline{B_2(x,\varepsilon)}$. Then, whenever $\eta \gg \theta$, $B(y,\eta) \cap B_2(x,\varepsilon) \neq \emptyset$. Pick $z \in B(y,\eta) \cap B_2(x,\varepsilon)$. Then $d(x,z) \leq \varepsilon$ and $d(z,y) \ll \eta$. It follows that $d(x,y) \leq d(x,z) + d(z,y) \ll \varepsilon + \eta$. Let $\eta \to \theta$. Then $d(x,y) \leq \varepsilon$. So $y \in B_2(x,\varepsilon)$. This proves that $\overline{B_2(x,\varepsilon)} \subseteq B_2(x,\varepsilon)$. On the other hand, it is clear that $\overline{B_2(x,\varepsilon)} \supseteq B_2(x,\varepsilon)$. So $\overline{B_2(x,\varepsilon)} = B_2(x,\varepsilon)$. □

The following example shows that any '$\subseteq$' in Proposition 2.15(1), (2) cannot be replaced by '='.

Example 2.16 Let the cone space (E,P) be defined as in [5], Example 1, *i.e.*, $E = \mathbb{R}^2 = \{(r,s) : r,s \in \mathbb{R}\}$ is the Euclidean plane and $P = \{(r,s) \in E : r,s \geq 0\}$. Let $X = \{x,y,z\}$. Define $d : X \times X \longrightarrow E$ as follows: $d(x,x) = d(y,y) = d(z,z) = (0,0)$, $d(x,y) = d(y,x) = d(y,z) = d(z,y) = (1,1)$, and $d(x,z) = d(z,x) = (1,0)$. It is not difficult to check that (X,d) is a cone metric space. Let $\varepsilon = (1,1) \gg \theta$.

(1) Note that $d(x,y) = (1,1) = \varepsilon$. By Remark 2.11, $d(x,y) \leq \varepsilon$, $d(x,y) \nless \varepsilon$, and $d(x,y) \not\ll \varepsilon$. So $y \notin B(x,\varepsilon)$, $y \notin B_1(x,\varepsilon)$, and $y \in B_2(x,\varepsilon)$. Also, $\varepsilon - d(x,z) = (1,1) - (1,0) = (0,1) \in P - (\{\theta\} \cup P^\circ)$. By Remark 2.11, $\varepsilon - d(x,z) > \theta$, and $\varepsilon - d(x,z) \not\gg \theta$, hence $d(x,z) < \varepsilon$ and $d(x,z) \not\ll \varepsilon$. So $z \notin B(x,\varepsilon)$, $z \in B_1(x,\varepsilon)$, and $y \in B_2(x,\varepsilon)$. It follows that $B(x,\varepsilon) = \{x\}$, $B_1(x,\varepsilon) = \{x,z\}$, and $B_2(x,\varepsilon) = \{x,y,z\}$. So any '$\subseteq$' in Proposition 2.15(1) cannot be replaced by '='.

(2) Note that (X,d) is Hausdorff ([5]). In fact, each cone metric space is metrizable ([8]). So $\overline{B(x,\varepsilon)} = B(x,\varepsilon)$, $\overline{B_1(x,\varepsilon)} = B_1(x,\varepsilon)$, and $\overline{B_2(x,\varepsilon)} = B_2(x,\varepsilon)$. By the above item (1), any '$\subseteq$' in Proposition 2.15(2) cannot be replaced by '='.

Remark 2.17

(1) By Example 2.16, Equality 1.3 is not true. Indeed, in Example 2.16, $\overline{B(x,\varepsilon)} = \{x\}$ and $B_2(x,\varepsilon) = \{x,y,z\}$. So $\overline{B(x,\varepsilon)} \neq B_2(x,\varepsilon)$.

(2) Let (X,d) be a cone metric space. In [5], the authors showed that $\overline{B(x,\varepsilon)}$ and $B_2(x,\varepsilon)$ are sequentially closed in (X,d) ([5], Proposition 2). In fact, (X,d) is metrizable, and hence (X,d) is a sequential space, *i.e.*, closed and sequentially closed in (X,d) are equivalent. On the other hand, indeed, the closure $\overline{B(x,\varepsilon)}$ of $B(x,\varepsilon)$ is closed and $B_2(x,\varepsilon)$ is closed by Proposition 2.15(2).

3 Conclusions

This paper discusses balls in partial b-metric spaces and cone metric spaces, respectively.

Conclusion 3.1 Let (X,p_b) be a partial b-metric space in the sense of [3]. For the family $\triangle$ of all p_b-open balls in (X,p_b), this paper proves that there are $x,y \in B \in \triangle$ such that $B' \nsubseteq B$ for all $B' \in \triangle$, where B and B' are with centers x and y, respectively. This result shows that $\triangle$ is not a base of any topology on X, which shows that [3], Proposition 4 and the claim following [3], Proposition 4, are not true.

Conclusion 3.2 Let (X,d) be a cone metric space. By $\ll$, $<$, and $\leq$ in (X,d), this paper establishes some relations among $\{y \in X : d(x,y) \ll \varepsilon\}$, $\{y \in X : d(x,y) < \varepsilon\}$, and $\{y \in X : d(x,y) \leq \varepsilon\}$. Furthermore, this paper also constructs a cone metric space (X,d) such that $\overline{\{y \in X : d(x,y) \ll \varepsilon\}} \neq \{y \in X : d(x,y) \leq \varepsilon\}$ for some $x,y \in X$ and $\varepsilon \gg 0$, which shows that the equality in [5], Proposition 2, is not true.

However, it must be emphasized that these corrections in this paper do not affect the rest of the results in the relevant papers.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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